MAE 524 - Elasticity

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Contents

1 Ch1 Mathematical preliminaries

1.1 Vectors

Vector

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3. \tag{1}$$

Basis vectors are unit vectors so that

$$|\mathbf{e}_1| = |\mathbf{e}_2| = |\mathbf{e}_3| = 1. \tag{2}$$

The basis vectors are mutually perpendicular so that

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{e}_3 = \mathbf{e}_2 \cdot \mathbf{e}_3 = 0. \tag{3}$$

In general

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \tag{4}$$

where Kronecker delta

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{5}$$

Therefore by definition these three vectors form an orthonormal basis.

We can rewrite $\mathbf{x}$ as

$$\mathbf{x} = x_i\mathbf{e}_i \tag{6}$$

where summation convention acts like

$$\delta_{ii} = \begin{cases} \delta_{11} + \delta_{22}, & \text{2D}, \\ \delta_{11} + \delta_{22} + \delta_{33}, & \text{3D}. \end{cases} \tag{7}$$

The Kronecker delta is the index notation form of identity matrix $\mathbf{I}$ and so $\delta_{ii} = \mathrm{tr}\mathbf{I}$ which is defined as the sum of the diagonal entries.

The basis vectors form a right handed orthogonal triad and so

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1, \quad \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2 \tag{8}$$

but

$$\mathbf{e}_1 \times \mathbf{e}_3 = -\mathbf{e}_2, \quad \mathbf{e}_3 \times \mathbf{e}_2 = -\mathbf{e}_1, \quad \mathbf{e}_2 \times \mathbf{e}_1 = -\mathbf{e}_3. \tag{9}$$

Generalizing in index notation,

$$\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk}\mathbf{e}_k \tag{10}$$

where

$$\epsilon_{ijk} = \begin{cases} 1, & ijk \Leftrightarrow 1 \to 2 \to 3 \to \\ -1, & ijk \Leftrightarrow 3 \to 2 \to 1 \to \\ 0, & ijk \Leftrightarrow \text{incohesive loop}. \end{cases} \tag{11}$$

For example

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1, \tag{12}$$

$$\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1, \tag{13}$$

$$\epsilon_{112} = \epsilon_{233} = \ldots = 0. \tag{14}$$

If there are two vectors

$$\mathbf{a} = a_i \mathbf{e}_i, \quad \mathbf{b} = b_i \mathbf{e}_i, \tag{15}$$

then their dot product is

$$\mathbf{a} \cdot \mathbf{b} = a_i \mathbf{e}_i \cdot b_j \mathbf{e}_j = a_i b_j \mathbf{e}_i \cdot \mathbf{e}_j = a_i b_j \delta_{ij} = a_i b_j \delta_{ji} = a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3. \tag{16}$$

The difference between two vectors

$$\mathbf{c} = \mathbf{a} - \mathbf{b} \tag{17}$$

has a magnitude which can be solved for using

$$(a - b\cos\theta)^2 + (b\sin\theta)^2 = c^2 \Longrightarrow a^2 - 2ab\cos\theta + b^2\cos^2\theta + b^2\sin^2\theta = c^2$$

$$\Longleftrightarrow a^2 + b^2 - 2ab\cos\theta = c^2 \tag{18}$$

where θ is the angle that separates $\mathbf{a}$ and $\mathbf{b}$. The length or magnitude of any $\mathbf{a}$ is

$$a = |\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_i a_i} = \sqrt{a_1^2 + a_2^2 + a_3^2}. \tag{19}$$

Substituting into Eq. 18,

$$a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2ab\cos\theta = c_1^2 + c_2^2 + c_3^2 \tag{20}$$

$$\Longleftrightarrow a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2ab\cos\theta = (a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 \tag{21}$$

$$\Longleftrightarrow a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2ab\cos\theta = a_1^2 - 2a_1 b_1 + b_1^2 + a_2^2 - 2a_2 b_2 + b_2^2 + a_3^2 - 2a_3 b_3 + b_3^2 \tag{22}$$

$$\Longleftrightarrow -2ab\cos\theta = -2a_1 b_1 - 2a_2 b_2 - 2a_3 b_3 \tag{23}$$

$$\Longleftrightarrow ab\cos\theta = a_1 b_1 + a_2 b_2 + a_3 b_3 \Leftrightarrow \mathbf{a} \cdot \mathbf{b}. \tag{24}$$

Therefore, dot product

$$\mathbf{a} \cdot \mathbf{b} = ab\cos\theta. \tag{25}$$

Cross product

$$\mathbf{a} \times \mathbf{b} = a_i \mathbf{e}_i \times \mathbf{b}_j \mathbf{e}_j = a_i b_j \mathbf{e}_i \times \mathbf{e}_j = a_i b_j \epsilon_{ijk} \mathbf{e}_k. \tag{26}$$

Expanding,

$$\mathbf{a} \times \mathbf{b} = \cancel{a_1 b_1 \epsilon_{111} \mathbf{e}_1} + \cancel{a_1 b_1 \epsilon_{112} \mathbf{e}_2} + \cancel{a_1 b_1 \epsilon_{113} \mathbf{e}_3} \tag{27}$$

$$+\cancel{a_1 b_2 \epsilon_{121} \mathbf{e}_1} + \cancel{a_1 b_2 \epsilon_{122} \mathbf{e}_2} + a_1 b_2 \epsilon_{123} \mathbf{e}_3 \tag{28}$$

$$+\cancel{a_1 b_3 \epsilon_{131} \mathbf{e}_1} + a_1 b_3 \epsilon_{132} \mathbf{e}_2 + \cancel{a_1 b_3 \epsilon_{133} \mathbf{e}_3} \tag{29}$$

$$+\cancel{a_2 b_1 \epsilon_{211} \mathbf{e}_1} + \cancel{a_2 b_1 \epsilon_{212} \mathbf{e}_2} + a_2 b_1 \epsilon_{213} \mathbf{e}_3 \tag{30}$$

$$+\cancel{a_2 b_2 \epsilon_{221} \mathbf{e}_1} + \cancel{a_2 b_2 \epsilon_{222} \mathbf{e}_2} + \cancel{a_2 b_2 \epsilon_{223} \mathbf{e}_3} \tag{31}$$

$$+a_2b_3\epsilon_{231}\mathbf{e}_1 + \cancel{a_2b_3\epsilon_{232}}\mathbf{e}_2 + \cancel{a_2b_3\epsilon_{233}}\mathbf{e}_3 \tag{32}$$

$$+\cancel{a_3b_1\epsilon_{311}}\mathbf{e}_1 + a_3b_1\epsilon_{312}\mathbf{e}_2 + \cancel{a_3b_1\epsilon_{313}}\mathbf{e}_3 \tag{33}$$

$$+a_3b_2\epsilon_{321}\mathbf{e}_1 + \cancel{a_3b_2\epsilon_{322}}\mathbf{e}_2 + \cancel{a_3b_2\epsilon_{323}}\mathbf{e}_3 \tag{34}$$

$$+\cancel{a_3b_3\epsilon_{331}}\mathbf{e}_1 + \cancel{a_3b_3\epsilon_{332}}\mathbf{e}_2 + \cancel{a_3b_3\epsilon_{333}}\mathbf{e}_3 \tag{35}$$

$$= \mathbf{e}_1(a_2b_3 - a_3b_2) + \mathbf{e}_2(a_3b_1 - a_1b_3) + \mathbf{e}_3(a_1b_2 - a_2b_1) \Leftrightarrow \det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}. \tag{36}$$

It is also provable that

$$\mathbf{a} \times \mathbf{b} = ab\sin\theta\hat{\mathbf{n}} \Leftrightarrow |\mathbf{a}||\mathbf{b}|\sin\theta\mathbf{e}_n \tag{37}$$

where $\mathbf{e}_n$ points in the direction normal to the plane formed by $\mathbf{a}$ and $\mathbf{b}$ and can be identified using the right hand rule. Volume of three new vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}) \cdot (|\mathbf{b}||\mathbf{c}|\sin\theta\mathbf{e}_n) = |\mathbf{a}||\mathbf{b}||\mathbf{c}|\sin\theta\cos\alpha \tag{38}$$

where θ is the angle between $\mathbf{b}$ and $\mathbf{c}$ and α is the angle between $\mathbf{a}$ and vector $\mathbf{b} \times \mathbf{c}$. In index notation

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot (b_ic_j\epsilon_{ijk}\mathbf{e}_k) = a_m\mathbf{e}_m \cdot b_ic_j\epsilon_{ijk}\mathbf{e}_k = a_mb_ic_j\epsilon_{ijk}\mathbf{e}_m \cdot \mathbf{e}_k \tag{39}$$

$$\Leftrightarrow a_mb_ic_j\epsilon_{ijk}\delta_{mk} = a_m\delta_{mk}b_ic_j\epsilon_{ijk} = a_kb_ic_j\epsilon_{ijk}. \tag{40}$$

Note indices are arbitrary in that

$$a_kb_ic_j\epsilon_{ijk} \Leftrightarrow a_ib_jc_k\underbrace{\epsilon_{jki}}_{\mathbf{I.}} = a_ib_jc_k\underbrace{\epsilon_{ijk}}_{\mathbf{I.}} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}). \tag{41}$$

Because of the arbitrariness of the indices it can be shown that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \tag{42}$$

so long as the permutation between a,b,c remains intact so that ϵ does not change sign (as evidenced by $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$). Note that a vector is direction times magnitude. So, a vector divided by its magnitude is just its direction ($\mathbf{a}/|\mathbf{a}|$). With that said the definition of the projection $\mathbf{p}$ of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\mathbf{p} = |\mathbf{b}|\cos\theta\frac{\mathbf{a}}{|\mathbf{a}|} \tag{43}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. In other words this is the horizontal component of $\mathbf{b}$ in the direction of $\mathbf{a}$. Note

$$|\mathbf{b}|\cos\theta\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{|\mathbf{a}||\mathbf{b}|\cos\theta}{|\mathbf{a}|}\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\frac{\mathbf{a}}{|\mathbf{a}|} \tag{44}$$

because of the definition of dot product.

1.2 Change of basis

A set of basis vectors is an arbitrary way to judge the location of a point. Sometimes it might be mathematically more simple to change the set of basis vectors as we desire, which because of its arbitrariness we are totally allowed to do. Consider vector

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 = v_i\mathbf{e}_i \tag{45}$$

where v_i are the components of $\mathbf{v}$ and $\mathbf{e}_i$ are the basis vectors. Let us define a new orthonormal basis

$$\mathbf{v} = \sum_{i=1}^{3} v_i\mathbf{e}_i = \sum_{i=1}^{3} \bar{v}_i\bar{\mathbf{e}}_i \tag{46}$$

where the barred quantities are also components of $\mathbf{v}$ and basis vectors. Then

$$v_i\mathbf{e}_i = \bar{v}_i\bar{\mathbf{e}}_i \Longleftrightarrow v_i \underbrace{\mathbf{e}_i \cdot \mathbf{e}_j}_{\textbf{I.}} = \bar{v}_i \underbrace{\bar{\mathbf{e}}_i \cdot \mathbf{e}_j}_{\textbf{II.}} \Longleftrightarrow \overbrace{v_i \underbrace{\delta_{ij}}_{\textbf{I.}}}^{\textbf{III.}} = \bar{v}_i \underbrace{R_{ij}}_{\textbf{II.}} \Longleftrightarrow \overbrace{v_j}^{\textbf{III.}} = \bar{v}_i R_{ij}. \tag{47}$$

where $R_{ij} = \bar{\mathbf{e}}_i \cdot \mathbf{e}_j$ is the dot product between the old and new coordinate systems. Note that this will not necessarily be identity $\mathbf{I}$ because it is not necessarily true that $\bar{\mathbf{e}}_1 \cdot \mathbf{e}_1 = \cos 0 = 1$, etc. Further,

$$v_j = \bar{v}_i R_{ij} = R^T_{ji}\bar{v}_i \Longleftrightarrow \mathbf{v} = \mathbf{R}^T\bar{\mathbf{v}}. \tag{48}$$

Instead of how we started with Eq. 47 which was to multiply both sides by $\mathbf{e}_j$, we could have also multiplied by $\bar{\mathbf{e}}_j$. What follows is

$$v_i\mathbf{e}_i = \bar{v}_i\bar{\mathbf{e}}_i \Longleftrightarrow v_i \underbrace{\mathbf{e}_i \cdot \bar{\mathbf{e}}_j}_{\textbf{I.}} = \bar{v}_i \underbrace{\bar{\mathbf{e}}_i \cdot \bar{\mathbf{e}}_j}_{\textbf{II.}} \Longleftrightarrow v_i \underbrace{R_{ji}}_{\textbf{I.}} = \overbrace{\bar{v}_i \underbrace{\delta_{ij}}_{\textbf{II.}}}^{\textbf{III.}} \Longleftrightarrow R_{ji}v_i = \overbrace{\bar{v}_j}^{\textbf{III}}. \tag{49}$$

$$\Longleftrightarrow \bar{\mathbf{v}} = \mathbf{R}\mathbf{v} \Longleftrightarrow \mathbf{v} = \mathbf{R}^{-1}\bar{\mathbf{v}}. \tag{50}$$

Combining Eq. 48 and Eq. 50,

$$\mathbf{R}^{-1} = \mathbf{R}^T \Leftrightarrow \mathbf{R}\mathbf{R}^T = \mathbf{R}\mathbf{R}^{-1} = \mathbf{I} \Longleftrightarrow R_{ik}R_{jk} = \delta_{ij} \tag{51}$$

where the indexing $R_{ik}R_{jk}$ defies the conventional rule of matrix multiplication that the dummy index k neighbors itself (as in $R_{ik}R_{kj}$) because of the transpose operation on $\mathbf{R}^T$.

