Lengths and Areas (I)

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1 Tangents and Pitot's theorem

1.1 Tangents

Tangents to a circle from a point have equal lengths.

1. Let the incircle of $\triangle ABC$ touch BC at D. Let the A-excircle touch BC at E. Show that $BD = CE$.

2. Let quadrilateral $ABCD$ have an inscribed circle. Show that $AB + CD = AD + BC$.

3. Let the incircle of $\triangle ABC$ have radius r. Show that

$$\text{Area } ABC = \frac{r}{2}(a + b + c)$$

4. Let $\triangle ABC$ have $\angle A = 90°$. Show that the inradius $r = \frac{AB+AC-BC}{2}$.

5. Hence or otherwise find all right angle triangles with integer side lengths whose area equals its perimeter.

6. Let $ABCD$ be a parallelogram. Let the incircle of $\triangle ACD$ and $\triangle ABC$ touch AC at K, M. Let the incircle of $\triangle BCD$ and $\triangle ABD$ meet BD at L, N. Show that $KLMN$ is a rectangle. (Hint: what defines a rectangle?)

1.2 Pitot's theorem

Given a quadrilateral $\square PQRS$, an inscribed circle ω is a circle tangent to the sides PQ, QR, RS, SP at at X, Y, Z, W respecitvely and inside the quadrilateral. Similarly, an escribed circle Ω is a circle tangent all four sides, but outside the quadrilateral.

1. Let ω touches sides PQ, QR, RS, SP . Show that XY, ZW, and PR are concurrent.

2. Let $\square ABCD$ be an convex quadrilateral. Show that $\square ABCD$ has an inscribed circle if and only if $AB + CD = AD + BC$. (Hint: What happens to AD when we draw a circle tangent to AB, BC, CD?)

3. Let $\square ABCD$ be a concave quadrilateral. Show that $\square ABCD$ has an inscribed circle if and only if $AB + CD = AD + BC$.

4. Let $\square ABCD$ be a convex quadrilateral. Show that $\square ABCD$ has an escribed circle if and only if $AB + BC = AD + CD$.

5. Let $\square ABCD$ be a concave quadrilateral. Show that $\square ABCD$ has an escribed circle if and only if $AB + BC = AD + CD$.

6. Let $\square ABCD$ be a quadrilateral that crosses over itself; i.e. $ACBD$ is a convex quadrilateral. Show that $\square ABCD$ has an escribed circle if and only if $AB + BC = AD + CD$.

7. (Australian team selection exam 2016) Let $\square ABCD$ be a convex quadrilateral such that $AC + BC = AD + BD$. Let $E = AC \cap BD$. Show that the angle bisector of $\angle CAD, \angle CBD, \angle CED$ meet at one point.

8. (2015 December camp prep problem G3) Let $\triangle ABC$ have three cevians AX, BY, CZ, meeting at point one D inside the triangle. Assume we have $AZ + DY = AY + ZD$ and $BZ + DX = BX + ZD$. Show that $CX + DY = CY + DX$.

2 Length ratios

2.1 Generalised angle bisector theorem

Consider $\triangle ABC$. Let D be a point on line BC. Then $\dfrac{BD}{DC} = \dfrac{AB \sin BAD}{AC \sin CAD}$.

1. Let A, B, C, D be on a line such that $AB/BC = AD/DC$. Let ℓ be another line, and P be a point not on either lines. Let PA, PB, PC, PD intersect ℓ at A', B', C', D'. Show that $A'B'/B'C' = A'D'/D'C'$.

2.2 Ceva

Let Cevians AD, BE, CF meet at a point P in $\triangle ABC$. Then $AF/FB \times BD/DC \times CE/EA = 1$.

1. Let X, Y be on sides AB, AC of $\triangle ABC$ such that $XY \parallel BC$. Let $BY \cap XC = T$. Let $AT \cap BC = M$. Show that M is the midpoint of BC.

