Length chasing

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1 Tangents to a circle from a point are equal

1. Let the incircle of $\triangle ABC$ touch sides BC, CA, AB at D, E, F respectively.
   (a) If $AB = 9, BC = 10, CA = 11$, what is the length of AE?
   (b) Let P, Q be points on side AB, AC such that $PQ \parallel BC$ and $I \in PQ$ where I is the incentre. What is the length of PQ?
   (c) Let the A-excircle touch BC, CA, AB at X, Y, Z respectively. Show that $BX = CD$.
   (d) Show that $AY = AZ = \frac{AB+BC+CA}{2}$.
2. Quadrilaterals with circles
   (a) Let a convex $ABCD$ have an inscribed circle. Show that $AB+CD = AD + BC$.
   (b) Let a concave $ABCD$ have an inscribed circle. Show that $AB+CD = AD + BC$.
   (c) Let a convex $ABCD$ have an escribed circle with non-parallel sides. Show that $AB + BC = AD + DC$.
   (d) Let a concave $ABCD$ have an escribed circle with non-parallel sides. Show that $AB + BC = AD + DC$.
   (e) Show that the converse of the statements above also hold.
3. Incircle of a right angle triangle
   (a) Let the incircle of $\triangle ABC$ have radius r. Show that
   $$\text{Area } ABC = \frac{r}{2}(a + b + c)$$
   (b) Let $\triangle ABC$ have $\angle A = 90°$. Show that the inradius $r = \frac{AB+AC-BC}{2}$.
   (c) Hence or otherwise find all right angle triangles with integer side lengths whose area equals its perimeter.
4. Let $ABCD$ be a parallelogram. Let the incircle of $\triangle ACD$ and $\triangle ABC$ touch AC at K, M. Let the incircle of $\triangle BCD$ and $\triangle ABD$ meet BD at L, N. Show that $KLMN$ is a rectangle. (Hint: what defines a rectangle?)
5. Let D be an arbitrary point on side BC of a given triangle ABC and let E be the intersection of AD and the second external common tangent of the incircles of $\triangle ABD$ and $\triangle ADC$. As D moves along side BC show that the locus of E is a circle.

6. Let $\triangle ABC$ have three cevians AX, BY, CZ, meeting at point one D inside the triangle. Assume we have $AZ + DY = AY + ZD$ and $BZ + DX = BX + ZD$. Show that $CX + DY = CY + DX$.

7. Let I be the incentre of $\triangle ABC$. Show that

$$\frac{AI^2}{bc} + \frac{BI^2}{ac} + \frac{CI^2}{ab} = 1$$

2 Menelaus and Ceva

Theorem 2.1. *(Menelaus) A line ℓ intersects the sides BC, CA, AB of $\triangle ABC$ at X, Y, Z respectively. Then we have*

$$\frac{AZ}{ZB} \times \frac{BX}{XC} \times \frac{CY}{YA} = 1$$

Theorem 2.2. *(Ceva) Let P be a point inside $\triangle ABC$. Let $AP \cap BC = X$, $BP \cap AC = Y$, $CP \cap AB = Z$. Then we have*

$$\frac{AZ}{ZB} \times \frac{BX}{XC} \times \frac{CY}{YA} = 1$$

Theorem 2.3. *(Generalised angle bisector theorem) Let $D \in BC$ in $\triangle ABC$. We have*

$$\frac{AB \sin BAD}{AC \sin CAD} = \frac{BD}{DC}$$

Theorem 2.4. *(Sin lemma) For triangle ABC with a circumradius of r,*

$$\frac{BC}{\sin(\angle BAC)} = \frac{CA}{\sin(\angle ABC)} = \frac{AB}{\sin(\angle CAB)} = 2r$$

1. Triangle centres

   (a) Show that the medians of a triangle meet at one point. I.e. the centroid exists.

   (b) Show that the angle bisectors of a triangle meet at one point. I.e. the incentre exists.

