Dilation

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June 16, 2025

Definition. *Given a point H and a factor $k \neq 0$, a dilation is a transformation that takes point A to point A' such that*

1. *H, A, A' are collinear*
2. *$\overrightarrow{HA} = k \cdot \overrightarrow{HA'}$*

We denote such dilation with $\mathcal{H}(H, k)$.

1 Dilation in general

1. Let $\mathcal{H}(H, k)$ be a dilation that takes A, B, C to A', B', C' respectively.
   (a) Show that $AB \parallel A'B'$. Show that $A'B' = kAB$.
   (b) Show that $\triangle ABC \sim \triangle A'B'C'$.
2. Let $ABCD$ be a trapezoid with $AB \parallel CD$ and let $E = AC \cap BD$. Let $\triangle ABF$, $\triangle CDG$ be equilateral triangles facing away from the trapezoid. Show that E, F, G collinear.
3. (ToT1984) Let $ABCD$ be a square and let P be a point inside the square. Show that the centroids of $\triangle ABP$, $\triangle BPC$, $\triangle CPD$, $\triangle DPA$ form a square.
4. Let $D \in BC$ of $\triangle ABC$. Let $E \in BC$ such that $BD = CE$. Let M be the midpoint of AD. Show that ME passes through a fixed point as point D varies along side BC.

2 Dilation of circles

Every pair of circles are similar. There exists two dilations, one with a positive factor and one with a negative factor. You can also think of circles as a polygon with infinitely many sides. Thus, a pair of corresponding points under dilaton on a pair of circles must have parallel tangents to their respective circles.

1. Let α, β be two circles that are internally tangent at T. Let $A \in \alpha$ and $B \in \beta$ such that T, A, B are collinear. Let the tangent to α at A meet β at X, Y. Show that TB bisects $\angle XTY$.

2. Let $\triangle ABC$ have its incircle touch the sides BC, CA, AB at D, E, F. Let P, Q, R be the midpoint of the arcs BC, CA, AB of its circumcircle. Show that DP, EQ, FR concurrent.

3. Let $\triangle ABC$ have the incircle touch side BC at D. Let the A-excircle touch side BC at E. Let AE intersect the incircle at F. Show that $FD \perp BC$.

4. Let α, β be two circles externally tangent at T. Let γ be a circle which is interally tangent to α, β at A, B respectively. Let AT intersect γ at P. Show that $\angle TBP = 90°$.

5. Let α, β be externally tangent at T and internally tangent to a bigger circle γ at points P, Q respectively. Let $PT \cap \gamma = X$. Let $QT \cap \gamma = Y$. Show that the common internal tangent at T is perpendicular to XY.

6. Let γ be a circle tangent to a line ℓ. Let M be a point on ℓ. Find all points P such that the tangents from P to the circle meet ℓ at points Q, R such that the midpoint of QR is M.

7. Let $AB + BC = 3AC$ in $\triangle ABC$ with incentre I. Let the incentre touch AB, BC at D, E. Let DK, EL be diametres of the incircle. Show that $ACKL$ is cyclic.

8. Let the incircle of $\triangle ABC$ meet side BC at D and the A-excircle of $\triangle ABC$ meet side BC at E. Let the incentre be I and the excentre be J. Let the altitude from A to BC be H.
   (a) Show that DJ meet on the midpoint of AH.
   (b) Let X be the intersection of DJ and the incircle. Let N be the midpoint of XD. Show that $ND \times DJ = BD \times DC$.
   (c) Hence or otherwise show that $XBKC$ is cyclic, where K is the midpoint of DJ
   (d) Hence show that the incircle and $XBKC$ must be tangent at X.

9. Let α, β be two disjoint circles with centres P, Q. Let consider a common external tangent that is tangent at points $A \in \alpha, B \in \beta$. Consider the circle with AB as a diameter, and let this meet α, β at X, Y respectively. Show that AY, BX, and PQ are concurrent.

3 Composition of dilations

1. Let $\mathcal{H}_1(X, k_1)$ and $\mathcal{H}_2(Y, k_2)$ be two dilations. Show that their composition is some dilation $\mathcal{H}(Z, k)$ where X, Y, Z are collinear and $k_1 k_2 k > 0$. This means, if you do the first dilation, then do the second dilation, it's the same thing as doing one dilation.

2. (Monge's theorem) Let α, β, γ be three disjoint circles outside of each other. Let the external common tangents of α, β meet at X, and those of β, γ meet at Y, and those of γ, α meet at Z. Show that α, β, γ are collinear.

3. Let M, N, P be the midpoints of sides BC, CA, AB of $\triangle ABC$. Let J, K, L be the incentres of $\triangle APN, BMP, CNM$.

   (a) Show that $\triangle JKL \sim \triangle ABC$.

   (b) Prove that JM, KN, LP, IG are concurrent, where I is the incentre, G is the centroid.

4. Consider three circles $\omega_1, \omega_2, \omega_3$ outside of each other. Let a circle γ be externally tangent to them at A_1, A_2, A_3 respectively. Let another circle Γ be internally tangent to them at B_1, B_2, B_3 respectively. Show that A_1B_1, A_2B_2, A_3B_3 are concurrent.

5. (a) Let D, E, F be the incentre touch points of sides BC, CA, AB. Let X, Y, Z be the midpoint of the arcs of BC, CA, AB of circumcircle of $\triangle ABC$. Show that DX, EY, FZ are concurrent at point H.

   (b) Let ω be a circle which is internally tangent to the circumcircle of $\triangle ABC$ at T and tangent to sides AB, AC. Show that A, H, T are collinear.

6. (a) Let β, γ be internally tangent to α at X, Y, and externally tangent to each other at Z. Let the common tangent at Z meet α at a point P. Let PX, PY meet β, γ at A, B respectively. Show that AB is tangent to β and γ.

   (b) Let XZ meet α again at K. Show that $KY \perp YZ$.

7. Let $\triangle ABC$ have incircle ω, A-excircle Γ. Recall that the nine point circle α is tangent to these two circles. Let $\alpha \cap \omega = X$, $\alpha \cap \Gamma = Y$. Let $XY \cap BC = D$. Show that AD is the angle bisector of $\angle BAC$.