MAE 507 - Engineering Analysis

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Contents

1 Mod1 Linear algebra

2 Mod2 ODEs

2.1 Lec 2a Physical prototypes and classification

2.2 Lec 2b Linear ODEs and power series

2.3 Lec 2c Linear ODEs analytic coefficients

2.4 Lec 2d Linear ODEs regular singular points

2.5 Lec 2e Bessel functions

2.6 Lec 2f Sturm Liouville eigenproblem

2.7 Lec 2g IVP numerical solutions

Consider the first order ODE

$$\frac{dy}{dx} = f(x, y). \tag{1}$$

First of all there is the numerical integration method where one considers

$$dy = f(x, y)dx \rightarrow \int_{y_i}^{y_{i+1}} dy = \int_{x_i}^{x_{i+1}} f(x, y)dx \rightarrow y_{i+1} - y_i = \int_{x_i}^{x_{i+1}} f(x, y)dx. \tag{2}$$

Then there are finite difference methods. Consider again

$$\frac{dy}{dx} = f(x, y). \tag{3}$$

Then Euler's method

$$dy/dx \approx \Delta y/\Delta x \approx y_{i+i} - y_i/x_{i+1} - x_i = f(x_i, y_i). \tag{4}$$

Let

$$x_{i+1} - x_i = h. \tag{5}$$

Then

$$y_{i+1} = y_i + hf(x_i, y_i). \tag{6}$$

h is called the step size. This is called Euler's method. It is known as a forward method because you are iterating forward. It is called explicit because the RHS only involves x_i and y_i where LHS involves y_{i+1}. So you are finding the unknown $i + 1$ term using the known i terms.

An alternative approach is the backward method. Let once again

$$dy/dx = f(x, y) \tag{7}$$

but

$$\nabla y/\nabla x \approx f(x, y). \tag{8}$$

Then
$$\frac{y_i - y_{i-1}}{x_i - x_{i-1}} = \frac{y_i - y_{i-1}}{h} \approx f(x_i, y_i). \tag{9}$$
So
$$y_i = y_{i-1} + hf(x_i, y_i). \tag{10}$$
This is a backward method because you are iterating backwards. It is called implicit because both the LHS and the RHS involve some i term. So there are unknowns on both sides, theoretically. Some assumption is required. Let us again consider Euler's forward method. If
$$dy/dx = f(x, y) \tag{11}$$
so that
$$y_{i+1} = y_i + hf(x_i, y_i), \tag{12}$$
we might ask, what is the error involved in this approach? We do not yet know exactly but we can approximate the order of magnitude of the error in the Taylor series expansion of $y(x)$ at x_i. That is,
$$y_{i+1} = y_i + \frac{h^1}{1!} h y_i' + \frac{h^2}{2!} y_i'' + \frac{h^3}{3!} y_i''' + \dots. \tag{13}$$
Recall that $\frac{dy}{dx} = y' = f(x_i, y_i)$. Then in approximating the infinite series into something containing only known values we might say
$$y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2!} y_i'' + \dots = y_{i+1} = y_i + hf(x_i, y_i) + \mathcal{O}(h^2) \tag{14}$$
where $\mathcal{O}$ is called the local truncation error and its order (h^2) is just the order of magnitude of the approximation error. That is error at the local level, or at this particular step in the iteration. However if this approximation is used over many steps then then error will accumulate. In this way it is then known that if local truncation error is $\mathcal{O}(h^2)$ then total truncation error is $\mathcal{O}(h)$. That applies to this particular example. Another example is: if local error is $\mathcal{O}(h^3)$ then total error is $\mathcal{O}(h^2)$. This generally applies.
This method has low accuracy. There is also potential for instability numerically. That is, the solve may not converge. How do we improve this accuracy? One way is to obtain steps at the beginning and end of the interval and then average. That is, let us improve
$$y_{i+1} = y_i + hf(x_i, y_i) \tag{15}$$
by saying instead that
$$y_{i+1} = y_i + \frac{h}{2}\Bigg(f(x_i, y_i) + f(x_{i+1}, y_{i+1})\Bigg). \tag{16}$$
But, note that terms on the RHS are unknown. So let us use Euler method to estimate the y_{i+1} term on that side. Let us create a value which serves as an approximation
$$y_{i+1}^e = y_i + hf(x_i, y_i). \tag{17}$$

Plugging Eq. 17 into the RHS of Eq. 16,

$$y_{i+1} = y_i + \frac{h}{2}\bigg(f(x_i, y_i) + f(x_{i+1}, y^e_{i+1}) \bigg) = y_i + \frac{h}{2}\bigg(f(x_i, y_i) + f(x_{i+1}, y_i + hf(x_i, y_i)) \bigg). \quad (18)$$

Now let us create a general model using as a characteristic example Eq. 18. First of all let us recall that

$$x_i + h = x_{i+1} \quad (19)$$

by virtue of Eq. 5. Now let

$$y_{i+1} = y_i + h\bigg(af(x_i, y_i) + bf(x_i + \alpha h, y_i + \beta hf(x_i, y_i)) \bigg) \quad (20)$$

where α, β, a, b are constants. Eq. 20 is a model of 18 if

$$\alpha = 1,\ \ \beta = 1,\ \ a = 1/2,\ \ b = 1/2. \quad (21)$$

2.8 Lec 2h Higher order methods

Reminiscent of Lec 2.7, consider the first order ODE

$$f(x, y) = \frac{dy}{dx} \quad (22)$$

and the forward Euler method

$$y_{i+1} = y_i + hf(x_i, y_i). \quad (23)$$

This essentially uses the slopes at x_i and at y_i (slopes because $\frac{dy}{dx}$ informs slope) to estimate y_{i+1}. Recall otherwise that for this example, truncation error $\mathcal{O} = \mathcal{O}(h^2)$ at the local (per step) level and that $\mathcal{O} = \mathcal{O}(h)$ at the total level.
Let us try to improve beyond the Euler method. Recall Eq. 20 with parameters α, β, a, b. It is generally true for this model that

$$\text{if } a + b = 1,\ \ \alpha b = 1/2,\ \ \beta b = 1/2, \quad (24)$$

then the resultant iteration will have $\mathcal{O}(h^3)$ locally and $\mathcal{O}(h^2)$ totally. We remember that Eq. 18 obeys Eq. 20 if

$$\alpha = 1, \beta = 1, a = 1/2, b = 1/2. \quad (25)$$

Let us move on to the Runge Kutta (run guh kud duh; RK) method. Let us rewrite the previous formula as what is called a second order Runge Kutta method

$$y_{i+1} = y_i + h(c_1 k_1 + c_2 k_2) \quad (26)$$

since $i = \{1, 2\}$ on $c_i k_i$. Eq. 26 equals Eq. 18 only if

$$c_1 = 1/2, \quad c_2 = 1/2, \quad k_1 = f(x_i, y_i), \quad k_2 = f(\underbrace{x_i + h}_{x_{i+1}}, y_i + hf(x_i, y_i)). \quad (27)$$

A step further, we see that k_1 is contained in part of k_2. So we can instead say

$$c_1 = 1/2, \quad c_2 = 1/2, \quad k_1 = f(x_i, y_i), \quad k_2 = f(\underbrace{x_i + h}_{x_{i+1}}, y_i + hk_2). \tag{28}$$

Note that k_i can only be a function of k_j for $i > j$. We can use this same Runge Kutta procedure to develop higher order methods. A fourth order Runge Kutta method

$$y_{i+1} = y_1 + h(c_1k_1 + c_2k_2 + c_3k_3 + c_4k_4) \tag{29}$$

equals the example

$$y_{i+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y^e_{i+1})\right) = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_i + hf(x_i, y_i))\right) \tag{30}$$

(which is Eq. 18 but repeated here for convenience) if

$$c_1 = 1/6, \quad c_2 = 1/3, \quad c_3 = 1/3, \quad c_4 = 1/6; \tag{31}$$

moreover that

$$k_1 = f(x_i, y_i), \quad k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}hk_1),$$
$$k_3 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}hk_2), k_4 = f(x_i + h, y_i + hk_3). \tag{32}$$

Now a fourth order Runge Kutta method admits a total $\mathcal{O}(h^4)$ (so local $\mathcal{O}(h^5)$). So the order of the method is the order of magnitude of the total truncation error. (A higher magnitude order for truncation error is actually good. Higher order terms are later on in the Taylor series expansion.)
So we know arbitrarily of our error but how do we better estimate it? Then, how do we control it? First we can try solving the problem multiple times with different h but with the same formula. This is "OK" but not preferred; some times you would need a very small h (Dargush). Instead you can use an adaptive algorithm that estimates the error at each step. Let us consider the "adaptive" Runge Kutta method. Here you combine a fourth order RK with a fifth order RK using the same evaluation points but with different coefficients. The psuedocode of this is

```
I.    Take a step of size h.
II.   Evaluate y_{i+1} and estimate the error e_{i+1}.
III.  Is e_{i+1} < e_{tolerance}?
    A. If yes, then accept y_{i+1} and perhaps increase h;
    B. if no, reduce h and repeat step.
```

