

# Hard problems

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## 1 Medium hard problems

1. Let  $ABCD$  be a cyclic quadrilateral with diameter  $AB$  and centre  $O$ . Let  $CD \cap AB = X$ . Let the circumcircles of  $\triangle AOD$  and  $\triangle BOC$  meet again at  $Y$ . Show that  $\angle OYX = 90^\circ$ .
2. (APMO 1998) Let  $ABC$  be a triangle and  $D \in BC$  such that  $AD \perp BC$ . Let  $E, D, F$  be collinear such that  $AE \perp BE$ , and  $AF \perp CF$ . Let  $M$  be the midpoint of  $BC$ , and  $N$  be the midpoint of  $EF$ . Show that  $AN \perp NM$ .
3. Let  $I$  be the incentre of  $\triangle ABC$ . Let  $\Gamma$  be the circumcircle of  $\triangle ABC$ . Let  $AI \cap \Gamma = \{A, D\}$ . Let  $E$  be a point on arc  $BDC$  and  $F \in BC$  such that  $\angle BAF = \angle EAC < \frac{1}{2}\angle BAC$ . Let  $G$  be the midpoint of  $IF$ . Show that  $EI \cap DG \in \Gamma$ .
4. Let  $\omega$  be the incircle of  $\triangle ABC$ . Let  $\omega$  touch sides  $BC, CA$  at  $D, E$  respectively. Let  $D'$  be the reflection of  $D$  over the midpoint of  $BC$ . Let  $E'$  be the reflection of  $E$  over the midpoint of  $AC$ . Let  $P = AD' \cap BE'$ . Let  $AD'$  intersect  $\omega$  at two points, the closer one to  $A$  to be labelled  $Q$ . Show that  $|AQ| = |D'P|$ .
5. Let  $\triangle ABC$  be inscribed in circle  $\Gamma$  with  $AB = AC$ . Circles  $\omega_B$  and  $\omega_C$  are inscribed in the circular segments given by  $AB$  and  $AC$  respectively.  $\omega_B$  and  $\omega_C$  are tangent to  $\Gamma$  at  $X, Y$  respectively. Let the common external tangent to  $\omega_B$  and  $\omega_C$  that is closer to  $A$  intersect  $AB, AC$  at  $P, Q$  respectively. Show that  $PX$  and  $QY$  intersect on the angle bisector of  $\angle BAC$ .

## 2 Hard problems

1. (IMO 2006 ISL G9) Points  $A_1, B_1, C_1$  are chosen on the sides  $BC, CA, AB$  of a triangle  $ABC$  respectively. The circumcircles of triangles  $AB_1C_1, BC_1A_1, CA_1B_1$  intersect the circumcircle of triangle  $ABC$  again at points  $A_2, B_2, C_2$  respectively ( $A_2 \neq A, B_2 \neq B, C_2 \neq C$ ). Points  $A_3, B_3, C_3$  are symmetric to  $A_1, B_1, C_1$  with respect to the midpoints of the sides  $BC, CA, AB$  respectively. Prove that the triangles  $A_2B_2C_2$  and  $A_3B_3C_3$  are similar.
2. (Dec camp 2024 G5) Let  $ABCD$  be a cyclic quadrilateral inscribed in circle  $\Gamma$  with centre  $O$ . Let  $AB \cap CD = E, BC \cap AD = F$ . Let the midpoint of  $EF$  be  $M$ . Let  $T \in \Gamma$  such that  $MT$  is a tangent to  $\Gamma$ . Show that  $MT = EM$ .
3. (IMO 2008 Q6) Let  $ABCD$  be a convex quadrilateral with  $BA \neq BC$ . Denote the incircles of triangles  $ABC$  and  $ADC$  by  $\omega_1$  and  $\omega_2$  respectively. Suppose that there exists a circle  $\omega$  tangent to ray  $BA$  beyond  $A$  and to the ray  $BC$  beyond  $C$ , which is also tangent to the lines  $AD$  and  $CD$ . Prove that the common external tangents to  $\omega_1$  and  $\omega_2$  intersect on  $\omega$ .
4. (IMO 2015 G5) Let  $ABC$  be a triangle with  $CA \neq CB$ . Let  $D, F$ , and  $G$  be the midpoints of the sides  $AB, AC$ , and  $BC$  respectively. A circle  $\Gamma$  passing through  $C$  and tangent to  $AB$  at  $D$  meets the segments  $AF$  and  $BG$  at  $H$  and  $I$ , respectively. The points  $H'$  and  $I'$  are symmetric to  $H$  and  $I$  about  $F$  and  $G$ , respectively. The line  $H'I'$  meets  $CD$  and  $FG$  at  $Q$  and  $M$ , respectively. The line  $CM$  meets  $\Gamma$  again at  $P$ . Prove that  $CQ = QP$ .
5. (IMO 2015) Let  $ABC$  be an acute triangle with  $AB > AC$ . Let  $\Gamma$  be its circumcircle,  $H$  its orthocenter, and  $F$  the foot of the altitude from  $A$ . Let  $M$  be the midpoint of  $BC$ . Let  $Q$  be the point on  $\Gamma$  such that  $\angle HQA = 90^\circ$  and let  $K$  be the point on  $\Gamma$  such that  $\angle HKQ = 90^\circ$ . Assume that the points  $A, B, C, K$ , and  $Q$  are all different and lie on  $\Gamma$  in this order. Prove that the circumcircles of triangles  $KQH$  and  $FKM$  are tangent to each other.

### 3 Hints to hard problems

1. Come on you got this
2. This is the most important diagram of all time.
3.  $DF \cap H'I'$  is an important point.
4. The circles are placed in a cringe way. How can we change this?
5.  $ABCD$  has an escribed circle. Find "top" and "bottom" points.