Algorithms and Invariants (S)

Jongmin Lim (December 2025)

Start from a desirable position, and perturb your position a bit and see what changes and what doesn't.

1 Example 1

Let n be a positive integer. Consider $2n$ points equally spaced around a circle. Suppose that n of them are coloured blue and the rest are coloured red. Now, write down the distance between each pair of blue points in a list, from shortest to longest. Do the same for each pair of red points. Show that the lists are the same.

1. Take a very simple configuration. Show that the claim is true for that configuration.
2. What happens to the lists of distances when you swap two points?
3. Show that any configuration can be reached with some swaps from your simple configuration.
4. Finally, prove the claim.

2 Example 2

Johnny works at the bubble tea shop. He currently has n orders. The k-th order is an order for x_k bubble teas. Each bubble tea takes 1 minute to make. We want to minimise the total amount of time that each order needs to wait. What is the ideal configuration?
For example, if there are orders of $(3, 7, 5)$ bubble teas each, making it in the order $(3, 7, 5)$ will cause a total wait time of $3 + 10 + 15 = 28$ minutes, while making it in the order $(7, 3, 5)$ will cause a total wait time of $7 + 10 + 15 = 32$ minutes.

1. Start with some configuration. What happens to the total wait time when you swap the order of two orders?
2. In which situation can you not make swaps to lower the wait time any more?
3. Finally, solve the problem.

3 Example 3

Johnny determines that it's probably fair for someone who ordered 5 bubble teas to wait 5 times as much as someone who ordered 1 bubble tea. Johnny defines the *relative wait time* to be the actual wait time, divided by the size of their order. How do we minimise the total sum of the *relative wait time* of everyone?
For example, if there are orders of $(3, 7, 5)$ bubble teas each, making it in the order $(3, 7, 5)$ will cause a total *relative wait time* of $\frac{3}{3} + \frac{10}{7} + \frac{15}{5} = \frac{38}{7}$ minutes, while making it in the order $(7, 3, 5)$ will cause a total *relative wait time* of $\frac{7}{7} + \frac{10}{3} + \frac{15}{5} = \frac{22}{3}$ minutes.

4 Example 4

Form a 2000×2002 screen with unit screens. Initially, there are more than 1999×2001 unit screens which are on. In any 2×2 screen, as soon as there are 3 unit screens which are off, the fourth screen turns off automatically. Prove that the whole screen can never be totally off.

1. What happens to the total amount of "on" screens after each move?
2. How many times do you need to make this move?

5 Problems

1. Let $\leftarrow$ denote the left arrow key on a standard keyboard. If one opens a text editor and types the keys "ab$\leftarrow$ cd $\leftarrow\leftarrow$ e $\leftarrow\leftarrow$ f", the result is "faecdb". We say that a string B is reachable from a string A if it is possible to insert some amount of $\leftarrow$'s in A such that typing the resulting characters produces B. Thus, our example shows that "faecdb" is reachable from "abcdef". Show that if A is reachable from B, then B is reachable from A.

2. Jongmin stands on the origin of a coordinate plane with his new pogo stick. At each iteration, Jongmin jumps 2^n units, in a direction parallel to either the x-axis or the y-axis. For which coordinates can Jongmin **not** visit in a finite amount of moves? For example, Jongmin can visit $(6, 3)$ with the following sequence of moves:

$$(0, 0) \to (0, -1) \to (-2, -1) \to (-2, 3) \to (6, 3)$$

3. In Lineland there are $n \geq 1$ towns, arranged along a road running from left to right. Each town has a left bulldozer (put to the left of the town and facing left) and a right bulldozer (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a left and right bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, bulldozers are quite unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes. Let A and B be two towns, with B to the right of A. We

say that town A can sweep town B away if the right bulldozer of A can move over to B pushing off all bulldozers it meets. Similarly town B can sweep town A away if the left bulldozer of B can move over to A pushing off all bulldozers of all towns on its way. Prove that there is exactly one town that cannot be swept away by any other one.

4. Let m, n be positive integers. If a given rectangle can be tiled by a combination of horizontal $1 \times m$ strips and vertical $n \times 1$ strips, prove that it can be tiled using only one of the two types.

5. Komi has N weights weighing $2^0, 2^1, \cdots, 2^{N-1}$ grams each. Her friend Tadano happens to have a balance scale. How many ways can Komi put weights on the balance scale such that the left side is never heavier than the right side? For example, for $N = 2$, there are 3 ways: (1 right, 2 right), or (2 right, 1 left), or (2 right, 1 right).

