Projective and Inversive Geometry (S)

Jongmin Lim (December 2023

1 Various inversions

1.1 Inversion

Definition. *An inversion with respect to a given circle with centre O and radius r is a function ϕ which takes a point X and outputs another point $X' = \phi(X)$ such that $X' \in \overrightarrow{OX}$ and $|OX| \times |OX'| = r^2$.*

Here are some properties of inversion. Try to prove them yourself!

1. Inversion is an involution (i.e. $f(f(x)) = x$)
2. If an inversion ϕ maps A to A' and B to B', then $ABA'B'$ is cyclic.
3. If A, B, C are collinear, then $OA'B'C'$ is cyclic. Thus, a line is mapped to a circle passing through the centre of inversion, under inversion.
4. If A, B, C, D is cyclic on circle which does not contain O, then $A'B'C'D'$ is also cyclic on a circle which does not contain O. Thus, a circle is mapped to another circle as long as it does not contain the centre.
5. If two circles are tangent, then they are tangent after inversion.
6. (Inversion preserves cross ratios) If $ABCD$ is on a circle/line such that $AB/BC = AD/DC$ (i.e. harmonic), then $A'B'C'D'$ is also harmonic.

1.2 $\sqrt{bc}$-inversion

Definition. *Given $\triangle ABC$, a $\sqrt{bc}$-inversion centred on A is an inversion with radius $\sqrt{|AB| \times |AC|}$ centred on A, composed with a reflection over the angle bisector of $\angle BAC$.*

Here are some properties of the $\sqrt{bc}$-inversion. Try to prove them yourself!

1. $\sqrt{bc}$-inversion is an involution.
2. If it maps X to Y, then $\triangle ABX \sim \triangle AYC$ as a spiral similarity.
3. B maps to C.
4. AB is mapped to AC.
5. BC is mapped to the circumcircle of $\triangle ABC$.

1.3 Poles and Polars

Definition. *Given a circle ω with centre O and a point X, if X inverts to X', then consider the line ℓ_X which is perpendicular to OX' and passes through X'. ℓ_X is the polar of point X, and vice versa.*

Here are some properties of poles and polars. Try to prove them yourself!

1. Poles and polars are an involution.
2. If ℓ_X passes through Y, then ℓ_Y passes through X.
3. If $\ell_X \cap \ell_Y = Z$, then $X, Y \in \ell_Z$.
4. Consider two points $A, B \in \omega$ such that A, B, X are collinear. Then $AB \cap \ell_X = Y$, where A, X, B, Y are harmonic.
   Hint: If M is the midpoint of AB, try prove $YB \times YA = YX \times YM$.

2 Problems

1. What happens to parallel lines after inversion?
2. Harmonic quadrilaterals
   (a) Let triangle ABC have circumcircle ω. Let the tangents at B, C intersect at T. Let AT intersect the circle again at D. Show that $ABDC$ is a harmonic quadrilateral.
   (b) Let us invert the diagram at point B and let the images of A, C, D be A', C', D' respectively. Show that $A'C' = C'D'$.
3. Let a circle γ be internally tangent to circle ω at point A, and tangent to chord XY of ω at point B. Let M be the midpoint of arc XY not containing point A. Show that M, A, B are collinear.
4. Let the circumcircle of $\triangle ABC$ be Γ. Consider a circle ω which is tangent to the circumcircle internally at T and sides AB and AC at X, Y.
   (a) What happens to ω under $\sqrt{bc}$-inversion?
   (b) Show that the incentre I of $\triangle ABC$ is the midpoint of XY.
   (c) Let E be the A-excircle touch point on BC. Show that $\triangle ABT \sim \triangle AEC$.
   (d) Let D be the incircle touch point on BC. Show that $\triangle TBD \sim \triangle TAC$.
   (e) TB is tangent to the circumcircle of ABE.
   (f) If $M \in \Gamma$ such that AM is the angle bisector of $\angle BAC$, then MD, AT, OI are concurrent.
5. Let $ABCD$ be a quadrilateral with an inscribed circle ω which is tangent to the sides AB, BC, CD, DA at points P, Q, R, S respectively. We want to show that AC, BD, PR, QS meet at one point.
   (a) Let $PR \cap QS = X$. Let CX meet the circle again at points G and H. Why is $QGRH$ a harmonic quadrilateral?

(b) Describe an inversion which maps $P \to R$, $Q \to S$. What happens to G and H?

(c) Why is A, C, G, H collinear? Finish the problem from here.

6. For quadrilateral $ABCD$, show that $AB \times CD + AD \times BC \geq AC \times BD$, where equality holds when $ABCD$ is cyclic.

7. Consider four points A, B, C, D on a semicircle with diameter AD and centre O. Let $BC \cap AD = K$. Let the intersection of the circumcircles of $\triangle ABO$ and $\triangle CDO$ be T. Show that $\angle OTK = 90°$.

8. (The Big Diagram) Let $ABCD$ be a cyclic quadrilateral on circle ω with centre O. $AB \cap CD = X$, $AD \cap BC = Y$, $AC \cap BD = Z$.

   (a) Let $XZ \cap AD = P$, $XZ \cap BC = Q$.
   Show $APDY$ harmonic. Show $BQCY$ harmonic.

   (b) Hence show that Y is the pole of XZ.

   (c) Hence show that XY is the pole of Y.

   (d) Show that $BODZ$, $AOCZ$ is cyclic (Hint: invert a line).

   (e) Show that OZ is the angle bisector of $\angle AZC$.

   (f) Show that $XADZ$, $XBCZ$, $YZAB$, $YZDC$ cyclic.

   (g) Show that Z is the Miquel point of $ABCD$.

9. Let $ABCD$ be a quadrilateral with an inscribed circle ω which is tangent to the sides AB, BC, CD, DA at points P, Q, R, S respectively.

   (a) What is the pole of lines AC and BD?

   (b) Let $PR \cap QS = Z$. What is the polar of Z?

   (c) Hence prove again that AC, BD, PR, QS are concurrent.

10. (Butterfly theorem) Consider a cyclic quadrilateral $ABCD$ on circle ω and centre O and let $AC \cap BD = Z$. Choose a line $\ell \perp OZ$ passing through Z. Let $\ell \cap AD = P$, $\ell \cap BC = Q$. Show that $PZ = QZ$.