Bijections (I)

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1 Things to know

1. A bijection is a function $f : X \to Y$. That is, injective and surjective. Which means:
   (a) Injective: For every $x \in X$, there exists a unique $f(x) \in Y$.
   (b) Surjective: For every $y \in Y$, there exists an $x \in X$ such that $f(x) = y$
2. Make sure you use some sick Combinotoric Algebra to clean things up.

2 Questions

1. (Basic Techniques) There are finite sets $X \subset \mathbb{N}$ and $Y \subset \mathbb{N}$ and $|X| \leq |Y|$. How many functions are there $f : X \to Y$ if....
   (a) There are no conditions?
   (b) f is injective?
   (c) f is strictly monotonic increasing?
   (d) f is monotonic increasing (not strict)?
2. There are $m \times n$ grid of squares. If I want to get from the bottom left corner to the top right corner only by moving right or up along the gridlines, how many ways are there?
3. There are $m \times n$ grid of squares (m wide, n high). If I want to get from the bottom left corner to the bottom right corner only by moving up, right, or left (but never going back the same way), how many ways are there? For example, for a 2×2 grid, there are 9 ways.
4. There are n people and k chairs in a line($k \geq 2n$). How many ways are there to sit such that none of the people are adjacent?
5. There are n people in line for lunch, including Seyoon and Wilson. If Wilson wants to get his lunch before Seyoon does, how many different ways are there to line up?
6. Two players play a game that involve flipping coins. If the coin shows heads, then A receives a point, and if the coin shows tails, then B receives a point. The game ends when either one player is 2 points ahead of the other, or a total of $2n$ flips completed without an outcome. How many ways can this game play out?
7. Let $f(n, k)$ be the number of ways to show n as a sum of at most k not necessarily distinct positive integers. Let $g(n, k)$ be the number of ways to show n as a sum of positive integers at most k. Show that $f(n, k) = g(n, k)$.
8. Show using (2) the following:
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$
9. Let a, b be coprime positive integers. Prove that $\sum_{k=1}^{b-1} \lfloor \frac{ak}{b} \rfloor = \sum_{k=1}^{a-1} \lfloor \frac{bk}{a} \rfloor$
10. Catalan numbers
    (a) Consider an $n \times n$ grid. How many paths are there from the bottom left to the top right without passing through the diagonal? (Touching the diagonal is fine)
    (b) Let there be n "left" brackets '(' and n "right" brackets ')'. How many ways are there to align these brackets?
    (c) Let there be an $n + 2$ sided regular polygon. How many different ways are there to divide them into triangles? (Rotation and reflection are counted differently)

(d) Let Andrew and Angelo take part in a mathematics fight to the death. We say 'Andrew dominated Angelo' if Andrew's score was always larger or equal to Angelo's. If they play n rounds, how many ways can they play the game such that Andrew dominates Angelo?

11. (Supermarket principle) At a candy store, there are n different kinds of candies, and an infinite supply of each. Your parents gave you enough money to only buy k candies overall. How many different ways can you buy your candies?

12. There's a river dividing two banks. There is an $m \times (m+1)$ grid of stepping stones in between, and each stepping stone that is horizontal or vertical to each other is connected by a wooden log. During the night, Donald Trump bombs the river, and each log breaks with a probability of $1/2$. What's the probability that you can cross the river the next morning?

13. (IMC 2015) Consider strings of 26 letters using the English alphabet (all lowercase). For each string, we give a score using the following formula, $Score = 1/(k+1)$ where k is the number of letters NOT used in the string. Prove that the total sum equals to 27^{25}

14. For question 5, prove that they both equal to $\frac{(a-1)(b-1)}{2}$

15. Let $S = \{1, 2, 3, \dots, 2000\}$. How many subsets of S has element sum divisible by 5?

3 Double Counting/Generating Functions

TECHNICALLY, it still counts as a bijection and it's a really fun topic so you should try it out. Look up Yufei Zhao's handouts for more interesting questions.

1. In a university, there are s students and c clubs. If each student is in exactly 3 clubs and each club has exactly 10 students, find the ratio c/s.

2. We choose n fruits out of apples, bananas, oranges, or pears. If there can only be an even number of apples, a multiple of 5 bananas, at most 4 oranges, and at most 1 pear, how many ways are there?

3. N players play a chess tournament where each player plays every other player. At each game, you can either win 1 point for winning, 0.5 points for a draw, or zero points for losing. At the end of the tournament, every player noticed that exactly half of their points were gained from their games with the k players with the least amount of points.

   (a) If $k = 10$, find N.

   (b) Find the conditions on k for there to exist N, and find the general solution to this problem.