

# Ninepoint circle

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## 1 Centroid

On  $\triangle ABC$ , let  $D, E, F$  be the midpoints of  $BC, CA, AB$  respectively.

1. Show that  $BC \parallel EF$ .
2. Show that  $\triangle ABC \sim \triangle DEF$ .
3. Show that  $AD, BE, CF$  meet at one point, say,  $G$ .
4. Show that  $AG/GD = BG/GE = CG/GF = 2$ .

## 2 Orthocentre

On  $\triangle ABC$ , let  $AD, BE, CF$  be the altitudes.

1. Show that  $BCEF$  is cyclic. Where is the centre of this circle?
2. Show that  $\triangle ABC \sim \triangle AEF$ .
3. Show that  $AD, BE, CF$  meet at one point, say,  $G$ .
4. Let  $M$  be the midpoint of  $BC$ . Show that  $ME = MF$ . Show that these are tangents to the circumcircle of  $\triangle AEF$ .

## 3 Nine point circle

Let  $AD, BE, CF$  be altitudes of the triangle concurrent at orthocentre  $H$ .  $O$  is the circumcentre of  $\triangle ABC$ . Let  $M$  be the midpoint of  $BC$ . Let  $L$  be the midpoint of  $AH$ .

1. Show that  $M, L$  lies on the circumcircle of  $\triangle DEF$ .
2. Let  $X$  be the reflection of  $H$  over  $BC$ . Show that  $X$  is on the circumcircle of  $\triangle ABC$ .
3. Let  $Y$  be the reflection of  $H$  over  $M$ . Show that  $Y$  is on the circumcircle of  $\triangle ABC$ .

4. Let the radius of the circumcircle of  $\triangle ABC$  be  $R$ . Let the radius of the circumcircle of  $\triangle DEF$  be  $r$ . What is  $R/r$ ?
5. Where is the circumcentre of the circumcircle of  $\triangle DEF$ ?
6. What's the ninepoint circle of  $\triangle HBC$ ?

## 4 Euler line

Continue the diagram from the Nine point circle section.

1. Show that  $AY$  is a diameter.
2. Show that  $AH = 2OM$ .
3. Hence show that  $O, H, G$  are collinear, where  $G$  is the centroid.
4. Find the ratio  $HN : NG : GO$ .

## 5 Problems

1. Let  $ABCD$  be a cyclic quadrilateral with  $AB$  as the diameter. Let  $AC, BD$  intersect at  $P$ . Let  $AD, BC$  intersect at  $Q$ . Let the tangent at  $C, D$  intersect at  $R$ . Prove that  $P, Q, R$  collinear. Say 2 more interesting facts about this diagram.
2. Do the Euler line section, but for when  $\angle BAC > 90^\circ$ .

## 6 Q-point

Continue the diagram from the Nine point circle section.

1. Let line  $HY$  meet the circle again at  $Q$ . Show that  $AQEF$  cyclic.
2. Show that  $AQ, EF, BC$  are concurrent at a point  $K$ .
3. Show that  $KQFB$  is cyclic. Show that  $KQEC$  is cyclic.
4. Let  $KH$  meet the circumcircle  $AEHF$  again at  $J$ . Show that  $A, J, M$  are collinear.
5. Show that  $HJBC$  cyclic.
6. Show that  $AM$  is the *symmedian* of triangle  $AEF$ .
7. Show that  $KFJC$  cyclic. Show that  $KEJB$  cyclic.
8. Show that  $KQJM$  cyclic. Show that  $KQHD$  cyclic.
9. Show that the midpoint of  $KA$  lies on the circumcircle of  $\triangle QDJ$ .

## 7 Generalisation

Let  $EFBC$  be a cyclic quadrilateral. Let  $BF$  and  $EC$  intersect at  $A$ . Let the circumcircle of  $\triangle AFE$  and  $\triangle ABC$  intersect again at  $Q$ .

1. Show that  $AQ$ ,  $EF$ , and  $CB$  are concurrent at  $K$ .
2. Show that  $KQFB$  is cyclic and that  $KQEC$  is cyclic.
3. Let  $H$  be the intersection of  $FC$  and  $BE$ . Let  $M$  be the center of the circumcircle of  $EFBC$ . Show that  $M$ ,  $H$ , and  $Q$  are collinear.
4. Show that  $r^2 = MH \times MQ$ , where  $r$  is the radius of the circumcircle of  $EFBC$ .
5. Show that  $AH \perp KM$ .
6. Show that  $AM \perp KH$ .
7. Show that  $QBME$  is cyclic and  $QFMC$  is cyclic.
8. Show that  $QM$  is the angle bisector of  $\angle BQE$ .