Introduction to Bayesian Methods

Machine Learning in Cognition Society

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Outline

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1. Essence of Bayesian methods

2. Model fitting

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3. Model comparison

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2. Model fitting
3. Model comparison
4. Extensions
5. Further reading

To begin

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Would I be successful at Bayesian modeling?

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Anybody can learn Bayesian modeling! The math might seem hard at first but after 10 to 50 hours of practice, depending on your background, it is more of the same.

With thanks to Wei Ji Ma

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- May re-describe things you know, but tailored to this context

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e.g. Bayesian model comparison, Bayesian decoding of neural activity (sensory stimulus, spatial position), Bayesian hierarchical modelling

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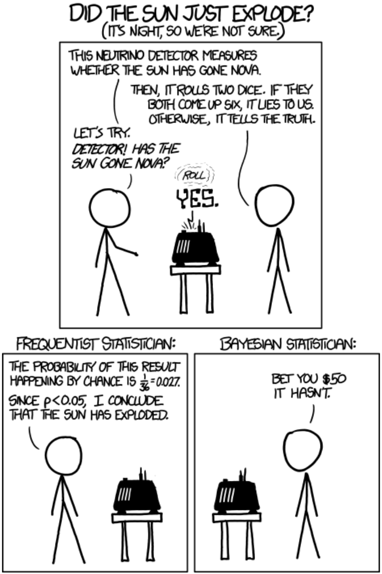

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Although use of priors tends to be the typical association/complaint with Bayesian statistics, this is often **not** so much what it is about!

Probability primer

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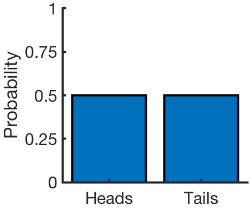
For a fair coin,

$$P(\mathit{flip} = \mathit{heads}) = P(\mathit{flip} = \mathit{tails}) = 0.5$$



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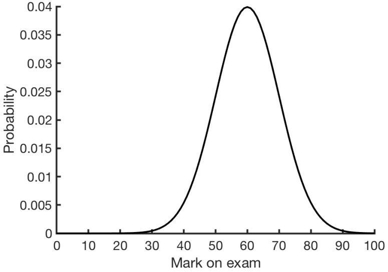

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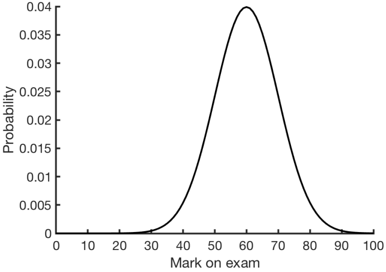
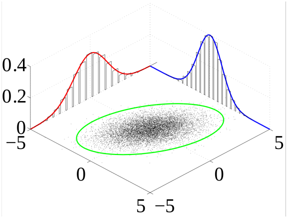


Chance of getting a First, $P(70 \leq \mathit{mark} \leq 100) \approx 0.16$

Probability primer

The *joint probability* of two outcomes is denoted $P(X = x_i, Y = y_i)$.



e.g. in a deck of cards, what is the probability that $P(\mathit{suit} = \mathit{hearts}, \mathit{number} = \mathit{even})$

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$$P(Y|X) = \frac{P(X, Y)}{P(X)}$$

Joint $P(X, Y)$ divided by marginal $P(X)$

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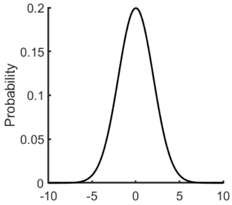

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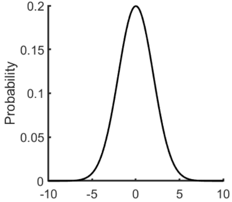

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FYI: Product of Gaussians is another Gaussian (add the exponents, rearrange)

Likelihoods as *hypotheses*

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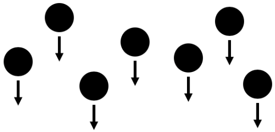

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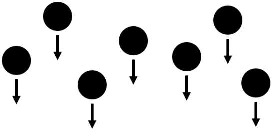

If you *observe* all seven dots moving downwards, and
$P(move = down) = P(move = up) = 0.5$,

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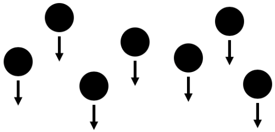

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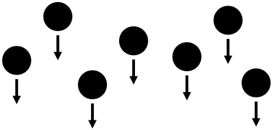

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What is the *likelihood* that (a) they have a common cause vs. (b) they are operating separately and *happen* to all move in the same direction?

Answer: (a) P(common cause) $= 0.5$ (b) P(separate) $= 0.5^7 \approx 0.008$

Bayes' rule:

Bayes' rule: Likelihood $\times$ prior $\propto$ Posterior

$$P(X|Y) = \frac{P(Y|X)P(X)}{P(Y)}$$

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Returning to the law of common fate

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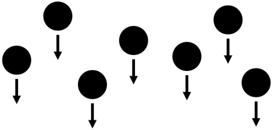

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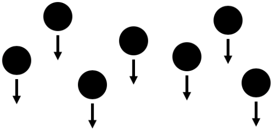

Say the *prior probability* of the common cause (a) is 1/4 (25%) and the dots being separate (b) is 3/4 (75%).