A matrix $\mathbf{R}$ that satisfies $\mathbf{R}^T = \mathbf{R}^{-1}$ is said to be orthogonal. Rotation matrices are always orthogonal. Consider

$$\mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{52}$$

This rotation matrix, when applied to a vector such that

$$\begin{Bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \bar{v}_3 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}, \tag{53}$$

is describing the component by component transformation

$$\bar{v}_1 = v_1 \cos\theta + v_2 \sin\theta, \quad \bar{v}_2 = -v_2 \sin\theta + v_2 \cos\theta, \quad \bar{v}_3 = v_3 \tag{54}$$

where θ is the angle of rotation. This particular transformation is describing a θ degrees counterclockwise rotation between orthonormal bases in the dimensions $\mathbf{e}_1$ and $\mathbf{e}_2$.

The rules of determinants for matrices are

$$\det \mathbf{S}^T = \det \mathbf{S}, \quad \det \mathbf{ST} = \det \mathbf{S} \det \mathbf{T}, \quad \det \mathbf{S}^{-1} = (\det \mathbf{S})^{-1}. \tag{55}$$

Accepting this,

$$1 = \det \mathbf{I} = \det \mathbf{RR}^T = \det \mathbf{R} \det \mathbf{R}^T = \det \mathbf{R} \det \mathbf{R} = (\det \mathbf{R})^2. \tag{56}$$

Therefore,

$$\det \mathbf{R} = \pm 1 \tag{57}$$

if $\mathbf{R}$ is orthogonal. The signage determines the functionality of $\mathbf{R}$. Particularly

$$\det \mathbf{R} = \begin{cases} 1, & \text{rotation, e.g.} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{(90 degree clockwise rotation in } xy) \\ -1, & \text{rotation and reflection, e.g.} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{(reflection in } z). \end{cases} \tag{58}$$

Note that under an orthogonal coordinate transformation the magnitude of the vector $\mathbf{v} \longrightarrow \bar{\mathbf{v}}$ does not change. Its square length

$$\bar{v}_i \bar{v}_i = R_{ij} v_j R_{ik} v_k = \delta_{jk} v_j v_k = v_j v_j. \tag{59}$$

A vector undergoing a transformation by $\mathbf{R}$ is considered a tensor with rank 1. A tensor of rank 0 is a scalar, and a tensor of rank 2 is an $m \times n$ matrix.

If the relationships

$$\bar{\mathbf{v}} = \mathbf{Rv} \to \mathbf{R}^T \bar{\mathbf{v}} = \mathbf{v}, \quad \bar{\mathbf{u}} = \mathbf{Ru} \to \mathbf{R}^T \bar{\mathbf{u}} = \mathbf{u}, \quad \mathbf{v} = \mathbf{Mu} \Longleftrightarrow \bar{\mathbf{v}} = \overbrace{\bar{\mathbf{M}}}^{\mathbf{I.}} \bar{\mathbf{u}} \tag{60}$$

hold, then

$$(\mathbf{v}) = \mathbf{M}(\mathbf{u}) \longrightarrow (\mathbf{R}^T \bar{\mathbf{v}}) = \mathbf{M}(\mathbf{R}^T \bar{\mathbf{u}}) \longrightarrow \mathbf{RR}^T \bar{\mathbf{v}} = \mathbf{RMR}^T \bar{\mathbf{u}} \longrightarrow \bar{\mathbf{v}} = \overbrace{\mathbf{RMR}^T}^{\mathbf{I.}} \bar{\mathbf{u}} \tag{61}$$

implies

$$\underbrace{\mathbf{RMR}^T = \bar{\mathbf{M}}}_{\mathbf{I.}} \Longleftrightarrow R_{ik} M_{kl} R_{jl} = \bar{M}_{ij}. \tag{62}$$

In general this is how to transform a second order tensor. In general for a tensor of any rank,

$$\bar{A}_{ij...k} = R_{ip}R_{jq}...R_{kr}A_{pq...r}. \tag{63}$$

The trace of a matrix

$$\mathrm{tr}\bar{\mathbf{M}} \Longleftrightarrow \bar{M}_{ii} = R_{ij}M_{jk}R_{ik} = \underbrace{R_{ij}R_{ik}}_{\mathbf{I}.} M_{jk} = \underbrace{\delta_{jk}}_{\mathbf{I}.} M_{jk} = M_{kk} \Longleftrightarrow \mathrm{tr}\mathbf{M}. \tag{64}$$

So the trace of a matrix under orthogonal transformation is invariant.

An isotropic tensor is one that does not change because of a coordinate transformation. For example the Kronecker delta is isotropic in that

$$\bar{\delta}_{ij} = R_{ik}R_{jl}\delta_{kl} = R_{il}R_{jl} = \delta_{ij}. \tag{65}$$

In matrix notation this is more obvious as

$$\bar{\mathbf{I}} = \mathbf{R}\mathbf{I}\mathbf{R}^T = \mathbf{R}\mathbf{R}^T = \mathbf{I}. \tag{66}$$

1.3 Symmetry and skew symmetry

A matrix $\mathbf{S}$ is symmetric if

$$\mathbf{S} = \mathbf{S}^T \longleftrightarrow S_{ij} = S_{ji} \longrightarrow [\mathbf{S}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}. \tag{67}$$

A matrix $\mathbf{A}$ is skew symmetric if

$$\mathbf{A} = -\mathbf{A}^T \longleftrightarrow A_{ij} = -A_{ji} \longrightarrow [\mathbf{A}] = \begin{bmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{bmatrix}. \tag{68}$$

Any matrix $\mathbf{M}$ has symmetric and skewsymmetric components

$$\mathbf{M} = \mathbf{S} + \mathbf{A} \quad \text{where} \quad \mathbf{S} = \frac{\mathbf{M} + \mathbf{M}^T}{2} = \mathbf{S}^T, \quad \mathbf{A} = \frac{\mathbf{M} - \mathbf{M}^T}{2} = -\mathbf{A}^T. \tag{69}$$

For example

$$\mathbf{M} = \begin{bmatrix} 3 & 5 & 7 \\ 1 & 2 & 8 \\ 9 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 8 \\ 3 & 2 & 7 \\ 8 & 7 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \mathbf{S} + \mathbf{A}. \tag{70}$$

Note that for skew symmetric $\mathbf{A}$,

$$\mathbf{x} \cdot \mathbf{A}\mathbf{x} = \mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{x}^T\mathbf{A}^T\mathbf{x}^{T^T} = \mathbf{x}^T\mathbf{A}^T\mathbf{x} = -\mathbf{x}^T\mathbf{A}\mathbf{x} \tag{71}$$

which implies

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = -\mathbf{x}^T\mathbf{A}\mathbf{x} \longrightarrow \mathbf{x} \cdot \mathbf{A}\mathbf{x} = 0 \forall \mathbf{x}. \tag{72}$$

Consider then a matrix $\mathbf{M}$ subjected to the matrix product

$$\mathbf{x} \cdot \mathbf{M}\mathbf{x} = \mathbf{x} \cdot (\mathbf{S} + \mathbf{A})\mathbf{x} = \mathbf{x} \cdot \mathbf{S}\mathbf{x} + \mathbf{x} \cdot \mathbf{A}\mathbf{x} = \mathbf{x} \cdot \mathbf{S}\mathbf{x}. \tag{73}$$

This is called the quadratic form of $\mathbf{M}$. Expanded, the quadratic form is

$$\mathbf{x}^T\mathbf{M}\mathbf{x} \Longleftrightarrow \mathbf{x} \cdot \mathbf{M}\mathbf{x} \Longleftrightarrow x_i M_{ij} x_j \tag{74}$$

$$= x_1(M_{11}x_1 + M_{12}x_2 + M_{13}x_3) + x_2(M_{21}x_1 + M_{22}x_2 + M_{23}x_3) + x_3(M_{31}x_1 + M_{32}x_2 + M_{33}x_3)$$

$$= x_1^2 M_{11} + x_2^2 M_{22} + x_3^2 M_{33} + x_1x_2(M_{12} + M_{21}) + x_1x_3(M_{13} + M_{31}) + x_2x_3(M_{23} + M_{32}). \tag{75}$$

$\mathbf{M}$ is positive definite if its quadratic form $\mathbf{x} \cdot \mathbf{M}\mathbf{x} > 0 \forall \mathbf{x}$ and positive semidefinite if $\mathbf{x} \cdot \mathbf{M}\mathbf{x} \geq 0 \forall \mathbf{x}$.

1.4 Derivatives and divergence

Consider the scalar function $\phi(x_j)$. The chain rule states

$$\frac{\partial \phi}{\partial \bar{x}_i} = \frac{\partial \phi}{\partial x_j}\frac{\partial x_j}{\partial \bar{x}_i}. \tag{76}$$

If $x_j = R_{kj}\bar{x}_k$, then

$$\frac{\partial x_j}{\partial \bar{x}_i} = \frac{\partial}{\partial \bar{x}_i}(R_{kj}\bar{x}_k) = R_{kj}\frac{\partial \bar{x}_k}{\partial \bar{x}_i} = R_{kj}\delta_{ki} = \delta_{ik}R_{kj} = R_{ij}. \tag{77}$$

Substituting this into Eq. 76,

$$\frac{\partial \phi}{\partial \bar{x}_i} = R_{ij}\frac{\partial \phi}{\partial x_j}. \tag{78}$$

The result of this is a tensor of rank 1. The del or nabla operator

$$\nabla = \mathbf{e}_1\frac{\partial}{\partial x_1} + \mathbf{e}_2\frac{\partial}{\partial x_2} + \mathbf{e}_3\frac{\partial}{\partial x_3} = \mathbf{e}_i\frac{\partial}{\partial x_i}. \tag{79}$$

Gradient

$$\nabla \phi = \frac{\partial \phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial \phi}{\partial x_2}\mathbf{e}_2 + \frac{\partial \phi}{\partial x_3}\mathbf{e}_3 = \frac{\partial \phi}{\partial x_i}\mathbf{e}_i = \phi_{,i}\mathbf{e}_i. \tag{80}$$

This turns the scalar ϕ into a vector. Gradient increases rank.

A directional derivative is the amount that a function's gradient aligns with direction $\mathbf{e}_s$. It is a scalar. It is

$$\frac{\partial \phi}{\partial s} = \mathbf{e}_s \cdot \nabla \phi \Leftrightarrow |\mathbf{e}_s||\nabla \phi| \cos\theta = |\nabla \phi| \cos\theta \tag{81}$$

where θ is the angle between vector $\mathbf{e}_s$ and vector $\nabla \phi$.

Now consider vector function $\mathbf{f}(x_j) = f_j\mathbf{e}_j$. Divergence of $\mathbf{f}$

$$\nabla \cdot \mathbf{f} = \left(\mathbf{e}_i\frac{\partial}{\partial x_i}\right) \cdot \left(f_j\mathbf{e}_j\right) = (\mathbf{e}_i \cdot \mathbf{e}_j)\frac{\partial f_j}{\partial x_i} = \delta_{ij}\frac{\partial f_j}{\partial x_i} = \frac{\partial f_i}{\partial x_i} = f_{i,i}. \tag{82}$$

This operation turns $\mathbf{f}$ from a vector into a scalar. Divergence decreases rank.

The Laplacian maintains rank. The Laplacian is the divergence of the gradient. Scalar ϕ has Laplacian

$$\nabla^2\phi := \nabla \cdot \nabla\phi = \nabla \cdot (\frac{\partial \phi}{\partial x_i}\mathbf{e}_i) = (\frac{\partial}{\partial x_j}\mathbf{e}_j) \cdot (\frac{\partial \phi}{\partial x_i}\mathbf{e}_i) = (\mathbf{e}_j \cdot \mathbf{e}_i)\frac{\partial^2 \phi}{\partial x_j \partial x_i} \tag{83}$$

$$= \delta_{ji}\frac{\partial^2 \phi}{\partial x_j \partial x_i} = \frac{\partial^2 \phi}{\partial^2 x_i} = \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = \phi_{,ii}. \tag{84}$$

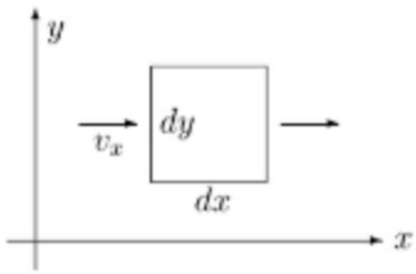
1.5 Divergence theorem

Consider a fluid with density $\rho = \rho(x, y, z)$ and velocity

$$\mathbf{v} = v_x\mathbf{e}_x + v_y\mathbf{e}_y + v_z\mathbf{e}_z \tag{85}$$

where $v_x = v_x(x, y, z)$, $v_y = v_y(x, y, z)$, $v_z = v_z(x, y, z)$. The meaning of this is that v_i and ρ can change in magnitude based on the specific point (x, y, z), but this has no bearing towards the directionality of the components of the velocity vector $(\circ)\mathbf{e}_i$ which always point in the x_i direction.

Imagine this fluid is flowing through a small cube.



