Angle chasing

Jongmin Lim

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1 Problems

1. (Circle properties) A circle is a set of points with equal distance from a fixed point.
   (a) Let A, B, C be three points on this circle. Prove that $2\angle BAC = \angle BOC$.
   (b) In particular, show that the angles on a semicircle is $90°$.
   (c) Let DE be a chord with M as its midpoint. Show that $OM \perp DE$, where O is the centre.
   (d) Show that if $ABCD$ is a cyclic quadrilateral, then $\angle BAC = \angle BDC$.
   (e) Show that $\angle ABC + \angle ADC = 180°$.
   (f) Let AC and BD intersect at X. Show that $AX \times CX = BX \times DX$.
   (g) Let AD and BC intersect at Y. Show that $YA \times YD = YC \times YB$.
   (h) Let $T \in BC$ such that AT is a tangent. Show that $\angle TAB = \angle BCA$.
   (i) Show that $TA^2 = TB \times TC$.
   (j) Show that (d) to (g) are all if and only if conditions. I.e. if $ABCD$ satisfies those conditions, then $ABCD$ is cyclic.
2. Show that the midpoint of the hypotenuse of a right angled triangle is the circumcentre of the triangle.
3. (Shadow theorem) Let $\triangle ABC$ have $\angle BAC = 90°$ and $D \in BC$ such that $AD \perp BC$. Show that
   (a) $AB^2 = BD \times BC$
   (b) $AC^2 = CD \times CB$
   (c) $AD^2 = BD \times CD$
   (d) Hence or otherwise prove the pythagorean theorem.
4. Consider $\triangle ABC$. Let $D \in AB, E \in AC$ such that $DE \parallel BC$. Let $BE \cap CD = X$. Let $AX \cap BC = M$. Show that $BM = CM$.
5. Let X be outside circle ω with centre O and let $P, Q \in \omega$ such that PX, QX are tangents. Let $PQ \cap OX = K$. Let $A, B \in \omega$ such that X, A, B is collinear. show that $ABKO$ is cyclic.

6. Let two circles α, β intersect at X, Y. Let $A \in \alpha$ and $B \in \beta$ such that AB is a common tangent. Show that XY meets AB at M, where M is the midpoint of AB.

7. Let A, B be two points on a circle. Let TA, TB be tangents to the circle. Let C be another point on the circle such that $BC \parallel TA$. Let TC meet the circle again at D. Let BD meet TA at M. Show that $TM = MA$.

8. (Spiral sym) Let two circles α, β meet at points X, Y. Let $A, C \in \alpha$ and $B, D \in \beta$ such that AXB and CXD are collinear.
   (a) Show that $\triangle YAB \sim \triangle YCD$.
   (b) Show that $\triangle YAC \sim \triangle YBD$.
   (c) Let $AM = BM$. Let $CN = DN$. Show that $\triangle YAC \sim \triangle YMN$.
   (d) Let AC intersect BD at Z. Show that $YAZB$ is cyclic. Show that $YCZD$ is cyclic.

9. (Incentre lemma) Consider D on the midpoint of the arc of BC not containing A. Show that $DB = DC = DI = DJ$, where I is the incentre and J is the excentre of triangle ABC.

10. (Angle bisector lengths) Let D be the midpoint of the arc BC not containing A. Let $AD \cap BC = X$. Show that $DB^2 = DC^2 = DX \times DA$. Also show that $AB \times AC = AX \times AD$.

11. (Shooting lemma) Consider D on the midpoint of the arc of BC not containing A. Choose any two points X, Y on the circle. Let DX, DY meet line BC at points P, Q. Show that $XYPQ$ is cyclic.

12. (Dilation) Let two circles α, β be tangent at point T. Let $A, C \in \alpha$ and $B, D \in \beta$ such that ATB and CTD are collinear. Show that $AC \parallel BD$.

13. (Dilation) Let two circles α, β be tangent at point T. Let $A \in \alpha$ and $B \in \beta$ such that A, T, B are collinear. Let the tangent at A intersect β at X, Y. Show that $BX = BY$.

14. (Nine point circle) Let D, E, F be midpoints of BC, CA, AB of $\triangle ABC$. Let $X \in BC, Y \in CA, Z \in AB$ such that AX, BY, CZ are altitudes of the triangle. Let P, Q, R be midpoints of AH, BH, CH. Show that $DEFXYZPQR$ all lie on one circle.

15. (Simson line) Let $ABCD$ be a cyclic quadrilateral. Let E, F, G be on lines AB, BD, DA such that $AB \perp CE$, $BD \perp CF$, $AD \perp CG$. Show that E, F, G are collinear.