EAS501 Notes:
https://www.joseph-marziale.com/notes/jjmarzia_eas501.pdf

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1.1 Matrices

1.1.1 Definition

A matrix is a 2-dimensional collection of numbers.

$$\mathbf{A} = \begin{bmatrix} 9 & 4 \\ -1 & 0 \\ 3 & 2 \end{bmatrix} \tag{1}$$

The number of rows in $\mathbf{A}$ is 3. The number of columns in $\mathbf{A}$ is 2. Therefore, we state that $\mathbf{A}$ is a $3\times$ 2 matrix. This is denoted by

$$\mathbf{A} \in \mathbb{R}^{3\times 2}. \tag{2}$$

Another matrix is

$$\mathbf{B} = \begin{bmatrix} 4 & 2 & 0 \\ 8 & 3 & -1 \end{bmatrix}, \quad \mathbf{B} \in \mathbb{R}^{2\times 3}. \tag{3}$$

There is a way to identify individual components of $\mathbf{B}$. The location of the element in the ith row and jth column of $\mathbf{B}$ is called B_{ij}. For example, to denote the element that is in the first row and second column of $\mathbf{B}$, we get

$$i = 1, \quad j = 2 \longrightarrow B_{ij} = B_{12} = 2. \tag{4}$$

All the elements are

$$\begin{aligned} B_{11} = 4, \quad B_{12} = 2, \quad B_{13} = 0, \\ B_{21} = 8, \quad B_{22} = 3, \quad B_{23} = -1. \end{aligned} \tag{5}$$

In general, a matrix $\mathbf{M} \in \mathbb{R}^{m\times n}$ consists of the elements

$$\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} & \dots & M_{1n} \\ M_{21} & M_{22} & \dots & M_{2n} \\ \dots & \dots & \dots & \dots \\ M_{m1} & M_{m2} & \dots & M_{mn} \end{bmatrix}, \tag{6}$$

where m denotes the total number of rows, and n denotes the total number of columns. A matrix can also be a collection of vectors. For example, let

$$\mathbf{a} = \begin{Bmatrix} 1 \\ 2 \\ 4 \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 9 \\ -1 \\ 0 \end{Bmatrix}, \quad \mathbf{d} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}. \tag{7}$$

Then, the statement

$$\mathbf{C} = \begin{bmatrix} 1 & 9 \\ 2 & -1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix} \tag{8}$$

is true. However, the "matrix"

$$\mathbf{E} = \begin{bmatrix} \mathbf{a} & \mathbf{d} \end{bmatrix} \tag{9}$$

is undefined. This is because **a** and **d** are different sizes, so together they do not build a complete matrix.

1.1.2 Transpose operator

The transpose operator on matrix $\mathbf{M} \in \mathbb{R}^{m\times n}$, written as $\mathbf{M}^T \in \mathbb{R}^{n\times m}$, works such that each element of the transposed matrix is

$$M^T_{ij} = M_{ji}. \tag{10}$$

For example,

$$\mathbf{C} = \begin{bmatrix} 1 & 9 \\ 2 & -1 \\ 4 & 0 \end{bmatrix} \longrightarrow \mathbf{C}^T = \begin{bmatrix} 1 & 2 & 4 \\ 9 & -1 & 0 \end{bmatrix}. \tag{11}$$

In other terms,

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix} \longrightarrow \mathbf{C}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \end{bmatrix} = \begin{bmatrix} C^T_{11} & C^T_{12} & C^T_{13} \\ C^T_{21} & C^T_{22} & C^T_{23} \end{bmatrix} = \begin{bmatrix} \mathbf{a}^T \\ \mathbf{b}^T \end{bmatrix}, \tag{12}$$

where $\mathbf{a}^T = \{1\ 2\ 4\}$, $\mathbf{b}^T = \{9\ \text{-}1\ 0\}$, according to Eq. 7.