On the left side, the fluid enters at a rate

$$\text{rate in} = (\rho)(v_x)(dydz) = \frac{\text{mass}}{\cancel{xyz \text{ vol}}}\frac{\cancel{x \text{ dist}}}{\text{time}}\cancel{yz \text{ area}} = \frac{\text{mass}}{\text{time}}. \tag{86}$$

On the right side, the fluid exits at a rate

$$\text{rate out} = (\rho v_x + \frac{\partial \rho v_x}{\partial x}dx)dydz \tag{87}$$

$$\rho v_x dydz + \frac{\partial \rho}{\partial x}v_x dxdydz + \rho\frac{\partial v_x}{\partial x}dxdydz \tag{88}$$

$$= \frac{\text{mass}}{\text{time}} + \frac{\text{mass}}{xyz \text{ vol} \times x \text{ dist}}\frac{x \text{ dist}}{\text{time}}xyz \text{ vol} + \frac{\text{mass}}{xyz \text{ vol}}\frac{x \text{ dist}}{\text{time} \times x \text{ dist}}xyz \text{ vol} = \frac{\text{mass}}{\text{time}}. \tag{89}$$

Then the total gain of mass per time is

$$\text{rate in} - \text{rate out} = \rho v_x dydz - \rho v_x dydz - \frac{\partial \rho v_x}{\partial x}dxdydz = -\frac{\partial}{\partial x}(\rho v_x)dxdydz. \tag{90}$$

where the total loss is the negative of the total gain. Considering all directions, total loss is

$$\frac{\partial}{\partial x}(\rho v_x)dxdydz + \frac{\partial}{\partial y}(\rho v_y)dxdydz + \frac{\partial}{\partial z}(\rho v_z)dxdydz \tag{91}$$

$$= \frac{\partial}{\partial x_i}(\rho v_i)dxdydz = \nabla \cdot (\rho v_i)dxdydz. \tag{92}$$

If bounded by volume V then this becomes

$$\text{total loss per time} = \int_V \nabla \cdot (\rho \mathbf{v})dV. \tag{93}$$

The divergence theorem is the relationship between the amount of fluid exiting with respect to the volume of the body and the amount of fluid crossing the outer surface across the perimeter. Physically they are the same thing. The relationship for this problem is

$$\int_V \nabla \cdot (\rho \mathbf{v})dV = \oint_S \rho \mathbf{v} \cdot \mathbf{n} dS. \tag{94}$$

In general for a vector $\mathbf{f}$, the divergence theorem

$$\int_V \nabla \cdot \mathbf{f} dV = \oint_S \mathbf{f} \cdot \mathbf{n} dS \Longleftrightarrow \int_V f_{i,i} dV = \oint_S f_i n_i dS. \tag{95}$$

Similar rules are the gradient theorem for scalar f

$$\int_V \nabla f dV = \oint_S f \mathbf{n} dS \tag{96}$$

and the curl theorem for vector $\mathbf{f}$

$$\int_V \nabla \times \mathbf{f} dV = \oint_S \mathbf{n} \times \mathbf{f} dS. \tag{97}$$

The divergence theorem can be approximated about a point P as

$$(\nabla \cdot \mathbf{f})_P \approx \frac{1}{\Delta V} \oint_S \mathbf{f} \cdot \mathbf{n} dS \tag{98}$$

where ΔV is a small volume element surrounding point P. This means that the divergence of $\mathbf{f}$ can be thought of as the outward flow of $\mathbf{f}$ normal to the surface per unit volume. It can be said about the divergence theorem that the sum of the sources and sinks (V) is equal to the net flow in and out of the surface (S).

The utility of the integral theorems can also be demonstrated by considering the applied pressure $p(x, y, z)$ on a body. Pressure is force/area, so force is pressure $\times$ area or (pressure/dist) $\times$ volume, and this force will act normal to the surface of the body. Then the gradient theorem dictates

$$\mathbf{F} = -\oint_S p \mathbf{n} DS = -\int_V \nabla p dV. \tag{99}$$

If pressure is constant then force is zero because a uniform pressure load across the entire body will result in no net force in any particular direction.

1.6 Eigenvalue problems

If for square matrix $\mathbf{A}$ there is some pair $\mathbf{x}, \lambda$ such that

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{100}$$

then $\mathbf{x}, \lambda$ are an eigenvector and eigenvalue pair of system matrix $\mathbf{A}$. Consider

$$\lambda\mathbf{x} = \lambda\mathbf{I}\mathbf{x} = \mathbf{A}\mathbf{x} \longrightarrow (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}. \tag{101}$$

For nontrivial $\mathbf{x} \neq \mathbf{0}$, it must be that $(\mathbf{A} - \lambda\mathbf{I})$ is singular, meaning that by definition

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0}. \tag{102}$$

This is called the characteristic equation or characteristic polynomial for the eigenvalue problem. If A $= n \times n$ then the polynomial will be of degree n, will have n roots, and thus will have n eigenvalues. For example suppose

$$[\mathbf{A}] = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \longrightarrow \det \mathbf{A} - \lambda\mathbf{I} = (2-\lambda)(1-\lambda) - (-1)(-1) = 1 - 3\lambda + \lambda^2 = 0. \tag{103}$$

Then

$$\lambda = \frac{3 \pm \sqrt{5}}{2} = \{0.382,\ 2.618\} \tag{104}$$

$$\longrightarrow \mathbf{0} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = (\mathbf{A} - \lambda\mathbf{x}) = \begin{bmatrix} 1.618 & -1 \\ -1 & 0.618 \end{bmatrix} \begin{Bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{Bmatrix} \longrightarrow \mathbf{x}^{(1)} = \begin{Bmatrix} 0.618 \\ 1 \end{Bmatrix}, \tag{105}$$

$$\longrightarrow \mathbf{0} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = (\mathbf{A} - \lambda\mathbf{x}) = \begin{bmatrix} -0.618 & -1 \\ -1 & -1.618 \end{bmatrix} \begin{Bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{Bmatrix} \longrightarrow \mathbf{x}^{(2)} = \begin{Bmatrix} -1.618 \\ 1 \end{Bmatrix}. \tag{106}$$

1.7 Even and odd functions

A function f is even if $f(-x) = f(x)$, meaning the y values on the left are the same as the y values on the right. It is odd if $f(-x) = -f(x)$, meaning the y values on the left are the opposite of the y values on the right. The following properties hold.

- f is even and smooth $\longrightarrow f'(0) = 0$.
- f is odd $\longrightarrow f(0) = 0$.
- f, g are even $\longrightarrow fg$ is even.
- f, g are odd $\longrightarrow fg$ is even.
- f is even, g is odd $\longrightarrow fg$ is odd.
- f is even $\longrightarrow f'$ is odd.
- f is odd $\longrightarrow f'$ is even.

- f is even $\longrightarrow \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.
- f is odd $\longrightarrow \int_{-a}^{a} f(x)dx = 0$.

Any f can be broken into even and odd parts f_e, f_o so that

$$f(x) = \underbrace{f_e(x)}_{\mathbf{I.}} + \underbrace{f_o(x)}_{\mathbf{II.}} = \underbrace{\frac{1}{2}[f(x) + f(-x)]}_{\mathbf{I.}} + \underbrace{\frac{1}{2}[f(x) - f(-x)]}_{\mathbf{II.}}. \tag{107}$$

2 Strain

2.1 Admissible deformation

Consider an object undergoing deformation so that $(x_1, x_2, x_3) \longrightarrow (\xi_1, \xi_2, \xi_3)$ is the transformation between the coordinates of a point P in its original state to the coordinates of the same point in a deformed state. We can express the deformed coordinates as a function of each of the original coordinates, so that $\xi_i = \xi_i(x_j)$. Inversely, we can say $x_i = x_i(\xi_j)$. The derivative operators are related by

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial \xi_1}\frac{\partial \xi_1}{\partial x_1} + \frac{\partial}{\partial \xi_2}\frac{\partial \xi_2}{\partial x_1} + \frac{\partial}{\partial \xi_3}\frac{\partial \xi_3}{\partial x_1}, \tag{108}$$

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial \xi_1}\frac{\partial \xi_1}{\partial x_2} + \frac{\partial}{\partial \xi_2}\frac{\partial \xi_2}{\partial x_2} + \frac{\partial}{\partial \xi_3}\frac{\partial \xi_3}{\partial x_2}, \tag{109}$$

$$\frac{\partial}{\partial x_3} = \frac{\partial}{\partial \xi_1}\frac{\partial \xi_1}{\partial x_3} + \frac{\partial}{\partial \xi_2}\frac{\partial \xi_2}{\partial x_3} + \frac{\partial}{\partial \xi_3}\frac{\partial \xi_3}{\partial x_3}. \tag{110}$$

As a system of equations,

$$\begin{Bmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} \partial\xi_1/\partial x_1 & \partial\xi_2/\partial x_1 & \partial\xi_3/\partial x_1 \\ \partial\xi_1/\partial x_2 & \partial\xi_2/\partial x_2 & \partial\xi_3/\partial x_2 \\ \partial\xi_1/\partial x_3 & \partial\xi_2/\partial x_3 & \partial\xi_3/\partial x_3 \end{bmatrix}}_{[\mathbf{J}]} \begin{Bmatrix} \partial/\partial \xi_1 \\ \partial/\partial \xi_2 \\ \partial/\partial \xi_3 \end{Bmatrix} = [\mathbf{J}] \begin{Bmatrix} \partial/\partial \xi_1 \\ \partial/\partial \xi_2 \\ \partial/\partial \xi_3 \end{Bmatrix}. \tag{111}$$

Jacobian matrix $\mathbf{J}$ is the 3×3 transformation tensor. A physically real transformation requires $\det \mathbf{J} > 0$. For the inverse Jacobian $\mathbf{J}^{-1}$ to exist, $\det \mathbf{J} \neq 0$. If the body is not deformed at all then $\det \mathbf{J} = 1$. This is the case where there is no distinction between x and ξ and so $\mathbf{J}$ becomes $\mathbf{I}$.

Displacement vector

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} \xi_1 - x_1 \\ \xi_2 - x_2 \\ \xi_3 - x_3 \end{Bmatrix} \tag{112}$$

implies

$$\begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Bmatrix} = \begin{Bmatrix} x_1 + u_1 \\ x_2 + u_2 \\ x_3 + u_3 \end{Bmatrix} \longrightarrow [\mathbf{J}] = \begin{bmatrix} 1 + \partial u_1/\partial x_1 & \partial u_2/\partial x_1 & \partial u_3/\partial x_1 \\ \partial u_1/\partial x_2 & 1 + \partial u_2/\partial x_2 & \partial u_3/\partial x_2 \\ \partial u_1/\partial x_3 & \partial u_2/\partial x_3 & 1 + \partial u_3/\partial x_3 \end{bmatrix}. \tag{113}$$

For example if

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} x_1 - 2x_2 \\ 3x_1 + 2x_2 \\ 5x_3 \end{Bmatrix} \tag{114}$$

then

$$\begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Bmatrix} = \begin{Bmatrix} 2x_1 - 2x_2 \\ 3x_1 + 3x_2 \\ 6x_3 \end{Bmatrix} \longrightarrow [\mathbf{J}] = \begin{bmatrix} 2 & 3 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \tag{115}$$

and $\det \mathbf{J} = 2(18) + -3(-12) = 72 > 0$, making it admissible.

$\mathbf{J}$ can be thought of as the ratio between the volume of the deformed configuration and the undeformed configuration (new/old). Vectors

$$\mathbf{dx}_1 = \begin{Bmatrix} \partial\xi_1/\partial x_1 \\ \partial\xi_2/\partial x_1 \\ \partial\xi_3/\partial x_1 \end{Bmatrix} dx_1, \quad \mathbf{dx}_2 = \begin{Bmatrix} \partial\xi_1/\partial x_2 \\ \partial\xi_2/\partial x_2 \\ \partial\xi_3/\partial x_2 \end{Bmatrix} dx_2, \quad \mathbf{dx}_3 = \begin{Bmatrix} \partial\xi_1/\partial x_3 \\ \partial\xi_2/\partial x_3 \\ \partial\xi_3/\partial x_3 \end{Bmatrix} dx_3 \tag{116}$$

are tangent to the coordinate curves of x_1, x_2, and x_3 respectively, where for example the x_1 coordinate curve is obtained by fixing x_2, x_3 and changing x_1. In the Cartesian coordinate system the coordinate curves are just the axes. For example consider Fig. 1. Going from $(x_1, x_2, x_3) \longrightarrow (x_1 + \Delta x_1, x_2, x_3)$ causes a change in both ξ_1, ξ_2 so that

$$\Delta x_1 \cos\theta = \Delta\xi_1 \longrightarrow \frac{\Delta x_1}{\Delta \xi_1} = \cos\theta, \tag{117}$$

$$\Delta x_1 \sin\theta = \Delta\xi_2 \longrightarrow \frac{\Delta x_1}{\Delta \xi_2} = \sin\theta. \tag{118}$$

