
EAS501 Notes

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September 7, 2023

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1 Lecture 1

1.1 Matrices

1.1.1 Definition

A matrix is a 2-dimensional collection of numbers.

$$\mathbf{A} = \begin{bmatrix} 9 & 4 \\ -1 & 0 \\ 3 & 2 \end{bmatrix} \quad (1)$$

The number of rows in \mathbf{A} is 3. The number of columns in \mathbf{A} is 2. Therefore, we state that \mathbf{A} is a 3×2 matrix. This is denoted by

$$\mathbf{A} \in \mathbb{R}^{3 \times 2}. \quad (2)$$

Another matrix is

$$\mathbf{B} = \begin{bmatrix} 4 & 2 & 0 \\ 8 & 3 & -1 \end{bmatrix}, \quad \mathbf{B} \in \mathbb{R}^{2 \times 3}. \quad (3)$$

There is a way to identify individual components of \mathbf{B} . The location of the element in the i th row and j th column of \mathbf{B} is called B_{ij} . For example, to denote the element that is in the first row and second column of \mathbf{B} , we get

$$i = 1, \quad j = 2 \longrightarrow B_{ij} = B_{12} = 2. \quad (4)$$

All the elements are

$$\begin{aligned} B_{11} &= 4, & B_{12} &= 2, & B_{13} &= 0, \\ B_{21} &= 8, & B_{22} &= 3, & B_{23} &= -1. \end{aligned} \quad (5)$$

In general, a matrix $\mathbf{M} \in \mathbb{R}^{m \times n}$ consists of the elements

$$\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} & \dots & M_{1n} \\ M_{21} & M_{22} & \dots & M_{2n} \\ \dots & \dots & \dots & \dots \\ M_{m1} & M_{m2} & \dots & M_{mn} \end{bmatrix}, \quad (6)$$

where m denotes the total number of rows, and n denotes the total number of columns. A matrix can also be a collection of vectors. For example, let

$$\mathbf{a} = \begin{Bmatrix} 1 \\ 2 \\ 4 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 9 \\ -1 \\ 0 \end{Bmatrix}, \quad \mathbf{d} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}. \quad (7)$$

Then, the statement

$$\mathbf{C} = \begin{bmatrix} 1 & 9 \\ 2 & -1 \\ 4 & 0 \end{bmatrix} = [\mathbf{a} \quad \mathbf{b}] \quad (8)$$

is true. However, the "matrix"

$$\mathbf{E} = [\mathbf{a} \quad \mathbf{d}] \quad (9)$$

is undefined. This is because \mathbf{a} and \mathbf{d} are different sizes, so together they do not build a complete matrix.

1.1.2 Transpose operator

The transpose operator on matrix $\mathbf{M} \in \mathbb{R}^{m \times n}$, written as $\mathbf{M}^T \in \mathbb{R}^{n \times m}$, works such that each element of the transposed matrix is

$$M_{ij}^T = M_{ji}. \quad (10)$$

For example,

$$\mathbf{C} = \begin{bmatrix} 1 & 9 \\ 2 & -1 \\ 4 & 0 \end{bmatrix} \longrightarrow \mathbf{C}^T = \begin{bmatrix} 1 & 2 & 4 \\ 9 & -1 & 0 \end{bmatrix}. \quad (11)$$

In other terms,

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix} \longrightarrow \mathbf{C}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \end{bmatrix} = \begin{bmatrix} C_{11}^T & C_{12}^T & C_{13}^T \\ C_{21}^T & C_{22}^T & C_{23}^T \end{bmatrix} = \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T \end{bmatrix}, \quad (12)$$

where $\mathbf{a}^T = \{1 \ 2 \ 4\}$, $\mathbf{b}^T = \{9 \ -1 \ 0\}$, according to Eq. 7.

1.1.3 Symmetric matrices

A matrix \mathbf{M} is symmetric if and only if $\mathbf{M} = \mathbf{M}^T$. For example, the matrix

$$\mathbf{F} = \begin{bmatrix} 1 & 9 & 4 \\ 9 & 3 & -1 \\ 4 & -1 & 2 \end{bmatrix} \longrightarrow \mathbf{F} = \begin{bmatrix} 1 & 9 & 4 \\ 9 & 3 & -1 \\ 4 & -1 & 2 \end{bmatrix} = \mathbf{F} \quad (13)$$

is symmetric. This means that

$$F_{ij} = F_{ji}. \quad (14)$$

To be symmetric, a matrix must be square. That is, it must have the same number of rows and columns. A nonsquare matrix can never be symmetric: the transpose of that matrix will not have the same shape, so it cannot be equal to its original configuration.