   (c) Show that the altitudes of a triangle meet at one point. I.e. the orthocentre exists.

   (d) Let the incircle of $\triangle ABC$ be tangent to BC, CA, AB at D, E, F. Show that AD, BE, CF are concurrent. I.e. the Gergonne point exists.

   (e) Let M, N, L be the midpoint of sides BC, CA, AB. Let X be a point on BC such that $\angle BAM = \angle CAX$. Similarly, let $Y \in AC$ such that $\angle CBN = \angle ABY$, and $Z \in AB$ such that $\angle ACL = \angle BCZ$. Show that AX, BY, CZ are concurrent. I.e. the symmedian centre exists.

   (f) Show that the internal angle bisector and two external angle bisectors meet at one point. I.e. the excentre exists.

   (g) Show that the perpendicular bisectors of the triangle sides meet at one point. I.e. the circumcentre exists.

2. (Apollonius Circle) Let A, B be two points. Find the locus of X that satisfies $|AX| = r|BX|$ for a given $r > 0$.
3. Consider $\triangle ABC$ where $\angle A = 60°$, $\angle B = 45°$, and $AC = 2$, what is AB?
4. Let convex quadrilateral $ABCD$ have an inscribed circle tangent to AB, BC, CD, DA at P, Q, R, S. Show that PQ, RS, AC meet at one point.
5. Let $\triangle ABC$ have incircle touching the sides BC, CA, AB at D, E, F. Let $EF \cap BC = X$, $FD \cap CA = Y$, $DE \cap AB = Z$. Show that X, Y, Z are collinear.
6. Let $\triangle ABC$ have side lengths $BC = a, CA = b, AB = c$. Let I be the incentre of $\triangle ABC$ and $AI \cap BC = D$. Show that $\frac{AI}{ID} = \frac{b+c}{a}$.
7. Let M be the midpoint of side BC of $\triangle ABC$. Let I be the incentre, and let the incircle touch side BC at D. Let N be the midpoint of AD. Show that N, I, M collinear.
8. Let $ABCDEF$ be a hexagon inscribed in a circle. Show that AD, BE, CF are concurrent if and only if $AB \times CD \times EF = BC \times DE \times FA$.
9. Consider a quadrilateral $ABCD$ such that the incircle of ABD touches sides AB, BD, DA at P, X, Q and the incircle of CBD touches sides CB, BD, DC at R, X, S. Show that P, Q, R, S are cyclic.
10. Let ABC be a triangle with incentre I. A straight line through I intersects sides AB and AC at points P and Q, respectively. Let $a = BC$, $b = AC$, $c = AB$, $p = \frac{PB}{PA}$ and $q = \frac{QC}{QA}$. Prove that if $a^2 = 4bcpq$, then lines AI, BQ and CP are concurrent.

3 Power of a point

Theorem 3.1. *(Power of a point) Let chords AB and CD of a circle γ intersect at X. Then*

$$AX \times BX = CX \times DX$$

1. (Harmonic) Let D be a point inside $\triangle ABC$. Let $AD \cap BC = X$, $BD \cap AC = Y$, $CD \cap AB = Z$. Let $YZ \cap BC = W$. Show that
   (a) $BX/CX = BW/CW$.
   (b) Let M be the midpoint of BC. Show that $BM^2 = MX \times MW$.
   (c) Show that $WB \times WC = WM \times WX$.
   (d) Show that $XB \times XC = XM \times WX$.
2. Let $\triangle ABC$ have incircle touching the sides BC, CA, AB at D, E, F. Let $EF \cap BC = X$, $FD \cap CA = Y$, $DE \cap AB = Z$.
   (a) Show that X, B, D, C is harmonic.
   (b) Let M be the midpoint of BC. Show that $XM \times XD = XB \times XC$.
   (c) Let P be the midpoint of XD. Show that $PD^2 = PB \times PC$.
   (d) Similarly let Q be the midpoint of YE, and R is the midpoint of ZF. Show that P, Q, R collinear.