# Similarity Transformations (S)

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## 1 Spiral similarity

$\triangle ABC$ and $\triangle DEF$ are spirally similar if there exists a dilation and a rotation that takes $\triangle ABC$ to $\triangle DEF$.

1. Let $\triangle ABC \sim \triangle AXY$ with spiral centre at $A$. Show that $\triangle ABX \sim \triangle ACY$.
2. Let $\triangle ABC \sim \triangle AXY$ be spirally similar. Let $BC \cap XY = Z$. Show that $ABZX$ is cyclic. Show that $ACZY$ is cyclic.
3. Let $\triangle ABC \sim \triangle AXY$ be spirally similar. Let $M$ bet the midpoint of $BX$, and $N$ be the midpoint of $CY$. Show that $\triangle AMN \sim \triangle ABC$.
4. Given two line segments $BC$ and $XY$, how do you construct point $A$ such that $\triangle ABC \sim \triangle AXY$ spirally similar?

## 2 Examples

### 2.1 Example 1

(IMO 2006 shortlist) Consider a pentagon $ABCDE$ such that $\angle BAC = \angle CAD = \angle DAE$ and $\angle CBA = \angle DCA = \angle EDA$. Let $P = BD \cap CE$. Show that $AP$ bisects $CD$.

1. Do you see any spirally similar triangles?
2. What circles can you get from these similarities?

### 2.2 Example 2

(Miquel point) Let $ABCD$ be a convex quadrilateral. Let $AB \cap CD = X$, $AD \cap BC = Y$. Show that the circumcircles of $ABY, CDY, ADX, BCX$ all pass through a common point.

1. There's lots of circles here. Can you find some spiral similarities?
2. Can we get more spiral similarities for free?
3. What circles do we get from those spiral similarities?

### 2.3 Example 3

Consider two squares $ABCD$ and $EDFH$. Let the midpoint of $AF, BH, CE$ be $P, Q, R$. Show that $PQRD$ is also a square.

1. Where is the spiral centre?
2. Where does the movie start, and where does the movie end?
3. What happens when the movie stops midway?

## 3 Exercise

1. Let $\triangle ABC$ and $\triangle CDE$ be isosceles triangles with $\angle ABC = \angle CDE = 90°$. Let the midpoint of $AE$ be $M$. Show that $BMD$ is also an isosceles right angle triangle.
2. (Napoleon's Theorem). Let $ABC$ be a triangle. Construct equilateral triangles using each side, such that the equilateral triangles are outside the triangle, and let the centres of those equilateral triangles be $P, Q, R$. Show that $PQR$ is also an equilateral triangle.
3. Let $ABCD$ be a quadrilateral where $AB = CD$. Let the midpoint of $BC$ be $M$ and the midpoint of $AD$ be $N$. Let $AB \cap MN = X$. Let $MN \cap CD = Y$. Show that $\angle AXM = \angle MYD$.
4. Let $ABCD$ be a quadrilateral where $\angle ABD = \angle BCD = 90°$. Also assume $BC$ is tangent to the circumcircle of $\triangle ABD$. Let $M$ be the midpoint of $AB$, and let $N$ be the midpoint of $BC$. Show that $\angle MND = 90°$.
5. A variable point $X$ lies on a semicircle $\omega$ with diameter $AB$. Let $Y$ be a point on ray $XA$ such that $XY = XB$. What is the locus of $Y$?
6. Let $ABCD$ be cyclic. Let $AC \cap BD = P$. Let $PE, PF$ be altitudes to $AB, CD$. Let $K, L$ be midpoints of $BC, DA$. Show that $KL \perp EF$.
7. (USAMO 2006) Let $ABCD$ be a quadrilateral with $AD$ not parallel to $BC$. Let $E \in AD$, $F \in BC$ such that $AE/ED = BF/FC$. Let $FE \cap AB = S$, $FE \cap CD = T$. Show that the four circumcircles $SAE$, $SBF$, $TCF$, $TDE$ all pass through a common point.
8. (Ptolemy's inequality) Let $ABCD$ be a quadrilateral. Show that
$$AB \times CD + AD \times BC \geq AC \times BD$$
9. Let $ABCD$ be a quadrilateral. Construct four squares using each side, such that the square is outside the quadrilateral, and let the centres of those squares be $P, Q, R, S$ in clockwise order. Show that $PR \perp QS$ and $PR = QS$.
10. (APMO 1998) Let $ABC$ be a triangle and $D \in BC$ such that $AD \perp BC$. Let $E, D, F$ be collinear such that $AE \perp BE$, and $AF \perp CF$. Let $M$ be the midpoint of $BC$, and $N$ be the midpoint of $EF$. Show that $AN \perp NM$.