Combinatorial Number Theory (I)

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1 Ideas

1.1 Combinatorial identities

Here are some basic ones.

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

You should also know

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

which by subbing in $x = 1$ should get you $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. Even more interesting is when you differentiate that expression and sub in $x = 1$, which gets you $n2^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}$, and so on.

Problem 1.1. *Let $n > 1$ be an odd integer. Show that the sequence*

$$\binom{n}{1}, \binom{n}{2}, \cdots, \binom{n}{\frac{n-1}{2}}$$

contains an odd number of odd numbers.

Problem 1.2. *Throughout the day, you'll get nine emails labelled 1 to 9 in this order. Whenever you have time, you will read the most recently arrived unread email. Before you go to lunch, you tell a friend that you've read email number 8. The friend wonders how many after-lunch email-reading experiences you can have. How many are there?*

1.2 Inclusion Exclusion principle

I want to count $A \cup B$. If I count A and B, then I counted things in $A \cap B$ twice. So $|A \cup B| = |A| + |B| - |A \cap B|$. I want to count $A \cup B \cup C$. If I count A, B, and C, then I have counted things in $A \cap B$, $B \cap C$, $C \cap A$ twice, so let's subtract them. Oops, I've subtracted $A \cap B \cap C$ three times so let's add those back. So $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$.

Problem 1.3. *How many positive integers $n \leq 2001$ are multiples of 3 or 4 but not 5?*

1.3 Pigeon Hole Principle

If I have k pigeon holes but at least mk pigeons, then there must exist a pigeon hole with at least m pigeons in it. If I have finitely many pigeon holes but infinitely many pigeons, then there must exist a pigeon hole with infinitely many pigeons in it.

Problem 1.4. *Prove that for every prime p, there exists a number of the form $777 \cdots 777$ that is divisible by p.*

1.4 Base k numbers

Instead of base 10, we can write numbers in base k. In base 10, we write numbers like 1203, which implicitly means $1 \times 10^3 + 2 \times 10^2 + 0 \times 10^1 + 3 \times 10^0$. In base k, we write numbers, but each digit is a power of k instead, and each digit is from $\{0, 1, \cdots, k-1\}$. These are often very useful.

Problem 1.5. *The increasing sequence $1, 3, 4, 9, 10, \cdots$ consists of all positive integers which are sums of distinct powers of 3. Find the 100th term of this sequence.*

1.5 Assume minimality

This is a bit like induction, but for the people who want things to go wrong instead of it going right. Instead of cutting down a big problem to a smaller problem, we assume that the problem is already small, and show that it can get even smaller. Then if things get too small, it's not an integer anymore, so things go wrong and you show that it doesn't exist.

Problem 1.6. *23 people of integer weights decide to play football. One is selected as the referee, and then they split up into two 11-person teams of equal total weights. Turns out, no matter who we choose to be the referee, we can always split the team in this way. Show that all 23 people have equal weights.*

1.6 Solving recurrences

I've emphasised this every camp, but knowing how to set up and solve recurrences is very important. Recurrences are pretty much induction written as an equation, so constructing the recurrence relation is not hard. Let's do some examples.

1. $a_{n+1} = 2a_n + 3$
2. $a_{n+1} = 3a_n - 4$
3. $a_{n+2} = 3a_{n+1} - 2a_n$
4. $6a_{n+2} = 5a_{n+1} - a_n$
5. $4a_{n+2} = 4a_{n+1} - a_n$

Problem 1.7. *Find the number of subsets of $\{1, 2, \cdots, 2000\}$ the sum of whose elements is divisible by 5.*

2 Problems

1. How many ways can you arrange the numbers 21, 31, 41, 51, 61, 71, 81 such that the sum of every four consecutive numbers is divisible by 3?

2. Let A, B two disjoint sets whose union is the set of natural numbers. Show that for every natural number n, there exist distinct $a, b > n$ such that $\{a, b, a+b\} \subseteq A$ or $\{a, b, a+b\} \subseteq B$.

3. 25 boys and 25 girls sit around a table. Prove that there always exists a person sitting between two girls.

4. Let n be a positive integer. Find the number of polynomials $P(x)$ with coefficients in $\{0, 1, 2, 3\}$ such that $P(2) = n$.

5. Let n be a positive integer. Find the number of polynomials $P(x)$ with coefficients in $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ such that $P(3) = n$.

6. Eren and Mikasa play a game where Eren thinks of a number from 1 to $2n$ and Mikasa has to guess what it is. If she guesses correctly, she wins. Every time Mikasa makes an incorrect guess, Eren increases or decreases his number by 1, but his number has to stay between 1 to $2n$. What's the smallest number k such that Mikasa can guarantee a win with in k guesses?

7. Komi has N weights weighing $2^0, 2^1, \cdots, 2^{N-1}$ grams each. Her friend Tadano happens to have a balance scale. How many ways can Komi put weights on the balance scale such that the left side is never heavier than the right side? For example, for $N = 2$, there are 3 ways: (1 right, 2 right), or (2 right, 1 left), or (2 right, 1 right).

8. Can we partition the natural numbers $\mathbb{Z}^{\geq 1}$ into two disjoint sets A, B such that A does not contain any arithmetic sequences of infinite length, and B does not contain any arithmetic sequences of length 3?

9. Consider an $n \times n$ grid with numbers in each cell, such that for each $k = 1, 2, \cdots, n$, the $2n - 1$ cells in either row k or column k contains the numbers $1, 2, \cdots, 2n - 1$ in some way. Show that this is impossible for $n = 2019$. Show that there are infinitely many n for which this is possible. For example, it is possible for $n = 4$ with the following.

1	2	3	4
5	1	7	6
6	4	2	5
7	3	1	2

10. Let n be a positive integer. Find the number of polynomials $P(x)$ with coefficients in $\{0, 1, \cdots, m^2 - 1\}$ such that $P(m) = n$.

11. 64 balls are separated into several piles. At each step, we are allowed to pick two piles, say a pile with p balls and another pile with q balls, with $p \geq q$. Then we take q balls from the first pile and put them in the

second pile, so there are $p - q$, and $2q$ resulting balls respectively. Show that we can put all the balls into one pile.

12. Let m and n be positive integers. If you can tile a rectangle with $1 \times m$ and $n \times 1$ tiles, then you can do it with only one of these.

13. In an arena, each row has 199 seats. One day, 1990 students come to attend a soccer match. We know that at most 39 students are from the same school, and every student from the same school must sit in the same row. What's the minimal number of rows that must be reserved for these students?