1.1.4 Identity matrix

The identity matrix is a square matrix with 1s (ones) on the diagonals and 0s (zeroes) everywhere else. It is denoted as

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}. \quad (15)$$

Properties of the identity matrix are

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A}, \quad \mathbf{Ix} = \mathbf{x}. \quad (16)$$

1.2 Matrix operations, Pt 1

1.2.1 Addition of equally sized matrices

If $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{m \times n}$, then

$$\mathbf{A} + \mathbf{B} = \mathbf{C} \longleftrightarrow A_{ij} + B_{ij} = C_{ij}, \quad (17)$$

where $\mathbf{C} \in \mathbb{R}^{m \times n}$.

1.2.2 Matrix-vector products

If matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and vector $\mathbf{x} \in \mathbb{R}^{n \times 1}$, then

$$\mathbf{Ax} = \mathbf{b} \longleftrightarrow A_{ij}x_j = b_i, \quad (18)$$

where $\mathbf{b} \in \mathbb{R}^{m \times 1}$. In Eq. 18, notice that the index j is repeated on the left hand side of the equation. If the index on one side of an equation is repeated, a summation is done on that index. In other terms,

$$A_{ij}x_j \longleftrightarrow \sum_{j=1}^n A_{ij}x_j = A_{i1}x_1 + A_{i2}x_2 + \dots + A_{in}x_n = b_i. \quad (19)$$

Let's look at an example in which $\mathbf{A} \in \mathbb{R}^{3 \times 2}$, $\mathbf{x} \in \mathbb{R}^{2 \times 1}$. The full multiplication is

$$\mathbf{Ax} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} A_{11}x_1 + A_{12}x_2 \\ A_{21}x_1 + A_{22}x_2 \\ A_{31}x_1 + A_{32}x_2 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} = \mathbf{b}, \quad (20)$$

where $\mathbf{b} \in \mathbb{R}^{3 \times 1}$. Notice that the "matrix product"

$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 9 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad (21)$$

is undefined.

Now, let us suppose that

$$\mathbf{A} = [\mathbf{a} \quad \mathbf{b} \quad \mathbf{c}], \quad \mathbf{x} = \begin{Bmatrix} d \\ e \\ f \end{Bmatrix}. \quad (22)$$

The product is

$$\mathbf{Ax} = d\mathbf{a} + e\mathbf{b} + f\mathbf{c}, \quad (23)$$

which is a linear combination of vectors.

1.2.3 Matrix-matrix products

If $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{p \times q}$, then \mathbf{AB} is defined if and only if $n = p$, and \mathbf{BA} is defined if and only if $q = m$. For example,

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \longleftrightarrow A_{ik}B_{kj} = C_{ij}. \quad (24)$$

1.2.4 More rules

- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.
- $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$, if c is a real valued constant.
- $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) \longleftrightarrow A_{ik}(B_{kl}C_{lj}) = (A_{ik}B_{kl})C_{lj}$.
- $\mathbf{AB} \neq \mathbf{BA}$ in general. For example... $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
 $\longrightarrow \mathbf{AB} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{BA} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.
- $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$.
- $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$.
- $(\mathbf{A} + \mathbf{B})\mathbf{x} = \mathbf{Ax} + \mathbf{Bx}$.
- $\mathbf{A}(\mathbf{x} + \mathbf{y}) = \mathbf{Ax} + \mathbf{Ay}$.
- $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.
- $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T \longleftrightarrow (A_{ik}B_{kj})^T = B_{kj}^T A_{ik}^T = B_{jk} A_{ki}$.
- $(\mathbf{Ax})^T = \mathbf{x}^T \mathbf{A}^T$.
- $(c\mathbf{A})^T = c\mathbf{A}^T$.

1.3 Mass spring system

Consider the mass spring system in Fig. 1.

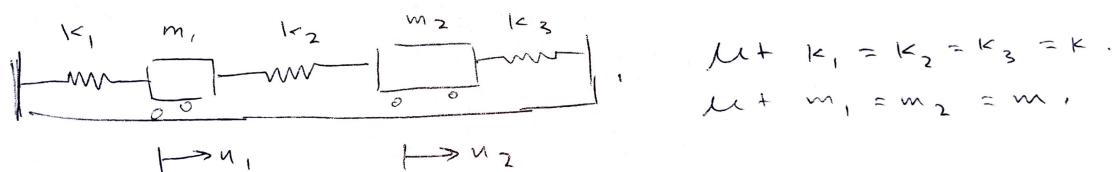


Figure 1: Mass spring system with two masses.

