

# Dilation

Jongmin Lim

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**Definition.** Given a point  $H$  and a factor  $k \neq 0$ , a dilation is a transformation that takes point  $A$  to point  $A'$  such that

1.  $H, A, A'$  are collinear
2.  $\overrightarrow{HA} = k \cdot \overrightarrow{HA'}$

We denote such dilation with  $\mathcal{H}(H, k)$ .

## 1 Dilation in general

1. Let  $\mathcal{H}(H, k)$  be a dilation that takes  $A, B, C$  to  $A', B', C'$  respectively.
  - (a) Show that  $AB \parallel A'B'$ . Show that  $A'B' = kAB$ .
  - (b) Show that  $\triangle ABC \sim \triangle A'B'C'$ .
2. Let  $\triangle ABC$  and  $\triangle A'B'C'$  be similar triangles such that  $AB \parallel A'B'$ ,  $AC \parallel A'C'$ ,  $BC \parallel B'C'$ . Show that  $AA'$ ,  $BB'$ ,  $CC'$  meet at one point.
3. Let  $ABCD$  be a trapezoid with  $AB \parallel CD$  and let  $E = AC \cap BD$ . Let  $\triangle ABF$ ,  $\triangle CDG$  be equilateral triangles facing away from the trapezoid. Show that  $E, F, G$  collinear.
4. (ToT1984) Let  $ABCD$  be a square and let  $P$  be a point inside the square. Show that the centroids of  $\triangle ABP$ ,  $\triangle BPC$ ,  $\triangle CPD$ ,  $\triangle DPA$  form a square.
5. Let  $D \in BC$  of  $\triangle ABC$ . Let  $E \in BC$  such that  $BD = CE$ . Let  $M$  be the midpoint of  $AD$ . Show that  $ME$  passes through a fixed point as point  $D$  varies along side  $BC$ .

## 2 Dilation of circles

Every pair of circles are similar. There exists two dilations, one with a positive factor and one with a negative factor. You can also think of circles as a polygon with infinitely many sides. **Thus, a pair of corresponding points under dilaton on a pair of circles must have parallel tangents to their respective circles**

1. Let  $\alpha, \beta$  be two circles that are internally tangent at  $T$ . Let  $A \in \alpha$  and  $B \in \beta$  such that  $T, A, B$  are collinear. Let the tangent to  $\alpha$  at  $A$  meet  $\beta$  at  $X, Y$ . Show that  $TB$  bisects  $\angle XTY$ .
2. Let  $\triangle ABC$  have its incircle touch the sides  $BC, CA, AB$  at  $D, E, F$ . Let  $P, Q, R$  be the midpoint of the arcs  $BC, CA, AB$  of its circumcircle. Show that  $DP, EQ, FR$  concurrent.
3. Let  $\triangle ABC$  have the incircle touch side  $BC$  at  $D$ . Let the  $A$ -excircle touch side  $BC$  at  $E$ . Let  $AE$  intersect the incircle at  $F$ . Show that  $FD \perp BC$ .
4. Let  $\alpha, \beta$  be two circles externally tangent at  $T$ . Let  $\gamma$  be a circle which is internally tangent to  $\alpha, \beta$  at  $A, B$  respectively. Let  $AT$  intersect  $\gamma$  at  $P$ . Show that  $\angle TBP = 90^\circ$ .
5. Let  $\alpha, \beta$  be externally tangent at  $T$  and internally tangent to a bigger circle  $\gamma$  at points  $P, Q$  respectively. Let  $PT \cap \gamma = X$ . Let  $QT \cap \gamma = Y$ . Show that the common internal tangent at  $T$  is perpendicular to  $XY$ .
6. Let  $\gamma$  be a circle tangent to a line  $\ell$ . Let  $M$  be a point on  $\ell$ . Find all points  $P$  such that the tangents from  $P$  to the circle meet  $\ell$  at points  $Q, R$  such that the midpoint of  $QR$  is  $M$ .
7. Let  $AB + BC = 3AC$  in  $\triangle ABC$  with incentre  $I$ . Let the incentre touch  $AB, BC$  at  $D, E$ . Let  $DK, EL$  be diameters of the incircle. Show that  $ACKL$  is cyclic.
8. Let  $\alpha, \beta$  be two disjoint circles with centres  $P, Q$ . Let consider a common external tangent that is tangent at points  $A \in \alpha, B \in \beta$ . Consider the circle with  $AB$  as a diameter, and let this meet  $\alpha, \beta$  at  $X, Y$  respectively. Show that  $AY, BX$ , and  $PQ$  are concurrent.
9. Let the incircle of  $\triangle ABC$  meet side  $BC$  at  $D$  and the  $A$ -excircle of  $\triangle ABC$  meet side  $BC$  at  $E$ . Let the incentre be  $I$  and the excentre be  $J$ . Let the altitude from  $A$  to  $BC$  be  $H$ .
  - (a) Show that  $DJ$  meet on the midpoint of  $AH$ .
  - (b) Let  $X$  be the intersection of  $DJ$  and the incircle. Let  $N$  be the midpoint of  $XD$ . Show that  $ND \times DJ = BD \times DC$ .
  - (c) Hence or otherwise show that  $XBKC$  is cyclic, where  $K$  is the midpoint of  $DJ$ .
  - (d) Hence show that the incircle and  $XBKC$  must be tangent at  $X$ .

