Functional Equations (I)

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1 Basic strats

- Try to get $f(0)$ or $f(1)$.
- Odd function? Even function? Periodic function? Injective function? Surjective function? Additive?
- Guess which functions might work, aim your strats towards them.
- If you can prove that f is linear, take $f(x) = ax + b$ and sub it in and solve for a, b.
- Watch out! Do my substitutions make sense? Did I sub in 0 when the domain is $R_{>0}$?
- Hey I think I solved it. Let's make sure to sub it back in and check if it really is a solution.

2 How to sub

Choose things you think would simplify the equation. Try subbing in zero here and there.

1. $(f : \mathbb{R} \to \mathbb{R})$ $f(x + y) = f(x) + y$
2. $(f : \mathbb{R} \to \mathbb{R})$ $f(x + xy + y) = f(x) + xf(y) + y$

3 Cauchy Functional Equation

The Cauchy functional equation is

$$(f : \mathbb{R} \to \mathbb{R})\ f(x + y) = f(x) + f(y)$$

We have the obvious solution $f(x) = cx$ for $c \in \mathbb{R}$, but there can exist super weird solutions. These super weird solutions are closely packed together in the whole coordinate plane. This means that I can always find a point $(x, f(x))$ in any small circle I draw in the plane. In other words, if I can draw a circle which does not contain any of the points on the graph, then the solution must be nice and linear.

1. Let $f : \mathbb{R} \to \mathbb{R}$ and $f(x + y) = f(x) + f(y)$. We also have $f(x) \geq 3$ for $x \geq 6$. Find all functions.
2. Let $f : \mathbb{R} \to \mathbb{R}$ and $f(x + y) = f(x) + f(y)$. We also have $f(x^2) = f(x)^2$. Find all functions.
3. Let $f : \mathbb{R} \to \mathbb{R}$ and $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$. Find all functions.

4 Twiddle Take

Take the given equation, and substitute something that would change only a small part of the equation. Then, expose the change by equating the non-changed things. For example, if you have a lot of x^2 terms, try subbing $-x$ to x. For example, if x and y are asymmetric, try swapping x and y.

1. $f : \mathbb{R} \to \mathbb{R}$ $f(f(x)) + f(f(y)) = f(xy) - xy + f(x) + y$
2. $f : \mathbb{R} \to \mathbb{R}$ $f(x^2 + y) = x^2 + xf(y) - y(x - 1)$

5 Cool Canceller

Make a substitution which makes the inside of two fs on opposite sides equal.

1. $f : \mathbb{R} \to \mathbb{R}$ $f(\frac{x+y}{2}) = f(f(x) + y) - \frac{1}{4}y$.

6 Injection Engine

An injective function is a function where $f(a) = f(b)$ implies $a = b$.
The injection engine is a twiddle take by assuming $f(a) = f(b)$ for some a, b and seeing what kind of relations arise between a and b. The best case scenario is when we get $a = b$, which is when f is injective.

1. $f : \mathbb{R} \to \mathbb{R}$ $f(x - f(y)) = f(f(x) + f(y)) + 2y$

Sometimes it's good to not be so greedy and show injectivity at zero.

7 Smashing Surjection

A surjective function is a function where an x exists for every y such that $f(x) = y$. For example, we have $f : \mathbb{R} \to \mathbb{R}$ is surjective if $f(x + f(y)) = y + f(x)$, since we can change y to get any value on the RHS.

1. $f : \mathbb{R} \to \mathbb{R}$ $f(xf(x) + f(x)f(y) + y - 1) = f(xf(x)) + xy + y - 1$

8 Warning

Say you've ended up with an equation like $(f(y) - y)f(y) = 0$. This does NOT mean $f(y) = y$ for all $y \in \mathbb{R}$ or $f(y) = 0$ for all $y \in \mathbb{R}$. It DOES mean $f(y) = y$ or $f(y) = 0$ for all $y \in \mathbb{R}$.

9 Angelo's Trick

From some equation like $f(f(x)) = 2x + 1$, we can take $f(f(f(x))) = f(2x + 1)$. But we also can substitute $f(x)$ into x and get $f(f(f(x))) = 2f(x) + 1$. Clearly $2f(x) + 1 = f(2x + 1)$ looks very nice.

10 Linear Homogeneous Recurrences

Consider a recurrence of the form

$$ax_{n+2} + bx_{n+1} + cx_n = 0$$

If the quadratic equation $ax^2 + bx + c = 0$ has...

1. distinct roots α, β, then the general term is of the form $x_n = A\alpha^n + B\beta^n$ for some constants A, B.
2. a double root α, then the general term is of the form $x_n = (An + B)\alpha^n$ for some constants A, B.