If the system is in equilibrium, the sum of the forces on each mass is zero. Let's consider all the physical forces acting on m_1 :

- If u_1 is positive, this elongates k_1 , and pushes m_1 backwards $\longleftrightarrow -k_1 u_1$.
- If u_1 is positive, this shortens k_2 , preventing m_1 from being pushed forward $\longleftrightarrow -k_2 u_1$.

- If u_2 is positive, this elongates k_2 , pushing m_1 forward. $\longleftrightarrow k_2 u_2$
- Some unknown magnitude of force f_1 brings the sum of the forces acting on m_1 to equilibrium.

Adding these together,

$$-k_1 u_1 - k_2 u_1 + k_2 u_2 + f_1 = 0. \quad (25)$$

Now let's consider all the physical forces acting on m_2 :

- If u_1 is positive, this shortens k_2 , preventing m_2 from being pushed backwards $\longleftrightarrow k_2 u_1$
- If u_2 is positive, this elongates k_2 , pushing m_2 backwards $\longleftrightarrow -k_2 u_2$
- If u_2 is positive, this shortens k_3 , preventing m_2 from being pushed forwards $\longleftrightarrow -k_3 u_2$
- Some unknown magnitude of force f_2 brings the sum of forces acting on m_2 to equilibrium.

Adding these together,

$$k_2 u_1 - k_2 u_2 - k_3 u_2 + f_2 = 0. \quad (26)$$

The compilation of Eqs. 25,26 is

$$-k_1 u_1 - k_2 u_1 + k_2 u_2 + f_1 = 0,$$

$$k_2 u_1 - k_2 u_2 - k_3 u_2 + f_2 = 0.$$

Rearranging for f_i , and substituting in $k_1 = k_2 = k_3 = k$,

$$2k u_1 - k u_2 = f_1,$$

$$-k u_1 + 2k u_2 = f_2.$$

In matrix form,

$$\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \longleftrightarrow \mathbf{K}\mathbf{u} = \mathbf{f}. \quad (27)$$

If we have k , u_1 , and u_2 , we can calculate \mathbf{f} .

1.4 Matrix operations, Pt. 2

1.4.1 Powers

A matrix \mathbf{M} can be raised to a power. For example,

$$\mathbf{M}^5 = \mathbf{M}\mathbf{M}\mathbf{M}\mathbf{M}\mathbf{M} \longleftrightarrow M_{ij}^p = M_{ik} M_{kl} M_{lp} M_{pq} M_{qj}. \quad (28)$$

Like with other quantities,

- $\mathbf{M}^p \mathbf{M}^q = \mathbf{M}^{p+q}$,
- $(\mathbf{M}^p)^q = \mathbf{M}^{pq}$.

1.4.2 Trace operator

The trace is the sum of the diagonal elements of a square matrix. (A nonsquare matrix does not have a trace.) For example,

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 9 \\ 0 & -1 & 2 \\ -1 & 0 & 8 \end{bmatrix} \longrightarrow \text{tr} \mathbf{A} = 1 - 1 + 8 = 8. \quad (29)$$

Note that

$$\text{tr} \mathbf{A} \longleftrightarrow A_{ii} = A_{11} + A_{22} + A_{33} \quad (30)$$

(remember that a repeated index in an expression indicates a summation). Some properties of the trace operator are

- $\text{tr}(\alpha \mathbf{A}) = \alpha \text{tr} \mathbf{A}$.
- $\text{tr}(\mathbf{A}^T) = \text{tr} \mathbf{A}$.
- $\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$.
- $\text{tr}(\mathbf{A}^T \mathbf{B}) = \text{tr}(\mathbf{B}^T \mathbf{A}) = \text{tr}(\mathbf{AB}^T) = \text{tr}(\mathbf{BA}^T)$.

1.4.3 Outer product operator

The outer product is an operation (written as \otimes) on two vectors that results in a matrix. For example,

$$\mathbf{a} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} \longrightarrow \mathbf{a} \otimes \mathbf{b} := \mathbf{ab}^T = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \begin{Bmatrix} b_1 & b_2 \end{Bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \\ a_3 b_1 & a_3 b_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix} = \mathbf{C}. \quad (31)$$

Note that

$$\mathbf{a} \otimes \mathbf{b} = \mathbf{C} \longleftrightarrow a_i b_j = C_{ij}. \quad (32)$$

1.4.4 Matrix determinant

1.4.5 Matrix inverse

1.4.6 Matrix norms..might not get to