The MATLAB implementation of this is

```
ode45.
```

Let us now consider the Runge Kutta Fehlberg method. Another type of fourth order RK is

$$\hat{y}_{i+1} = y_i + h(c_1k_1 + 0 + c_3k_3 + c_4k_4 + c_5k_5); \tag{33}$$

similarly, a fifth order RK can be

$$y_{i+1} = y_i + h(c_1k_1 + 0 + c_3k_3 + c_4k_4 + c_5k_5 + c_6k_6), \tag{34}$$

noticing for both cases that the $i = 2$ term is diminished. For these RK,

$$k_1 = f(x_i, y_i), \tag{35}$$

$$k_2 = f(x_i + a_2h, y_i + b_1hk_1), \tag{36}$$

$$k_3 = f(x_i + a_3h, y_i + b_2hk_1 + b_3hk_2), \tag{37}$$

$$k_4 = f(x_i + a_4h, y_i + b_4hk_1 + b_5hk_2 + b_6hk_3), \tag{38}$$

$$k_5 = f(x_i + a_5h, y_i + b_7hk_1 + b_8hk_2 + b_9hk_3 + b_{10}hk_4), \tag{39}$$

$$k_6 = f(x_i + a_6h, y_i + b_{11}hk_1 + b_{12}hk_2 + b_{13}hk_3 + b_{14}hk_4 + b_{15}hk_5). \tag{40}$$

For the coefficients a_i, b_i, see Rao (2002). Then the error can be computed as

$$e = y_{i+1} - \hat{y}_{i+1}. \tag{41}$$

We can compare this error to some tolerance that we have established and modify h accordingly (increase with success, decrease with failure).
Let us consider a multi step method. Here we use information from several previous steps to inform y_{i+1}. We interpolate over the previous steps $\Bigg(x_i, x_{i-1}, x_{i-2}, \dots\Bigg) + \Bigg(y_i, y_{i-1}, y_{i-2}, \dots\Bigg)$. Then we extrapolate to estimate y_{i+1} at x_{i+1}. This process is guided by a Taylor series expansion.
The Adams Bashford formulas are examples of this. The fourth order method (total $\mathcal{O}(h^4)$)

$$y_{i+1} = y_i + \frac{h}{24}\Bigg(55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}\Bigg) \tag{42}$$

where $f_i = f(x_i, y_i)$ is explicit. It only uses known values on RHS. On the other hand the Adams Moulton formulas $\mathcal{O}(h^4)$

$$y_{i+1} = y_i + \frac{h}{24}\Bigg(9f_{i+1} + 19f_{i-1} - 5f_{i-1} + f_{i-2}\Bigg) \tag{43}$$

is implicit in that f_{i+1} is unknown. So that term is estimated or found simultaneously. Lastly a predictor-corrector method uses an explicit formula to predict y_{i+1}, then uses an implicit formula to correct in order to improve the solution. The $y_{i+1}^{(1)}$ term in

$$y_{i+1}^{(1)} = y_i + \frac{h}{24}\Bigg(55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}\Bigg) \tag{44}$$

is used in the $f_{i+1}^{(j)}$ term in

$$y_{i+1}^{(j+1)} = y_i + \frac{h}{24}\Bigg(9f_{i+1}^{(j)} + 19f_i - 5f_{i-1} + f_{i-2}\Bigg). \tag{45}$$

This is called the Adams predictor-corrector. You can iterate over j until convergence is achieved.

2.9 Lec 2i Simultaneous ODEs

Now consider a set of first order ODEs

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, ..., y_n) \tag{46}$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, ..., y_n) \dots \tag{47}$$

$$\frac{dy_i}{dx} = f_i(x, y_1, y_2, ..., y_n) \dots \tag{48}$$

$$\frac{dy_n}{dx} = f_n(x, y_1, y_2, ..., y_n) \tag{49}$$

with the corresponding set of initial conditions

$$y_1(x_0) = y_{1,0}, \quad y_2(x_0) = y_{2,0}, \quad ..., y_n(x_0) = y_{n,0}. \tag{50}$$

We can rewrite this in vector notation as

$$\frac{d\mathbf{y}}{dx} = \mathbf{f}(x, \mathbf{y}), \quad \mathbf{y}(x_0) = \mathbf{y}_0, \tag{51}$$

$$\mathbf{y} = \begin{Bmatrix} y_1(x) \\ y_2(x) \\ \dots \\ y_n(x) \end{Bmatrix}. \tag{52}$$

We can use all previous methodologies to solve a system. The extension is straightforward. For example the Euler forward method for the system is

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h\mathbf{f}(x_i, \mathbf{y}_i) \tag{53}$$

and the Adams predictor-corrector is

$$\mathbf{y}_{i+1}^{(1)} = \mathbf{y}_i + \frac{h}{24}\bigg(55\mathbf{f}_i - 59\mathbf{f}_{i-1} + 37\mathbf{f}_{i-2} - 9\mathbf{f}_{i-3}\bigg) \tag{54}$$

where the $\mathbf{y}_{i+1}^{(1)}$ term is plugged into $\mathbf{f}_{i+1}^{(j)} = \mathbf{f}^{(j)}(x_{i+1}, \mathbf{y}_{i+1})$ in

$$\mathbf{y}_{i+1}^{(j+1)} = \mathbf{y}_i + \frac{h}{24}\bigg(9\mathbf{f}_{i+1}^{(j)} + 19\mathbf{f}_i - 5\mathbf{f}_{i-1} + \mathbf{f}_{i-2}\bigg). \tag{55}$$

As far as the Runge Kutta Fehlberg method for the ODE set, we have

$$\hat{\mathbf{y}}_{i+1} = \mathbf{y}_i + h(c_1\mathbf{k}_1 + 0 + c_3\mathbf{k}_3 + c_4\mathbf{k}_4 + c_5\mathbf{k}_5); \tag{56}$$

and

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h(c_1\mathbf{k}_1 + 0 + c_3\mathbf{k}_3 + c_4\mathbf{k}_4 + c_5\mathbf{k}_5 + c_6\mathbf{k}_6), \tag{57}$$

where

$$\mathbf{k}_1 = \mathbf{f}(x_i, \mathbf{y}_i), \tag{58}$$

$$\mathbf{k}_2 = \mathbf{f}(x_i + a_2 h, \mathbf{y}_i + b_1 h\mathbf{k}_1), \tag{59}$$

$$\mathbf{k}_3 = \mathbf{f}(x_i + a_3 h, \mathbf{y}_i + b_2 h\mathbf{k}_1 + b_3 h\mathbf{k}_2), \tag{60}$$

$$\mathbf{k}_4 = \mathbf{f}(x_i + a_4 h, \mathbf{y}_i + b_4 h\mathbf{k}_1 + b_5 h\mathbf{k}_2 + b_6 h\mathbf{k}_3), \tag{61}$$

$$\mathbf{k}_5 = \mathbf{f}(x_i + a_5 h, \mathbf{y}_i + b_7 h\mathbf{k}_1 + b_8 h\mathbf{k}_2 + b_9 h\mathbf{k}_3 + b_{10} h\mathbf{k}_4), \tag{62}$$

$$\mathbf{k}_6 = \mathbf{f}(x_i + a_6 h, \mathbf{y}_i + b_{11} h\mathbf{k}_1 + b_{12} h\mathbf{k}_2 + b_{13} h\mathbf{k}_3 + b_{14} h\mathbf{k}_4 + b_{15} h\mathbf{k}_5). \tag{63}$$

Which is the same formulation as earlier except $\mathbf{f}, \{\mathbf{k}_i\}, \mathbf{y}_i$ are all vectors. Then the error is the L2 norm of Eq. 41, or

$$e = ||\mathbf{y}_{i+1} - \hat{\mathbf{y}}_{i+1}||_2. \tag{64}$$

So far we have discussed first order ODEs. Let us now consider the higher, second order ODE

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = P(t), \quad u(0) = u_0, \quad \frac{du}{dt}(0) = v_0. \tag{65}$$