You can find the constants A, B using the inital conditions.

10.1 Example 1

Solve $a_{n+2} = 3a_{n+1} - 2a_n$, where $a_0 = 0$, $a_1 = 1$.
We solve the quadratic

$$x^2 - 3x + 2 = 0$$
$$(x - 1)(x - 2) = 0$$

The general solution to the recurrence is $a_n = A2^n + B1^n$ for some constants A and B. Since $a_0 = A+B = 0$ and $a_1 = 2A+B = 1$, we conclude that $A = 1$, $B = -1$. Hence, the solution to this recurrence is $a_n = 2^n - 1$.

10.2 Example 2

Solve $a_{n+2} = 4a_{n+1} - 4a_n$, where $a_0 = 1$, $a_1 = 6$.
The quadratic is $(x - 2)^2 = 0$. The general solution is $a_n = (A + Bn)2^n$ for some constants A, B, where we deduce $A = 1$ and $B = 2$ via the initial conditions. Thus, the solution is $a_n = (1 + 2n)2^n$.

10.3 Problems:

1. $a_{n+2} = -a_{n+1} + 2a_n$ and $a_n > 0$ for all $n \in \mathbb{N}$. Show that a_n is a constant sequence.
2. $2a_{n+2} = 5a_{n+1} - 3a_n$ and $n \in \mathbb{Z}$. Show that a_n must be a constant sequence.

11 Problems

1. $(f : \mathbb{R} \to \mathbb{R})$ $2f(x) + f(1-x) = 3x^2 - 2x + 1$
2. $(f : \mathbb{R} \to \mathbb{R})$ $f(x^2 + y) = f(x)^2 + f(y)$
3. $(f : \mathbb{R} \to \mathbb{R})$ $f\left(\frac{x + f(x)}{2} + y\right) = f(x) + y$
4. $(f : \mathbb{R} \to \mathbb{R})$ $f\left(\frac{x + f(x)}{2} + f(y)\right) = f(x) + y$
5. $(f : \mathbb{R} \to \mathbb{R})$ $f(y)^2 = f(x^2 + yf(y)) - xf(x) + x^2(f(y) - y)$
6. $(f : \mathbb{R} \to \mathbb{R})$ $f(y + 2xf(x)) = f(2f(x)^2 - y) + 2y$
7. $(f : \mathbb{Z} \to \mathbb{Z})$ $f(f(n)) = n + 1$
8. $(f : \mathbb{Z} \to \mathbb{Z})$ $7f(x) = 3f(f(x)) + 2x$
9. $(f : \mathbb{R}^+ \to \mathbb{R}^+)$ Let $a, b \in \mathbb{R}^+$ be fixed numbers. Solve $f(f(x)) + af(x) = b(a + b)x$
10. $(f : \mathbb{R}^+ \to \mathbb{R}^+)$ $(x + y)f(2yf(x) + f(y)) = x^3 f(yf(x))$ (2016 Tset)
11. $(f : \mathbb{R} \to \mathbb{R})$ $f(x^2 + yf(x)) = xf(x + y)$
12. $(f : \mathbb{Z}^+ \to \mathbb{Z}^+)$ $f(n + 1) > f(n)$ and $f(f(n)) = 3n$
13. $(f : \mathbb{R} \to \mathbb{R})$ $f(xf(x) + f(y)) = f(x)^2 + y$
14. $(f : \mathbb{R} \to \mathbb{R})$ $f(x + y) = f(x)f(y) + xy$
15. $(f : \mathbb{Q}^+ \to \mathbb{Q}^+)$ $f(f(x)^2 y) = x^3 f(xy)$ (2010 ISL A5)
16. $(f : \mathbb{R} \to \mathbb{R})$ $f(x) + f\left(1 - \frac{1}{x}\right) = \arctan x$ (Putnam 2016)
17. $(f : \mathbb{R} \to \mathbb{R})$ $f(x + f(y)) = f(y - x) + 2x(f(y + 1) - y)$
18. $(f : \mathbb{Z} \to \mathbb{Z})$ $f(y + f(x)) = f(x - y) + y(f(y + 1) - f(y - 1))$ (2016 TST)
19. $(f : \mathbb{R} \to \mathbb{R})$ $f(x^3 + y^3 + xy) = x^2 f(x) + y^2 f(y) + f(xy)$
20. $(f : \mathbb{R} \to \mathbb{R})$ $f(f(x) + f(y) + f(z)) = f(f(z) + 2xy) + f(f(x) - f(y)) + 2f(xz - yz)$