

Bounding arguments (I)

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1 Example 1

For an integer x , when is $x^2 + 5x + 15$ a square number?

1. What is a square number that is obviously smaller/bigger than this?
2. This brings us to just a few cases. Finish the problem.

2 Example 2

Find all positive integers a, b, c such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$.

1. Let us assume $a \geq b \geq c$. Why are we allowed to assume this? What does this tell us about c ?
2. This brings us to just a few cases. Finish the problem.

3 Example 3

Find all $n \in \mathbb{N}$ such that $n^3 + 6n^2 + 2n + 1$ is divisible by $n^2 + 3n + 1$.

1. What's close to $n^3 + 6n^2 + 2n + 1$ and already divisible by $n^2 + 3n + 1$?
2. Let's look at what's left. What happens when n gets really big?
3. This brings us to just a few cases. Finish the problem.

4 Example 4

Let $\{a_n\}$ be a sequence of positive integers such that $a_{n+2} = -a_{n+1} + 2a_n$. Show that a_n must be a constant sequence.

1. Can you find some clever ways to rearrange the recurrence relation?
2. What happens when n gets really big?

5 Example 5

Find integers x, y such that $3^x - 2^y = 1$.

6 Questions

- Find all $n \in \mathbb{N}$ such that $n^2 + 5n + 1$ is a perfect square.
- Find all positive integers x, y such that $x^3 = y^3 + 2y^2 + 8$.
- Let n be a positive integer such that $n + 10$ divides $n^3 + 100$. Find the biggest such number.
- (AMC) $\frac{1}{3} + \frac{1}{n}$ has a denominator less than n . How many such integer n are there from 0 to 2018?
- Find all $x, y \in \mathbb{Z}$ such that $x^3 - y^3 = xy + 61$.
- Find all $n \in \mathbb{N}$ such that $n! + 5$ is a perfect cube.
- (Hungary 1995) The product of a few primes is ten times as much as the sum of the primes. What are these (not necessarily distinct) primes?
- (Russia 1999) Do there exist 19 distinct positive integers that add up to 1999 and have the same sum of digits?
- (Bulgaria 1995) find all primes p, q such that pq divides $(5^p - 2^p)(5^q - 2^q)$.
- Find all positive integers x, y such that $x^2 + 3y$ and $y^2 + 3x$ are both perfect squares.
- Suppose $a, b, n \in \mathbb{N}$ where $n^2 + 1 = ab$. Prove that $|a - b| \geq \sqrt{4n - 3}$.
- How many $a, b, n \in \mathbb{N}$ such that $0 \leq n \leq 2018$ such that there exists a, b where $n^2 + 1 = ab$ and $a - b = \sqrt{4n - 3}$?
- (SMMC) Find all primes p, q such that $p^{q-1} + q^{p-1}$ is a perfect square.
- (SMMC) Find all even x , odd q , and integer y such that $x^3 + x^2 + x + 1 = y^q$.
- Show that there are infinitely many integer solutions to $a^2 + b^2 + c^2 = a^3 + b^3 + c^3$.
- (TST 2016) Let $m > n$ be positive integers. Let $x_k = \frac{m+k}{n+k}$ for $k = 1, 2, \dots, n+1$. Prove that if all the numbers x_1, x_2, \dots, x_{n+1} are integers, then $x_1 x_2 \dots x_{n+1} - 1$ is divisible by an odd prime.
- Find all triples (x, y, z) of positive integers such that $x \leq y \leq z$ and

$$x^3(y^3 + z^3) = 2012(xyz + 2)$$

- Find the number of ordered triples (x, y, z) of positive integers such that $x, y, z \leq 2013$ and

$$x|(yz - 1), \quad y|(xz - 1), \quad z|(xy - 1)$$

7 Linear Homogeneous Recurrences

Example 1: $a_{n+2} = 3a_{n+1} - 2a_n$, where $a_0 = 0, a_1 = 1$.

We start with the *ansatz* $a_n = x^n$. This gets us

$$\begin{aligned}x^{n+2} &= 3x^{n+1} - 2x^n \\x^2 - 3x + 2 &= 0 \\(x-1)(x-2) &= 0\end{aligned}$$

Hence we conclude that $a_n = 2^n$ and $a_n = 1^n$ are solutions to the recurrence relation. Notice that any linear combination of these two solutions is also a solution to the recurrence. So the general solution to the recurrence is $a_n = A2^n + B1^n$ for some constants A and B . Since $a_0 = A + B = 0$ and $a_1 = 2A + B = 1$, we conclude that $A = 1, B = -1$. Hence, the solution to this recurrence is $a_n = 2^n - 1$.

Example 2: $a_{n+2} = 4a_{n+1} - 4a_n$, where $a_0 = 1, a_1 = 6$.

Using the same *ansatz*, we get the equation $(x-2)^2 = 0$. Notice that we now have a root with multiplicity. So, alongside the usual solution of $a_n = 2^n$, we also have the solution $a_n = n2^n$ (If it was a triple root, then we have the solution $a_n = n^22^n$, etc (check it!)). As before, the general solution is $a_n = (A + Bn)2^n$ for some constants A, B , where we deduce $A = 1$ and $B = 2$ via the initial conditions. Thus, the solution is $a_n = (1 + 2n)2^n$.

Exercises: Find the general term.

1. $a_{n+3} = 6a_{n+2} - 11a_{n+1} + 6a_n$, where $(a_0, a_1, a_2) = (-1, 0, 4)$
2. $a_{n+2} = a_{n+1} - a_n$, where $(a_0, a_1) = (0, 1)$
3. $a_{n+3} = a_{n+2} + a_{n+1} - a_n$, where $(a_0, a_1, a_2) = (2, 2, 6)$
4. $F_{n+2} = F_{n+1} + F_n$, where $(a_0, a_1) = (0, 1)$.

Problems:

1. Let $\{a_n\}$ be an integer sequence such that $2a_{n+2} = 5a_{n+1} - 3a_n$. Show that a_n must be a constant sequence.
2. Find all functions $f: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ such that $f(f(x)) = 4f(x) + 5x$.
3. Find all functions $f: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ such that $f(x) > x$ and $f(f(x) - x) = 2x$.
4. (ISL 2010 A5) Find all functions $f: \mathbb{Q}^+ \rightarrow \mathbb{Q}^+$ such that $f(f(x)^2y) = x^3f(xy)$.
5. (China 2017) The sequences $\{u_n\}$ and $\{v_n\}$ are defined by $u_{n+2} = 2u_{n+1} - 3u_n$, where $(u_0, u_1) = (1, 1)$ and $v_{n+3} = v_{n+2} - 3v_{n+1} + 27v_n$, where $(v_0, v_1, v_2) = (a, b, c)$. There exists a positive integer N such that for any $n > N$, u_n divides v_n . Prove that $3a = 2b + c$.