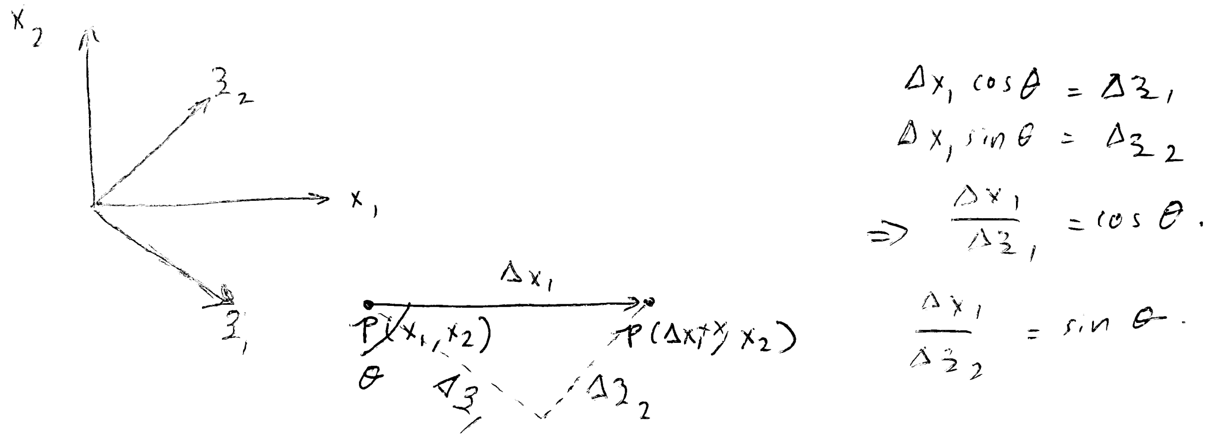


Figure 1: 2D coordinate transformation

Because of the triple scalar product identity

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}, \tag{119}$$

we find

$$\mathbf{dx}_1 \cdot \mathbf{dx}_2 \times \mathbf{dx}_3 = \det \begin{bmatrix} \partial\xi_1/\partial x_1 & \partial\xi_2/\partial x_1 & \partial\xi_3/\partial x_1 \\ \partial\xi_1/\partial x_2 & \partial\xi_2/\partial x_2 & \partial\xi_3/\partial x_2 \\ \partial\xi_1/\partial x_3 & \partial\xi_2/\partial x_3 & \partial\xi_3/\partial x_3 \end{bmatrix} dx_1 dx_2 dx_3 = \det \mathbf{J} dx_1 dx_2 dx_3 = dV. \tag{120}$$

This justifies the claim that $\det \mathbf{J}$ must be positive because it leads to a volume element that is positive and one cannot have negative volume.

Since the determinant is a norm or magnitude of the matrix, $\det \mathbf{J}$ can be thought of as a ratio between new and old volume, in the sense that it is the magnitude of the change in new coordinates with respect to the old coordinates. Therefore,

$$\det \mathbf{J} = \frac{V + \Delta V}{V} = 1 + \frac{\Delta V}{V}, \tag{121}$$

where the change in volume with respect to the original volume $\Delta V/V$ is called the volumetric strain. Recalling the representation of $\mathbf{J}$ that is Eq. 113,

$$\det \mathbf{J} = \det \begin{bmatrix} 1 + \partial u_1/\partial x_1 & \partial u_2/\partial x_1 & \partial u_3/\partial x_1 \\ \partial u_1/\partial x_2 & 1 + \partial u_2/\partial x_2 & \partial u_3/\partial x_2 \\ \partial u_1/\partial x_3 & \partial u_2/\partial x_3 & 1 + \partial u_3/\partial x_3 \end{bmatrix} \Leftrightarrow \det \begin{bmatrix} 1 + u_{1,1} & u_{2,1} & u_{3,1} \\ u_{1,2} & 1 + u_{2,2} & u_{3,2} \\ u_{1,3} & u_{2,3} & 1 + u_{3,3} \end{bmatrix}$$

$$= (1 + u_{1,1})[(1 + u_{2,2})(1 + u_{3,3}) - u_{2,3}u_{3,2}] \tag{122}$$

$$-u_{2,1}[u_{1,2}(1 + u_{3,3}) - u_{1,3}u_{3,2}] + u_{3,1}(u_{1,2}u_{2,3} - u_{1,3}(1 + u_{2,2})) \tag{123}$$

$$= 1 + u_{1,1} + u_{2,2} + u_{3,3} + \underbrace{\ldots}_{\text{combinations of product terms}}. \tag{124}$$

If the displacement is small then the product terms are negligible because small times small is extremely small. So in this case we can simplify to say

$$\det \mathbf{J} = 1 + \frac{\Delta V}{V} = 1 + u_{i,i}, \tag{125}$$

of course meaning

$$\frac{\Delta V}{V} = u_{i,i}. \tag{126}$$

2.2 Affine transformations

Let x_i be the original coordinates and $x_i'(x_j)$ be the new coordinates. Here we are only concerned with the deformation behavior itself and not how it happens (temperature, force, etc.). A special type of deformation is an affine transformation, which is when the function describing the relationship between the deformed coordinates and the original coordinates is linear. That is,

$$x_i' = \underbrace{x_i}_{\text{original coordinate vector}} + \underbrace{\alpha_{i0}}_{\text{translation vector}} + \underbrace{\alpha_{ij}x_j}_{\text{rotation and stretch}} \Leftrightarrow \begin{Bmatrix} x_1' = x_1 + \alpha_{10} + \alpha_{1j}x_j \\ x_2' = x_2 + \alpha_{20} + \alpha_{2j}x_j \\ x_3' = x_3 + \alpha_{30} + \alpha_{3j}x_j \end{Bmatrix} \tag{127}$$

which implies

$$x_i' = \delta_{ij}x_j + \alpha_{i0} + \alpha_{ij}x_j \longrightarrow x_i' = \alpha_{i0} + (\delta_{ij} + \alpha_{ij})x_j \tag{128}$$

or

$$\mathbf{x}' = \boldsymbol{\alpha}_0 + (\mathbf{I} + \boldsymbol{\alpha})\mathbf{x}. \tag{129}$$

So the term $\boldsymbol{\alpha}$ would have to be $\mathbf{0}$, not $\mathbf{I}$ if it was assumed that there was no rotation and no stretch. As another example the matrix

$$\mathbf{I}+\boldsymbol{\alpha}=\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \longrightarrow \boldsymbol{\alpha}=(c-1)\mathbf{I} \tag{130}$$

represents uniform stretch by the factor c, NOT by the factor $c-1$. The matrix

$$\mathbf{I}+\boldsymbol{\alpha}+\mathbf{I}=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \longrightarrow \boldsymbol{\alpha}=\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \tag{131}$$

represents a 90 degree CCW rotation in that

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} x_1' \\ x_2' \end{Bmatrix} \tag{132}$$

implies

$$\begin{Bmatrix} x_1' \\ x_2' \end{Bmatrix} = \begin{Bmatrix} -x_2 \\ x_1 \end{Bmatrix} \tag{133}$$

which can be thought of visually, turning the axes x_1, x_2 90 degrees counterclockwise. Like earlier, note that this does NOT imply the transformation $\boldsymbol{\alpha}\mathbf{x}=\mathbf{x}'$ but rather $(\boldsymbol{\alpha}+\mathbf{I})\mathbf{x}=\mathbf{x}'$. In the same way we can solve for $\mathbf{x}'$ in terms of $\mathbf{x}$ using Eq. 128, we also can solve for $\mathbf{x}$ in terms of $\mathbf{x}'$. So there must exist some $\boldsymbol{\beta}_0, \boldsymbol{\beta}$ such that

$$x_i=\beta_{i0}+(\delta_{ij}+\beta_{ij})x_j', \tag{134}$$

and this is also an affine/linear coordinate transformation.

Affine transformations have two interesting properties. First it transforms planes into other planes. The general equation for a plane is

$$Ax+By+Cz=D$$

and if we plug in the affine transformations into this equation then we receive another linear equation for a plane. The second interesting property of affine transformations is that straight lines transform into other straight lines. This is a consequence of (1) since lines are just intersections of planes. If planes turn into planes, then the straight lines on that plane turn into other straight lines.

As a consequence of (2), a vector

$$\mathbf{A}=A_i\mathbf{e}_i \Longrightarrow^{\text{affine}} A_i'\mathbf{e}_i'=\mathbf{A}' \tag{135}$$

turns into another vector under an affine transformation. Let $\mathbf{A}$ be a vector within a body that goes from one point x_{i0} to another point x_i. Note this is NOT the displacement vector that maps the undeformed coordinates to the deformed coordinates. This is simply a vector that travels across the body in its undeformed state from one point to another point. So

$$A_i=x_i-x_{i0}. \tag{136}$$

Then suppose the body undergoes an affine transformation. Then

$$A_i' = x_i' - x_{i0}' = (x_i + \cancel{\alpha_{i0}} + \alpha_{ij}x_j) - (x_{i0} + \cancel{\alpha_{i0}} + \alpha_{ij}x_{j0}) \tag{137}$$

$$= (x_i - x_{i0}) + \alpha_{ij}(x_j - x_{j0}) = A_i + \alpha_{ij}A_j. \tag{138}$$

Let

$$\delta A_i = \alpha_{ij}A_j = A_i' - A_i, \tag{139}$$

where $\delta A_i = \{\delta A_1, \delta A_2, \delta A_3\}$ are the components of the change between the vector before and after deformation. It is also defined as the components of the rotation/stretch vector. Both of these definitions follow from reading Eq. 139. Also, the length of $\mathbf{A}$ is

$$\sqrt{\mathbf{A}\cdot\mathbf{A}} = \sqrt{A^2\cos 0} = A \Longrightarrow A^2 = \mathbf{A}\cdot\mathbf{A} = A_iA_i, \tag{140}$$

and the length of δA_i is

$$\delta A = \sqrt{\delta A_i \delta A_i}. \tag{141}$$

Then

$$2A\delta A = 2\sqrt{A_iA_i}\sqrt{\delta A_i\delta A_i} = 2\sqrt{A_i\delta A_i}\sqrt{A_i\delta A_i} = 2A_i\delta A_i \Longrightarrow A\delta A = A_i\delta A_i. \tag{142}$$

Substituting in Eq. 139,

$$A\delta A = A_i \underbrace{\delta A_i}_{\mathbf{I.}} = A_i \underbrace{\alpha_{ij}A_j}_{\mathbf{I.}} = \alpha_{ij}A_iA_j. \tag{143}$$

If there is rotation but no stretch, then the change in the length of vector $\mathbf{A}$ does not change. Therefore $\delta A = 0$ and

$$\alpha_{ij}A_iA_j = 0 \quad \forall A_i. \tag{144}$$

Expanding,

$$0 = \alpha_{11}A_1A_1 + \alpha_{12}A_1A_2 + \alpha_{13}A_1A_3 \tag{145}$$

$$+\alpha_{21}A_2A_1 + \alpha_{22}A_2A_2 + \alpha_{23}A_2A_3 \tag{146}$$

$$+\alpha_{31}A_3A_1 + \alpha_{32}A_3A_2 + \alpha_{33}A_3A_3 \tag{147}$$

$$= 0 = \alpha_{11}A_1^2+\alpha_{22}A_2^2+\alpha_{33}A_3^2+A_1A_2(\alpha_{12}+\alpha_{21})+A_1A_3(\alpha_{13}+\alpha_{31})+A_2A_3(\alpha_{23}+\alpha_{32}). \tag{148}$$

If this is true for any $A_1, A_2, A_3 = A_i$, then

$$\alpha_{ii} = 0, \quad \alpha_{ij} = -\alpha_{ji}. \tag{149}$$

This means α_{ij} is skew symmetric. Please note that this is in the specific case where there is rotation but no stretch. This is not to say that α_{ij} in general is skew. Speaking more generally, the tensor $\boldsymbol{\alpha}$ like any tensor can be broken up into symmetric and skew parts

$$\alpha_{ij} = \frac{1}{2}(\alpha_{ij} + \alpha_{ji}) + \frac{1}{2}(\alpha_{ij} - \alpha_{ji}) = \epsilon_{ij} + \omega_{ij} \tag{150}$$

where $\boldsymbol{\epsilon}$ is solely dedicated to deformation and $\boldsymbol{\omega}$ is solely dedicated to rotation. $\boldsymbol{\epsilon} \Leftrightarrow \epsilon_{ij}$ is called the strain tensor.

2.3 Geometrical interpretations of strain components

Recalling Eq. 143,

$$A\delta A = \alpha_{ij}A_iA_j = (\epsilon_{ij} + \omega_{ij})A_iA_j = \epsilon_{ij}A_iA_j + \omega_{ij}A_iA_j. \tag{151}$$

Consider the last term, recalling that ω_{ij} is skew. This means

$$\omega_{ij}A_iA_j = -\omega_{ji}A_iA_j \Leftrightarrow -\omega_{ij}A_jA_i \Longrightarrow 2\omega_{ij}A_iA_j = 0 \Longrightarrow \omega_{ij}A_iA_j = 0. \tag{152}$$

Therefore,

$$A\delta A = \epsilon_{ij}A_iA_j \Longrightarrow \frac{\delta A}{A} = \frac{\epsilon_{ij}A_iA_j}{A^2}. \tag{153}$$

This represents the amount that the vector $\mathbf{A}$ has changed divided by its length. It is rating of relative length change. For example suppose $\mathbf{A}$ only had a component in the direction x_1. This means the length $A = A_1$ and

$$\frac{\delta A}{A_1} = \frac{\epsilon_{11}A_1^2}{A_1^2} \Longrightarrow \frac{\delta A}{A} = \epsilon_{11}. \tag{154}$$

So, the physical interpretation of the diagonal strain components ϵ_{ii} is that they are a measure of the change in length per unit length in the direction x_i. As for off diagonal components, consider two vectors that exist in the body

$$\mathbf{A} = A_2\mathbf{e}_2, \quad \mathbf{B} = B_3\mathbf{e}_3. \tag{155}$$

Here $\mathbf{A}$ only has a component in the direction x_2 and likewise $\mathbf{B}$ in x_3. Because of Eq. 139 ($\delta A_i = \alpha_{ij}A_j$),

$$\delta A_3 = \alpha_{32}A_2, \quad \delta B_2 = \alpha_{23}B_3. \tag{156}$$

The correct interpretation of this equation set is this. Initially B_2 is zero but a deformation in the body changes B_2 from zero to something that is not zero by the amount δB_2. This amount is equal to the initial component B_3 transformed under the tensor α_{23}. The same is true of $\mathbf{A}$. Both $\mathbf{B}$ and $\mathbf{A}$ change orientation, and this means they have a change in angle in relationship to one another, and the quantity of this change is

$$\alpha_{23} + \alpha_{32} = 2 * \frac{1}{2}(\alpha_{23} + \alpha_{32}) = 2\epsilon_{23} = \text{change in angle between } \mathbf{A} \text{ and } \mathbf{B}. \tag{157}$$

Note that $2\epsilon_{23} = \gamma_{23}$, where $\boldsymbol{\gamma}$ is the engineering strain tensor. So to recap, the diagonal strain components represent the change in length of a vector in a body with respect to its original length, and off-diagonal components represent the shear-induced change in angle between two vectors pointing in the two directions that correspond to the particular component of interest.

