Spiral Similarity

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1 Spiral similarity

A spiral similarity $\xi = \phi \circ \psi$ is a transformation which is a composition of a dilation ϕ and a rotation ψ from the same focal point. We call the focal point the spiral centre. Two triangles $\triangle ABC$ and $\triangle DEF$ are spirally similar if $A = \phi(\psi(D)), B = \phi(\psi(E))$, and $C = \phi(\psi(F))$. In this case, it's easy to see $\triangle ABC \sim \triangle DEF$.

1. Let $\triangle ABC \sim \triangle AXY$ with spiral centre at A. Show that $\triangle ABX \sim \triangle ACY$.
2. Let $\triangle ABC \sim \triangle AXY$ be spirally similar. Let $BC \cap XY = Z$. Show that $ABZX$ is cyclic. Show that $ACZY$ is cyclic.
3. Let $\triangle ABC \sim \triangle AXY$ be spirally similar. Let M bet the midpoint of BX, and N be the midpoint of CY. Show that $\triangle AMN \sim \triangle ABC$.
4. Given two line segments BC and XY, how do you construct point A such that $\triangle ABC \sim \triangle AXY$ spirally similar?

2 Examples

2.1 Example 1

(IMO 2006 shortlist) Consider a pentagon $ABCDE$ such that $\angle BAC = \angle CAD = \angle DAE$ and $\angle CBA = \angle DCA = \angle EDA$. Let $P = BD \cap CE$.

1. Do you see any spirally similar triangles?
2. What circles can you get from these similarities?
3. Show that AP bisects CD.

2.2 Example 2

(Miquel point) Let $ABCD$ be a convex quadrilateral. Let $AB \cap CD = X$, $AD \cap BC = Y$.

1. There's lots of circles here. Can you find some spiral similarities?
2. Can we get more spiral similarities for free?

3. What circles do we get from those spiral similarities?
4. Show that the circumcircles of $\triangle ABY, \triangle CDY, \triangle ADX, \triangle BCX$ all pass through a common point.

2.3 Example 3

Consider two squares $ABCD$ and $EDFG$. Let the midpoint of AF, BG, CE be P, Q, R.

1. Where is the spiral centre?
2. How do we describe a "movie" which involves $\square PQRD$?
3. Show that $\square PQRD$ is also a square.

2.4 Example 4

Consider two squares $ABCD$ and $EDFG$. Let the midpoint of AE, BD, CF, DG be P, Q, R, S.

1. Where is the spiral centre?
2. How do we describe a "movie" which involves $\square PQRS$?
3. Show that $\square PQRS$ is also a square.

3 Exercise

1. Let P be a point inside ABC such that $\triangle ABP \sim \triangle CAP$. Let AP intersect the circumcircle of $\triangle ABC$ again at X. Show that $AP = PX$.
2. Let $\triangle ABC$ and $\triangle CDE$ be isosceles triangles with $\angle ABC = \angle CDE = 90°$. Let the midpoint of AE be M. Show that BMD is also an isosceles right angle triangle.
3. (Napoleon's Theorem). Let ABC be a triangle. Construct equilateral triangles using each side, such that the equilateral triangles are outside the triangle, and let the centres of those equilateral triangles be P, Q, R. Show that PQR is also an equilateral triangle.
4. Let $ABCD$ be a quadrilateral with $AB = CD$. Let M, N be the midpoint of AD, BC respectively. Let $AB \cap MN = X$, $MN \cap CD = Y$. Show that $\angle AXM = \angle MYD$.
5. Let $ABCD$ be a quadrilateral where $\angle ABD = \angle BCD = 90°$. Also assume BC is tangent to the circumcircle of $\triangle ABD$. Let M be the midpoint of AB, and let N be the midpoint of BC. Show that $\angle MND = 90°$.
6. A variable point X lies on a semicircle ω with diameter AB. Let Y be a point on ray XA such that $XY = XB$. What is the locus of Y?
7. Let $ABCD$ be cyclic. Let $AC \cap BD = P$. Let PE, PF be altitudes to AB, CD. Let K, L be midpoints of BC, DA. Show that $KL \perp EF$.

8. (USAMO 2006) Let $ABCD$ be a quadrilateral with AD not parallel to BC. Let $E \in AD$, $F \in BC$ such that $AE/ED = BF/FC$. Let $FE \cap AB = S$, $FE \cap CD = T$. Show that the four circumcircles SAE, SBF, TCF, TDE all pass through a common point.

9. (Ptolemy's inequality) Let $ABCD$ be a quadrilateral. Show that

$$AB \times CD + AD \times BC \geq AC \times BD$$

10. Let $ABCD$ be a quadrilateral. Construct four squares using each side, such that the square is outside the quadrilateral, and let the centres of those squares be P, Q, R, S in clockwise order. Show that $PR \perp QS$ and $PR = QS$.

11. (APMO 1998) Let ABC be a triangle and $D \in BC$ such that $AD \perp BC$. Let E, D, F be collinear such that $AE \perp BE$, and $AF \perp CF$. Let M be the midpoint of BC, and N be the midpoint of EF. Show that $AN \perp NM$.

12. (IMO 2006 ISL G9) Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2 respectively ($A_2 \neq A, B_2 \neq B, C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

13. (IMO 2025) Let Ω and Γ be circles with centres M and N, respectively, such that the radius of Ω is less than the radius of Γ. Suppose Ω and Γ intersect at two distinct points A and B. Line MN intersects Ω at C and Γ at D, so that C, M, N, D lie on MN in that order. Let P be the circumcentre of triangle ACD. Line AP meets Ω again at $E \neq A$ and meets Γ again at $F \neq A$. Let H be the orthocentre of triangle PMN. Prove that the line through H parallel to AP is tangent to the circumcircle of triangle BEF.

14. (IMO 1985) A circle with center O passes through the vertices A and C of the triangle ABC and intersects the segments AB and BC again at distinct points K and N respectively. Let M be the point of intersection of the circumcircles of triangles ABC and KBN (apart from B). Prove that $\angle OMB = 90°$.

15. (IMO 2014) Points P and Q lie on side BC of acute-angled $\triangle ABC$ so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on lines AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that lines BM and CN intersect on the circumcircle of $\triangle ABC$.

16. (JL) Let $\triangle ABC$ be an acute triangle with $AB < AC$. Let the angle bisector of $\angle BAC$ meet BC and the circumcircle at D and M. Let Y be the reflection of D over C. Let $J \neq C$ be the intersection of the circumcircle of $\triangle ACY$ and CM. Let BM and JY meet at X. Show that $CX \perp BC$.