Power of a point

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Definition. *Consider a circle γ with centre O, radius r. Let P be a point not on the circle. The power of P with respect to γ is defined as $f(P,\gamma) = (OP + r)(OP - r) = OP^2 - r^2$.*

1 Power of a point and Radical axis theorem

1. Assume P is outside the circle and let A, B, C, D on the circle such that P, A, B and P, C, D are collinear. Let $T \in \gamma$ such that PT is a tangent. Show $PA \times PB = PC \times PD = PT^2 = f(P, \gamma)$.
2. Assume P is inside the circle and let A, B, C, D on the circle such that A, P, B and C, P, D are collinear. Show $PA \times PB = PC \times PD = -f(P, \gamma)$.
3. Let P, Q be two points. $f(P, \gamma) = f(Q, \gamma)$, if and only if $OP = OQ$.
4. Let α, β be two circles. If $f(P, \alpha) = f(P, \beta)$, then the locus of P is a line perpendicular to the line that passes through the centre of α and β.
   (a) This is called the *radical axis* between α and β.
   (b) In particular, if α and β intersect at X, Y, the locus is line XY.
5. Let α, β, γ be three circles. Let ℓ be the radical axis between α, β. Let m be the radical axis between β, γ. Let n be the radical asxis between γ, α. Then l, m, n are concurrent at a point or all parallel.

1.1 Exercises

1. (JL) Let P be a point outside of a circle with centre O. Draw two tangents PX, PY to the circle. Let XY meet OP at point Q. Draw a line through P such that it meets the circle at two points A, B. Show that $OQAB$ is a cyclic quadrilateral.
2. (Folklore) Two circles α, β intersect at points X, Y. A common tangent touches the two circles at A, B. Let XY intersect line AB at point D. Show that $AD = BD$.
3. (YZ) Let ABC be an acute triangle. Let the line through B perpendicular to AC meet the circle with diameter AC at points P and Q, and let the line through C perpendicular to AB meet the circle with diameter AB at points R and S. Prove that P, Q, R, S are concyclic.

4. (AMO?) Consider $\triangle ABC$ such that $AB = AC$, and circumcircle γ. Let D be a point such that DA is a tangent to γ. Let DB intersect γ again at point E. Let DC intersect the γ again at point F. Let EF meet DA at point X. show that $DX = AX$.

5. (YZ) Let C be a point on a semicircle of diameter AB and let D be the midpoint of arc AC. Let E be on line BC such that $DE \perp BC$. Let AE meet the semicircle at F. Show that BF bisects DE.

6. (Euler) Let a triangle have circumcentre O and circumradius R, and incentre I and inradius r. Show that $R^2 - 2Rr = OI^2$.

7. (USAMO 1998) Let C_1 and C_2 be concentric circles, with C_2 in the interior of C_1. Let A be a point on C_1 and B a point on C_2 such that AB is tangent to C_2. Let C be the second point of intersection of AB and C_1, and let D be the midpoint of AB. A line passing through A intersects C_2 at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.

8. Let ABC be a triangle and let D and E be points on the sides AB and AC, respectively, such that DE is parallel to BC. Let P be any point interior to triangle ADE, and let F and G be the intersections of DE with the lines BP and CP, respectively. Let Q be the second intersection point of the circumcircles of triangles PDG and PFE. Prove that the points A, P, and Q are collinear.

9. (IMO 2000) Two circles Γ_1 and Γ_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on Γ_1 and D on Γ_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

2 Various inversions

Definition. *An inversion with respect to a given circle with centre O and radius r is a function ϕ which takes a point X and outputs another point $X' = \phi(X)$ such that $X' \in \overrightarrow{OX}$ and $|OX| \times |OX'| = r^2$.*

Here are some properties of inversion. Try to prove them yourself!

1. Inversion is an involution (i.e. $f(f(x)) = x$)
2. If an inversion ϕ maps A to A' and B to B', then $ABA'B'$ is cyclic.
3. If A, B, C are collinear, then $OA'B'C'$ is cyclic. Thus, a line is mapped to a circle passing through the centre of inversion, under inversion.
4. If A, B, C, D is cyclic on circle which does not contain O, then $A'B'C'D'$ is also cyclic on a circle which does not contain O. Thus, a circle is mapped to another circle as long as it does not contain the centre.
5. If two circles are tangent, then they are tangent after inversion.
6. What happens to parallel lines after inversion?

3 Problems

1. Consider circle γ and a chord BC. Let α be a circle internally tangent to γ at X and tangent to BC at P. Let β be a circle internally tangent to γ at Y and tangent to BC at Q and tangent to α at I. Let the common tangent of α, β at point I meet the circle at A, which is on the same side of BC as points X, Y.