This models a spring mass system with spring constant k, mass m, displacement u, damping constant c, external force P, and initial conditions on displacement and velocity $u_0, \quad v_0$. This is visualized in Fig. 1. Now let us redefine some parameters in

$$t \to x, u(t) \to y_1(x), \frac{du}{dt} \to y_2(x), \tag{66}$$

which is called the state space approach. Then Eq. 65 becomes

$$m\frac{dy_2}{dx} + cy_2 + ky_1 = P(x). \tag{67}$$

Rearranging,

$$\frac{dy_2}{dx} = \frac{1}{m}\Big(-cy_2 - ky_1 + P(x)\Big) = -\frac{cy_2}{m} - \frac{ky_1}{m} + \frac{P}{m}. \tag{68}$$

By virtue of Eq. 66 we already know that

$$\frac{dy_1}{dx} = y_2; \tag{69}$$

also that the initial conditions become

$$y_1(0) = u_0, y_2(0) = v_0. \tag{70}$$

The new representations that are Eqs. 69 and 68 can be integrated into the matrix equation

$$\begin{Bmatrix} \frac{dy_1}{dx} \\ \frac{dy_2}{dx} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ P/m \end{Bmatrix} \iff \frac{d\mathbf{y}}{dx} = \underbrace{\mathbf{A}\mathbf{y} + \mathbf{b}}_{\mathbf{f}}. \tag{71}$$

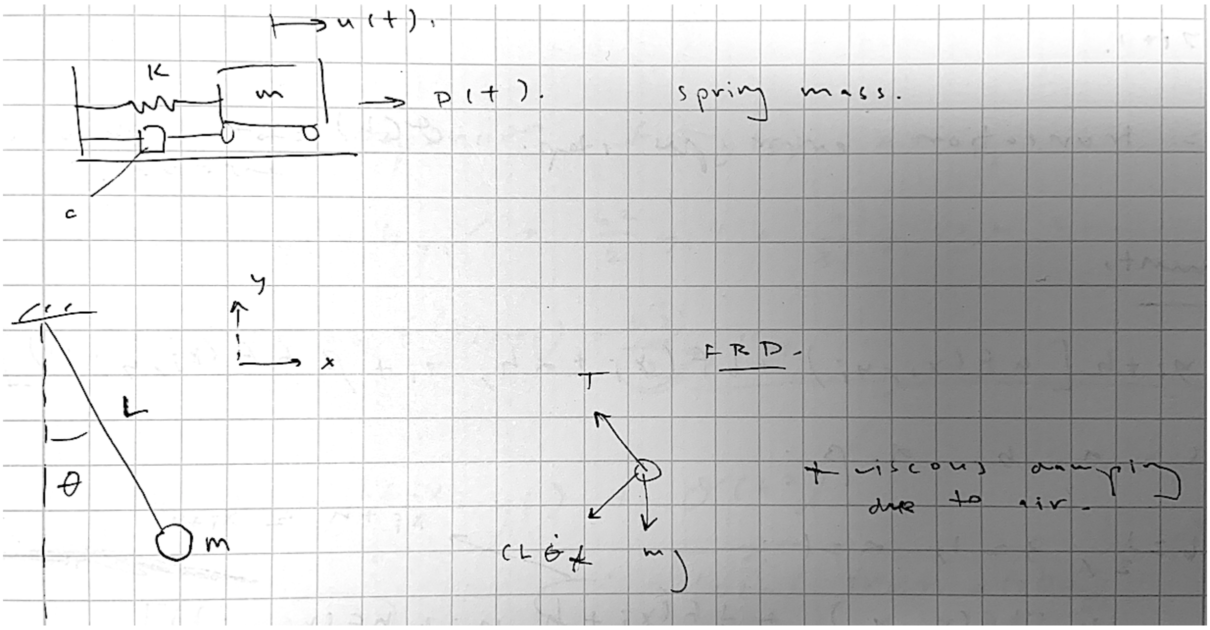

Figure 1: Spring-mass system and pendulum.

$\mathbf{f} = \frac{d\mathbf{y}}{dx}$ because of Eq. 51. It was our original consideration. Also we can generalize the initial conditions as

$$\mathbf{y}(0) = \mathbf{y}_0 \Longleftrightarrow \begin{Bmatrix} y_1(0) \\ y_2(0) \end{Bmatrix} = \begin{Bmatrix} u_0 \\ v_0 \end{Bmatrix} = \begin{Bmatrix} y_{1,0} \\ y_{2,0} \end{Bmatrix}. \tag{72}$$

We wish to state the problem in terms of physical quantities that are meaningful. In particular we want natural frequency ω and damping ratio ξ. Therefore let

$$\omega^2 = k/m, \quad 2\xi\omega = c/m. \tag{73}$$

Then

$$\begin{Bmatrix} \frac{dy_1}{dx} \\ \frac{dy_2}{dx} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\xi\omega \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ P/m \end{Bmatrix}, \quad \begin{Bmatrix} y_1(0) \\ y_2(0) \end{Bmatrix} = \begin{Bmatrix} u_0 \\ v_0 \end{Bmatrix}. \tag{74}$$

Also, letting

$$P = 0 \tag{75}$$

permits the system to freely vibrate. Now recalling from Eq. 71 that $\mathbf{f} = \frac{d\mathbf{y}}{dx} = \mathbf{A}\mathbf{y} + \mathbf{b}$, we can use Euler's forward method to say that

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h\mathbf{f} = \mathbf{y}_i + h\mathbf{A}\mathbf{y}_i + h\mathbf{b}. \tag{76}$$

We have just considered as an example a spring mass system. As another example let us consider the pendulum in Fig. 1. Here the pendulum is swinging through a viscoous fluid and so it experiences a resisting force proportional to its velocity $cL\dot{\theta}$ as well as the usual tension force T and gravitational force mg.
The sum of the torques (moments) at origin O is equal to the time rate of change of the angular momentums at that same point. That is,

$$\sum M_{0_z} = \dot{H}_{0_z}. \tag{77}$$

Recall that generally speaking torque

$$\tau = M = r \times F = rF\sin\phi \tag{78}$$

where ϕ is the angle between the contributing force and the distance between the body and some axis of interest. It is useful to make $\phi = 90^o$. So for example, since gravity acts straight down, r will be the distance between the ball and the vertical axis so that r is a horizontal space between two vertical lines and

$$M_{mg} = rF\sin\phi = (L\sin\theta)(-mg)\underbrace{\sin\phi}_{\phi=90^o} = -mgL\sin\theta. \tag{79}$$

Viscous force acts perpendicularly to the wire and the distance between the ball and the origin O with respect to the perpendicular axis is L. So

$$M_{viscous} = -(cl\dot{\theta})L\sin\phi = -cL^2\dot{\theta}. \tag{80}$$

These two equations comprise LHS. Then on the right hand side is the time derivative of the angular momentum, where angular momentum H is linear momentum times radius. Generally speaking

$$\rho = mv = m(L\dot{\theta}) \tag{81}$$

because the velocity of a point on the wire increases as you go further down it. Then

$$H_0 = (mL\dot{\theta})L \rightarrow \dot{H}_0 = \frac{d}{dt}\Big[(mL\dot{\theta})L\Big] \tag{82}$$

which makes up RHS. Together,

$$-cL^2\dot{\theta} - mgL\sin\theta = mL^2\ddot{\theta} \tag{83}$$

implies

$$mL^2\ddot{\theta} + mgL\sin\theta + cL^2\dot{\theta} = 0 \tag{84}$$

implies

$$\ddot{\theta} + \frac{c}{m}\dot{\theta} + \frac{g}{L}\sin\theta = 0. \tag{85}$$

Eq. 85 can be simplified with the Taylor series

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots \approx \theta \tag{86}$$

so that now,

$$\ddot{\theta} + \frac{c}{m}\dot{\theta} + \frac{g}{L}\theta = 0. \tag{87}$$

Substituting physical quantities in from Eq. 73,

$$\ddot{\theta} + 2\xi\omega\dot{\theta} + \omega^2\theta = 0. \tag{88}$$

Using the state space approach, we invoke a process similar to Eq. 66, which is

$$x \leftarrow t, \quad \theta \leftarrow y_1, \dot{\theta} \leftarrow y_2. \tag{89}$$

Then

$$\frac{dy_1}{dx} = y_2 \tag{90}$$

and

$$\frac{dy_2}{dx} = \ddot{\theta} = -2\xi\omega\dot{\theta} - \omega^2\theta = -2\xi\omega y_2 - \omega^2 y_1. \tag{91}$$

This information is sufficient to build the matrix equation

$$\begin{Bmatrix} \frac{dy_1}{dx} \\ \frac{dy_2}{dx} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\xi\omega \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \mathbf{A}\mathbf{y} + \mathbf{b} = \mathbf{f} = \frac{d\mathbf{y}}{dx} \quad (\mathbf{b} = \mathbf{0}). \tag{92}$$

Not using the Taylor series approximation and thus leaving the matrix equation system as

$$\frac{dy_1}{dx} = y_2 \tag{93}$$

and

$$\frac{dy_2}{dx} = -2\xi\omega y_2 - \omega^2 \sin y_1 \tag{94}$$

admits a set of nonlinear ODEs. Even one nonlinear equation in a set redefines the whole set as nonlinear.