### 3 Composition of dilations

1. Let  $\mathcal{H}_1(X, k_1)$  and  $\mathcal{H}_2(Y, k_2)$  be two dilations. Show that their composition is some dilation  $\mathcal{H}(Z, k)$  where  $X, Y, Z$  are collinear and  $k_1 k_2 k > 0$ . This means, if you do the first dilation, then do the second dilation, it's the same thing as doing one dilation.

2. (Monge's theorem) Let  $\alpha, \beta, \gamma$  be three disjoint circles outside of each other. Let the external common tangents of  $\alpha, \beta$  meet at  $X$ , and those of  $\beta, \gamma$  meet at  $Y$ , and those of  $\gamma, \alpha$  meet at  $Z$ . Show that  $\alpha, \beta, \gamma$  are collinear.
3. Let  $M, N, P$  be the midpoints of sides  $BC, CA, AB$  of  $\triangle ABC$ . Let  $J, K, L$  be the incentres of  $\triangle APN, BMP, CNM$ .
  - (a) Show that  $\triangle JKL \sim \triangle ABC$ .
  - (b) Prove that  $JM, KN, LP, IG$  are concurrent, where  $I$  is the incentre,  $G$  is the centroid.
4. Consider three circles  $\omega_1, \omega_2, \omega_3$  outside of each other. Let a circle  $\gamma$  be externally tangent to them at  $A_1, A_2, A_3$  respectively. Let another circle  $\Gamma$  be internally tangent to them at  $B_1, B_2, B_3$  respectively. Show that  $A_1B_1, A_2B_2, A_3B_3$  are concurrent.
5. (a) Let  $D, E, F$  be the incentre touch points of sides  $BC, CA, AB$ . Let  $X, Y, Z$  be the midpoint of the arcs of  $BC, CA, AB$  of circumcircle of  $\triangle ABC$ . Show that  $DX, EY, FZ$  are concurrent at point  $H$ .
  - (b) Let  $\omega$  be a circle which is internally tangent to the circumcircle of  $\triangle ABC$  at  $T$  and tangent to sides  $AB, AC$ . Show that  $A, H, T$  are collinear.
6. (a) Let  $\beta, \gamma$  be internally tangent to  $\alpha$  at  $X, Y$ , and externally tangent to each other at  $Z$ . Let the common tangent at  $Z$  meet  $\alpha$  at a point  $P$ . Let  $PX, PY$  meet  $\beta, \gamma$  at  $A, B$  respectively. Show that  $AB$  is tangent to  $\beta$  and  $\gamma$ .
  - (b) Let  $XZ$  meet  $\alpha$  again at  $K$ . Show that  $KY \perp YZ$ .
7. Let  $\triangle ABC$  have incircle  $\omega$ ,  $A$ -excircle  $\Gamma$ . Recall that the nine point circle  $\alpha$  is tangent to these two circles. Let  $\alpha \cap \omega = X$ ,  $\alpha \cap \Gamma = Y$ . Let  $XY \cap BC = D$ . Show that  $AD$  is the angle bisector of  $\angle BAC$ .