Angle chasing (I)

Jongmin Lim

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Here are some tricks you should know, which we will prove with angle chasing.

1 Incentre/Excentre Lemma

Let ABC be three points on a circle Γ. Let M be the midpoint of the arc BC not containing A. Let I, J be on line AM such that $MI = MJ = MB = MC$. Then I, J are the incentre and the excentre of ABC.

2 Dilation tricks

Let Γ_1 and Γ_2 be two circles internally tangent at T. (Γ_1 is smaller)

1. Let X, Y be on Γ_1, and let TX and TY meet Γ_2 at P, Q. Then $XY \parallel PQ$.
2. The tangent at X on Γ_1 and the tangent at P on Γ_2 are parallel.
3. Let the tangent at X on Γ_1 meet Γ_2 at A, B. Then TX is an angle bisector of $\triangle ATB$.

3 Spiral similarity

Let Γ_1 and Γ_2 be two circles which meet at two distinct points A, B.

1. Let $P \in \Gamma_1$. Let AP meet Γ_2 at Q. If the radius of Γ_1 and Γ_2 are $15, 20$ respectively and $AB = 24$. What is the longest that PQ can be?
2. Let $P' \in \Gamma_1$. Let AP' meet Γ_2 at Q'. Show that $\triangle BPP' \sim BQQ'$.
3. Let ℓ_1, ℓ_2 be two lines in the plane meeting at point X. Choose two points $A, B \in \ell_1$ and $C, D \in \ell_2$ ($X \notin \{A, B, C, D\}$). Can you construct a point P such that $\triangle PAB \sim PCD$?
4. (Miquel point) Let A, B, C, D be a convex quadrilateral such that $AB \cap CD = P$, $AD \cap BC = Q$. Then the circumcircles of PAB, PCD, QAD, QBC all meet at one point M.
5. (Movie theorem) Let $\triangle PA_1B_1$ and $\triangle PA_2B_2$ be sprially similar. Let $A \in A_1A_2$ and $B \in B_1B_2$ such that $A_1A : AA_2 = B_1B : BB_2$. Then $\triangle PAB$ is also spirally similar to the other two triangles.

4 Nine point circle

Let AD, BE, CF be altitudes of the triangle concurrent at orthocentre H. O is the circumcentre of $\triangle ABC$.

1. Let X be the intersection of AD with the circumcircle again. Then $HD = DX$.
2. Let M be the midpoint of BC. Let HM intersect the circle again at Y such that Y and A are on the opposite sides of BC. Then $HM = MY$.
3. Let K be the midpoint of AH. The circle KDM has a radius half of the circle ABC.
4. If N is the centre of circle KDM, then N is the midpoint of OH.
5. AY is a diameter. $2OM = AH$. Hence AM meets OH at the centroid G.
6. Hence the four points O, G, N, H are on the *Euler line*, with a $OG : GN : NH = 2 : 1 : 3$.
7. H is the incentre of $\triangle DEF$. K is the midpoint of the arc of EF in circle DEF and A is the excentre.
8. $AFHE$ is cyclic with K as the centre.
9. Let Z be the intersection of HM with the circle such that Z and A are on the same side as BC. Then Z is also on the circle AFE.
10. MF, ME are tangents to the circumcircle of AFE.
11. KF, KE are tangents to the circumcircle of $BCEF$.
12. Notice that $\triangle ABC \sim \triangle AEF$ in an opposite orientation. Thus, if we let M' be the midpoint of EF, then AM' and AM are reflections over the angle bisector of $\angle BAC$. Thus, AM is the symmedian of $\triangle AFE$ (which proves that the A-symmedian and the tangents to the circumcircle from B and C are concurrent).
13. Z is the Miquel point of $BCFE$. From here, refer to the *big diagram*.

5 Exercises

1. This guy has great handouts `https://yufeizhao.com/olympiad/`. I highly recommend the geometry handouts. Start with *Cyclic quadrilaterals - The big picture*. If you do all of the geometry handouts, you'll be able to take an any camp question ezpz.
2. (Sawayama) Let ABC be points on a circle Γ. Let $D \in BC$. Let α be a circle that is tangent to circle Γ at T, tangent to line BC at P, and tangent to line AD at Q. Then the incentre of ABC is on the line PQ.
3. (Christmas tree) Let $ABCD$ be cyclic, and CD is the diameter. Let $BC \cap AD = P$. Let $AC \cap BD = Q$. Let the tangents to the circle at A, B meet at R. Show that PQR is collinear, perpendicular to CD, and $PR = QR$.
4. Let P, Q be on sides AB and AC of $\triangle ABC$ such that $PQCB$ are cyclic with centre O. Let circle ABC and APQ meet at point Z. Show that $AZ \perp ZO$.