6. Eren and Mikasa play a game where Eren thinks of a number from 1 to $2n$ and Mikasa has to guess what it is. If she guesses correctly, she wins. Every time Mikasa makes an incorrect guess, Eren increases or decreases his number by 1, but his number has to stay between 1 to $2n$. What's the smallest number k such that Mikasa can guarantee a win with in k guesses?

7. Let G be a finite simple graph and let k be the largest number of vertices of any clique in G. Suppose that we label each vertex of G with a non-negative real number so that the sum of all such labels is 1. Define the value of an edge to be the product of the labels of the two vertices at its ends. define the value of a labelling to be the sum of the values of the edges. Prove that the maximum possible value of a labelling of G is $\frac{k-1}{2k}$.

8. Consider an $n \times n$ grid with numbers in each cell, such that for each $k = 1, 2, \cdots, n$, the $2n - 1$ cells in either row k or column k contains the numbers $1, 2, \cdots, 2n - 1$ in some way. Show that this is impossible for $n = 2019$. Show that there are infinitely many n for which this is possible. For example, it is possible for $n = 4$ with the following.

1	2	3	4
5	1	7	6
6	4	2	5
7	3	1	2

9. A sequence of polynomials is defined as follows: $K_0 = 1$, $K_1(x) = x_1$, and

$$K_n(x_1, x_2, \cdots, x_n) := x_n K_{n-1}(x_1, \cdots, x_{n-1}) + (x_n^2 + x_{n-1}^2) K_{n-2}(x_1, \cdots, x_{n-2})$$

for every integer $n \geq 2$. Prove that $K_n(x_1, x_2, \cdots, x_n) = K_n(x_n, x_{n-1}, \cdots, x_1)$.

10. Let n be a positive integer. Find the number of polynomials $P(x)$ with coefficients in $\{0, 1, \cdots, m^2 - 1\}$ such that $P(m) = n$.

11. Let $G = (V, E)$ be a graph. Each edge $e \in E$ is labelled by a real number $0 \leq x_e \leq 1$ such that for each vertex, the sum of the labels of its adjacent

edges sum to at most 1. Now, consider the set of all such labellings L, which can be regarded as a subset of $\mathbb{R}^E$. If a labelling $v \in L$ can be expressed as $v = (u + w)/2$ some labellings $u, w \in L$ with $u \neq w$, then we say v is *nice*. Show that all labellings that are not *nice* consist of the numbers $0, 1$, or $\frac{1}{2}$.

12. An $m \times n$ array is filled with the numbers $\{1, 2, \cdots, n\}$ each used exactly m times. Show that one can permute the numbers within columns such that each row contains every number $\{1, 2, \cdots n\}$ exactly once.

13. A pile of n pebbles is placed in a vertical column. A pebble can be moved if it is at the top of a column which contains at least two more pebbles than the column immediately to its right. If there are no pebbles to its right, think of it as being next to a column of zero pebbles. At each stage, choose a pebble that can be moved according to this rule and move it to the top of the column to its right. For each n, show that no matter what choices were made at each stage, the final configuration obtained is unique. Describe this configuration in terms of n.

14. For some positive integer n, a coin will be flipped n times to obtain a sequence of n heads and tails. For each flip of the coin, there is a probability p of obtaining a head and probability of $1 - p$ of obtaining a tail, where $0 < p < 1$ is a rational number. Komi writes all 2^n possible sequences of n heads and tails in two columns, with some sequences in the left and the remaining sequences in the right. Komi would like the sequence produced by the coin flips to appear on the left column with probability $\frac{1}{2}$. Determine all pairs (n, p) for which this is possible.

15. Find all integers n for which each cell of $n \times n$ table can be filled with one of the letters I, M and O in such a way that

    (a) in each row and each column, one third of the entries are I, one third are M and one third are O; and

    (b) in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are I, one third are M and one third are O.

    Note: The rows and columns of an $n \times n$ table are each labelled 1 to n in a natural order. Thus each cell corresponds to a pair of positive integer (i, j) with $1 \leq i, j \leq n$. For $n > 1$, the table has $4n - 2$ diagonals of two types. A diagonal of first type consists all cells (i, j) for which $i + j$ is a constant, and the diagonal of this second type consists all cells (i, j) for which $i - j$ is constant.

16. (IMC 2015) Consider strings of 26 letters using the English alphabet (all lowercase). For each string, we give a score using the following formula, $Score = 1/(k+1)$ where k is the number of letters NOT used in the string. Prove that the total sum equals to 27^{25}