

Basic combinatorics (J)

Jongmin Lim

March Camp 2019

1 How to count things

1. Number of ways to permute things without replacement: ${}_nP_r$
2. Number of ways to combine things without replacement: ${}_nC_r$
3. Number of ways to permute things with replacement: ${}_n\Pi_r$
4. Number of ways to combine things with replacement: ${}_nH_r$

1.1 What are they really?

Let there be 10 people and 5 chairs.

1. How many different people can sit in the first chair?
2. Let's say the person to sit in the first chair has been chosen. How many people to sit in the second chair?
3. Find an expression for ${}_nP_r$. (watch out how many you're counting!)
4. How many ways can 5 people stand in line?
5. 20 people have a race. How many ways can the gold, silver, bronze prizes be awarded?
6. How many ways are there to choose a president and a vice president out of a club of 20 members?

Now let's say we don't really care about the order they sit, but only whether someone gets a chair.

1. How many ways can a certain 5 people A, B, C, D, E sit on these 5 chairs?
2. This should mean that ${}_nP_r$ counted each combination that many times each. So how many ways are there to choose 5 people from 10 people?
3. Find an expression for ${}_nC_r$.
4. How many ways are there to choose 5 representatives out of a club of 20 members?
5. How many ways are there to line up 5 apples and 4 books?
6. How many ways are there to line up 3 apples, 4 books, and 5 carrots?
7. How many ways are there to put 6 people into a red team and a blue team, 3 people each?
8. How many ways are there to put 6 people into two teams?
9. How many ways are there to put 6 people into three teams?

What if we can repeat things now? There are 10 people and 5 kinds of sandwiches.

1. How many ways can these people choose their sandwiches?

What if we really don't care about the order at all? Let's say we have 10 dollars, and each sandwich costs 1 dollar. There are 5 types of sandwiches.

1. How many ways can we buy these sandwiches? (Here's a hint: Let's *sort* these sandwiches and put them in a line. We can put four *dividers* where the sandwiches change kinds.)

1.2 Summary

$${}_nP_r = n!/(n-r)!, \quad {}nC_r = {}nP_r/r!, \quad {}n\Pi_r = n^r, \quad {}nH_r = {}_{n+r-1}C_r$$

2 Inclusion exclusion principle

There's not much to tell you. Try doing it with questions. Make sure to draw your venn diagram.

1. In a class of 40 boys and girls, 20 people wear glasses. There are 6 more boys than girls. 5 girls don't wear glasses. How many boys don't wear glasses?
2. How many numbers are either multiples of 2, 3, or 5 that are less than or equal to 300?

3 Making sequences

Try to reduce the n -th problem to the $n - 1$ -th problem. If solving the 0-th problem is easy, that means solving the n -th problem shouldn't be so bad.

1. What is the maximal number of regions can n lines can divide the plane?
2. Let there be 10 stairs. You can go up either 1 or 2 steps at a time. How many ways are there to go up 10 stairs?

4 Questions

1. How many non-negative integer solutions are there to $x_1 + x_2 + x_3 = 10$?
2. Five pairs of people go to a raffle. There can be only five winners. How many different ways can the winning raffle be distributed if there is exactly one pair such that neither person did not win the prize?
3. A, B are opposite corners of a 5×4 grid. How many ways can you follow the gridlines to go from A to B ?
4. How many ways can 6 people sit around a round table?
5. How many ways can 6 people sit around a triangular table, two people on each side?
6. How many ways can A, B, C, D, E, F be in line for popcorn, but A, B want to stand next to each other?
7. How many ways can A, B, C, D, E, F be in line for popcorn, but A gets his popcorn before B does?
8. How many ways can A, B, C, D, E, F be in line for popcorn, but A, B doesn't stand next to each other?
9. How many ways can A, B, C, D, E, F be in line for popcorn, but A, B stand next to each other but C stands next to B ?
10. How many ways can A, B, C, D, E, F sit in a round table, but A, B sits next to each other, C doesn't sit next to A , and E sits next to D ?
11. Let there be $n + 1$ people in this class. We plan to choose $r + 1$ people. You are one of the people in this class.
 - (a) How many ways are there to choose $r + 1$ people such that it includes you?
 - (b) How many ways are there to choose $r + 1$ people such that it does not include you?
 - (c) Hence show that ${}_nC_r + {}_nC_{r+1} = {}_{n+1}C_{r+1}$
12. Let there be n different marbles in the bag. You reach and grab a handful.
 - (a) How many ways can your hand consist of 0 marbles? 1 marble? 2 marbles? $n - 1$ marbles? n marbles?
 - (b) Hence show that $\sum_{r=0}^n {}_nC_r = 2^n$.