Bounding arguments (I)

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1 Example 1

For an integer x, when is $x^2 + 5x + 15$ a square number?

1. What is a square number that is obviously smaller/bigger than this?
2. This brings us to just a few cases. Finish the problem.

2 Example 2

Find all positive integers a, b, c such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$.

1. Let us assume $a \geq b \geq c$. Why are we allowed to assume this? What does this tell us about c?
2. This brings us to just a few cases. Finish the problem.

3 Example 3

Find all $n \in \mathbb{N}$ such that $n^3 + 6n^2 + 2n + 1$ is divisible by $n^2 + 3n + 1$.

1. What's close to $n^3 + 6n^2 + 2n + 1$ and already divisible by $n^2 + 3n + 1$?
2. Let's look at what's left. What happens when n gets really big?
3. This brings us to just a few cases. Finish the problem.

4 Example 4

Let $\{a_n\}$ be a sequence of positive integers such that $a_{n+2} = -a_{n+1} + 2a_n$. Show that a_n must be a constant sequence.

1. Can you find some clever ways to rearrange the recurrence relation?
2. What happens when n gets really big?

5 Example 5

Find integers x, y such that $3^x - 2^y = 1$.

6 Questions

1. Find all positive integers x, y such that $x^3 = y^3 + 2y^2 + 8$.

2. Let n be a positive integer such that $n + 10$ divides $n^3 + 100$. Find the biggest such number.

3. (AMC) $\frac{1}{3} + \frac{1}{n}$ has a denominator less than n. How many such integer n are there from 0 to 2018?

4. Find all $x, y \in \mathbb{Z}$ such that $x^3 - y^3 = xy + 61$.

5. Find all $n \in \mathbb{N}$ such that $n! + 5$ is a perfect cube.

6. (Hungary 1995) The product of a few primes is ten times as much as the sum of the primes. What are these (not necessarily distinct) primes?

7. (Russia 1999) Do there exist 19 distinct positive integers that add up to 1999 and have the same sum of digits?

8. (Bulgaria 1995) find all primes p, q such that pq divides $(5^p - 2^q)(5^q - 2^p)$.

9. Find all positive integers x, y such that x^2+3y and y^2+3x are both perfect squares.

10. Suppose $a, b, n \in \mathbb{N}$ where $n^2 + 1 = ab$. Prove that $|a - b| \geq \sqrt{4n - 3}$.

11. How many $0 \leq n \leq 2018$ such that there exists a, b where $n^2 + 1$ and $a - b = \sqrt{4n - 3}$?

12. (SMMC) Find all primes p, q such that $p^{q-1} + q^{p-1}$ is a perfect square.

13. (SMMC) Find all even x, odd q, and integer y such that $x^3+x^2+x+1 = y^q$.

14. Find all $n \in \mathbb{N}$ such that $n^2 + 5n + 1$ is a perfect square.

15. Show that there are infinitely many integer solutions to $a^2 + b^2 + c^2 = a^3 + b^3 + c^3$.

16. (TST 2016) Let $m > n$ be positive integers. Let $x_k = \frac{m+k}{n+k}$ for $k = 1, 2, ..., n + 1$. Prove that if all the numbers $x_1, x_2, ..., x_{n+1}$ are integers, then $x_1 x_2 ... x_{n+1} - 1$ is divisible by an odd prime.

17. Find all triples (x, y, z) of positive integers such that $x \leq y \leq z$ and
$$x^3(y^3 + z^3) = 2012(xyz + 2)$$

18. Find the number of ordered triples (x, y, z) of positive integers such that $x, y, z \leq 2013$ and
$$x|(yz - 1), \quad y|(xz - 1), \quad z|(xy - 1)$$

7 Linear Homogeneous Recurrences

Example 1: $a_{n+2} = 3a_{n+1} - 2a_n$, where $a_0 = 0$, $a_1 = 1$.
We start with the *ansatz* $a_n = x^n$. This gets us

$$\begin{aligned} x^{n+2} &= 3x^{n+1} - 2x^n \\ x^2 - 3x + 2 &= 0 \\ (x-1)(x-2) &= 0 \end{aligned}$$

Hence we conclude that $a_n = 2^n$ and $a_n = 1^n$ are solutions to the recurrence relation. Notice that any linear combination of these two solutions is also a solution to the recurrence. So the general solution to the recurrence is $a_n = A2^n + B1^n$ for some constants A and B. Since $a_0 = A + B = 0$ and $a_1 = 2A + B = 1$, we conclude that $A = 1$, $B = -1$. Hence, the solution to this recurrence is $a_n = 2^n - 1$.
Example 2: $a_{n+2} = 4a_{n+1} - 4a_n$, where $a_0 = 1$, $a_1 = 6$.
Using the same *ansatz*, we get the equation $(x-2)^2 = 0$. Notice that we now have a root with multiplicity. So, alongside the usual solution of $a_n = 2^n$, we also have the solution $a_n = n2^n$ (If it was a triple root, then we have the solution $a_n = n^2 2^n$, etc (check it!)). As before, the general solution is $a_n = (A + Bn)2^n$ for some constants A, B, where we deduce $A = 1$ and $B = 2$ via the initial conditions. Thus, the solution is $a_n = (1 + 2n)2^n$.
Exercises: Find the general term.

1. $a_{n+3} = 6a_{n+2} - 11a_{n+1} + 6a_n$, where $(a_0, a_1, a_2) = (-1, 0, 4)$

2. $a_{n+2} = a_{n+1} - a_n$, where $(a_0, a_1) = (0, 1)$

3. $a_{n+3} = a_{n+2} + a_{n+1} - a_n$, where $(a_0, a_1, a_2) = (2, 2, 6)$

4. $F_{n+2} = F_{n+1} + F_n$, where $(a_0, a_1) = (0, 1)$.

Problems:

1. Let $\{a_n\}$ be an integer sequence such that $2a_{n+2} = 5a_{n+1} - 3a_n$. Show that a_n must be a constant sequence.

2. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $f(f(x)) = 4f(x) + 5x$.

3. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $f(x) > x$ and $f(f(x) - x) = 2x$.

4. (ISL 2010 A5) Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that $f(f(x)^2 y) = x^3 f(xy)$.

5. (China 2017) The sequences $\{u_n\}$ and $\{v_n\}$ are defined by $u_{n+2} = 2u_{n+1} - 3u_n$, where $(u_0, u_1) = (1, 1)$ and $v_{n+3} = v_{n+2} - 3v_{n+1} + 27v_n$, where $(v_0, v_1, v_2) = (a, b, c)$. There exists a positive integer N such that for any $n > N$, u_n divides v_n . Prove that $3a = 2b + c$.