2. (angle ceva) Let Cevians AD, BE, CF meet at a point P in $\triangle ABC$. Then

$$\frac{\sin BAD}{\sin DAC} \times \frac{\sin ACF}{\sin FCB} \times \frac{\sin CBE}{\sin EBA} = 1$$

2.3 Menelaus

Let $\triangle ABC$ have sides BC, CA, AB meet a line ℓ at points D, E, F respectively. Then $AF/FB \times BD/DC \times CE/EA = 1$.

1. Let AD, BE, CF be Cevians. Let $EF \cap BC = X$. Show that $BD/CD = BX/CX$.

3 Other tricks

3.1 Parallel lines

Consider $\triangle ABC$. Let ℓ pass through A and be parallel to BC. Then for any $A' \in \ell$, the area of $\triangle A'BC$ equals the area of $\triangle ABC$.

1. Consider a convex quadrilateral $ABCD$. Construct a line (using straightedge and compass) that bisects the area of this quadrilateral.

3.2 Power of a point

Given a point P and a circle ω with centre O and radius r, we define the power of the point P with respect to ω as

$$p(P, \omega) = OP^2 - r^2$$

Notice that when P is outside of the circle, this is equal to the length of the tangent to ω from P. When P is inside of the circle, this value is negative.

1. Let A, B on circle ω and X be outside the circle such that XA, XB are tangents. Let C, D also be on the circle such that XCD is collinear in this order.
   (a) Show that $\triangle XAC \sim \triangle XDA$.
   (b) Hence or otherwise, show that $\frac{AC}{AD} = \frac{BC}{BD}$.

4 Problems

1. Show that $OI^2 = R(R - 2r)$ for circumradius R and inradius r for a given triangle with circumcentre O and incentre I.

2. Let $ABCD$ be a convex quadrilateral. Let K, L, M, N be the midpoints of AB, BC, CD, DA respectively. Let NL intersect KM at T. Show that

$$\frac{8}{3} Area(DNTM) < Area(ABCD) < 8Area(DNTM)$$

3. Let a, b, c be the side lengths of a triangle. Show that $abc \geq (a+b-c)(a-b+c)(-a+b+c)$

4. The incircle of a non-isosceles triangle ABC touches the sides AC, BC at P, Q respectively. The excircles to the sides AC and BC touch the line AB at points M and N respectively. It is known that M, N, P, Q are cyclic. Prove that $\angle ACB = 90°$.

5. (USAMO 1998) Let ω_1, ω_2 be concentric circles with ω_2 in the interior of ω_1. Let A be a point on ω_1 and B a point on ω_2 such that AB is tangent to ω_2. Let C be the second point of intersection of AB and ω_1, and let D be the midpoint of AB. A line passing through A intersects ω_2 at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. What is AM/MC?

6. Let ω be the incircle of triangle ABC, where AB is the longest side. Let L, N, E be the points of tangency of ω with the sides AB, BC, CA respectively. Lines LE and BC intersect at the point H and lines LN and AC intersect at the point J. Let O, P be the midpoints of EJ and NH respectively. If $Area(ABOP) = u^2$ and $Area(COP) = v^2$, show that $Area(HJNE) = 4uv$.

7. Let $\triangle ABC$ have points D, E, F on sides BC, CA, AB such that AD, BE, CF are concurrent. Show that if $BDPF$ has an incircle and $CDPE$ has an incircle, then $AEPF$ must have an incircle.

8. Let ABC be a triangle with incentre I. A straight line through I intersects AB and AC at points P, Q respectively. Let $a = BC$, $b = AC$, $c = AB$, $p = \frac{PB}{PA}$, $q = \frac{QC}{QA}$. Prove that if $a^2 = 4bcpq$,then AI, BQ, CP are concurrent.

9. Show that the isogonal conjugate exists. I.e. Let P be a point and let ABC be a triangle. Reflect AP by the angle bisector of $\angle A$, reflect BP by the angle bisector of $\angle B$, and reflect CP by the angle bisector of $\angle C$. Show that the three lines meet at a point P'.

10. For a convex quadrilateral $ABCD$, show that the locus of the point P such that $Area(PAB) + Area(PCD) = Constant$ is a line.

11. Hence or otherwise prove that for a quadrilateral $ABCD$ with an incircle centred at O, then the midpoint M of AC, and the midpoint N of BD are collinear with O.