2.4 Strain as a tensor

Let us prove that strain $\boldsymbol{\epsilon}$ is a tensor. Recall Eq. 153, which is

$$A\delta A = \epsilon_{ij} A_i A_j = A_i \underbrace{\epsilon_{ij} A_j}_{\text{some vector } v_i} = \mathbf{A} \cdot \boldsymbol{\epsilon}\mathbf{A} = \mathbf{A}^T \boldsymbol{\epsilon}\mathbf{A}. \tag{158}$$

The change in length δA times the length A is invariant under the transformation of coordinates. Therefore

$$A\delta A = \bar{A}\delta\bar{A} \iff \mathbf{A}^T \boldsymbol{\epsilon}\mathbf{A} = \bar{\mathbf{A}}^T \bar{\boldsymbol{\epsilon}}\bar{\mathbf{A}}. \tag{159}$$

Let us define $\bar{\mathbf{A}}$ as the transformation of $\mathbf{A}$ due to $\mathbf{R}$. Then $\bar{\mathbf{A}} = \mathbf{R}\mathbf{A} \implies \mathbf{A} = \mathbf{R}^T \bar{\mathbf{A}}$ and

$$\bar{\mathbf{A}}^T \bar{\boldsymbol{\epsilon}}\bar{\mathbf{A}} = \mathbf{A}^T \boldsymbol{\epsilon}\mathbf{A} = (\mathbf{R}^T \bar{\mathbf{A}})^T \boldsymbol{\epsilon}(\mathbf{R}^T \bar{\mathbf{A}}) = \bar{\mathbf{A}}^T \mathbf{R}\boldsymbol{\epsilon}\mathbf{R}^T \bar{\mathbf{A}}. \tag{160}$$

Therefore

$$\bar{\boldsymbol{\epsilon}} = \mathbf{R}\boldsymbol{\epsilon}\mathbf{R}^T \implies \bar{\epsilon}_{ij} = R_{ik}\epsilon_{kl}R_{jl} = R_{ik}R_{jl}\epsilon_{kl} \tag{161}$$

which satisfies the definition of a transformed second order tensor.

2.5 General infinitesimal deformation

A major consequence of the affine transformation is Eq. 139, which is

$$\delta A_i = \alpha_{ij} A_j = A'_i - A_i. \tag{162}$$

Here A_j is a vector within some body and α_{ij} is a tensor that represents the rotation and deformation of that body. The result is the vector δA_i which represents the change between the original vector A and the new vector A'_i.

Similarly to the concept of $\mathbf{A}$, which is a vector in the undeformed body, let us consider two points in the undeformed configuration x_{i0} and x_i. After deformation, the corresponding displacements are

$$u_{i0} = \overbrace{x'_{i0} - x_{i0}}^{\mathbf{I}}, \quad u_i = \underbrace{x'_i - x_i}_{\mathbf{II}}. \tag{163}$$

If vector $\mathbf{A}$ represents the distance between x_i and x_{i0}, so that

$$A_i = x_i - x_{i0}, \tag{164}$$

$$\delta A_i = A'_i - A_i = (x'_i - x'_{i0}) - (x_i - x_{i0}) = \underbrace{x'_i - x_i}_{\mathbf{II}} \overbrace{-x'_{i0} + x_{i0}}^{-\mathbf{I}} = u_i - u_{i0}. \tag{165}$$

Now we will represent displacement as a Taylor series. In general for some function f,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \tag{166}$$

Neglecting higher order terms,

$$f(x) = f(x_0) + \underbrace{f'(x_0)(x - x_0)} + \dots . \tag{167}$$

For two variables,

$$f(x, y) = f(x_0, y_0) + \underbrace{\frac{\partial f(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0)} + \dots . \tag{168}$$

For many variables

$$f(x_i) = f(x_{i0}) + \frac{\partial f(x_{j0})}{\partial x_j}(x_j - x_{j0}) + \dots . \tag{169}$$

Substituting in displacement,

$$u_i = u_i(x_i) = u_{i0} + \frac{\partial u_{i0}}{\partial x_j}(x_j - x_{j0}) + \dots . \tag{170}$$

Recalling the assumption Eq. 164 ($A_i = x_i - x_{i0}$),

$$u_i = u_{i0} + u^{?}_{i,j} A_j + \dots . \tag{171}$$

or

$$u_i - u_{i0} = u_{i,j} A_j. \tag{172}$$

Because of Eq. 165 ($\delta A_i = u_i - u_{i0}$),

$$\delta A_i = u_{i,j} A_j. \tag{173}$$

Substituting in Eq. 139 ($\delta A_i = \alpha_{ij} A_j$),

$$\alpha_{ij} = u_{i,j}. \tag{174}$$

The decomposition of α_{ij} in Eq. 150 implies

$$u_{i,j} = \frac{1}{2}(u_{i,j} + u_{j,i}) + \frac{1}{2}(u_{i,j} - u_{j,i}) = \epsilon_{ij} + \omega_{ij}, \tag{175}$$

and the relationship between strain and displacement ($(u_{i,j} + u_{j,i})/2 = \epsilon_{ij}$) is called the strain/displacement equation.

For clarification purposes we now write u, v, w in place of u_x, u_y, u_z. Then for instance

$$\epsilon_{xx} = \frac{\partial u}{\partial x}, \quad \epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}\gamma_{yz}, \quad \text{etc.} \tag{176}$$

The diagonal components are called the normal strains and the off diagonal components are called the shear strains.

2.6 Compatibility equations

Given displacement field $\mathbf{u}$, strain components

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{177}$$

However it is not necessarily true that given a number of strain components we can calculate displacement. To answer this let us make the modification

$$2e_{rip}\epsilon_{ij} = e_{rip}(u_{i,j} + u_{j,i}) \tag{178}$$

where e here is the Levi Civita symbol. Differentiating with respect to x_p,

$$2e_{rip}\epsilon_{ij,p} = e_{rip}u_{i,jp} + e_{rip}u_{j,ip}. \tag{179}$$

The last term vanishes because

$$e_{rip}u_{j,ip} = -e_{rpi}u_{j,ip} \Leftrightarrow \underbrace{-e_{rip}u_{j,pi}}_{\text{arbitrary indices}} = \underbrace{-e_{rip}u_{j,ip}}_{\text{arbitrary derivative order}} \implies 2e_{rip}u_{j,ip} = 0 \implies e_{rip}u_{j,ip} = 0. \tag{180}$$

Therefore

$$2e_{rip}\epsilon_{ij,p} = e_{rip}u_{i,jp} \tag{181}$$

which implies

$$2e_{rip}e_{sjq}\epsilon_{ij,p} = e_{rip}e_{sjq}u_{i,jp} \implies 2e_{rip}e_{sjq}\epsilon_{ij,pq} = e_{rip}e_{sjq}u_{i,jpq}. \tag{182}$$

However this RHS term also vanishes for the same reason as Eq. 180, which is that an arbitrary switch of indices changes the sign of the Levi Civita constant but not the derivative terms, meaning the whole term must be equal to be its own negative, meaning the term must be zero. Therefore

$$e_{rip}e_{sjq}\epsilon_{ij,pq} = 0 \tag{183}$$

is a true set of equations called the compatibility equations. If you are given a number of strain components you must be able to solve for this set of equations. Otherwise it is impossible to infer a displacement solution from what strains you are given.

Indices r, s occur once, and so these are free indices. Each equation is unique to one free index, meaning one r and one s. The other indices i, j, p, q occur multiple times and so are dummy indices. Each equation has every version of the dummy index among 1,2,3.

Because of the many combinations of e_{rip}, e_{sjq} that are null, and also because of the symmetry properties of $\boldsymbol{\epsilon}$, there are nine total equations based on different r, s but only six of them are unique. The set of index pairs r, s that correspond to each unique equation is $r = s = 1,\ r = 1, s = 2,\ r = 1, s = 3,\ r = 2, s = 2,\ r = 2, s = 3,\ r = 3, s = 3$. The set of equations that correspond to this set is

$$2\epsilon_{32,23} = \epsilon_{22,33} + \epsilon_{33,22} \tag{184}$$

$$2\epsilon_{21,12} = \epsilon_{11,22} + \epsilon_{22,11} \tag{185}$$

$$2\epsilon_{31,13} = \epsilon_{11,33} + \epsilon_{33,11} \tag{186}$$

$$\epsilon_{11,23} = -\epsilon_{23,11} + \epsilon_{13,12} + \epsilon_{12,13} \tag{187}$$

$$\epsilon_{22,13} = -\epsilon_{13,22} + \epsilon_{23,21} + \epsilon_{21,23} \tag{188}$$

$$\epsilon_{33,12} = -\epsilon_{12,33} + \epsilon_{32,31} + \epsilon_{31,32} \tag{189}$$

or in general

$$\epsilon_{ij,kl} = -\epsilon_{kl,ij} + \epsilon_{jl,ik} + \epsilon_{ik,jl} \Longrightarrow \epsilon_{ij,kl} + \epsilon_{kl,ij} - \epsilon_{jl,ik} - \epsilon_{ik,jl} = 0. \tag{190}$$

Eq. 190 is another way to write the set of compatibility equations. It is necessary and sufficient for all of these to be true in order for there to exist a displacement solution given the strains.

What follow from the sufficiency of the compatibility equations are two things. First, zero strains imply no deformation, and this is called rigid body motion, meaning there is only translation and rotation. Second, a set of strains together with a particular set of translation and rotation parameters yields a unique displacement solution.

2.7 Integrating the strain displacement equations

We have proven that if a set of given strains satisfies the compatibility equations, then from that set we can infer a displacement solution. For example consider the 2D case

$$\epsilon_{xx} = A = \frac{\partial u}{\partial x},\ \epsilon_{yy} = 0 = \frac{\partial v}{\partial y},\ 2\epsilon_{xy} = 0 = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \tag{191}$$

Then taking antiderivatives of the diagonals leads to

$$u(x) = Ax + f(y), \quad v(y) = g(x). \tag{192}$$

Substituting into the off diagonal,

$$0 = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(\frac{\partial(Ax + f(y))}{\partial y} + \frac{\partial(g(x))}{\partial x}\right) = f'(y) + g'(x), \tag{193}$$

meaning

$$f'(y) = -g'(x). \tag{194}$$

If a function of y is a function of $x \forall x, y$ then the function cannot depend on either x or y, meaning it is a constant. So

$$f'(y) = -g'(x) = B \tag{195}$$

which implies

$$f(y) = By + C, \quad g(x) = -Bx + D, \tag{196}$$

which implies

$$u(x) = Ax + By + C, \quad v(y) = Bx + D. \tag{197}$$

As a system of equations,

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} A & B \\ -B & 0 \end{Bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} C \\ D \end{Bmatrix}. \tag{198}$$

Separating the stretch component associated with ϵ_{xx}, which is A, away from rigid body components B, C, D,

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}}_{\text{deformation}} \begin{Bmatrix} x \\ y \end{Bmatrix} + \underbrace{\begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix}}_{\text{rotation (skew)}} \begin{Bmatrix} x \\ y \end{Bmatrix} + \underbrace{\begin{Bmatrix} C \\ D \end{Bmatrix}}_{\text{translation}}. \tag{199}$$

2.8 Principal axes of strain

Given strain tensor

$$[\boldsymbol{\epsilon}] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix}, \tag{200}$$

we wish to know if there exists some coordinate rotation $\mathbf{R}$ such that the strain tensor is diagonalized, i.e.

$$\boldsymbol{\epsilon}' \Leftrightarrow \begin{bmatrix} \epsilon'_{11} & 0 & 0 \\ 0 & \epsilon'_{22} & 0 \\ 0 & 0 & \epsilon'_{33} \end{bmatrix} = \mathbf{R}\boldsymbol{\epsilon}\mathbf{R}^T \Longrightarrow \mathbf{R}^T\boldsymbol{\epsilon}' = \mathbf{R}\mathbf{R}^T\boldsymbol{\epsilon}\mathbf{R}^T = \boldsymbol{\epsilon}\mathbf{R}^T. \tag{201}$$