2.10 Lec 2j State space dynamics and stability

To summarize Fig. 1, the spring mass system can be written as

$$m\ddot{u} + c\dot{u} + ku = P \rightarrow \ddot{u} = -\frac{c}{m}\dot{u} - \frac{k}{m}u + P/m \tag{95}$$

which implies

$$\begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} + \begin{Bmatrix} 0 \\ P/m \end{Bmatrix} \tag{96}$$

provided $\dot{u} = v \rightarrow \ddot{u} = \dot{v}$. Likewise the pendulum is written as

$$\ddot{\theta} + \frac{c}{m}\dot{\theta} + \frac{g}{L}\sin\theta = 0 \tag{97}$$

which implies

$$\begin{Bmatrix} \dot{\theta} \\ \dot{\Omega} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -c/m & -g/L \end{bmatrix} \begin{Bmatrix} \theta \\ \Omega \end{Bmatrix} \tag{98}$$

provided $\dot{\theta} = \Omega \rightarrow \ddot{\theta} = \dot{\Omega}$. The general structure of these equations (especially seen in the spring mass system) is

$$\mathbf{M}\dot{\mathbf{v}} + \mathbf{C}\mathbf{v} + \mathbf{K}\mathbf{u} = \mathbf{P}(t). \tag{99}$$

The conversion of this system to state space is

$$\mathbf{y} = \begin{Bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{u} \\ \mathbf{v} \end{Bmatrix}. \tag{100}$$

We understand

$$\dot{\mathbf{u}} = \mathbf{v}; \tag{101}$$

also that, after rearranging Eq. 99,

$$\dot{\mathbf{v}} = \mathbf{M}^{-1}\Big(-\mathbf{K}\mathbf{u} - \mathbf{C}\mathbf{v} + \mathbf{P}\Big) = -\mathbf{M}^{-1}\mathbf{K}\mathbf{u} - \mathbf{M}^{-1}\mathbf{C}\mathbf{v} + \mathbf{M}^{-1}\mathbf{P}. \tag{102}$$

Then

$$\underbrace{\begin{Bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{v}} \end{Bmatrix}}_{\mathbf{f}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{Bmatrix} \mathbf{u} \\ \mathbf{v} \end{Bmatrix}}_{\mathbf{y}} + \underbrace{\begin{Bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{P} \end{Bmatrix}}_{\mathbf{b}}. \tag{103}$$

Let us apply this to Euler integration. Generally,

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h\mathbf{f}_i = \mathbf{y}_i + h\mathbf{A}\mathbf{y}_i + h\mathbf{b}_i. \tag{104}$$

This implies

$$\mathbf{y}_{i+1} = \Big(\mathbf{I} + h\mathbf{A}\Big)\mathbf{y}_i + h\mathbf{b}_i. \tag{105}$$

Starting at the first step:

$$\mathbf{y}_1 = \Big(\mathbf{I} + h\mathbf{A}\Big)\mathbf{y}_0 + h\mathbf{b}_0. \tag{106}$$

Then,

$$\mathbf{y}_2 = \Big(\mathbf{I} + h\mathbf{A}\Big)(\mathbf{y}_1) + h\mathbf{b}_1$$

$$= \Big(\mathbf{I} + h\mathbf{A}\Big)(\Big(\mathbf{I} + h\mathbf{A}\Big)\mathbf{y}_0 + h\mathbf{b}_0) + h\mathbf{b}_1$$

implies

$$\mathbf{y}_2 = \Big(\mathbf{I} + h\mathbf{A}\Big)^2\mathbf{y}_0 + \Big(\mathbf{I} + h\mathbf{A}\Big)h\mathbf{b}_0 + h\mathbf{b}_1. \tag{107}$$

In general,

$$\mathbf{y}_n = \Big(\mathbf{I} + h\mathbf{A}\Big)^n\mathbf{y}_0 + \mathbf{g}_n(\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_n). \tag{108}$$

$\mathbf{g}$ is some function of $\mathbf{b}_i$. Matrix

$$\hat{\mathbf{A}} = \Big(\mathbf{I} + h\mathbf{A}\Big) \tag{109}$$

when expressed in

$$\mathbf{y}_n = \hat{\mathbf{A}}^n\mathbf{y}_0 + \mathbf{g} \tag{110}$$

is called the Jordan canonical form. The spectral decomposition

$$\hat{\mathbf{A}} = \mathbf{PJP}^{-1} \tag{111}$$

admits the Jordan normal form $\mathbf{J}$ which is a matrix containing the eigenvalues of $\hat{\mathbf{A}}$ on its diagonal and sometimes the superdiagonal (the parallel line of elements right above the diagonal). Raised to the power n,

$$\hat{\mathbf{A}}^n = \left(\mathbf{PJP}^{-1}\right)\left(\mathbf{PJP}^{-1}\right)\left(\mathbf{PJP}^{-1}\right)\ldots = \mathbf{PJ}^n\mathbf{P}^{-1}. \tag{112}$$

We do not want $\mathbf{J}^n$ to increase to infinity as $n \to \infty$. That is, we want $\mathbf{J}^n$ to be bounded. This is true if the spectral radius of $\hat{\mathbf{A}} = \rho(\hat{\mathbf{A}}) \leq 1$. Spectral radius is defined as the maximum of the absolute values of its eigenvalues. That is,

$$\rho(\hat{\mathbf{A}}) = \max\{|\lambda_1|, |\lambda_2|, \ldots |\lambda_n|\}. \tag{113}$$

We consider an example of using the Jordan canonical form. Let

$$\dot{x}_1 = -3x_1 - 2x_2, \quad \dot{x}_2 = 2x_1 + x_2; \quad x_1(0) = 1, \quad x_2(0) = 0. \tag{114}$$

Then

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}}_{\mathbf{A}} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}. \tag{115}$$

For a linear system

$$\mathbf{x}' = \mathbf{Ax}, \tag{116}$$

if $\{\lambda, \mathbf{r}\}$ is an eigenpair for $\mathbf{A}$, then the solution of the system is

$$\mathbf{x} = e^{\lambda t}\mathbf{r} \to \mathbf{x}' = \lambda e^{\lambda t}\mathbf{r} \tag{117}$$

because

$$\mathbf{Ax} = \mathbf{A}e^{\lambda t}\mathbf{r} = e^{\lambda t}\mathbf{Ar} = \lambda e^{\lambda t}\mathbf{r} = \mathbf{x}' \tag{118}$$

implies the eigenproblem

$$e^{\lambda t}\mathbf{Ar} = e^{\lambda t}\lambda\mathbf{r} \to \mathbf{Ar} = \lambda\mathbf{r}. \tag{119}$$

So we need to find the eigenpairs of the matrix $\mathbf{A}$. This requires

$$0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{bmatrix} -3-\lambda & -2 \\ 2-\lambda & 1 \end{bmatrix} = (-3-\lambda)(1-\lambda) - (2)(-2) = 0. \tag{120}$$

Solving for λ,

$$\lambda^2 + 2\lambda + 1 = 0 \to (\lambda+1)(\lambda+1) = 0 \to \lambda_1, \lambda_2 = -1. \tag{121}$$

The corresponding eigenvector $\mathbf{r}$ can be found in

$$\mathbf{Ar} = \lambda\mathbf{r} \to \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \begin{Bmatrix} -r_1 \\ -r_2 \end{Bmatrix} \tag{122}$$

by considering algebraically

$$-3r_1 - 2r_2 = -r_1 \rightarrow -2r_2 = 2r_1 \rightarrow r_2 = -r_1$$

$$\rightarrow \mathbf{r} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \rightarrow \text{normalizing if desired/necessary} \rightarrow \mathbf{r} = \begin{Bmatrix} 1\sqrt{2} \\ -1\sqrt{2} \end{Bmatrix}. \tag{123}$$

Noonberg (2010) pg. 166-167 proves that if $\lambda_1 = \lambda_2$ for $[\mathbf{A}]_{2\text{x}2}$ then the general solution of $\mathbf{A}\mathbf{x} = \mathbf{x}'$ can be written in the form

$$\mathbf{x} = c_1 e^{\lambda t}\mathbf{r} + c_2 e^{\lambda t}(t\mathbf{r} + \mathbf{r}^*), \quad \text{where} \quad (\mathbf{A} - \lambda \mathbf{I})\mathbf{r}^* = \mathbf{r}. \tag{124}$$