   (a) Let M be the midpoint of the arc BC. Show that X, P, M are collinear.

   (b) Hence or otherwise show that I is the incentre of $\triangle ABC$.

2. Let triangle ABC have circumcircle ω. Let the tangents at B, C intersect at T. Let AT intersect the circle again at D. Show that $ABDC$ is a harmonic quadrilateral. Let us invert the diagram at point B and let the images of A, C, D be A', C', D' respectively. Show that $A'C' = C'D'$.

3. An angle of fixed magnitude θ revolves around a fixed vertex A and meets a fixed line ℓ at points B, C. Show that the circumcircles of $\triangle ABC$ are all tangent to a fixed circle.

4. Let P be a point inside triangle ABC such that $\angle APB - \angle ACB = \angle APC - \angle ABC$. Show that the angle bisector of $\angle ABP$, the angle bisector of $\angle ACP$ intersect on line AP.

5. (Prove Ptolemy's theorem using inversion) For quadrilateral $ABCD$, show that $AB \times CD + AD \times BC \geq AC \times BD$, where equality holds when $ABCD$ is cyclic. Invert at A.

6. Consider four points A, B, C, D on a semicircle with diameter AD and centre O. Let $BC \cap AD = K$. Let the intersection of the circumcircles of $\triangle ABO$ and $\triangle CDO$ be T. Show that $\angle OTK = 90°$.

7. Let $ABCD$ be a cyclic quadrilateral on circle ω with centre O. $AB \cap CD = X$, $AD \cap BC = Y$.

   (a) Show that circles XAD, XBC, YAB, YCD meet at one point, Z. Show that Z lies on line XY.

   (b) Show that $BODZ$, $AOCZ$ is cyclic.

   (c) Hence or otherwise show that Z inverts E, where $E = AC \cap BD$.

   (d) Show that $OE \perp XY$.

8. Let the circumcircle of $\triangle ABC$ be Γ. Consider a circle ω which is tangent to the circumcircle internally at T and sides AB and AC at X, Y.

   (a) What happens to ω under $\sqrt{bc}$-inversion? I.e. invert from A with radius $\sqrt{AB \times AC}$, then reflect over the angle bisector.

   (b) Show that the incentre I maps to the excenter J under this inversion.

   (c) Show that the incentre I of $\triangle ABC$ is the midpoint of XY.

   (d) Let E be the A-excircle touch point on BC. Show that $\triangle ABT \sim \triangle AEC$.

(e) Let D be the incircle touch point on BC. Show that $\triangle TBD \sim \triangle TAC$.

(f) TB is tangent to the circumcircle of ABE.

(g) If $M \in \Gamma$ such that AM is the angle bisector of $\angle BAC$, then MD, AT, OI are concurrent.

9. (Kazakhstan 2012) Points K, L on the side of BC of $\triangle ABC$ satisfy $\angle BAK = \angle CAL < \frac{1}{2}\angle BAC$. Let ω_1 be any circle tangent to lines AB, AL, and let ω_2 be any circle tangent to lines AK, AC. Let $\omega_1 \cap \omega_2 = \{P, Q\}$. Show that $\angle BAP = \angle QAC$.

10. Let $\triangle ABC$ be scalene. The angle bisector of $\angle BAC$ intersects BC at D and circumcircle γ at E. A circle with diameter DE intersects γ again at F. Show that AF is the symmedian of $\triangle ABC$. I.e. if M is the midpoint of BC, then $\angle BAF = \angle MAC$.

11. (Poles and polars) Given a circle ω with centre O and a point X, if X inverts to X', then consider the line ℓ_X which is perpendicular to OX' and passes through X'. ℓ_X is the polar of point X, and vice versa. Here are some properties of poles and polars. Try to prove them yourself!

    (a) Poles and polars are an involution.

    (b) If ℓ_X passes through Y, then ℓ_Y passes through X.

    (c) If $\ell_X \cap \ell_Y = Z$, then $X, Y \in \ell_Z$.

12. (IMO 2015 shortlist) Let ABC be a triangle with $CA \neq CB$. Let D, F, and G be the midpoints of the sides AB, AC, and BC respectively. A circle Γ passing through C and tangent to AB at D meets the segments AF and BG at H and I, respectively. The points H' and I' are symmetric to H and I about F and G, respectively. The line $H'I'$ meets CD and FG at Q and M, respectively. The line CM meets Γ again at P. Prove that $CQ = QP$.

13. (IMO 2015) Let ABC be an acute triangle with $AB > AC$. Let Γ be its circumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = 90°$ and let K be the point on Γ such that $\angle HKQ = 90°$. Assume that the points A, B, C, K, and Q are all different and lie on Γ in this order. Prove that the circumcircles of triangles KQH and FKM are tangent to each other.