Suppose

$$\mathbf{R} = \begin{Bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \mathbf{v}_3^T \end{Bmatrix} \Longleftrightarrow \mathbf{R}^T = \begin{Bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{Bmatrix}, \tag{202}$$

where $\mathbf{v}_i$ are the i columns of $\mathbf{R}^T$. Then because of Eq. 201 ($\boldsymbol{\epsilon}\mathbf{R}^T = \mathbf{R}^T\boldsymbol{\epsilon}'$),

$$\boldsymbol{\epsilon}\begin{Bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{Bmatrix} = \begin{Bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{Bmatrix} \begin{bmatrix} \epsilon'_{11} & 0 & 0 \\ 0 & \epsilon'_{22} & 0 \\ 0 & 0 & \epsilon'_{33} \end{bmatrix} = \begin{Bmatrix} \epsilon'_{11}\mathbf{v}_1 & \epsilon'_{22}\mathbf{v}_2 & \epsilon'_{33}\mathbf{v}_3 \end{Bmatrix}. \tag{203}$$

Therefore

$$\boldsymbol{\epsilon}\mathbf{v}_i = \epsilon'_{ii}\mathbf{v}_i \Longleftrightarrow \boldsymbol{\epsilon}\mathbf{v} = \epsilon'\mathbf{v}. \tag{204}$$

This is called an eigenproblem, where ϵ' are the eigenvalues and $\mathbf{v}$ are the eigenvectors. The goal in solving Eq. 204 is to find nonzero $\mathbf{v}$ which, when transformed by $\boldsymbol{\epsilon}$ (i.e. $\boldsymbol{\epsilon}\mathbf{v}$), produce vectors parallel to $\mathbf{v}$ that are scaled by magnitude ϵ' (i.e. $\epsilon'\mathbf{v}$). Eq. 204 implies

$$(\boldsymbol{\epsilon} - \epsilon'\mathbf{I})\mathbf{v} = \mathbf{0}. \tag{205}$$

The solution to this is either the trivial solution $\mathbf{v} = \mathbf{0}$, which is uninteresting, or nontrivial solutions where $\boldsymbol{\epsilon} - \epsilon'\mathbf{I}$ is singular, meaning

$$0 = \det \begin{bmatrix} \epsilon_{11} - \epsilon' & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} - \epsilon' & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} - \epsilon' \end{bmatrix} \tag{206}$$

$$= (\epsilon_{11} - \epsilon')[(\epsilon_{22} - \epsilon')(\epsilon_{33} - \epsilon') - \epsilon_{32}\epsilon_{23}]$$
$$-\epsilon_{12}[\epsilon_{21}(\epsilon_{33} - \epsilon') - \epsilon_{31}\epsilon_{23}] + \epsilon_{13}[\epsilon_{21}\epsilon_{32} - \epsilon_{31}(\epsilon_{22} - \epsilon')] \tag{207}$$
$$= -\epsilon'^3 + \theta_1\epsilon'^2 - \theta_2\epsilon' + \theta_3 = 0 \tag{208}$$

where

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} \mathrm{tr}\boldsymbol{\epsilon} \\ (\epsilon_{ii}\epsilon_{jj} - \epsilon_{ij}\epsilon_{ji})/2 \\ \det\boldsymbol{\epsilon} \end{Bmatrix}. \tag{209}$$

The roots $\epsilon' = \{\epsilon'_{11}, \epsilon'_{22}, \epsilon'_{33}\}$ to Eq. 208 are the principal eigenstrains, and the resulting strain tensor is

$$\boldsymbol{\epsilon}' = \begin{bmatrix} \epsilon'_{11} & 0 & 0 \\ 0 & \epsilon'_{22} & 0 \\ 0 & 0 & \epsilon'_{33} \end{bmatrix}, \tag{210}$$

and the corresponding $\mathbf{v}$ are the principal coordinates.

The principal strains ϵ' are coordinate independent. Therefore so must be the coefficients θ of the characteristic equation of the eigenproblem Eq. 208, which are called the principal invariants. If ϵ' are known, they can be solved as

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} \epsilon'_{11} + \epsilon'_{22} + \epsilon'_{33} \\ \epsilon'_{22}\epsilon'_{33} + \epsilon'_{11}\epsilon'_{33} + \epsilon'_{11}\epsilon'_{22} \\ \epsilon'_{11}\epsilon'_{22}\epsilon'_{33} \end{Bmatrix}. \tag{211}$$

This process is true of all second order tensors such as $\boldsymbol{\epsilon}$.

2.9 Properties of the real symmetric eigenvalue problem

Note that in the eigenproblem Eq. 204 ($\boldsymbol{\epsilon}\mathbf{v} = \epsilon'\mathbf{v}$), strain $\boldsymbol{\epsilon}$ is symmetric because it is strain, defined as the symmetric part of the displacement gradient. Suppose the components of system matrix $\mathbf{M}$ in

$$\mathbf{M}\mathbf{x} = \lambda\mathbf{x} \tag{212}$$

are real and symmetric. Here λ are the eigenvalues and $\mathbf{x}$ are the eigenvectors. Real symmetric eigenproblems have two properties of interest. The first property is that it must yield real eigenvalues. To prove this recall the general definition of a complex conjugate

$$z = a + bi \Longrightarrow z^* = a - bi. \tag{213}$$

Taking the complex conjugate of the eigenproblem as a whole,

$$\mathbf{M}\mathbf{x}^* = \lambda^*\mathbf{x}^*. \tag{214}$$

Note that $\mathbf{M}$ is real, so $\mathbf{M} = \mathbf{M}^*$ necessarily. Respectively from Eq. 212 and Eq. 214 we can deduce

$$\mathbf{x}^{*T}\mathbf{M}\mathbf{x} = \mathbf{x}^{*T}\lambda\mathbf{x}, \qquad \mathbf{x}^T\mathbf{M}\mathbf{x}^* = \mathbf{x}^T\lambda^*\mathbf{x}^*. \tag{215}$$

The two Eqs. 215 are actually equal because

$$(\mathbf{x}^{*T}\mathbf{M}\mathbf{x})^T = \mathbf{x}^T\mathbf{M}\mathbf{x}^*. \tag{216}$$

These equations are producing scalars, and all scalars are equal to their own transpose. Therefore

$$\mathbf{x}^{*T}\lambda\mathbf{x} \Leftrightarrow \begin{Bmatrix} x_1^* & x_2^* & x_3^* \end{Bmatrix} \lambda \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \lambda x_1 x_1^* + \lambda x_2 x_2^* + \lambda x_3 x_3^*$$

$$= \lambda^* x_1 x_1^* + \lambda^* x_2 x_2^* + \lambda^* x_3 x_3^* = \begin{Bmatrix} x_1 & x_2 & x_3 \end{Bmatrix} \lambda^* \begin{Bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{Bmatrix} = \mathbf{x}^T \lambda^* \mathbf{x}^*. \tag{217}$$

Rearranged,

$$(\lambda - \lambda^*)(x_1 x_1^* + x_2 x_2^* + x_3 x_3^*) = 0. \tag{218}$$

If this is true of all x_i, x_i^*, then

$$\lambda = \lambda^*, \tag{219}$$

meaning there is no complex part to the eigenvalues and so they are real.

That was the first property of interest of a real symmetric eigenproblem. The second property is that if the eigenvalues are distinct, then the eigenvectors are orthogonal. Consider two of the possible three eigenvalue/eigenvector pairs that serve as solutions to the same system matrix $\mathbf{M}$ in

$$\mathbf{M}\mathbf{x}_1 = \lambda_1 \mathbf{x}_1, \qquad \mathbf{M}\mathbf{x}_2 = \lambda_2 \mathbf{x}_2. \tag{220}$$

These imply

$$\mathbf{x}_2^T \mathbf{M}\mathbf{x}_1 = \mathbf{x}_2^T \lambda_1 \mathbf{x}_1, \qquad \mathbf{x}_1^T \mathbf{M}\mathbf{x}_2 = \mathbf{x}_1^T \lambda_2 \mathbf{x}_2, \tag{221}$$

or

$$\mathbf{x}_2^T \mathbf{M}\mathbf{x}_1 = \lambda_2 \mathbf{x}_2 \cdot \mathbf{x}_1, \qquad \mathbf{x}_1^T \mathbf{M}\mathbf{x}_2 = \lambda_2 \mathbf{x}_1 \cdot \mathbf{x}_2. \tag{222}$$

Subtracting,

$$\begin{aligned}(\lambda_1 - \lambda_2)(\mathbf{x}_1 \cdot \mathbf{x}_2) &= \mathbf{x}_2^T \mathbf{M}\mathbf{x}_1 - \mathbf{x}_1^T \mathbf{M}\mathbf{x}_2 \qquad (223)\\ &= \mathbf{x}_2^T \mathbf{M}\mathbf{x}_1 - (\mathbf{x}_2^T \mathbf{M}\mathbf{x}_1)^T = 0\end{aligned}$$

where we know the whole expression is zero because the transpose of a scalar is itself. Therefore

$$(\lambda_1 - \lambda_2)(\mathbf{x}_1 \cdot \mathbf{x}_2) = 0, \tag{224}$$

and if it is assumed that the eigenvalues are distinct so that $\lambda_1 \neq \lambda_2$, then it must be that $\mathbf{x}_1 \cdot \mathbf{x}_2 = 0$, which is the definition of the two eigenvectors being orthogonal to one another.

2.10 Geometrical interpretation of the first invariant

Recall the principal invariants Eq. 209. The first of them is

$$\theta_1 = \mathrm{tr}\boldsymbol{\epsilon} \Leftrightarrow \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = \epsilon_{ii}. \tag{225}$$

Remember that a diagonal strain component indicates the stretch in the direction of the coordinate the component represents. Note also that shear components do not induce volume change. So if the original volume of a cube is

$$V = l_1 l_2 l_3 \tag{226}$$

where l are the side lengths, then the volume change is

$$\Delta V = l_1 \epsilon_{11} l_2 \epsilon_{22} l_3 \epsilon_{33}, \tag{227}$$

and the new volume is

$$\begin{aligned} V + \Delta V &= l_1(1+\epsilon_{11}) l_2(1+\epsilon_{22}) l_3(1+\epsilon_{33}) \\ &= l_1 l_2 l_3 (1+\epsilon_{11})(1+\epsilon_{22})(1+\epsilon_{33}) \\ &= l_1 l_2 l_3 (1+\epsilon_{22}+\epsilon_{11}+\epsilon_{11}\epsilon_{22})(1+\epsilon_{33}) \\ &= l_1 l_2 l_3 (1+\epsilon_{33}+\epsilon_{22}+\epsilon_{22}\epsilon_{33}+\epsilon_{11}+\epsilon_{11}\epsilon_{33}+\epsilon_{11}\epsilon_{22}+\epsilon_{11}\epsilon_{22}\epsilon_{33}) \\ &\approx l_1 l_2 l_3 (1+\epsilon_{11}+\epsilon_{22}+\epsilon_{33}) \end{aligned}$$

where such an approximation is made because products of small strain components are very small and so are considered negligible. Then

$$\begin{aligned} \Delta V = V + \Delta V - V &= l_1 l_2 l_3 (1+\epsilon_{11}+\epsilon_{22}+\epsilon_{33}) - l_1 l_2 l_3 \\ &= l_1 l_2 l_3 (\epsilon_{11}+\epsilon_{22}+\epsilon_{33}) = V\epsilon_{ii} = \Delta V. \end{aligned} \tag{228}$$

Therefore

$$\epsilon_{ii} = \frac{\Delta V}{V} = \theta_1 \tag{229}$$

which is the first invariant. So the first invariant can be interpreted as the volumetric strain, or the change in volume with respect to the original volume.