This is also called variation of parameters. In this case,

$$\underbrace{\begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}}_{\mathbf{A}-\lambda\mathbf{I}} \underbrace{\begin{Bmatrix} 1 \\ -3/2 \end{Bmatrix}}_{\mathbf{r}^*} = \underbrace{\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}}_{\mathbf{r}}. \tag{125}$$

So we can write the general solution as

$$\mathbf{x} = c_1 e^{-t} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} + c_2 e^{-t} \begin{Bmatrix} t+1 \\ -t-3/2 \end{Bmatrix} = \mathbf{x}(t). \tag{126}$$

That means

$$\mathbf{x}(0) = c_1 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} + c_2 \begin{Bmatrix} 1 \\ -3/2 \end{Bmatrix} = \begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \tag{127}$$

where $\{x_1(0), x_2(0)\}^{\mathrm{T}}$ comes from Eq. 114. Recall that we want to bound $\mathbf{J}^n$ by causing the spectral radius of $\hat{\mathbf{A}}$ - that is, its maximum eigenvalue - to be less than or equal to 1. This is true if, using Eq. 109 and 113,

$$\max\{|\lambda_1|, |\lambda_2|\} \leq 1, \tag{128}$$

where λ_i are recovered in letting

$$0 = \det(\hat{\mathbf{A}} - \lambda I) = \det(\mathbf{I} + h\mathbf{A} - \lambda I) = \det \begin{bmatrix} 1 - 3h - \lambda & -2h \\ 2h & 1 + h - \lambda \end{bmatrix}$$

$$= (1 - 3h - \lambda)(1 + h - \lambda) - (2h)(-2h)$$

$$= -h^2 - 2h\lambda + 2h - \lambda^2 + 2\lambda - 1 = 0 \rightarrow \lambda_1 = \lambda_2 = 1 - h. \tag{129}$$

To satisfy the enforcement of Eq. 128,

$$|1 - h| \leq 1 \rightarrow -2 \leq h \leq 2. \tag{130}$$

Now recall once again the spring mass system

$$\begin{Bmatrix} \dot{u} \\ \dot{v} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} + \begin{Bmatrix} 0 \\ P/m \end{Bmatrix}. \tag{131}$$

Consider the undamped case. That is, $c = 0$. Remembering also that natural frequency $\omega^2 = k/m$,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}; \tag{132}$$

eigenvalues are brought out of

$$0 = \det \begin{bmatrix} -\lambda & 1 \\ -\omega^2 & -\lambda \end{bmatrix} = \lambda^2 + \omega^2 = 0 \to \lambda_1, \lambda_2 = \pm i\omega. \tag{133}$$

Eigenvalues of the Jordan canonical form are in

$$0 = \det(\mathbf{I} + h\mathbf{A} - \lambda\mathbf{I}) = \det \begin{bmatrix} 1-\lambda & h \\ -h\omega^2 & 1-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda) - (-h\omega^2)(h)$$

$$= 1 - 2\lambda + \lambda^2 + h^2\omega^2 = 0 \to \begin{Bmatrix} \lambda_1 \\ \lambda_2 \end{Bmatrix} = \begin{Bmatrix} i(h\omega - i) = 1 + h\omega i \\ -i(h\omega + i) = 1 - h\omega i \end{Bmatrix}. \tag{134}$$

$\lambda = 1 + ih\omega$ implies

$$|\lambda| = \sqrt{1^2 + i^2h^2\omega^2} = \sqrt{1 + h^2\omega^2}. \tag{135}$$

$h^2\omega^2$ is always positive, so $\lambda \geq 1$ always, which means the rule can never be enforced, making $\mathbf{J}^n$ unbounded and $\hat{\mathbf{A}}$ unstable always.
We consider a stiff ODE system as another example. Let

$$\frac{du}{dt} = (\beta - 2)u + (2\beta - 2)v, \tag{136}$$

$$\frac{dv}{dt} = (1 - \beta)u + (1 - 2\beta)v, \tag{137}$$

as well as

$$u(0) = 1, \quad v(0) = 0. \tag{138}$$

The exact solution for $\beta > 2$ is

$$u(t) = 2e^{-t} - e^{-\beta t}, \tag{139}$$

$$v(t) = -e^{-t} + e^{-\beta t}. \tag{140}$$

where $\lambda_1 = -1, \lambda_2 = -\beta$. For a large β, the terms containing β will decay very fast with time. Now, for any explicit (Euler forward) method, one must resolve the faster scale. Otherwise severe instability will be brought into the system. For this problem consider a very general linear set of ODEs with constant coefficients

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} \Longleftrightarrow \begin{Bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{Bmatrix} = \begin{bmatrix} \beta - 2 & 2\beta - 2 \\ 1 - \beta & 1 - 2\beta \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}. \tag{141}$$

As played out in Eqs. 116-119, we can assume the solution

$$\mathbf{y} = \boldsymbol{\phi} e^{\lambda t} \tag{142}$$

leading to

$$\dot{\mathbf{y}} = \lambda \boldsymbol{\phi} e^{\lambda t} \tag{143}$$

and then, substituting into the original linear set,

$$\lambda \boldsymbol{\phi} e^{\lambda t} = \mathbf{A} \boldsymbol{\phi} e^{\lambda t} \tag{144}$$

which implies the standard eigenproblem

$$\lambda \boldsymbol{\phi} = \mathbf{A} \boldsymbol{\phi}. \tag{145}$$

For the previous problem,

$$\mathbf{A} = \begin{bmatrix} \beta - 2 & 2\beta - 2 \\ 1 - \beta & 1 - 2\beta \end{bmatrix} \tag{146}$$

and $\lambda_1 = -1, \lambda_2 = -\beta$. For a large $\beta = 100$,

```
% Input file
clear
clc
beta = 100
A = [(beta-2), (2*beta -2); (1-beta), (1-2*beta)];
[Lambda, Phi] = eig(A)

% Command window
beta =

   100


Lambda =

    0.8944   -0.7071
   -0.4472    0.7071


Phi =

    -1     0
     0  -100
```

The assumed solution

$$\mathbf{y} = \boldsymbol{\phi} e^{\lambda t} \tag{147}$$

for two eigenvalues requires a general solution in the form of

$$\mathbf{y} = c_1 \boldsymbol{\phi}_1 e^{\lambda_1 t} + c_2 \boldsymbol{\phi}_2 e^{\lambda_2 t}. \tag{148}$$

Now suppose we want to find c_1, c_2 which satisfies the initial conditions

$$y_1(0) = 0, \quad y_2(0) = 0 \leftrightarrow u(0) = 0, \quad v(0) = 0. \tag{149}$$

To do this we have at our disposal Euler's method or one of the Runge Kutta methods. For a large β the second term $\lambda_2 = -\beta$ decays very fast and thus requires treatment to avoid instability.
Evaluation of the eigenvalues of the Jordan canonical form of $\mathbf{A}$ - that is, $\hat{\mathbf{A}}$, brings

$$0 = \det(\underbrace{\mathbf{I} + h\mathbf{A}}_{\hat{\mathbf{A}}} - \lambda\mathbf{I}) \rightarrow \hat{\lambda}_1 = 1 - h, \hat{\lambda}_2 = 1 - \beta h. \tag{150}$$

Then to satisfy

$$\max\{|\lambda_1|, |\lambda_2|\} \leq 1, \tag{151}$$

we determine

$$|1 - h| \leq 1 \rightarrow -2 \leq h \leq 2 \tag{152}$$

and

$$|1 - \beta h| \leq 1 \rightarrow -2 \leq \beta h \leq 2 \rightarrow \frac{-2}{\beta} \leq h \leq \frac{2}{\beta}. \tag{153}$$

Therefore one would need a very small h for a large β; as β grows, $2/\beta$ shrinks and still h must be lesser than $2/\beta$.

Finally let us consider Euler's backward method for a linear system, which is a generalization and reindex of Eq. 10. That is,

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h\mathbf{f}_{i+1}. \tag{154}$$

This implies

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h(\mathbf{A}\mathbf{y}_{i+1} + \mathbf{b}_{i+1}). \tag{155}$$

Rearranging,

$$\mathbf{y}_{i+1} - h\mathbf{A}\mathbf{y}_{i+1} = \Big(\mathbf{I} - h\mathbf{A}\Big)\mathbf{y}_{i+1} = \mathbf{y}_i + h\mathbf{b}_{i+1}. \tag{156}$$

We can isolate

$$\mathbf{y}_{i+1} = \Big(\mathbf{I} - h\mathbf{A}\Big)^{-1}(\mathbf{y}_i + h\mathbf{b}_{i+1}) \tag{157}$$

and let the Jordan canonical form be differently defined for the backward problem as

$$\hat{\mathbf{A}} = \Big(\mathbf{I} - h\mathbf{A}\Big)^{-1}. \tag{158}$$

Similarly to the forward problem, for stability we require that

$$\rho(\hat{\mathbf{A}}) \leq 1. \tag{159}$$

Finding the eigenvalues of this expression even as it is inverted is not difficult. One must only find the eigenvalues of the expression $\mathbf{I} - h\mathbf{A}$ and then invert each eigenvalue individually to receive those of $(\mathbf{I} - h\mathbf{A})^{-1}$.

