1 Problems

1. Eren and Mikasa play a game where Eren thinks of a number from 1 to $2n$ and Mikasa has to guess what it is. If she guesses correctly, she wins. Every time Mikasa makes an incorrect guess, Eren increases or decreases his number by 1, but his number has to stay between 1 to $2n$. What's the smallest number k such that Mikasa can guarantee a win within k guesses?

2. Let n be a positive integer. Consider $2n$ points equally spaced around the circle, n of them coloured blue, and n of them coloured red. Write down the shortest arc-distance between each pair of blue points, from shortest to longest. Do the same for each pair of red points. Show that the two lists are identical.
   Example: When $n = 4$, let the points be coloured *bbrbbrrr* around a circle. Then the blue distances are $B = (1, 1, 2, 3, 3, 4)$, and the red distances are also $R = (1, 1, 2, 3, 3, 4)$. Obviously $R = B$. Why?

3. Find the number of polynomials $P(x)$ such that $P(3) = n$, and each coefficient of P is in the set $\{0, 1, ...8\}$. For example, there are 4 polynomials with $P(3) = 10$: $x + 7$, $2x + 4$, $3x + 1$, and $x^2 + 1$.

4. Komi has N weights weighing $2^0, 2^1, \cdots, 2^{N-1}$ grams each. Her friend Tadano happens to have a balance scale. How many ways can Komi put the weights on the balance scale such that the left side is never heavier than the right side? For example, for $N = 2$, there are 3 ways: (1 right, 2 right), or (2 right, 1 left), or (2 right, 1 right).

5. Can we partition the natural numbers $\mathbb{N}$ into two disjoint sets A, B, such that A does not contain any arithmetic sequences of infinite length, and B does not contain any arithmetic sequences of length 3? In other words, does there exist $A, B \subseteq \mathbb{N}$ such that $A \cap B = \emptyset$, $A \cup B = \mathbb{N}$ and $\nexists a, d \in \mathbb{N}$ such that $\{a + (n-1)d \mid n \in \mathbb{N}\} \subseteq A$ nor $\{a, a+d, a+2d\} \subseteq B$.

6. Tell a clever story to prove
$$\sum_{k=0}^{n} 2^k \binom{n}{k} \binom{n-k}{\lfloor (n-k)/2 \rfloor} = \binom{2n+1}{n}$$

7. Consider an $n \times n$ grid with numbers in each cell, such that for each $k = 1, 2, \cdots, n$, the $2n-1$ cells in either k or column k contains the numbers $1, 2, \cdots, 2n-1$ in some way. Show that this is impossible for $n = 2019$. Show that there are infinitely many n for which this is possible. For example, it is possible for $n = 4$ like the following

1	2	3	4
5	1	7	6
6	4	2	5
7	3	1	2

8. n people play a chess tournament where everyone plays each other once. If you win a game, you win 1 point, if you draw, you get 0.5, and if you lose, you get 0. After all $\binom{n}{2}$ rounds, everyone noticed that they gained exactly half of their points with their games against the lowest scoring 10 players. (The lowest players each gained exactly half of their points with their games against the other 9 lowest scoring players, too). What is n?

Hints are on the next page, so don't scroll/turn over if you don't it.

2 Hints

1. Try small cases. How can you corner your opponent?
2. Start from an obviously symmetric configuration. What happens when you swap things bit by bit?
3. Looks like a good ol base 3 problem but the coefficients are a bit weird... can you do something about the coefficients? Hint: $8 = 3 \times 3 - 1$
4. How can you make an $N + 1$-sequence from an N-sequence? Do exactly the opposite of your first thought, if your first attempt didn't work.
5. Notice that every arithmetic sequence can be defined by a pair $(a, d) \in \mathbb{N}^2$. Do you remember how $\mathbb{N}^2$ is countable? If you don't, you should try looking up why the rationals $\mathbb{Q}$ is countable. Use induction.
6. Obviously the story should start with choosing n things out of $2n + 1$ things. Now the 2-power and the 2-floor should smell like some things pairing up...
7. How many times should each number appear on the grid? The main diagonal plays a big part in this.
8. Count the scores in two different ways, set up an equation, get two possible values for n, and choose the right one.