Harmonic points

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June 16, 2025

1 Warm up

1. On $\triangle ABC$, let D be a point on BC such that AD is the internal angle bisector of $\angle BAC$. Show that $AB/AC = DB/DC$.

2. Let E be a point on BC such that AE is the external angle bisector of $\angle BAC$. Show that $AB/AC = EB/EC$.

3. Let X be a random point on line BC. Show that $\frac{AB \sin BAX}{AC \sin CAX} = \frac{XB}{XC}$.

2 Harmonic points

Definition. *A, X, B, Y are harmonic points on a line if $AX/BX = AY/BY$.*

1. (Apollonious circle) Let A, B be fixed points. Choose a number $r > 1$. Let the locus of P such that $AP/BP = r$ be γ. Show that γ is a circle.

2. In the above diagram, let line AB meet γ at points X, Y. Show that A, X, B, Y are harmonic.

3. In this diagram, let AT be a tangent to γ. Show that $AB \perp BT$.

4. Using this diagram, show that in general, if A, X, B, Y are harmonic points and if M is the midpoint of XY,

   (a) Show that $XM^2 = MA \times MB$.

   (b) Show that $AX \times AY = AB \times AM$

   (c) Show that $BX \times BY = BA \times BM$

5. On this diagram again, choose a point B_0 on segment BT. Let AB_0 meet the circle again at X_0 and Y_0. Show that A, X_0, B_0, Y_0 is harmonic.

6. Let X be a point in the interior of $\triangle ABC$. Let AX, BX, CX intersect the sides of $\triangle ABC$ again at D, E, F respectively. Let $EF \cap BC = T$. Show that B, D, C, T is harmonic.

3 Harmonic Pencil

Definition. *For four points on the line A, X, B, Y, a cross ratio is the value $(AX/BX)/(AY/BY)$.*

1. In general, let A, X, B, Y be harmonic points on line ℓ_0 and let J be a point not on the line.
   (a) Show that the cross ratio of $AXBY$ is independent of the lengths JA, JX, JB, JY, but only on the angles $\angle AJX, \angle XJB, \angle BJY$.
   (b) Draw another line ℓ_1 such that AJ, XJ, BJ, YJ meets ℓ_1 at A_1, X_1, B_1, Y_1. Show that A_1, X_1, B_1, Y_1 are harmonic.
   (c) Let Γ be a circle passing through J. Let AJ, BJ, XJ, YJ meet the circle at A_2, X_2, B_2, Y_2. Then A_2, X_2, B_2, Y_2 are harmonic, that is, $A_2X_2/B_2X_2 = A_2Y_2/B_2Y_2$ (line segment lengths).

2. Consider triangle PAB and let X be the midpoint of side AB. Let $PY \parallel AB$. Show that PA, PB, PX, PY is a harmonic pencil.

3. Let BT, DT be tangents to the circumcircle of $\triangle ABD$. Let AT meet the circle again at C.
   (a) Show that $AB/BC = AD/CD$.
   (b) Let the circle centred at T with radius $BT = DT$ meet lines AB, AD again at E, F. Show that E, T, F are colliear.
   (c) Let M be the midpoint of BD. Show that $\triangle ABM \sim \triangle AFT$. Thus, show that AT is a symmedian of $\triangle ABD$.

4. Show that a cyclic quadrilateral $ABCD$ is harmonic if and only if A, C, T are collinear, where T is the intersection of the tangents at B, D. Show that $ABCD$ is harmonic if and only if AC is a symmedian of $\triangle ABD$.

3.1 Example

1. Let $ABCD$ be a cyclic quadrilateral with $AB \cap CD = X$, $AD \cap BC = Y$, $AC \cap BD = Z$. Show that XB, XC, XZ, XY is a harmonic pencil.

4 Harmonics and inversions

1. Let X be a point outside circle γ. Let $A, B \in \gamma$ such that X, A, B collinear. Let the intersection of XA and the pole of X be Y. Then $XAYB$ is harmonic.

2. Let A, B, C, D be on a line ℓ and $AB/BC = AD/DC$. Let X be a point not on ℓ. Show that the inversion of A, B, C, D forms a cyclic harmonic quadrilateral.

3. Let A, B, C, D be a cyclic harmonic quadrilateral. Let X be a point on the circle. Show that the inversion of A, B, C, D forms harmonic points on a line.

4. Let A, B, C, D be a cyclic harmonic quadrilateral. Let X be a point not on the circle. Show that the inversion of A, B, C, D forms a cyclic harmonic quadrilateral.

4.1 Example

1. Let $ABCD$ be a quadrilateral with an inscribed circle ω which is tangent to the sides AB, BC, CD, DA at points P, Q, R, S respectively. We want to show that AC, BD, PR, QS meet at one point.
   (a) Let $PR \cap QS = X$. Let CX meet the circle again at points G and H. Show that $QGRH$ a harmonic quadrilateral.
   (b) Describe an inversion which maps $P \to R$, $Q \to S$.
   (c) Show that A, C, G, H collinear.

5 Harmonic points in the Big diagram

1. (The Big Diagram) Let $ABCD$ be a cyclic quadrilateral on circle ω with centre O. $AB \cap CD = X$, $AD \cap BC = Y$, $AC \cap BD = Z$. Let K be the inversion of Z over ω. Let $XZ \cap AD = P$, $XZ \cap BC = Q$.
   (a) Show $APDY$ harmonic. Show $BQCY$ harmonic.
   (b) Hence show that Y is the pole of XZ.
   (c) Hence show that XY is the pole of Z.
   (d) Show that $BODK$, $AOCK$ is cyclic.
   (e) Show that OK is the angle bisector of $\angle AKC$.
   (f) Show that $XADK$, $XBCK$, $YKAB$, $YKDC$ cyclic.
   (g) Show that K is the Miquel point of $ABCD$.
2. Let $ABCD$ be a quadrilateral with an inscribed circle ω which is tangent to the sides AB, BC, CD, DA at points P, Q, R, S respectively.
   (a) Show that AC, PQ, RS concurrent. Show that they concur at the pole of BD.
   (b) Let $PR \cap QS = Z$. What is the polar of Z?
   (c) Hence prove again that AC, BD, PR, QS are concurrent.
3. (Butterfly theorem) Consider a cyclic quadrilateral $ABCD$ on circle ω and centre O and let $AC \cap BD = Z$. Choose a line $\ell \perp OZ$ passing through Z. Let $\ell \cap AD = P$, $\ell \cap BC = Q$. Show that $PZ = QZ$.