Equations Functions Graphs (I)

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1 Basics

Let $f : X \to Y$ be a function. We call X the domain, and Y the codomain. Notice that the *range* is different to the codomain. The range is defined as $f(X) = \{f(x)|x \in X\}$. A graph is defined as $\{(x, f(x)|x \in X\}$. Here are some special functions.

1. Injective functions: For each $x \in X$, it has a unique $f(x)$ value. In other words, $x \neq y$ implies $f(x) \neq f(y)$.
2. Surjective functions: This is when the codomain coincides with the range. In other words, for every $y \in Y$, there exists an $x \in X$ such that $f(x) = y$.
3. Bijective functions: Functions that are injective and surjective
4. Inverse functions: When $f : X \to Y$ is bijective, then f creates a one-to-one correspondence between the sets X and Y. This means that this correspondence can be reversed to a function $g : Y \to X$ with the property that $f(g(x)) = g(f(x)) = x$. In this case, we call g the inverse function of f and we denote $g = f^{-1}$.

Most of the time we won't be dealing with functions like f : Cats $\to$ Dogs. We'll mostly be working with functions that take numbers, say, $f : \mathbb{R} \to \mathbb{R}$. Here are some special functions that deal with numbers.

1. Even functions: When $f(x) = f(-x)$. In other words, the function is symmetric about the $y - axis$.
2. Odd functions: When $f(x) = -f(-x)$. In other words, the function is symmetric about the origin.
3. Periodic functions: When there exists a $c > 0$ such that $f(x) = f(x + c)$
4. Involution: When $f(f(x)) = x$.

We say a function has a *fixed point* x_0 if $f(x_0) = x_0$. Fixed points are useful because we can apply f as many times as we want and its value won't change. I.e. $f(f(f(f(f(f(f(x_0))))))) = x_0$. Speaking of which, we can compose two functions to get a new function. For example, take $f(x) = 1 - \frac{1}{x}$ and $g(x) = \frac{1}{1-x}$.

1. Show that $f(f(x)) = g(x)$
2. Show that $g(g(x)) = f(x)$
3. Show that $f(g(x)) = x$
4. Find an expression for $f(f(f(f(f(f(f(f(g(g(g(f(g(f(g(f(g(g(x)))))))))))))))))$.

As you can see, writing a lot of brackets doesn't look very nice. From now on, we'll write function composition nicely:

$$f(g(x)) = (f \circ g)(x)$$

Then the fourth question would look like

4. Find an expression for $(f \circ f \circ f \circ f \circ f \circ f \circ f \circ f \circ g \circ g \circ g \circ f \circ g \circ f \circ g \circ f \circ g \circ g)(x)$.

There are also other functions according to how they behave locally.

1. Increasing functions
   (a) Weakly increasing functions: $f(x) \geq f(y)$ for all $x \geq y$.
   (b) Strictly increasing functions: $f(x) > f(y)$ for all $x > y$.

2. Decreasing functions
   (a) Weakly decreasing functions: $f(x) \leq f(y)$ for all $x \geq y$.
   (b) Strictly decreasing functions: $f(x) < f(y)$ for all $x > y$.
3. Monotonic functions are either increasing functions or decreasing functions. You can also say weakly/strictly monotonic.
4. **Continuous functions** A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point x_0 if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$. A continuous function is a function that is continuous at every point.

That last one was a bit hard. To put it into words, if you can find a reasonably small neighbourhood of x so that their function values are all near $f(x)$, then it is continuous. To put it into the simplest terms, if you can draw the graph without taking your pen off the paper, then it is continuous.
Most of the functions you've seen are continuous. Polynomial functions $f(x) = x^2 + 2x + 3$, and exponential functions $f(x) = 2^x$ are all continuous.

Theorem 1.1 *(Intermediate Value Theorem) If $f : [a, b] \to \mathbb{R}$ is a continuous function, then for any $f(a) < c < f(b)$, there exists an $x \in [a, b]$ such that $f(x) = c$.*

Here are some questions.

1. You walk from point A to point B from 8am to 5pm, at a random speed. You can take breaks, slow down, go backwards for a bit, but you always get to point B by 5pm. The next day at 8am to 5pm you make your journey from point B to point A, taking your time as you wish. Show that there must exist a time and space where you were at the same spot on both days.
2. Show that if f is defined on $[a, b]$ and $f(a) < 0 < f(b)$, f must have a root between a and b.
3. Show that if $f : \mathbb{R} \to \mathbb{R}$ is strictly monotonic, then f must be injective.
4. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is strictly monotonic, but not surjective.
5. Show that even functions cannot be involutions.
6. Show that involutions must be bijective.
7. Find a bijective function $f : (0, 1) \to \mathbb{R}$.
8. Find a non-constant periodic function $f : \mathbb{R} \to \mathbb{R}$ with arbitrarily small periods. In other words, given any $\epsilon > 0$ there exists $0 < c < \epsilon$ such that $f(x + c) = f(x)$ for all $x \in \mathbb{R}$.
9. Find all functions $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ such that $3f(x) + 2f\left(\frac{1}{x}\right) = 4x$.
10. Find all real numbers x such that $10^x + 11^x + 12^x = 13^x + 14^x$
11. Find all functions $f : \mathbb{R} \setminus \{0, 1\} \to \mathbb{R}$ $f(x) + f(\frac{1}{1-x}) = x$
12. $f(x) = (ax + b)/(cx + d)$ for $a, b, c, d \in \mathbb{R} \setminus \{0\}$. If $f(19) = 19$, $f(97) = 97$ and $f(f(x)) = x$, find $f(x)$.