Invariants (I)

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1 General strats

1. Start from a nice configuration
2. What makes this configuration nice? Come up with a number which represents how nice a configuration is.
3. Tweak the configuration little by little
4. Calculate what happens to your number when you do these tweaks. Find a pattern.
5. Show that you can get to the problematic position by tweaking.
6. Write up your proof by induction on your number.

2 Problems

1. Let there be 2020 players in a chess tournament. Every round, the players pair up and 1010 games are simultaneously played, where the winner receives 1 point, loser receives zero points, and drawers get 0.5 points each. What is the average number of points of the 2020 players after 1234 games?
2. Elongated Muskrat invents another electric vehicle, called the *Pogsla*. Assume Elongated Muskrat is at coordinate $(0,0)$. Pogsla is a pogo stick which hops 2^{n-1} metres in its n-th jump in one of the four directions, North, East, South, West. For example, he would be able to reach $(4,1)$ by the following series of moves.

   $$(0,0) \to (0,-1) \to (0,1) \to (4,1)$$

   Which coordinates can Elongated Muskrat reach using his Pogsla?
3. Form a 2000×2002 screen with unit screens. Initially, there are more than 1999×2001 unit screens which are on. In any 2×2 screen, as soon as there are 3 unit screens which are off, the fourth screen turns off automatically. Prove that the whole screen can never be totally off.
4. There are 23 people. Their ages have the property that whoever we choose to leave out, the remaining 22 people can be divided into two teams of 11 people so that the sum of the ages of one team is the same as the sum of the ages of the other team. Show that the 23 people must be the same age.
5. Let n be a positive integer. Consider $2n$ points equally spaced around a circle. Suppose that n of them are coloured blue and the rest are coloured red. Now, write down the distance between each pair of blue points in a list, from shortest to longest. Do the same for each pair of red points. Show that the lists are the same.
6. Komi writes 0 on a piece of paper. Then at every step, Komi copies the sequence of numbers twice, adding 1 to each number for each copy. This would be the first 4 steps. If 0 is the *zeroth* term, what is the 2020th term?

(a) 0

(b) 0 / 1 / 2

(c) 0, 1, 2 / 1, 2, 3 / 2, 3, 4

(d) 0, 1, 2, 1, 2, 3, 2, 3, 4 / 1, 2, 3, 2, 3, 4, 3, 4, 5 / 2, 3, 4, 3, 4, 5, 4, 5, 6

7. Let $\leftarrow$ denote the left arrow key. If one opens a text editor and pushes the keys "ab$\leftarrow$ cd $\leftarrow\leftarrow$ e $\leftarrow\leftarrow$ f", the result is "faecdb". We say that a string B is reachable from a string A if it is possible to insert some amount of $\leftarrow$'s in A such that typing the resulting characters produces B. Thus, our example shows that "faecdb" is reachable from "abcdef". Show that if A is reachable from B, then B is reachable from A.

8. Let G be a finite undirected graph such that the edges are labelled by a positive real number. For each vertex, the sum of the labels of its adjacent edges is at most 1. If X, Y are such labellings, then $\frac{X+Y}{2}$ is another labelling, where each edge is labelled by the average of its labels in X and Y. Show that if Z is a labelling such that there does **not** exist distinct X and Y such that $Z = \frac{X+Y}{2}$, then every label of Z must be from the set $\{0, 0.5, 1\}$.

9. Eren and Mikasa play a game where Eren thinks of a number from 1 to $2n$ and Mikasa has to guess what it is. If she guesses correctly, she wins. Every time Mikasa makes an incorrect guess, Eren increases or decreases his number by 1, but his number has to stay between 1 to $2n$. What's the smallest number k such that Mikasa can guarantee a win with in k guesses?

10. Consider an $n \times n$ grid with numbers in each cell, such that for each $k = 1, 2, \cdots, n$, the $2n - 1$ cells in either row k or column k contains the numbers $1, 2, \cdots, 2n - 1$ in some way. Show that this is impossible for $n = 2019$. Show that there are infinitely many n for which this is possible. For example, it is possible for $n = 4$ with the following.

1	2	3	4
5	1	7	6
6	4	2	5
7	3	1	2

11. Let G be a finite simple graph and let k be the largest number of vertices of any clique in G. Suppose that we label each vertex of G with a non-negative real number so that the sum of all such labels is 1. Define the value of an edge to be the product of the labels of the two vertices at its ends. define the value of a labelling to be the sum of the values of the edges. Prove that the maximum possible value of a labelling of G is $\frac{k-1}{2k}$.

12. Find all integers n for which each cell of $n \times n$ table can be filled with one of the letters I, M and O in such a way that

    (a) in each row and each column, one third of the entries are I, one third are M and one third are O; and

    (b) in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are I, one third are M and one third are O.

    Note: The rows and columns of an $n \times n$ table are each labelled 1 to n in a natural order. Thus each cell corresponds to a pair of positive integer (i, j) with $1 \le i, j \le n$. For $n > 1$, the table has $4n - 2$ diagonals of two types. A diagonal of first type consists all cells (i, j) for which $i + j$ is a constant, and the diagonal of this second type consists all cells (i, j) for which $i - j$ is constant.