Harmonic points

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Definition. *A, X, B, Y are harmonic points on a line if $AX/BX = AY/BY$.*

1 Properties

1. Let K be a point in the interior of $\triangle PAB$. Let PK, AK, BK intersect AB, PB, PA at X, Q, R. Let QR intersect AB at Y. Show that A, X, B, Y are harmonic.

2. (Harmonic quadrilateral) Let $\triangle AXY$ have circumcircle ω. If the tangents at X, Y to ω intersect at T, and AT intersects ω again at B, show that A, X, B, Y are harmonic, that is, $AX/BX = AY/BY$ (line segment lengths). Show that AB is a symmedian of $\triangle AXY$.

3. If A, X, B, Y are harmonic points and if M is the midpoint of XY,
   (a) Show that $XM^2 = MA \times MB$.
   (b) Show that $AX \times AY = AB \times AM$.
   (c) Show that $BX \times BY = BA \times BM$.
   (d) Show that the converses hold.

4. Let A be a point outside of circle ω. Let $S, T, X, Y \in \omega$ such that AS, AT are tangents to ω and A, X, Y are collinear. Let $AY \cap ST = B$. Show that A, X, B, Y are harmonic. (i.e. the *pole* of A is the locus of the point that is harmonic with X, Y as A, X, Y varies as a secant).

5. (Harmonic pencil) Let A, X, B, Y be harmonic points on line ℓ_0 and let J be a point not on the line.
   (a) Show that the value $c = (AX/BX)/(AY/BY)$ is independent of the lengths JA, JX, JB, JY, but only on the angles $\angle AJX, \angle XJB, \angle BJY$.
   (b) Draw another line ℓ_1 such that AJ, XJ, BJ, YJ meets ℓ_1 at A_1, X_1, B_1, Y_1. Show that A_1, X_1, B_1, Y_1 are harmonic.
   (c) Let Γ be a circle passing through J. Let AJ, BJ, XJ, YJ meet the circle at A_2, X_2, B_2, Y_2. Then A_2, X_2, B_2, Y_2 are harmonic, that is, $A_2X_2/B_2X_2 = A_2Y_2/B_2Y_2$ (line segment lengths).

6. Consider triangle PAB and let X be the midpoint of side AB. Let $PY \parallel AB$. Show that PA, PB, PX, PY is a harmonic pencil.

7. Let A, X, B, Y be harmonic and let A', B', C', D' be the image of the inversion with centre P with some radius. Show that A', B', C', D' is harmonic under the following conditions:
   (a) A, X, B, Y is harmonic on a line, and P is not on the line.
   (b) A, X, B, Y is harmonic on a line, and P is on the line.
   (c) A, X, B, Y is harmonic on a circle, and P is not on the circle.
   (d) A, X, B, Y is harmonic on a circle, and P is on the circle.

2 Problems

1. Let $\triangle ABC$ be an isosceles triangle with $AB = BC$ and circumcircle ω. Let the tangents to ω at B, C intersect at T. Let AT intersect ω at X. Show that CX bisects BT.

2. (USAJMO 2011) Let A, B, C, D, E lie on circle ω such that the tangents to ω at B and D intersect at P, and P, A, C are collinear, and $DE \parallel AC$. Show that BE bisects AC.

3. Let ABC be a triangle and let the tangents at B, C meet at point X. We also have $XB \parallel AC$. Let AX meet the circle again at point D. Let $BD \cap AC = E$. Show that $AC = CE$.

4. (2021 Dec prep G7) Let M be the midpoint of side BC of $\triangle ABC$. The circle with centre M and radius AM intersects AB, AC again at P, Q. The tangents to this circle at P, Q intersect at D. Show that the prependicular bisector of BC bisects AD.

5. (USA TST 2011) Let $\triangle ABC$ be an acute triangle with altitudes AD, BE, CF and H as the orthocentre. Let $P, Q \in EF$ such that $AP \perp EF$, $HQ \perp EF$. Let $DP \cap QH = R$. Show that $HQ = HR$.

6. (APMO 2013) Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of AC such that PB and PD are tangent to ω. The tangent at C intersects PD at Q and the line AD at R. Let E be the second point of intersection between AQ and ω. Prove that B, E, R are collinear.

7. Let $\triangle ABC$ have incircle γ touch sides BC, CA, AB at D, E, F respectively. Let $EF \cap BC = X$, $DF \cap AC = Y$, $DE \cap AB = Z$. Let the midpoint of XD, YE, ZF be L, M, N. Show that L, M, N collinear.

8. Let $ABCD$ be a quadrilateral with an inscribed circle ω which is tangent to the sides AB, BC, CD, DA at points P, Q, R, S respectively. We want to show that AC, BD, PR, QS meet at one point.
   (a) Let $PR \cap QS = X$. Let CX meet the circle again at points G and H. Show that $QGRH$ a harmonic quadrilateral.
   (b) Describe an inversion which maps $P \to R$, $Q \to S$.
   (c) Show that A, C, G, H collinear.

9. (The Big Diagram) Let $ABCD$ be a cyclic quadrilateral on circle ω with centre O. $AB \cap CD = X$, $AD \cap BC = Y$, $AC \cap BD = Z$. Let K be the inversion of Z over ω.

   (a) Hence show that Y is the pole of XZ.

   (b) Hence show that XY is the pole of Z.

   (c) Show that $BODK$, $AOCK$ is cyclic.

   (d) Show that $XADK$, $XBCK$, $YKAB$, $YKDC$ cyclic.

   (e) Show that K is the Miquel point of $ABCD$.

10. Let $ABCD$ be a quadrilateral with an inscribed circle ω which is tangent to the sides AB, BC, CD, DA at points P, Q, R, S respectively.

   (a) Show that AC, PQ, RS concurrent. Show that they concur at the pole of BD.

   (b) Let $PR \cap QS = Z$. What is the polar of Z?

   (c) Hence prove again that AC, BD, PR, QS are concurrent.

11. (Butterfly theorem) Consider a cyclic quadrilateral $ABCD$ on circle ω and centre O and let $AC \cap BD = Z$. Choose a line $\ell \perp OZ$ passing through Z. Let $\ell \cap AD = P$, $\ell \cap BC = Q$. Show that $PZ = QZ$.

12. Let the circumcircle of $\triangle ABC$ be Γ. Consider a circle ω which is tangent to the circumcircle internally at T and sides AB and AC at X, Y. Let Z be the second intersection of the circumcircle of $\triangle AXY$ and Γ. Show that $ZTBC$ is a harmonic quadrilateral.

13. (ISL 2016 G2) Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of $\overline{BC}$. The points D, E, F are selected on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.