Sequences (S)

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Here are some powerful techniques in solving questions with sequences.

1 Linear recurrences

1.1 Linear Homogeneous Recurrences

Example 1: $a_{n+2} = 3a_{n+1} - 2a_n$, where $a_0 = 0$, $a_1 = 1$.

We start with the *ansatz* $a_n = x^n$. This gets us

$$x^{n+2} = 3x^{n+1} - 2x^n$$
$$x^2 - 3x + 2 = 0$$
$$(x-1)(x-2) = 0$$

Hence we conclude that $a_n = 2^n$ and $a_n = 1^n$ are solutions to the recurrence relation. Notice that any linear combination of these two solutions is also a solution to the recurrence. So the general solution to the recurrence is $a_n = A2^n + B1^n$ for some constants A and B. Since $a_0 = A + B = 0$ and $a_1 = 2A + B = 1$, we conclude that $A = 1$, $B = -1$. Hence, the solution to this recurrence is $a_n = 2^n - 1$.

Example 2: $a_{n+2} = 4a_{n+1} - 4a_n$, where $a_0 = 1$, $a_1 = 6$.

Using the same *ansatz*, we get the equation $(x-2)^2 = 0$. Notice that we now have a root with multiplicity. So, alongside the usual solution of $a_n = 2^n$, we also have the solution $a_n = n2^n$ (If it was a triple root, then we have the solution $a_n = n^2 2^n$, etc (check it!)). As before, the general solution is $a_n = (A + Bn)2^n$ for some constants A, B, where we deduce $A = 1$ and $B = 2$ via the initial conditions. Thus, the solution is $a_n = (1 + 2n)2^n$.

Exercises: Find the general term.

1. $a_{n+3} = 6a_{n+2} - 11a_{n+1} + 6a_n$, where $(a_0, a_1, a_2) = (-1, 0, 4)$
2. $a_{n+2} = a_{n+1} - a_n$, where $(a_0, a_1) = (0, 1)$
3. $a_{n+3} = a_{n+2} + a_{n+1} - a_n$, where $(a_0, a_1, a_2) = (2, 2, 6)$
4. $F_{n+2} = F_{n+1} + F_n$, where $(a_0, a_1) = (0, 1)$.

Problems:

1. Let $\{a_n\}$ be a positive sequence such that $a_{n+2} = -a_{n+1} + 2a_n$. Show that a_n must be a constant sequence.

2. Let $\{a_n\}$ be an integer sequence such that $2a_{n+2} = 5a_{n+1} - 3a_n$. Show that a_n must be a constant sequence.

3. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(f(x)) = 4f(x) + 5x$.

4. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(x) > x$ and $f(f(x) - x) = 2x$.

5. (ISL 2010 A5) Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that $f(f(x)^2 y) = x^3 f(xy)$.

6. (China 2017) The sequences $\{u_n\}$ and $\{v_n\}$ are defined by $u_{n+2} = 2u_{n+1} - 3u_n$, where $(u_0, u_1) = (1, 1)$ and $v_{n+3} = v_{n+2} - 3v_{n+1} + 27v_n$, where $(v_0, v_1, v_2) = (a, b, c)$. There exists a positive integer N such that for any $n > N$, u_n divides v_n . Prove that $3a = 2b + c$.

7. (SMMC sample) Consider a circle with circumference $\frac{1+\sqrt{5}}{2}$ and mark a point P_0 on it. Going clockwise, you mark the points $P_1, P_2, \cdots$ for every unit arc length. Suppose you terminate at step n, after you mark the point P_n. Show that if P_i, P_j are adjacent on the circle, then $|i - j|$ is a Fibonacci number.

1.2 Linear Inhomogeneous Recurrences

These are sequences with linear a_n terms, with along with some functions of n. The homogeneous version of the recurrence gives the general part of the solution, while you need a particular solution to take care of the inhomogeneous part (this is where you need to be smart).

Example 1: $a_{n+1} = 3a_n - 2$, where $a_0 = 0$.

The homogeneous version, $a_{n+1} = 3a_n$ has an easy general solution $a_n = A3^n$. We can guess a particular solution to the original equation, which is $a_n \equiv 1$. Thus, the general solution is $a_n = A3^n + 1$, where we tweak $A = -1$ to match the initial condition. Thus, the solution is $a_n = 1 - 3^n$.