1.1.3 Symmetric matrices

A matrix $\mathbf{M}$ is symmetric if and only if $\mathbf{M} = \mathbf{M}^T$. For example, the matrix

$$\mathbf{F} = \begin{bmatrix} 1 & 9 & 4 \\ 9 & 3 & -1 \\ 4 & -1 & 2 \end{bmatrix} \longrightarrow \mathbf{F} = \begin{bmatrix} 1 & 9 & 4 \\ 9 & 3 & -1 \\ 4 & -1 & 2 \end{bmatrix} = \mathbf{F} \tag{13}$$

is symmetric. This means that

$$F_{ij} = F_{ji}. \tag{14}$$

To be symmetric, a matrix must be square. That is, it must have the same number of rows and columns. A nonsquare matrix can never be symmetric: the transpose of that matrix will not have the same shape, so it cannot be equal to its original configuration.

1.1.4 Identity matrix

The identity matrix is a square matrix with 1s (ones) on the diagonals and 0s (zeroes) everywhere else. It is denoted as

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}. \tag{15}$$

Properties of the identity matrix are

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A}, \quad \mathbf{Ix} = \mathbf{x}. \tag{16}$$

1.2 Matrix operations, Pt 1

1.2.1 Addition of equally sized matrices

If $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{m\times n}$, then

$$\mathbf{A} + \mathbf{B} = \mathbf{C} \longleftrightarrow A_{ij} + B_{ij} = C_{ij}, \tag{17}$$

where $\mathbf{C} \in \mathbb{R}^{m\times n}$.

1.2.2 Matrix-vector products

If matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ and vector $\mathbf{x} \in \mathbb{R}^{n\times 1}$, then

$$\mathbf{Ax} = \mathbf{b} \longleftrightarrow A_{ij}x_j = b_i, \tag{18}$$

where $\mathbf{b} \in \mathbb{R}^{m\times 1}$. In Eq. 18, notice that the index j is repeated on the left hand side of the equation. If the index on one side of an equation is repeated, a summation is done on that index. In other terms,

$$A_{ij}x_j \longleftrightarrow \sum_{j=1}^{n} A_{ij}x_j = A_{i1}x_1 + A_{i2}x_2 + \ldots + A_{in}x_n = b_i. \tag{19}$$

Let's look at an example in which $\mathbf{A} \in \mathbb{R}^{3\times 2}, \mathbf{x} \in \mathbb{R}^{2\times 1}$. The full multiplication is

$$\mathbf{Ax} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} A_{11}x_1 + A_{12}x_2 \\ A_{21}x_1 + A_{22}x_2 \\ A_{31}x_1 + A_{32}x_2 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} = \mathbf{b}, \tag{20}$$

where $\mathbf{b} \in \mathbb{R}^{3\times 1}$. Notice that the "matrix product"

$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 9 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \tag{21}$$

is undefined.

Now, let us suppose that

$$\mathbf{A} = \begin{bmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{bmatrix}, \quad \mathbf{x} = \begin{Bmatrix} d \\ e \\ f \end{Bmatrix}. \tag{22}$$

The product is

$$\mathbf{Ax} = d\mathbf{a} + e\mathbf{b} + f\mathbf{c}, \tag{23}$$

which is a linear combination of vectors.

1.2.3 Matrix-matrix products

If $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{p\times q}$, then $\mathbf{AB}$ is defined if and only if $n = p$, and $\mathbf{BA}$ is defined if and only if $q = m$. For example,

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21}, & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21}, & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \longleftrightarrow A_{ik}B_{kj} = C_{ij}. \tag{24}$$