2.11 Lec 2k Qualitative theory of ODEs

One must look at nonlinear problems with a different perspective. There is no superposition with nonlinear ODEs. Analytical solutions are also difficult or impossible to obtain. We can however use numerical methods. We do this with the goal of gathering information that provides insight into the character of general solutions. Poincaré initiated the qualitative theory of ODEs in the late 19th century. We can use qualitative theory for nonlinear phenomena in solids and fluids, and for control of nonlinear systems. The key issue as far as numerical methods go is stability.

The qualitative theory is a highly geometric approach as opposed to solving many equations. We consider the trajectory of a system for specified initial conditions in the phase space (or, with two dependent variables, a phase plane). A set of trajectories provides a phase portrait.

We can then characterize the nonlinear ODE according to the geometric pattern of the phase portrait. We can attempt to investigate the trajectories as $t \to \infty$.

We begin with a special linear case. Consider the 2nd order linear homogeneous ODE with constant coefficients

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}. \tag{160}$$

We define the critical points as those in which

$$\frac{d\mathbf{x}}{dt} = 0 \tag{161}$$

and thus

$$\mathbf{A}\mathbf{x} = \mathbf{0}. \tag{162}$$

At the critical (also called equilibrium) points, for det $\mathbf{A} \neq 0$ (i.e. if we have two linearly independent equations; i.e. if $\mathbf{A}$ is nonsingular; i.e. if $\mathbf{A}$ is invertible), then the only solution is at the origin $\mathbf{x} = \mathbf{0}$. That is the trivial solution.

Now let us classify the solution. Let

$$\mathbf{x}(t) = \mathbf{k}e^{\lambda t}. \tag{163}$$

From here we form the characteristic equation and find roots (eigenvalues). The nature of these eigenvalues determines the type of general solution.

If (I) both eigenvalues are real and of the same sign, then the general solution

$$\mathbf{x}(t) = c_1\mathbf{k}_1 e^{\lambda_1 t} + c_2\mathbf{k}_2 e^{\lambda_2 t}, \qquad \lambda_1 \neq \lambda_2 \in \mathcal{R}. \tag{164}$$

If

$$\lambda_1 < \lambda_2 < 0, \tag{165}$$

then all trajectories approach the origin as $t \to \infty$. That is, $\mathbf{x}$ approaches $\mathbf{0}$ because if both λ_1, λ_2 are negative, then both terms $e^{\lambda_1 t}, e^{\lambda_2 t}$ will go to zero. Suppose we define $\mathbf{x}_0$ as the initial condition or the point in space in the phase plane. If $\mathbf{x}$ is on $\mathbf{k}_1$, then $c_2 = 0$ and, assuming still that λ_1 is negative, the trajectory will approach the origin in a straight line along $\mathbf{k}_1$. The converse is true if $\mathbf{x}$ is on $\mathbf{k}_2$: $c_1 = 0$ and the trajectory will approach $\mathbf{0}$ along $\mathbf{k}_2$.

Suppose we rewrote

$$\mathbf{x}(t) = c_1\mathbf{k}_1 e^{\lambda_1 t} + c_2\mathbf{k}_2 e^{\lambda_2 t} \tag{166}$$

as

$$\mathbf{x}(t) = e^{\lambda_2 t}[c_1\mathbf{k}_1 e^{(\lambda_1-\lambda_2)t} + c_2\mathbf{k}_2]. \tag{167}$$

Now as $t \to \infty$, the $c_1\mathbf{k}_1 e^{(\lambda_1-\lambda_2)t}$ term becomes negligible compared to the $c_2\mathbf{k}_2$ term because λ_1 and λ_2 are competing. This means essentially that

$$\lim_{t\to\infty} \mathbf{x}(t) = e^{\lambda_2 t}[c_2\mathbf{k}_2], \tag{168}$$

or

$$\lim_{t\to\infty} \mathbf{x}(t)e^{-\lambda_2 t} = c_2\mathbf{k}_2 \tag{169}$$

provided $c_2 \neq 0$. As time goes to infinity, our solutions will go to the critical or equilibrium point that is the origin. So we see that based on Eq. 169, trajectories near the critical point align with $\mathbf{k}_2$, as in Fig. 2. Here the equilibrium solution $\mathbf{x} = \mathbf{0}$ is stable asymptotically. We also say that the critical point is an improper node.
Now if

$$\lambda_1, \lambda_2 > 0, \tag{170}$$

then the same behavior occurs as in Fig. 2 except that the trajectories are in opposite directions. So they bend outward to infinity from alignment with $\mathbf{k}_2$. The equilibrium solution is unstable and the critical point is still an improper node.

Our first case (I) was that where the two eigenvalues was of the same sign, either positive or negative. Now let us consider if (II) the eigenvalues are real but of opposite sign. The general solution is still

$$\mathbf{x}(t) = c_1\mathbf{k}_1 e^{\lambda_1 t} + c_2\mathbf{k}_2 e^{\lambda_2 t}, \qquad \lambda_1 \neq \lambda_2 \in \mathcal{R}. \tag{171}$$

If

$$\lambda_1 > 0 \text{ and } \lambda_2 < 0, \tag{172}$$

then for the initial conditions along $\mathbf{k}_1$ or $\mathbf{k}_2$, the trajectories are still straight along, but on $\mathbf{k}_1$ the trajectories point away from the origin (positive trajectory as $t \to \infty$) while on $\mathbf{k}_2$ the trajectories point toward the origin (negative trajectory as $t \to \infty$). The $c_1\mathbf{k}_1 e^{\lambda_1 t}$ (λ_1 positive) term begins to dominate as $t \to \infty$ while the $c_2\mathbf{k}_2 e^{\lambda_2 t}$ (λ_2 negative) term tends to zero. So, all other points tend asymptotically towards $\mathbf{k}_1$. The equilibrium solution is unstable because all lines tend away from the origin. Moreover the equilibrium point is called a saddle point. This case is shown in Fig. 3.

Now let us consider the case (III) where the two eigenvalues are equal. If (IIIi) we have two independent eigenvectors such that

$$\lambda_1 = \lambda_2 = \lambda, \quad \mathbf{x}(t) = c_1\mathbf{k}_1 e^{\lambda t} + c_2\mathbf{k}_2 e^{\lambda t}, \tag{173}$$

then for $\lambda < 0$, all trajectories approach the origin along a straight line as $t \to \infty$. The equilibrium point in this case is called a proper node. The phase space is asymptotically stable. For $\lambda > 0$ the trajectories approach infinity along the same straight lines. Now

