Divisibility and Congruences (I)

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November 5, 2023

1 Fun factorisations

1.1 Example 1

Find all $n \in \mathbb{Z}^+$ such that $n + 10 | n^3 + 100$

1. What else is divisible by $n + 10$ that has the term n^3 in it?
2. Can you simplify the question a bit more using this fact?

1.2 Example 2

Find all $x, y \in \mathbb{Z}^+$ such that $\frac{1}{x} + \frac{1}{y} = \frac{1}{2022}$

1. Can we change the equation into a from of something $\times$ something $=$ number?
2. Since the number only has finitely many factors, this tells us what the "something"s can be.

1.3 Fun factorisations

Factorise fully with rational coefficients.

1. $x^2 - y^2 + 2y - 1$
2. $x^2 - y^2 - 4x + 2y + 3$
3. $x^4 + x^2 + 1$
4. $x^4 + 4$
5. $x^5 + x^4 + 1$

1.4 Problems

1. Find all primes where $p^3 - 4p + 9$ is a perfect square.
2. Find all $n \in \mathbb{Z}^+$ such that $\frac{1}{3} + \frac{1}{n}$ can be expressed as a fraction with a denominator less than n.
3. Find all $n \in \mathbb{Z}$ such that $n^2 + 3n + 1$ divides $n^3 + 6n^2 + 2n + 1$.

4. Find $\sqrt{1000 \times 1001 \times 1002 \times 1003 + 1}$
5. For which n does $x^2 + x + 1$ divide $x^{2n} + x^n + 1$?
6. Show that if $4^n + 2^n + 1$ is prime, then n must be a power of 3.
7. Let $P(x) = x^2 + x + 1$ for positive integers x. Let $Q(x)$ be the largest prime divisor of $P(x)$. Show that $Q(x)$ is never eventually monotonically increasing.

2 Euclidean algorithm

Given $a = bq + r$, we can show that $gcd(a, b) = gcd(b, r)$.

1. Show that $\frac{12n+1}{30n+2}$ is irreducible for all $n \in \mathbb{Z}^+$.
2. Show that two consecutive Fibnacci numbers are always coprime.
3. Find the greatest common divisor between $7^{610} - 3^{610}$ and $7^{377} - 3^{377}$.

3 GCD trick

3.1 Example 1

Let positive integers a, b, c satisfy $c(ac + 1)^2 = (5c + 2b)(2c + b)$, and c is odd. Show that c must be a perfect square.

1. Let $g = gcd(b, c)$. Why? Because then I can factorise out g and cancel it out. The fact that I take out the gcd also means I get to set $b = gx$ and $c = gy$, and use the fact that $gcd(x, y) = 1$.
2. Show that $g|y$.
3. Show that $y|g$. Hence, conclude $y = g$ and solve the problem.

3.2 Example 2

Find square numbers that can be expressed in the form of $t(t + 1)$.

1. Notice that t and $t + 1$ are coprime. What can we deduce from this?
2. Solve the problem.

3.3 Problems

1. Show that $(36a + b)(36b + a)$ cannot be a power of 2.
2. Find all integer solutions to $y^2(x^2 + y^2 - 2xy - x - y) = (x + y)^2(x - y)$.
3. Find all triples of positive integers a, b, c such that $\frac{(at+1)(bt+1)(ct+1)-1}{lcm(at,bt,ct)}$ is an integer, for any integer t.
4. Let positive integers a, b, c satisfy $c(ac + 1)^2 = (5c + 2b)(2c + b)$, and c is even. Show that there are no solutions.