2.11 Finite deformation

In the past we have neglected products of strain components because of the assumption that they were so small that they could be considered negligible. Now though we wish to consider a more general derivation for larger deformations. Displacement

$$u_i = x'_i - x_i \tag{230}$$

If dl is a small distance between x_i and a neighboring point, then

$$dl^2 = dx_i dx_i, \qquad dl'^2 = dx'_i dx'_i. \tag{231}$$

From Eq. 230,

$$dx'_i = dx_i + \underbrace{du_i} = dx_i + \underbrace{\frac{\partial u_i}{\partial x_j}} dx_j = dx_i + \underbrace{u_{i,j} dx_j} = dx'_i. \tag{232}$$

Substituting this into Eq. 231,

$$
\begin{aligned}
dl'^2 &= (dx_i + u_{i,j}dx_j)(dx_i + u_{i,k}dx_k) \\
&= dx_i dx_i + dx_i u_{i,k} dx_k + dx_i u_{i,j} dx_j + u_{i,j} u_{i,k} dx_j dx_k \\
&= dl^2 + u_{i,j} dx_i dx_j + \underbrace{u_{i,k} dx_i dx_k} + \underbrace{u_{i,j} u_{i,k} dx_j dx_k} \\
&= dl^2 + u_{i,j} dx_i dx_j + \underbrace{u_{j,i} dx_j dx_i}_{k\to i,\ i\to j} + \underbrace{u_{k,i} u_{k,j} dx_i dx_j}_{i\to k,\ k\to j,\ j\to i}.
\end{aligned}
\tag{233}
$$

Therefore

$$
\begin{aligned}
dl'^2 - dl^2 &= u_{i,j} dx_i dx_j + u_{j,i} dx_j dx_i + u_{k,i} u_{k,j} dx_i dx_j \\
&= (u_{i,j} + u_{j,i} + u_{k,i} u_{k,j}) dx_i dx_j.
\end{aligned}
\tag{234}
$$

Think of the physical meaning of LHS. It is measuring the squared difference in length between two neighboring points before and after a deformation. This is a rating of the deformation itself. It is measuring to what extent points in the body are separating or stretching. Units wise,

$$
dl'^2 - dl^2 \sim \text{meters}^2, \qquad \epsilon \sim \text{dimensionless},
\tag{235}
$$

so we multiply strain by a representative small box of area. Particularly

$$
dl'^2 - dl^2 = 2\epsilon_{ij} dx_i dx_j \sim \text{meters}^2.
\tag{236}
$$

Strain in this case is

$$
\epsilon_{ij} = \frac{1}{2} \frac{dl'^2 - dl^2}{dx_i dx_j}
\tag{237}
$$

which is one half the squared change in distance between two points with respect to the area of a square with sides defined by a small unit distance dx. Conceptually it is a rating of the extent that points on a body have separated after a deformation process with respect to the original configuration. Substituting Eq. 236 into Eq. 234,

$$
\epsilon_{ij} = \frac{1}{2}\Big(u_{i,j} + u_{j,i} + u_{k,i} u_{k,j} \Big)
\tag{238}
$$

where i, j are free and k is dummy. Remember, dummy means you sum over all indices. Free is independent. So for example

$$
\begin{aligned}
\epsilon_{ij}\big|_{i=x,j=x} = \epsilon_{xx} &= \frac{1}{2}\Big(\underbrace{u_{x,x}}_{u_{i,j}} + \underbrace{u_{x,x}}_{u_{j,i}} + \underbrace{(u_{x,x}u_{x,x} + u_{y,x}u_{y,x} + u_{z,x}u_{z,x})}_{u_{k,i}u_{k,j}} \Big) \\
&= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 \right)
\end{aligned}
\tag{239}
$$

and

$$
\begin{aligned}
\epsilon_{zy} &= \frac{1}{2}\Big(u_{z,y} + u_{y,z} + u_{x,z}u_{x,y} + u_{y,z}u_{y,y} + u_{z,z}u_{z,y} \Big) \\
&= \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial y} \right) = \epsilon_{yz} \quad \text{(symmetric)}.
\end{aligned}
\tag{240}
$$

Recall that for the small strain assumption the product terms can be neglected.

3 Stress

In terms of force, a body can be acted upon by

- body forces, which act at every point on the body, such as gravity, and
- surface forces, which act only on surface points, such as traction or hydrostatic pressure.

If a body's density is ρ and its volume is V, then its mass is ρV or

$$m = \int_V \rho dV. \tag{241}$$

Therefore gravitational force

$$\mathbf{F}_g = m\mathbf{g} = \int_V \rho \mathbf{g} dV. \tag{242}$$

In general the net body force is $\mathbf{f}$ is

$$\int_V \rho \mathbf{f} dV \tag{243}$$

and torque/moment is

$$\mathbf{r} \times \mathbf{F} = \int_V \mathbf{r} \times \rho \mathbf{f} dV = ||r||||\rho f|| \sin\theta \hat{\mathbf{n}} \tag{244}$$

if $\mathbf{r} = x_i \mathbf{e}_i$ is the position of a point on the body with respect to the origin and θ is the angle between $\mathbf{r}$ and $\mathbf{F}$ (really $\mathbf{f}$).

Stress is force per area. If a stress vector is $\mathbf{t}$ and the area of a surface is S, then surface force is $\mathbf{t}S$ or

$$\oint_S \mathbf{t} dS. \tag{245}$$

Summing the surface forces with the body forces, the net force is

$$\int_V \rho \mathbf{f} dV + \oint_S \mathbf{t} dS \tag{246}$$

and the net torque/moment is

$$\int_V \mathbf{r} \times \rho \mathbf{f} dV + \oint_S \mathbf{r} \times \mathbf{t} dS. \tag{247}$$

Surface force $\mathbf{t} = \mathbf{t}(\mathbf{x}, \mathbf{n})$ depends on the location of a point on the surface $\mathbf{x}$ and on the vector which is normal to the surface $\mathbf{n}$. Let σ_{ij} (NO SUM, not a tensor), be the jth component of $\mathbf{t}$ if the surface was normal to $\mathbf{e}_i$ (NO SUM). That is

$$\sigma_{ij} = \mathbf{e}_j \cdot \mathbf{t}(\mathbf{x}, \mathbf{e}_i) \quad \text{(no sum)}. \tag{248}$$

For example

$$\sigma_{12} = \mathbf{e}_2 \cdot \begin{Bmatrix} t_{1,(1)} \\ t_{2,(1)} \\ t_{3,(1)} \end{Bmatrix} = \begin{Bmatrix} 0 & 1 & 0 \end{Bmatrix} \begin{Bmatrix} t_{1,(1)} \\ t_{2,(1)} \\ t_{3,(1)} \end{Bmatrix} = t_{2,(1)} \tag{249}$$

where in this example, the result $t_{2,(1)}$ denotes the 2nd component of the stress vector acting on a surface which is normal to $\mathbf{e}_1$. In three dimensions

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} t_{1,(1)} & t_{2,(1)} & t_{3,(1)} \\ t_{1,(2)} & t_{2,(2)} & t_{3,(2)} \\ t_{1,(3)} & t_{2,(3)} & t_{3,(3)} \end{bmatrix} \quad \text{where } t \sim t_{\text{component of } t,\ (\text{direction of vector normal to surface})}. \tag{250}$$

A visualization of the stress components in 2D is Fig. 2. For example consider the right face. The direction of the vector normal to this surface is $\mathbf{e}_1$ or $x \rightarrow t_{\_\_\_\_,(x)}$. Then $\sigma_{xx} = t_{1,(1)}$ is the 1st or $x-$ component of $\mathbf{t}$ for that surface $\rightarrow t_{x,(\_\_\_\_)}$. On the other hand the stress component $\sigma_{x,y} = t_{2,(1)}$ is the 2nd or $y-$ component of $\mathbf{t}$ for the face whose normal points in the direction $\mathbf{e}_1$ or x. For normal components the stress vector always

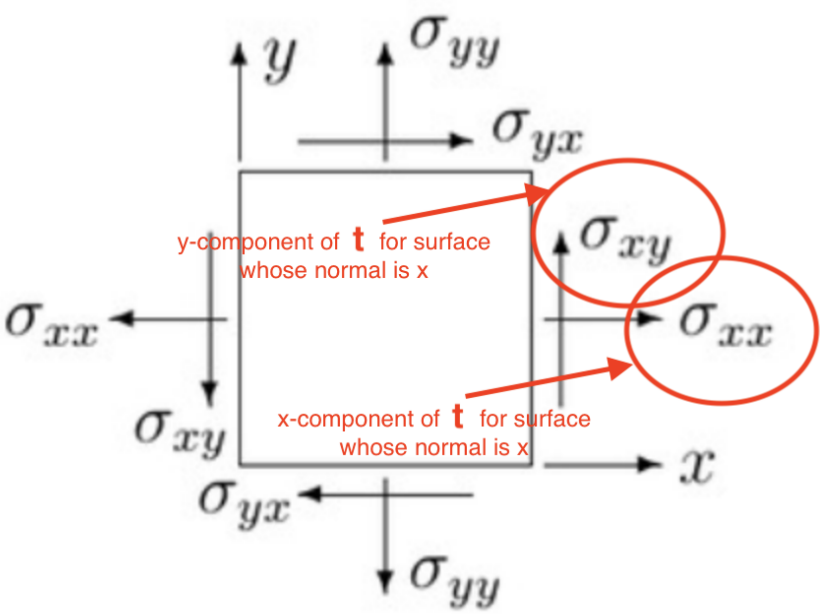


Figure 2: Stress components in 2D and their associated signs (+/-)

points away from the applied surface. Whether or not a shear component is positive is directly related to whether or not the vector normal to the surface is positive. So, the bottom face will have a negative shear component because the face's normal vector points in the direction $-y$. However the right face will have a positive shear component because the face's normal vector points in the direction $+x$.

Thus far we have only considered $\mathbf{n} = \{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$, but we wish to establish a general relationship between $\mathbf{t}(\mathbf{x}, \mathbf{n})$ and σ_{ij} (no sum) for any $\mathbf{n}$. Consider Fig. 3, a four faced pyramid shape (which is called a tetrahedron) with vertices $\{\mathcal{O}, (x_1, 0, 0), (0, x_2, 0), (0, 0, x_3)\}$.

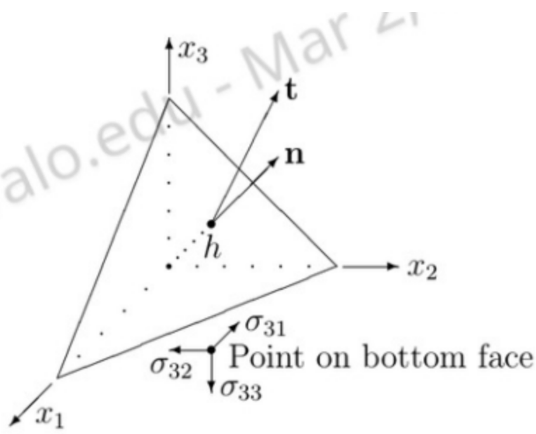


Figure 3: Stresses on trirectangular (three right triangle) tetrahedron

Let the height h $(\neq x_3)$ be the shortest distance between origin $\mathcal{O}$ and the closest point on the "inclined" surface, which is the triangle formed by $(x_1, 0, 0), (0, x_2, 0), (0, 0, x_3)$. If ΔS is the area of this triangle and ΔS_i is the area of each of the three right triangles with unit normal $\mathbf{e}_i$, then

$$\Delta S_i = n_i \Delta S \Leftrightarrow \begin{Bmatrix} \text{area of triangle with unit normal } x \\ \text{area of triangle with unit normal } y \\ \text{area of triangle with unit normal } z \end{Bmatrix} = \begin{Bmatrix} \Delta S_1 \\ \Delta S_2 \\ \Delta S_3 \end{Bmatrix} = \begin{Bmatrix} n_1 \Delta S \\ n_2 \Delta S \\ n_3 \Delta S \end{Bmatrix}, \tag{251}$$

where $\mathbf{n}$ is the vector normal to the inclined surface ΔS, and the components of $\mathbf{n}$ are called the direction cosines.

Suppose h is small. In this case the volume is small. We assume density is constant. Therefore mass is also small. Therefore net force is also small. If we assume there is no net force, then using Eq. 246 (net force $= \int_V \rho \mathbf{f} dV + \oint_S \mathbf{t} dS =$ body + surfaces),

$$\Leftrightarrow \rho f_i V + t_i \Delta S -^{?} \underbrace{\sigma_{ji} \Delta S_j}_{i\text{th component of } \mathbf{t} \text{ for surface whose normal is } \mathbf{e}_j} = 0. \tag{252}$$

For all tetrahedra, $V = h\Delta S/3$. Making this substitution as well as Eq. 251 ($\Delta S_j = n_j \Delta S$),

$$\begin{aligned} &\rho f_i h \Delta S/3 + t_i \Delta S - \sigma_{ji}(n_j \Delta S) = 0 \\ &\Leftrightarrow \rho f_i h \Delta S/3 + \Delta S(t_i - \sigma_{ji} n_j) = 0. \end{aligned} \tag{253}$$

We have supposed h is small, so $h \longrightarrow 0$. Because of this,

$$\Delta S(t_i - \sigma_{ji} n_j) = 0 \longrightarrow t_i - \sigma_{ji} n_j = 0$$

$$\longrightarrow t_i = \sigma_{ji} n_j \Leftrightarrow \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 \\ \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3 \\ \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3 \end{Bmatrix}. \tag{254}$$

The way to interpret Eq. 254 is this. On an arbitrary surface, the components of the stress vector for that surface is determined by the set of the individual stress components on that surface and the direction in which the vector normal to the surface points. This relationship is important because even though it is easy to infer σ_{ij} based on $\mathbf{t}(\mathbf{x}, \mathbf{e}_i)$, we are not usually given $\mathbf{t}$. Instead we are usually given $[\sigma_{ij}] \Leftrightarrow \boldsymbol{\sigma}$ and from that are needing to figure out $\mathbf{t}(\mathbf{x}, \mathbf{n})$.

3.1 Momentum equation

The net force on the body is the product of the body's mass and acceleration, according to Newton. Getting the net force from Eq. 246,

$$\int_V \rho \mathbf{f} dV + \oint_S \mathbf{t} dS = m\mathbf{a} = \int_V \ddot{\mathbf{u}} \rho dV. \tag{255}$$

In index notation,

$$\int_V \rho f_i dV + \oint_S \underbrace{t_i} dS = \int_V \ddot{u}_i \rho dV. \tag{256}$$

Substituting in Eq. 254,

$$\int_V \rho f_i dV + \oint_S \underbrace{\sigma_{ji} n_j} dS = \int_V \ddot{u}_i \rho dV. \tag{257}$$

Because of the divergence theorem Eq. 95 ($\oint_S [\circ]_j n_j dS = \int_V [\circ]_{j,j} dV$),

$$\int_V \rho f_i dV + \int_V \sigma_{ji,j} dV = \int_V \ddot{u}_i \rho dV. \tag{258}$$

Rearranged,

$$\int_V \rho f_i dV + \int_V \sigma_{ji,j} dV - \int_V \ddot{u}_i \rho dV \tag{259}$$

implies

$$\int_V (\sigma_{ji,j} + \rho f_i - \rho \ddot{u}_i) = 0 \tag{260}$$

implies

$$\sigma_{ji,j} + \rho f_i - \rho \ddot{u}_i = 0 \longrightarrow \sigma_{ji,j} + \rho f_i = \rho \ddot{u}_i. \tag{261}$$

This is called the momentum equation because it arises from the balance of net force, which is related to linear momentum.