1.2.4 More rules

- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.
- $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$, if c is a real valued constant.
- $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) \longleftrightarrow A_{ik}(B_{kl}C_{lj}) = (A_{ik}B_{kl})C_{lj}$.
- $\mathbf{AB} \neq \mathbf{BA}$ in general. For example... $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
  $\longrightarrow \mathbf{AB} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{BA} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.
- $\mathbf{A}(\mathbf{B}+\mathbf{C}) + \mathbf{AB} + \mathbf{AC}$.
- $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$.
- $(\mathbf{A}+\mathbf{B})\mathbf{x} = \mathbf{Ax}+\mathbf{Bx}$.
- $\mathbf{A}(\mathbf{x}+\mathbf{y}) = \mathbf{Ax}+\mathbf{Ay}$.
- $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.
- $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \longleftrightarrow (A_{ik}B_{kj})^T = B^T_{kj}A^T_{ik} = B_{jk}A_{ki}$.
- $(\mathbf{Ax})^T = \mathbf{x}^T\mathbf{A}^T$.
- $(c\mathbf{A})^T = c\mathbf{A}^T$.

1.3 Mass spring system

Consider the mass spring system in Fig. 1.

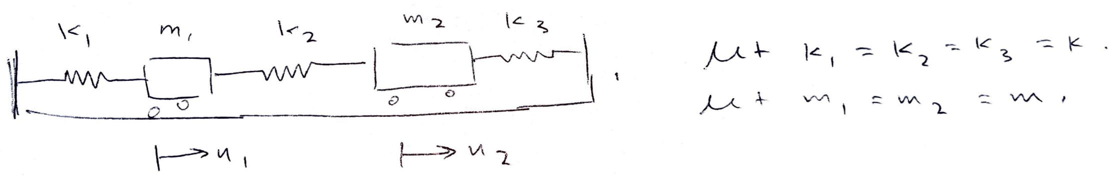

Figure 1: Mass spring system with two masses.

If the system is in equilibrium, the sum of the forces on each mass is zero. Let's consider all the physical forces acting on m_1:

- If u_1 is positive, this elongates k_1, and pushes m_1 backwards $\longleftrightarrow -k_1u_1$.
- If u_1 is positive, this shortens k_2, preventing m_1 from being pushed forward $\longleftrightarrow$ $-k_2u_1$.

- If u_2 is positive, this elongates k_2, pushing m_1 forward. $\longleftrightarrow k_2u_2$
- Some unknown magnitude of force f_1 brings the sum of the forces acting on m_1 to equilibrium.

Adding these together,

$$-k_1u_1 - k_2u_1 + k_2u_2 + f_1 = 0. \tag{25}$$

Now let's consider all the physical forces acting on m_2:

- If u_1 is positive, this shortens k_2, preventing m_2 from being pushed backwards $\longleftrightarrow k_2u_1$
- If u_2 is positive, this elongates k_2, pushing m_2 backwards $\longleftrightarrow -k_2u_2$
- If u_2 is positive, this shortens k_3, preventing m_2 from being pushed forwards $\longleftrightarrow -k_3u_2$
- Some unknown magnitude of force f_2 brings the sum of forces acting on m_2 to equilibrium.

Adding these together,

$$k_2u_1 - k_2u_2 - k_3u_2 + f_2 = 0. \tag{26}$$

The compilation of Eqs. 25,26 is

$$-k_1u_1 - k_2u_1 + k_2u_2 + f_1 = 0,$$

$$k2u_1 - k_2u_2 - k_3u_2 + f_2 = 0.$$

Rearranging for f_i, and substituting in $k_1 = k_2 = k_3 = k$,

$$2ku_1 - ku_2 = f_1,$$

$$-ku_1 + 2ku_2 = f_2.$$

In matrix form,

$$\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \longleftrightarrow \mathbf{Ku} = \mathbf{f}. \tag{27}$$

If we have k, u_1, and u_2, we can calculate $\mathbf{f}$.

1.4 Matrix operations, Pt. 2

1.4.1 Powers

A matrix $\mathbf{M}$ can be raised to a power. For example,

$$\mathbf{M}^5 = \mathbf{MMMMM} \longleftrightarrow M_{ij}^p = M_{ik}M_{kl}M_{lp}M_{pq}M_{qj}. \tag{28}$$

Like with other quantities,

- $\mathbf{M}^p\mathbf{M}^q = \mathbf{M}^{p+q}$,
- $(\mathbf{M}^p)^q = \mathbf{M}^{pq}$.