What is the *posterior probability* of (a) and (b)?

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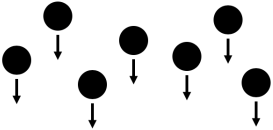

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Answer: since the *likelihood* of (b) was very small, (a) still wins.

$$(a) 0.5 \times 0.25 = 0.125 \qquad (b) \approx 0.008 \times 0.75 = 0.006$$

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As will often be the case, here, **most of the action is in the likelihood.**

This waves away objections about the prior (and therefore Bayesian methods) being "too subjective".

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Recall: product of Gaussians (here, product is the posterior) is another Gaussian.

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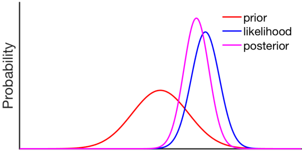

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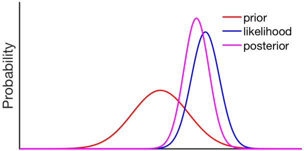

After rearranging, likelihood $\times$ prior is another Gaussian with:

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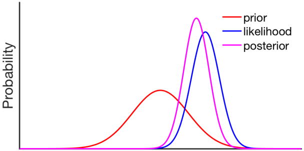

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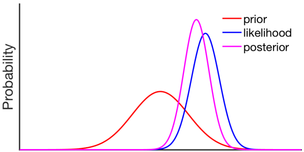

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Intuition: Prior "pulls posterior away" from the observations, depending on the relative narrowness (inverse variance, "certainty") of each.

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- Compare for different models.

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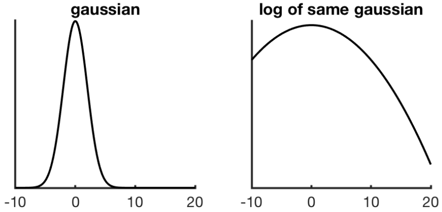

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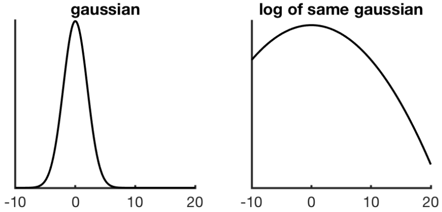

In fact, use the *negative* log likelihood (because standard packages usually *minimize* – "*convex* optimization").

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Maximum a posteriori (“MAP”) estimate

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Maximum a posteriori ("MAP") estimate
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If the prior is uniform (flat), multiplying it with the likelihood literally does nothing, i.e. reduces to MLE anyway.

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- Gradient descent
- Other standard optimization packages e.g. `fminunc` for MATLAB

Model evaluation: Using the model evidence

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Notice a big problem: Integrating over all parameter settings is difficult if you have many parameters!

Model comparison: metrics

How to determine *which model* best explains the data?

In practice: the more metrics the merrier!

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By eye: Meh...

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Using maximum likelihood:

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(log) likelihood ratio for nested models, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC)

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Penalizes for number of parameters k.

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Interesting fact: *implicitly* penalizes over-fitting (Bayesian "Occam's Razor")

Getting the full posterior

Analytic/closed-form solution

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If it's fair to model your likelihood function as a Gaussian, can pick a "conjugate prior" to make computing the posterior very easy.

Getting the full posterior
Approximations

If your function is not literally a Gaussian, just approximate it as one.

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Commonly, a *Laplace approximation* is used.
(It is a second-order Taylor series expansion: just means it will have the same first and second derivative, i.e. same slope and curvature)

Extensions

- **Hierarchical model-fitting**
  What if your participants are using different strategies? Group-level distributions over *models*. e.g. use expectation maximization to find.

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  Don't use a flat/uniform prior!
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  https://github.com/stan-dev/stan/wiki/Prior-Choice-Recommendations
- **Sampling-based methods**: Approximating the full posterior by *sampling*
  (look up: Markov Chain Monte Carlo (MCMC), Metropolis-Hastings, Gibbs sampling)

Model-fitting in practice

Model-fitting in practice

It is useful to simulate data with known parameters and try to *recover* those parameters. You may discover you *can't* recover parameters because they “trade-off” in unexpected ways.

For Further Reading

Specific articles

- Trial-by-trial data analysis using computational models (Daw, 2011)
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More resources

- Wei Ji Ma's course notes on Bayesian modeling http://www.cns.nyu.edu/malab/courses.html
- Zoubin Ghahramani's talks and tutorials http://mlg.eng.cam.ac.uk/zoubin
- Michael Betancourt's blog posts https://betanalpha.github.io/writing/
- Quentin Huys' teaching material https://quentinhuys.com/teaching.html
- Hanneke Den Ouden's tutorial on fitting reinforcement learning models https://hannekedenouden.ruhosting.nl/RLtutorial/Instructions.html

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Textbooks

- Pattern Recognition and Machine Learning (Bishop)
- Information Theory, Inference, and Learning Algorithms (MacKay)