3.2 Angular momentum

The angular analog to force is torque, and the angular analog to linear momentum is angular momentum. So the sum of the torques implies a balance of angular momentum. Getting the net torque from Eq. 247,

$$\int_V \mathbf{r} \times \rho \mathbf{f} dV + \oint_S \mathbf{r} \times \mathbf{t} dS = \mathbf{r} \times m\mathbf{a} = \mathbf{r} \times m \frac{d}{dt}(\mathbf{v}) = \mathbf{r} \times m \frac{d}{dt}(\dot{\mathbf{u}}) = \frac{d}{dt} \int_V \mathbf{r} \times \rho \dot{\mathbf{u}} dV \tag{262}$$

where u is displacement and $\mathbf{r} = x_i\mathbf{e}_i$. In index notation,

$$\int_V \mathbf{r} \times \rho\mathbf{f}dV + \oint_S \mathbf{r} \times \mathbf{t}dS = \frac{d}{dt}\int_V \mathbf{r} \times \rho\dot{\mathbf{u}}dV$$

$$\Longleftrightarrow \int_V \rho x_i f_j \epsilon_{ijk} dV + \oint x_i(t_j)\epsilon_{ijk}dS = \int_V \rho x_i \ddot{u}_j \epsilon_{ijk} dV \tag{263}$$

$$\Longleftrightarrow \int_V \rho x_i f_j \epsilon_{ijk} dV + \oint x_i(\sigma_{lj}n_l)\epsilon_{ijk}dS = \int_V \rho x_i \ddot{u}_j \epsilon_{ijk} dV \tag{264}$$

$$\Longleftrightarrow \int_V \rho x_i f_j \epsilon_{ijk} dV + \int_V (x_i\sigma_{lj}\epsilon_{ijk})_{,l}dV = \int_V \rho x_i \ddot{u}_j \epsilon_{ijk} dV \tag{265}$$

$$\Longleftrightarrow \int_V \rho x_i f_j \epsilon_{ijk} dV + \int_V (x_{i,l}\sigma_{lj}\epsilon_{ijk} + x_i\sigma_{lj,l}\epsilon_{ijk})dV = \int_V \rho x_i \ddot{u}_j \epsilon_{ijk} dV \tag{266}$$

$$\Longleftrightarrow \int_V \rho x_i f_j \epsilon_{ijk} dV + \int_V (\delta_{il}\sigma_{lj}\epsilon_{ijk} + x_i\sigma_{lj,l}\epsilon_{ijk})dV = \int_V \rho x_i \ddot{u}_j \epsilon_{ijk} dV \tag{267}$$

$$\Longleftrightarrow \int_V \rho x_i f_j \epsilon_{ijk} dV + \int_V (\sigma_{ij}\epsilon_{ijk} + x_i\sigma_{lj,l}\epsilon_{ijk})dV = \int_V \rho x_i \ddot{u}_j \epsilon_{ijk} dV \tag{268}$$

$$\Longleftrightarrow \int_V [\rho x_i f_j \epsilon_{ijk} + \sigma_{ij}\epsilon_{ijk} + x_i\sigma_{lj,l}\epsilon_{ijk} - \rho x_i \ddot{u}_j \epsilon_{ijk}]dV = 0 \tag{269}$$

$$\Longleftrightarrow \int_V [x_i\epsilon_{ijk}(\rho f_j + \sigma_{lj,l} - \rho\ddot{u}_j) + \sigma_{ij}\epsilon_{ijk}]dV = 0. \tag{270}$$

Then because of Eq. 261 ($\sigma_{lj,l} + \rho f_j = \rho\ddot{u}_j$), the parenthetical term cancels, leaving

$$\Longleftrightarrow \int_V \sigma_{ij}\epsilon_{ijk}dV = 0 \longrightarrow \sigma_{ij}\epsilon_{ijk} = 0. \tag{271}$$

Switching indices,

$$\sigma_{ji}\epsilon_{jik} = 0. \tag{272}$$

By definition,

$$\epsilon_{jik} = -\epsilon_{ijk}. \tag{273}$$

Therefore

$$\sigma_{ji}\epsilon_{jik} = -\sigma_{ji}\epsilon_{ijk} = \sigma_{ij}\epsilon_{ijk} \longrightarrow \sigma_{ij} = \sigma_{ji}. \tag{274}$$

This means stress $\boldsymbol{\sigma}$ is symmetric. Reconsidering then Eqs. 261 and 254,

$$\sigma_{ij,j} + \rho f_i = \rho\ddot{u}_i \Leftrightarrow \begin{Bmatrix} \sigma_{xx,x} + \rho f_x + \sigma_{xy,y} + \sigma_{xz,z} \\ \sigma_{yx,x} + \rho f_y + \sigma_{yy,y} + \sigma_{yz,z} \\ \sigma_{zx,x} + \rho f_z + \sigma_{zy,y} + \sigma_{zz,z} \end{Bmatrix} = \begin{Bmatrix} \rho\ddot{u} \\ \rho\ddot{v} \\ \rho\ddot{w} \end{Bmatrix}, \qquad t_i = \sigma_{ij}n_j \Leftrightarrow \mathbf{t} = \boldsymbol{\sigma}\mathbf{n}. \tag{275}$$

3.3 Stress as a tensor

Here is proof that $\boldsymbol{\sigma}$ is a rank two tensor. If traction $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$ in the original coordinate system x_i, then in another coordinate system $\hat{x}_i$,

$$\hat{\mathbf{t}} = \hat{\boldsymbol{\sigma}}\hat{\mathbf{n}}. \tag{276}$$

For rank one tensors,

$$\hat{\mathbf{t}} = \mathbf{R}\mathbf{t}, \quad \hat{\mathbf{n}} = \mathbf{R}\mathbf{n} \longrightarrow \mathbf{R}^T\hat{\mathbf{t}} = \mathbf{t}, \quad \mathbf{R}^T\hat{\mathbf{n}} = \mathbf{n}. \tag{277}$$

Substituting back into Eq. 275b ($\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$),

$$\mathbf{R}^T\hat{\mathbf{t}} = \boldsymbol{\sigma}\mathbf{R}^T\hat{\mathbf{n}} \longrightarrow \mathbf{R}\mathbf{R}^T\hat{\mathbf{t}} = \hat{\mathbf{t}} = \mathbf{R}\boldsymbol{\sigma}\mathbf{R}^T\hat{\mathbf{n}}. \tag{278}$$

Substituting this result into Eq. 276,

$$\hat{\boldsymbol{\sigma}} = \mathbf{R}\boldsymbol{\sigma}\mathbf{R}^T\hat{\mathbf{n}}, \tag{279}$$

which establishes $\boldsymbol{\sigma}$ as a rank two/second order tensor. $\boldsymbol{\sigma}$ is symmetric and real. As as shown in Sec. 2.9, the two properties that follow from this are (1) the characteristic eigenproblem must yield real eigenvalues and (2) if the eigenvalues are distinct then the eigenvectors are mutually orthogonal. The characteristic eigenproblem to determine principal stresses/eigenvalues $\hat{\sigma}$ is

$$\boldsymbol{\sigma}\mathbf{x} = \hat{\sigma}\mathbf{x}. \tag{280}$$

The stress vector $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$ is a vector normal to the surface $\mathbf{n}$ which is transformed by tensor $\boldsymbol{\sigma}$. Therefore this vector $\mathbf{t}$ will not necessarily point in the direction of $\mathbf{n}$. To find out what component of $\mathbf{t}$ points in $\mathbf{n}$, normal stress

$$\sigma_n = \mathbf{n}\cdot\mathbf{t} = \mathbf{n}\cdot\boldsymbol{\sigma}\mathbf{n} = \mathbf{n}^T\boldsymbol{\sigma}\mathbf{n} \tag{281}$$

$$= \begin{Bmatrix} n_1 & n_2 & n_3 \end{Bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} n_1 & n_2 & n_3 \end{Bmatrix} \begin{Bmatrix} \sigma_{11}n_1 + \sigma_{12}n_2 + \sigma_{13}n_3 \\ \sigma_{21}n_1 + \sigma_{22}n_2 + \sigma_{23}n_3 \\ \sigma_{31}n_1 + \sigma_{32}n_2 + \sigma_{33}n_3 \end{Bmatrix} \tag{282}$$

$$= n_1(\sigma_{11}n_1+\sigma_{12}n_2+\sigma_{13}n_3)+n_2(\sigma_{21}n_1+\sigma_{22}n_2+\sigma_{23}n_3)+n_3(\sigma_{31}n_1+\sigma_{32}n_2+\sigma_{33}n_3) \tag{283}$$

$$\Leftrightarrow n_i\sigma_{ij}n_j = \sigma_n \tag{284}$$

is the normal component of stress on that surface. This calculation of $\boldsymbol{\sigma}$ is permissible for all second order tensors, so normal strain

$$\epsilon_n = n_i\epsilon_{ij}n_j. \tag{285}$$

3.4 Mean stress in a deformed body

If for a body subject to Eq. 275a ($\sigma_{ij,j} + \rho f_i = \rho \ddot{u}_i$) there are no body forces and if the body is static, then $\ddot{u} = 0$ and $f_i = 0$, meaning

$$\sigma_{ij,j} = 0. \tag{286}$$

This implies

$$0 = \int_V \sigma_{ij,j} x_k dV = \int_V [(\sigma_{ij} x_k)_{,j} - \sigma_{ij} x_{k,j}] dV \tag{287}$$

$$= \int_V [(\sigma_{ij} x_k)_{,j} - \sigma_{ij} \delta_{kj}] dV \tag{288}$$

$$= \int_V (\sigma_{ij} x_k)_{,j} dV - \int_V \sigma_{ik} dV = 0 \longrightarrow \int_V (\sigma_{ij} x_k)_{,j} dV = \int_V \sigma_{ik} dV \tag{289}$$

$$\longrightarrow \frac{1}{V} \int_V (\sigma_{ij} x_k)_{,j} dV = \frac{1}{V} \int_V \sigma_{ik} dV. \tag{290}$$

We define mean stress over volume as

$$\bar{\sigma}_{ik} = \frac{1}{V} \int_V \sigma_{ik} dV. \tag{291}$$

Substituting,

$$\bar{\sigma}_{ik} = \frac{1}{V} \int_V (\sigma_{ij} x_k)_{,j} dV = \frac{1}{V} \oint_S \sigma_{ij} x_k n_j dS = \frac{1}{V} \oint_S t_i x_k dS. \tag{292}$$

Changing indices,

$$\bar{\sigma}_{ij} = \frac{1}{V} \oint_S t_i x_j dS. \tag{293}$$

Note that the product $t_i x_j \leftrightarrow \mathbf{t} \otimes \mathbf{x}$ produces a second order tensor. Also, we understand that $[\bar{\sigma}_{ij}]$ is symmetric. Therefore the symmetric component of this tensor is the only component, and that is

$$\bar{\sigma}_{ij} = \mathrm{sym}(\bar{\sigma}_{ij}) = \frac{1}{2}\left(\frac{1}{V} \oint_S [(t_i x_j) + (t_i x_j)^T] dS\right) = \frac{1}{2V} \oint_S (t_i x_j + t_j x_i) dS = \bar{\sigma}_{ij}. \tag{294}$$

The utility of this is that you can solve for the mean value of the stress tensor using only surface tractions.

3.5 Fluid structure interface condition

The stress normal to an elastic structure in contact with an inviscid fluid is just the inward pressure. That is,

$$\sigma_n = -p. \tag{295}$$

4 Equations of elasticity

4.1 Hooke's law

4.2 Strain energy

4.3 Material symmetry

4.4 Isotropic materials

5 Simplest problems of elastostatics

5.1 Simple shear

5.2 Simple tension

5.3 Uniform compression

5.4 Stress and strrain deviators

5.5 Stable reference states

6 Boundary value problems in elastostatics

6.1 Uniqueness

6.2 Uniqueness for thr traction problem

6.3 Uniqueness for the mixed problem

7 Torsion

7.1 Circular shaft

7.2 Noncircular shaft

7.3 Uniqueness of warrping function in torsion problem

7.4 Existence of warping function in torsion problem

7.5 Some properties of harmonic functions

7.6 Stress function for torsion

7.7 Torsion of elliptical cylinder

7.8 Torsion of rectangular bars: warping function

7.9 Torsion of rectangular bars: stress function

8 Plane deformation

8.1 Plane strain

8.2 Plane stress

8.3 Formal equivalence between plane stress/plane strain

8.4 Compatibility equation in terms of stress

8.5 Airy stress function

8.6 Polynomial solutions of the biharmonic equation

8.7 Bending of a narrow cantilever of rectangular cross section under end load

8.8 Bending of a beam by uniform load

9 General theorems of infinitesimal elastostatics

9.1 Work theorem

9.2 Betti's reciprocal theorem

9.3 Variational principles

9.4 Theorem of minimum potential energy

9.5 Minimum complementary energy