1.4.2 Trace operator

The trace is the sum of the diagonal elements of a square matrix. (A nonsquare matrix does not have a trace.) For example,

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 9 \\ 0 & -1 & 2 \\ -1 & 0 & 8 \end{bmatrix} \longrightarrow \mathrm{tr}\mathbf{A} = 1 - 1 + 8 = 8. \tag{29}$$

Note that

$$\mathrm{tr}\mathbf{A} \longleftrightarrow A_{ii} = A_{11} + A_{22} + A_{33} \tag{30}$$

(remember that a repeated index in an expression indicates a summation). Some properties of the trace operator are

- $\mathrm{tr}(\alpha\mathbf{A}) = \alpha\mathrm{tr}\mathbf{A}$.
- $\mathrm{tr}(\mathbf{A}^T) = \mathrm{tr}\mathbf{A}$.
- $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$.
- $\mathrm{tr}(\mathbf{A}^T\mathbf{B}) = \mathrm{tr}(\mathbf{B}^T\mathbf{A}) = \mathrm{tr}(\mathbf{AB}^T) = \mathrm{tr}(\mathbf{BA}^T)$.

1.4.3 Outer product operator

The outer product is an operation (written as $\otimes$) on two vectors that results in a matrix. For example,

$$\mathbf{a} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}, \ \mathbf{b} = \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} \longrightarrow \mathbf{a}\otimes\mathbf{b} := \mathbf{ab}^T = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \begin{Bmatrix} b_1 & b_2 \end{Bmatrix} = \begin{bmatrix} a_1b_1 & a_1b_2 \\ a_2b_1 & a_2b_2 \\ a_3b_1 & a_3b_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix} = \mathbf{C}. \tag{31}$$

Note that

$$\mathbf{a} \otimes \mathbf{b} = \mathbf{C} \longleftrightarrow a_ib_j = C_{ij}. \tag{32}$$

1.4.4 Matrix determinant

The determinant is a scalar that encodes certain properties of a square matrix ($\mathbf{A} \in \mathbb{R}^{n\times n}$). The best way to define the determinant is with recursive examples.

- If $\mathbf{A} \in \mathbb{R}^{1\times 1}$, then $\det \mathbf{A} = \det \begin{bmatrix} A_{11} \end{bmatrix} = A_{11}$.
- If $\mathbf{A} \in \mathbb{R}^{2\times 2}$, then $\det \mathbf{A} = \det \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = A_{11} \det \begin{bmatrix} A_{22} \end{bmatrix} - A_{12} \det \begin{bmatrix} A_{12} \end{bmatrix} = A_{11}A_{22} - A_{12}A_{21}$.
- If $\mathbf{A} \in \mathbb{R}^{3\times 3}$, then $\det \mathbf{A} = \det \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$
  $= A_{11} \det \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} - A_{12} \det \begin{bmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{bmatrix} + A_{13} \det \begin{bmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}$
  $= A_{11}(A_{22}A_{33} - A_{32}A_{23}) - A_{12}(A_{21}A_{33} - A_{31}A_{23}) + A_{13}(A_{21}A_{32} - A_{31}A_{22})$.

In general,

$$\det \mathbf{A} = \sum_{j=1}^{n} A_{ij}(-1)^{i+j} M_{ij}, \tag{33}$$

where M_{ij} is the determinant of the submatrix. The submatrix is the set of elements not in row i or column j.

As seen in this definition, to do the determinant you can pick any row i and do the summation over the columns j. In the recursive examples above, we picked the first row, but you can pick any row.