The methodical way to do this is to rearrange the original equation to $a_{n+1} - 1 = 3(a_n - 1)$, and define $b_n = a_n - 1$. Then the b_n recurrence is $b_{n+1} = 3b_n$, etc. Sometimes guessing is not that easy, so these substitutions are useful.

Example 2: $a_{n+1} - 5a_n = 5^{n+1}$, where $a_0 = 0$

Divide both sides with 5^{n+1} to get $\frac{a_{n+1}}{5^{n+1}} - \frac{a_n}{5^n} = 1$. Define $b_n = \frac{a_n}{5^n}$, then we get $b_{n+1} - b_n = 1$, with $b_0 = \frac{a_0}{5^0} = 0$. This is yields $b_n = n$, so we conclude $a_n = n5^n$.

Problems:

1. $a_{n+1} - 5a_n = (2n + 1)5^{n+1}$, where $a_0 = 0$.

2. (AMO 2015) Define the sequence $\{a_n\}$ such that $a_{n+2} = 2a_{n+1} - a_n + 2$ with $(a_0, a_1) = (4, 7)$. Show that $a_n a_{n+1}$ is always a term of the sequence.

3. (102 PiC) How many subsets of $\{1, 2, 3, \cdots, 2020\}$ has the property that the sum of its elements is a multiple of 5?

1.3 First Order Multi-variable Homogeneous Recurrences

Sometimes you have more than one sequence working in tandem to define each other.

Example: Find the number of binary sequences of length n such that there does not exist consecutive zeros.

Let a_n, b_n be the number of binary sequences of length n ending with 0 and 1 respectively. As there cannot be consecutive zeros, we have $a_{n+1} = b_n$ (every sequence counted in a_{n+1} is a 0 subtended to a b_n sequence) and

$b_{n+1} = a_n + b_n$ (every sequence counted in b_{n+1} is 1 subtended to a a_n or b_n sequence).
Since we're only interested in the sum $x_n := a_n + b_n$, let us inspect this instead.

$$x_n = a_n + b_n = b_{n-1} + b_n = (a_{n-2} + b_{n-2}) + (a_{n-1} + b_{n-1}) = x_{n-2} + x_{n-1}$$

Hence the recurrence is the Fibonacci recurrence.
What if the double-recurrence isn't nice enough so that such manipulations are not immediately obvious?
Our double recurrence can be expressed as $\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}$. Let $\mathbf{x_n} := \begin{pmatrix} a_n \\ b_n \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.
Then $\mathbf{x_{n+1}} = A\mathbf{x_n}$. Thus, our general form becomes $\mathbf{x_n} = A^n \mathbf{x_0}$. The real task is figuring out how to find A^n in a reasonable amount of time. You can *diagonalise* the matrix such that $A = PDP^{-1}$ for some diagonal matrix D (which means we can find D^n easily), so we get $A^n = PD^nP^{-1}$, which is actually solvable (details not included for sanity; look up how to diagonalise a matrix).

1.4 Pell equations

In number theory, this is a Pell equation: $x^2 - dy^2 = 1$ for a non-square number d.
Example: $x^2 - 2y^2 = 1$.
We need an initial solution. There is no method to find this, so good luck with your guesses. In this case, $(x, y) = (3, 2)$ is a solution. Let (x_n, y_n) be a solution to the equation. Then we do a weird factorisation.

$$(x_n - y_n\sqrt{2})(x_n + y_n\sqrt{2}) = 1$$
$$(3 - 2\sqrt{2})(3 + 2\sqrt{2}) = 1$$

Multiplying the terms at their respective locations,

$$((3x_n + 4y_n) - (2x_n + 3y_n)\sqrt{2})((3x_n + 4y_n) + (2x_n + 3y_n)\sqrt{2}) = 1$$

So we see that $(x_{n+1}, y_{n+1}) = (3x_n + 4y_n, 2x_n + 3y_n)$ is a new solution to this Pell equation.
Before you start to write matrices, look a bit closer on how we generated the solutions. $x_n \pm y_n\sqrt{2} = (3 \pm 2\sqrt{2})^n$.

$$x_n = \frac{(3+2\sqrt{2})^n + (3-2\sqrt{2})^n}{2} \qquad y_n = \frac{(3+2\sqrt{2})^n - (3-2\sqrt{2})^n}{2\sqrt{2}}$$