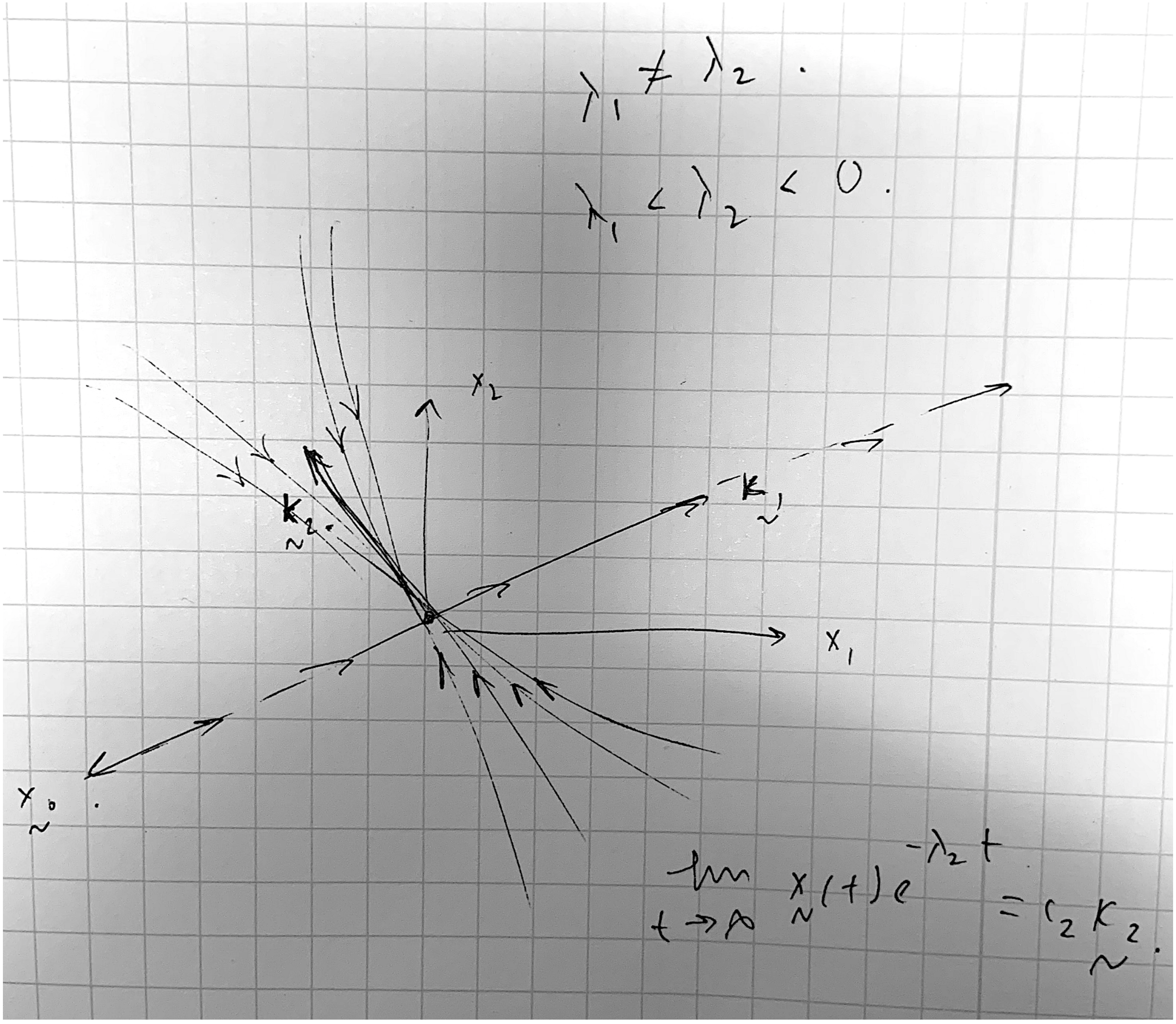

Figure 2: $\lambda_1 < \lambda_2 < 0, \quad \lim_{t\to\infty} \mathbf{x}(t)e^{-\lambda_2 t} = c_2\mathbf{k}_2$

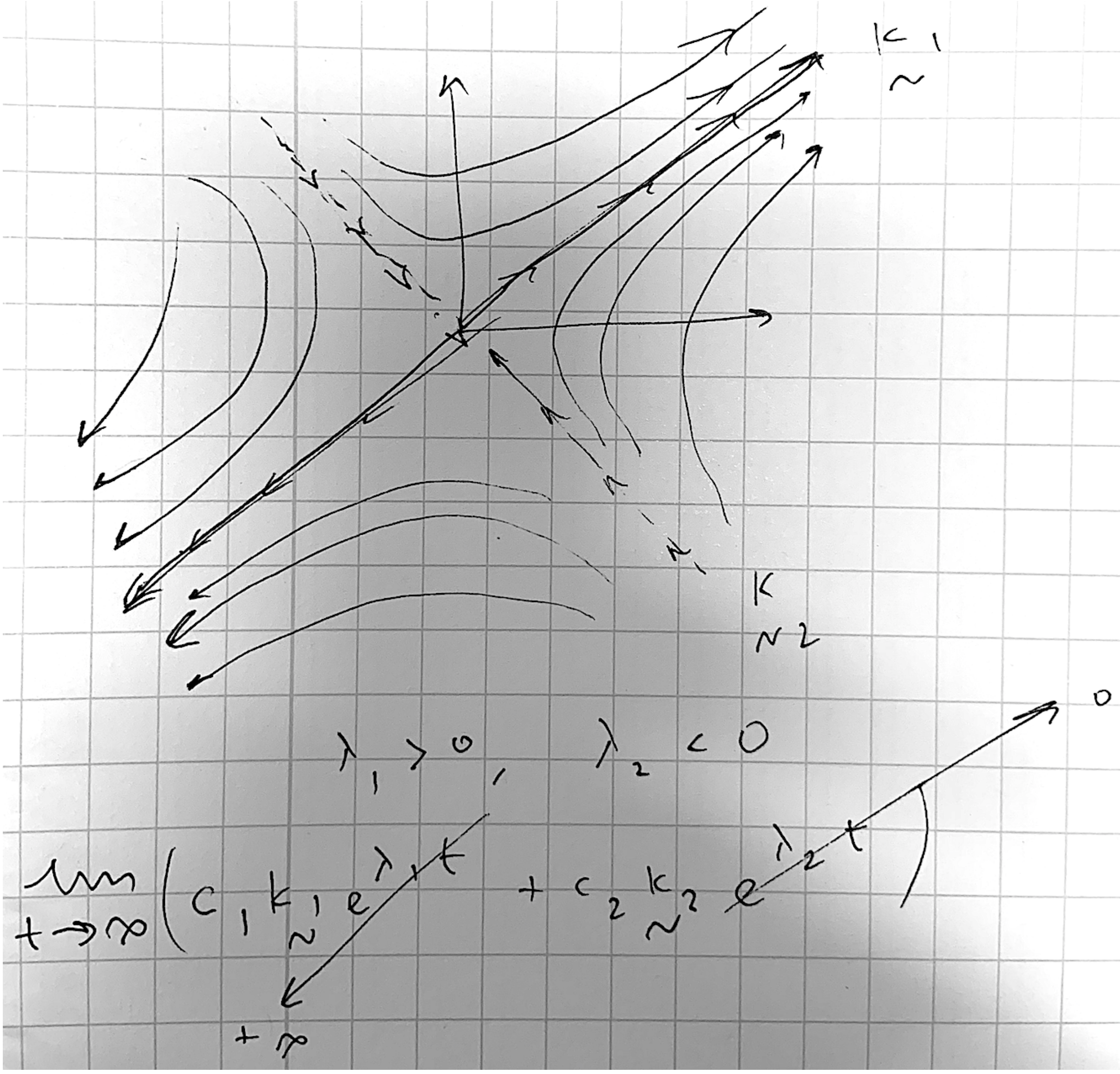

Figure 3: $\lambda_1 > 0, \quad \lambda_2 < 0$

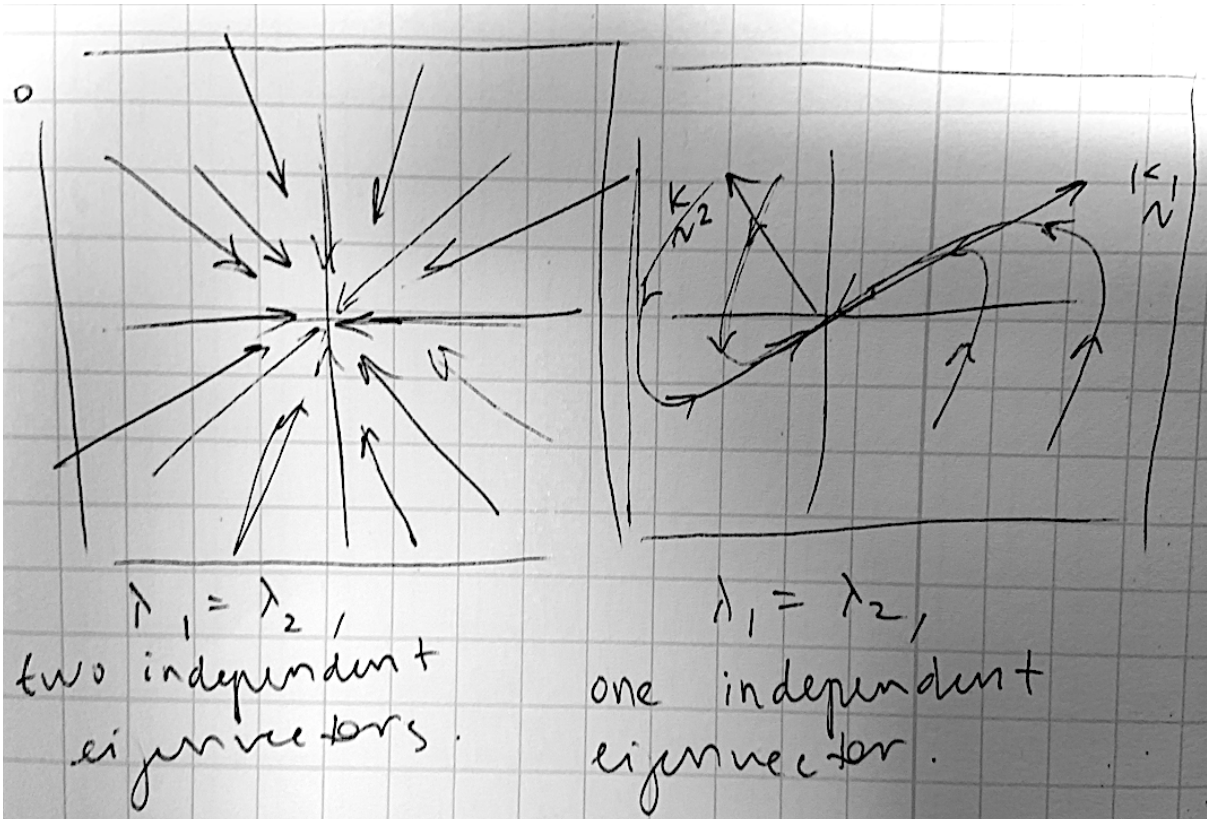

Figure 4: $\lambda_1 = \lambda_2 = \lambda < 0,$ one independent eigenvector

if (IIIii) we have only one independent eigenvector, then the solution is derived through variation of parameters as in Eq. 124 and is written as