Let $\mathbf{I}_n$ be the $n \times n$ identity matrix, and let α be a real valued scalar. Then, some properties of the determinant are

- $\det(\mathbf{AB}) = \det \mathbf{A} \det \mathbf{B}$.
- $\det(\alpha \mathbf{I}_n) = \alpha^n$.
- $\det(\alpha \mathbf{A}) = \det(\alpha \mathbf{I}_n \mathbf{A}) = \det(\alpha \mathbf{I}_n) \det \mathbf{A} = \alpha^n \det \mathbf{A}$.
- $\det(\mathbf{A}^T) = \det \mathbf{A}$.

1.4.5 Matrix inverse

Square matrices have inverses. If $\mathbf{A} \in \mathbb{R}^{n \times n}$, and if $\mathbf{I}$ is the $n \times n$ identity matrix, then we say that $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$ if and only if $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

Let's look at the closed form solution of the inverses of 2×2 and 3×3 matrices. If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\longrightarrow \mathbf{A}^{-1}\mathbf{A} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} da - bc & db - bd \\ -ca + ca & -cb + ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{34}$$

$$\longrightarrow \mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\frac{1}{\det \mathbf{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) = \frac{1}{ad - bc} \begin{bmatrix} ad - bc & -ab + ba \\ cd - dc & -bc + ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{35}$$

One rule of determinants is that

$$\det(\mathbf{A}^{-1}) = \frac{1}{\det \mathbf{A}}. \tag{36}$$

This rule implies that if $\det \mathbf{A} \neq 0$, then $\det(\mathbf{A}^{-1})$ exists, and therefore $\mathbf{A}^{-1}$ exists. On the other hand, if $\det \mathbf{A} = 0$, then $\det(\mathbf{A}^{-1})$ does not exist, and therefore $\mathbf{A}^{-1}$ does not exist. For example, consider

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 6 \end{bmatrix} \longrightarrow \det \mathbf{A} = 0 * 6 - 1 * 0 = 0, \quad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} 6 & -1 \\ 0 & 0 \end{bmatrix} = \frac{1}{0} \begin{bmatrix} 6 & -1 \\ 0 & 0 \end{bmatrix} \quad \ldots? \tag{37}$$

And so the inverse matrix only exists if the determinant of the original matrix is nonzero.

Now, let us return to the mass-spring system (Eq. 27), in which

$$\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \longleftrightarrow \mathbf{Ku} = \mathbf{f}.$$

Earlier, we supposed that we knew $\mathbf{K}$ and $\mathbf{u}$, and thus we could find $\mathbf{f}$. But let us instead suppose that we know $\mathbf{K}$ and $\mathbf{f}$, but not $\mathbf{u}$. In that case, can we determine $\mathbf{u}$?

To answer to this question let us isolate $\mathbf{u}$. We can do this with a series of steps:

$$\mathbf{Ku} = \mathbf{f} \longrightarrow \mathbf{K}^{-1}\mathbf{Ku} = \mathbf{K}^{-1}\mathbf{f} \longrightarrow \mathbf{Iu} = \mathbf{K}^{-1}\mathbf{f} \longrightarrow \mathbf{u} = \mathbf{K}^{-1}\mathbf{f}. \tag{38}$$

Now this equation ($\mathbf{u} = \mathbf{K}^{-1}\mathbf{f}$) is only valid if $\mathbf{K}^{-1}$ exists. And to determine if $\mathbf{K}^{-1}$ exists, we can check if the determinant of $\mathbf{K}$ is nonzero:

$$\det \mathbf{K} = 2k * 2k - (-k * -k) = 4k^2 - k^2 = 3k^2 \neq 0. \tag{39}$$

Since the determinant is nonzero, $\mathbf{K}^{-1}$ does exist, and so we can find $\mathbf{u}$ if we are given $\mathbf{K}$ and $\mathbf{f}$, using Eq. 38.

Because I like you guys, you can get these notes at:
https://www.joseph-marziale.com/notes/jjmarzia_eas501.pdf.