Problems:

1. (AMO 2020) Sequences $\{a_n\}$ and $\{b_n\}$ satisfy $a_1 = b_1 = 1$ and $a_{n+1} = \frac{a_n+2}{a_n+1}$ and $b_{n+1} = \frac{b_n}{2} + \frac{1}{b_n}$. Show that $b_{n+1} = a_{2^n}$.
2. (Crux) $x_{n+2} = x_{n+1}\sqrt{x_n^2 + 1} + x_n\sqrt{x_{n+1}^2 + 1}$ satisfies $x_0 = x_1 = 1$. Find its general form.
3. (2016 April prep) Positive integers a, b, c satisfy $c(ac+1)^2 = (5c+2b)(2c+b)$. Show that if c is odd, then c is a perfect square. Does there exist a solution with an even c? Show that there are infinitely many solutions for this equation (Hint: Try $\gcd(b, c) = 1$).

2 Generating functions

Given a sequence $\{a_n\}$, we consider a formal power series $f(x) = a_0 + a_1x + a_2x^2 + \cdots$. This is actually completely meaningless, but it somehow works under some situations and nobody knows why (which is why it's cool).

Under a combinatorial setting a_n is thought of as the number of ways to choose n things.

Example 1: You have 1 apples, 2 bananas, 3 carrots. How many ways can you choose 4 things?

The gf for apples is $a(x) = 1+x$, the gf for bananas is $b(x) = 1+x+x^2$ and the gf for carrots is $1+x+x^2+x^3$. The gf for the whole thing is $a(x)b(x)c(x) = (1+x)(1+x+x^2)(1+x+x^2+x^3) = x^6+3x^5+5x^4+6x^3+5x^2+3x+1$. Since the coefficient for x^4 is 5, the answer is 5.

The multiplication of the gfs make perfect sense. To choose n things, you must count the number of ways to choose k things, and multiply it with the number of ways to choose $n-k$ things, for all $k = 0, 1, \cdots, n$, and this is exactly how polynomial multiplication works. It even works when it's not a polynomial.

Example 2: You have apples, bananas, carrots, and durians. You have to choose a multiple of 5 apples, even number of bananas, at most 1 carrot and at most 4 durians. How many ways can you choose 2020 things?

The gf for apples is $a(x) = 1 + x^5 + x^{10} + \cdots = \frac{1}{1-x^5}$.

The gf for bananas is $b(x) = 1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2}$.

The gf for carrots is $c(x) = 1 + x$.

The gf for durians is $d(x) = 1 + x + x^2 + x^3 + x^4$.

The gf for the whole thing is $a(x)b(x)c(x)d(x) = \frac{1}{(1-x)^2}$. To retrieve the power series form, write out the Taylor series $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$. Probably the easier way in this particular case is to use the commonly known $\frac{1}{1-x} = 1 + x + x^2 + \cdots$ and differentiate both sides. The answer is 2021.

Try subbing in $x = 1/2$ and show off to all your friends that if a_n is the general answer to the problem above, then $\sum_{n=0}^{\infty} \frac{a_n}{2^n} = 2$.

Problems:

1. Consider an $n \times n$ grid. Let c_n be the number of shortest paths along the gridlines from the bottom left corner to the top right corner. Find the general form for c_n. What is $\sum_{n=0}^{\infty} \frac{c_n}{4^n}$?.

2. (SMMC 2018) For each positive integer n, consider a cinema with n seats in a row, numbered left to right from 1 up to n. There is a cup holder between any two adjacent seats and there is a cup holder on the right of seat n. So seat 1 is next to one cup holder, while every other seat is next to two cup holders. There are n people, each holding a drink, waiting in line to sit down. In turn, each person chooses an available seat uniformly at random and does the following:

   (a) If they sit next to two empty cup holders, then they choose a cup holder at random and put their drink in it

   (b) If they sit next to one empty cup holder, then they place their drink in that cup holder.

   (c) Otherwise, they get angry.

   Let p_n be the probability that no one gets angry. What is $p_1 + p_2 + p_3 + \cdots$?

3. An alternating permutation is a permutation of $\{1, 2, \cdots, n\}$ such that $a_1 < a_2 > a_3 < a_4 > a_5 \cdots$. How many alternating permutations are there?