

Dilation

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Definition. Given a point H and a factor $k \neq 0$, a dilation is a transformation that takes point A to point A' such that

1. H, A, A' are collinear
2. $\overrightarrow{HA} = k \cdot \overrightarrow{HA'}$

We denote such dilation with $\mathcal{H}(H, k)$.

1 Dilation in general

1. Let $\mathcal{H}(H, k)$ be a dilation that takes A, B, C to A', B', C' respectively.
 - (a) Show that $AB \parallel A'B'$. Show that $A'B' = kAB$.
 - (b) Show that $\triangle ABC \sim \triangle A'B'C'$.
2. Let $\triangle ABC$ and $\triangle A'B'C'$ be similar triangles such that $AB \parallel A'B'$, $AC \parallel A'C'$, $BC \parallel B'C'$. Show that AA' , BB' , CC' meet at one point.
3. Let $ABCD$ be a trapezoid with $AB \parallel CD$ and let $E = AC \cap BD$. Let $\triangle ABF$, $\triangle CDG$ be equilateral triangles facing away from the trapezoid. Show that E, F, G collinear.
4. (ToT1984) Let $ABCD$ be a square and let P be a point inside the square. Show that the centroids of $\triangle ABP$, $\triangle BPC$, $\triangle CPD$, $\triangle DPA$ form a square.
5. Let $D \in BC$ of $\triangle ABC$. Let $E \in BC$ such that $BD = CE$. Let M be the midpoint of AD . Show that ME passes through a fixed point as point D varies along side BC .

2 Dilation of circles

Every pair of circles are similar. There exists two dilations, one with a positive factor and one with a negative factor. You can also think of circles as a polygon with infinitely many sides. **Thus, a pair of corresponding points under dilaton on a pair of circles must have parallel tangents to their respective circles**

1. Let α, β be two circles that are internally tangent at T . Let $A \in \alpha$ and $B \in \beta$ such that T, A, B are collinear. Let the tangent to α at A meet β at X, Y . Show that TB bisects $\angle XTY$.
2. Let $\triangle ABC$ have its incircle touch the sides BC, CA, AB at D, E, F . Let P, Q, R be the midpoint of the arcs BC, CA, AB of its circumcircle. Show that DP, EQ, FR concurrent.
3. Let $\triangle ABC$ have the incircle touch side BC at D . Let the A -excircle touch side BC at E . Let AE intersect the incircle at F . Show that $FD \perp BC$.
4. Let α, β be two circles externally tangent at T . Let γ be a circle which is internally tangent to α, β at A, B respectively. Let AT intersect γ at P . Show that $\angle TBP = 90^\circ$.
5. Let α, β be externally tangent at T and internally tangent to a bigger circle γ at points P, Q respectively. Let $PT \cap \gamma = X$. Let $QT \cap \gamma = Y$. Show that the common internal tangent at T is perpendicular to XY .
6. Let γ be a circle tangent to a line ℓ . Let M be a point on ℓ . Find all points P such that the tangents from P to the circle meet ℓ at points Q, R such that the midpoint of QR is M .
7. Let $AB + BC = 3AC$ in $\triangle ABC$ with incentre I . Let the incentre touch AB, BC at D, E . Let DK, EL be diametres of the incircle. Show that $ACKL$ is cyclic.
8. Let α, β be two disjoint circles with centres P, Q . Let consider a common external tangent that is tangent at points $A \in \alpha, B \in \beta$. Consider the circle with AB as a diameter, and let this meet α, β at X, Y respectively. Show that AY, BX , and PQ are concurrent.
9. Let the incircle of $\triangle ABC$ meet side BC at D and the A -excircle of $\triangle ABC$ meet side BC at E . Let the incentre be I and the excentre be J . Let the altitude from A to BC be H .
 - (a) Show that DJ meet on the midpoint of AH .
 - (b) Let X be the intersection of DJ and the incircle. Let N be the midpoint of XD . Show that $ND \times DJ = BD \times DC$.
 - (c) Hence or otherwise show that $XBKC$ is cyclic, where K is the midpoint of DJ .
 - (d) Hence show that the incircle and $XBKC$ must be tangent at X .

3 Composition of dilations

1. Let $\mathcal{H}_1(X, k_1)$ and $\mathcal{H}_2(Y, k_2)$ be two dilations. Show that their composition is some dilation $\mathcal{H}(Z, k)$ where X, Y, Z are collinear and $k_1 k_2 k > 0$. This means, if you do the first dilation, then do the second dilation, it's the same thing as doing one dilation.

2. (Monge's theorem) Let α, β, γ be three disjoint circles outside of each other. Let the external common tangents of α, β meet at X , and those of β, γ meet at Y , and those of γ, α meet at Z . Show that α, β, γ are collinear.
3. Let M, N, P be the midpoints of sides BC, CA, AB of $\triangle ABC$. Let J, K, L be the incentres of $\triangle APN, BMP, CNM$.
 - (a) Show that $\triangle JKL \sim \triangle ABC$.
 - (b) Prove that JM, KN, LP, IG are concurrent, where I is the incentre, G is the centroid.
4. Consider three circles $\omega_1, \omega_2, \omega_3$ outside of each other. Let a circle γ be externally tangent to them at A_1, A_2, A_3 respectively. Let another circle Γ be internally tangent to them at B_1, B_2, B_3 respectively. Show that A_1B_1, A_2B_2, A_3B_3 are concurrent.
5. (a) Let D, E, F be the incentre touch points of sides BC, CA, AB . Let X, Y, Z be the midpoint of the arcs of BC, CA, AB of circumcircle of $\triangle ABC$. Show that DX, EY, FZ are concurrent at point H .
 - (b) Let ω be a circle which is internally tangent to the circumcircle of $\triangle ABC$ at T and tangent to sides AB, AC . Show that A, H, T are collinear.
6. (a) Let β, γ be internally tangent to α at X, Y , and externally tangent to each other at Z . Let the common tangent at Z meet α at a point P . Let PX, PY meet β, γ at A, B respectively. Show that AB is tangent to β and γ .
 - (b) Let XZ meet α again at K . Show that $KY \perp YZ$.
7. Let $\triangle ABC$ have incircle ω , A -excircle Γ . Recall that the nine point circle α is tangent to these two circles. Let $\alpha \cap \omega = X$, $\alpha \cap \Gamma = Y$. Let $XY \cap BC = D$. Show that AD is the angle bisector of $\angle BAC$.
8. (IMO 2008 paraphrased) Let a convex quadrilateral $ABCD$ ($BA \neq BC$) have an escribed circle γ that is tangent to ray BA beyond A , ray BC beyond C , and lines AD and CD . Let the incircle of $\triangle ABC$ be ω_1 and let the incircle of $\triangle ACD$ be ω_2 . Show that the positive dilation centre of ω_1 and ω_2 lies on γ .