Hard problems

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1 Medium hard problems

1. Let $ABCD$ be a cyclic quadrilateral with diameter AB and centre O. Let $CD \cap AB = X$. Let the circumcircles of $\triangle AOD$ and $\triangle BOC$ meet again at Y. Show that $\angle OYX = 90°$.

2. (APMO 1998) Let ABC be a triangle and $D \in BC$ such that $AD \perp BC$. Let E, D, F be collinear such that $AE \perp BE$, and $AF \perp CF$. Let M be the midpoint of BC, and N be the midpoint of EF. Show that $AN \perp NM$.

3. Let I be the incentre of $\triangle ABC$. Let Γ be the circiumcircle of $\triangle ABC$. Let $AI \cap \Gamma = \{A, D\}$. Let E be a point on arc BDC and $F \in BC$ such that $\angle BAF = \angle EAC < \frac{1}{2}\angle BAC$. Let G be the midpoint of IF. Show that $EI \cap DG \in \Gamma$.

4. Let ω be the incircle of $\triangle ABC$. Let ω touch sides BC, CA at D, E respectively. Let D' be the reflection of D over the midpoint of BC. let E' be the reflection of E over the midpoint of AC. Let $P = AD' \cap BE'$. Let AD' intersect ω at two points, the closer one to A to be labelled Q. Show that $|AQ| = |D'P|$.

5. Let $\triangle ABC$ be inscribed in circle Γ with $AB = AC$. Circles ω_B and ω_C are inscribed in the circular segments given by AB and AC respectively. ω_B and ω_C are tangent to Γ at X, Y respectively. Let the common external tangent to ω_B and ω_C that is closer to A intersect AB, AC at P, Q respectively. Show that PX and QY intersect on the angle bisector of $\angle BAC$.

2 Hard problems

1. (IMO 2006 ISL G9) Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2 respectively ($A_2 \neq A, B_2 \neq B, C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

2. (Dec camp 2024 G5) Let $ABCD$ be a cyclic quadrilateral inscribed in circle Γ with centre O. Let $AB \cap CD = E$, $BC \cap AD = F$. Let the midpoint of EF be M. Let $T \in \Gamma$ such that MT is a tangent to Γ. Show that $MT = EM$.

3. (IMO 2008 Q6) Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to the lines AD and CD. Prove that the common external tangents to ω_1 and ω_2 intersect on ω.

4. (IMO 2015 G5) Let ABC be a triangle with $CA \neq CB$. Let D, F, and G be the midpoints of the sides AB, AC, and BC respectively. A circle Γ passing through C and tangent to AB at D meets the segments AF and BG at H and I, respectively. The points H' and I' are symmetric to H and I about F and G, respectively. The line $H'I'$ meets CD and FG at Q and M, respectively. The line CM meets Γ again at P. Prove that $CQ = QP$.

5. (IMO 2015) Let ABC be an acute triangle with $AB > AC$. Let Γ be its circumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = 90°$ and let K be the point on Γ such that $\angle HKQ = 90°$. Assume that the points A, B, C, K, and Q are all different and lie on Γ in this order. Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

3 Hints to hard problems

1. Come on you got this
2. This is the most important diagram of all time.
3. $DF \cap H'I'$ is an important point.
4. The circles are placed in a cringe way. How can we change this?
5. $ABCD$ has an escribed circle. Find "top" and "bottom" points.