$$\lambda_1 = \lambda_2 = \lambda, \quad \mathbf{x}(t) = c_1\mathbf{k}_1 e^{\lambda t} + \underbrace{c_2(\mathbf{k}_1 t e^{\lambda t}}_{\text{dominates, } t\to\infty} + \mathbf{k}_2 e^{\lambda t}). \tag{174}$$

In this case one of the $\mathbf{k}_1$ terms dominate and so all trajectories will tend toward $\mathbf{k}_1$. Fig. 4 portrays the case of $\lambda < 0$ where the system is stable and trajectories tend toward the origin. Conversely though if $\lambda > 0$, the trajectories would tend toward infinitely far from the origin and so the system would be classified as unstable.

Now let us consider the case (IV) where the eigenvalues are complex with a real part. That is,

$$\lambda = \alpha \pm i\beta. \tag{175}$$

If the eigenvalues are complex then they will always be complex conjugate pairs. In this case, in the phase space there exists a spiral pattern where the general solution has some $e^{\alpha t}e^{i\beta t}$ term. The critical point is the spiral point or the center of the spiral. If $\alpha < 0$, the trajectories spiral inward towards the critical point and the system is stable; if $\alpha > 0$, the trajectories spiral outward towards it and the system is unstable.
In the special case (V) of $\alpha = 0$, so that the eigenvalues are strictly complex with no real part, i.e.

$$\lambda = \pm i\beta, \tag{176}$$

then the spiral will neither tend inward nor outward. Actually, the trajectories do not comprise a spiral but a set of closed loops. The system is then stable because the trajectories do not tend to infinity. The equilibrium point is called a center.
We have considered all possible classifications of the critical points, which are: improper node, proper node, saddle point, spiral point, or center.
We have let the critical point be the origin but this does not have to be the case. Let us generalize and call the critical point $\mathbf{x}^*$. As we have seen, the stability of the critical point has to do with how trajectories behave around $\mathbf{x}^*$. If you begin at $t = 0$ within some circle of radius δ centered at $\mathbf{x}^*$, then the trajectory must remain within some circle of radius ϵ for all t. If we can identify some circle then the solution is stable. If we cannot identify any circle - that is, if the trajectories tend toward infinity - then the solution is unstable.
If in addition to satisfying stability,

$$\lim_{t\to\infty} \mathbf{x} = \mathbf{x}^*, \tag{177}$$

then the critical point is called asymptotically stable.
Let us reconsider the damped spring mass system in this context. The same as in Eq. 65 and Eq. 73, it is

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = 0 \tag{178}$$

where

$$\omega^2 = k/m, \quad 2\xi\omega = c/m, \tag{179}$$

so that

$$\ddot{u} + 2\xi\omega\dot{u} + \omega^2 u = 0. \tag{180}$$

At the global level, the sign and magnitude of $2\xi\omega$ and ω^2 affects the stability. One can draw a stability diagram with axes $c/m = 2\xi\omega$ and $k/m = \omega^2$ a line drawn where the solution crosses from stability to instability. Static stability in this case is associated with a negative stiffness $k < 0$; dynamic instability is associated with a negative damping $c < 0$. The sensitivity of system behavior to parameter changes is noteworthy. In this case, for example, small changes in ξ influentially pushes the system from stable to asymptotically stable to unstable.

2.12 Lec 2l Autonomous systems

Suppose

$$\frac{dx}{dt} = F(x, y), \quad x(t_0) = x_0, \tag{181}$$

$$\frac{dy}{dt} = G(x, y), \quad y(t_0) = y_0, \tag{182}$$

where F and G are arbitrary functions. Of course we can summarize this to say that

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0. \tag{183}$$

This is an autonomous system, which means the independent variable t does not appear anywhere in explicit form in the equation. That is, F and G do not have a direct contribution from the variable t. Note that for constant $\mathbf{A}$, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is autonomous.
Autonomous systems have restrictions. This simplifies the analysis. That is because there is only one trajectory which passes through a point in phase space. In this case the critical points are exactly the $\mathbf{x}$ where

$$\mathbf{f}(\mathbf{x}) = \mathbf{0}. \tag{184}$$

There exist some nonlinear systems that are called an "almost linear system" (ALS). Suppose

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}). \tag{185}$$

Let us find trajectories near the critical point $\mathbf{x}^* = \mathbf{0}$. Although this is assumed to be the origin it does not have to be. If it is not the origin then one can shift the coordinates by introducing the translation $\mathbf{u} = \mathbf{x} - \mathbf{x}^*$.
Now, if (I) it is true that we can rewrite Eq. 185 as

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{g}, \tag{186}$$

and if (II) $\mathbf{x}^*$ is an isolated critical point of $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{g}$, and if (III) $\det \mathbf{A} \neq 0$ (so that $\mathbf{x} = \mathbf{0}$ is the only critical point of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$), and if (IV) $\frac{||\mathbf{g}||}{||\mathbf{x}||} \to 0$ as $\mathbf{x} \to 0$, then the system is "almost linear." In (II), an isolated critical point means that one can draw a circle around $\mathbf{x} = \mathbf{0}$ with no other critical points in that circle. Also, the nonlinear terms that are $\mathbf{g}$ should become smaller and smaller as we approach the critical point. That is basically (IV).
For ALS's, one can say something about the trajectories near the critical point. The behavior of the ALS near the critical point is the same as that of corresponding linear system, *except* for (I) centers, where the eigenvalues are imaginary, and (2) nodes, where the eigenvalues are equal. For (I), a center in the linear system can become a spiral in the ALS. The stability also is compromised. For (II), the node in the linear system to a spiral point in the ALS. However, the stability does not change.
A perfect nonlinear example is the damped pendulum that is Eq. 87, or

$$\ddot{\theta} + \frac{c}{m}\dot{\theta} + \frac{g}{L}\sin\theta = 0. \tag{187}$$

We maintain the claim that is Eq. 77, or

$$\sum M_{0_z} = \dot{H}_{0_z}. \tag{188}$$

We can convert this to a first order system by letting

$$x \leftarrow \theta, y \leftarrow \dot{\theta}, \tag{189}$$

making

$$\dot{x} = y, \tag{190}$$

$$\dot{y} + \frac{c}{m}y + \frac{g}{L}\sin x = 0 \quad \longrightarrow \quad \dot{y} = -\frac{c}{m}y - \frac{g}{L}\sin x. \tag{191}$$

The critical points are

$$0 = \dot{x} = y, \tag{192}$$

$$0 = \dot{y} = -\frac{c}{m}y - \frac{g}{L}\sin x. \tag{193}$$

Therefore

$$y = 0, \quad \sin x = 0 \quad \to \quad x = \pm n\pi, \quad n = 0,\ 1,\ 2,\ \ldots . \tag{194}$$

The physical meaning of $x = \pm n\pi$ is related to the pendulum diagram. if $x = 0$, the pendulum hangs straight down; if $x = \pi$, the pendulum shoots straight up; if $x = 2\pi$, the pendulum has traversed all the way to its original position and hangs straight down; etc. These are the equilibrium positions. However, this does not necessarily imply stability. For instance, if $x = \pi$ and the mass is directly above the support (really any n odd), then this is not a stable position. But, if $x = 0$ and the mass is directly below the support (really any n even), then this is a stable position.
Given a nonlinear system we attempt to convert it to an ALS. For the pendulum, let

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots . \tag{195}$$

Still

$$\dot{x} = y \tag{196}$$

but now, substituting Eq. 195 into Eq. 191,

$$\dot{y} = -\frac{c}{m}y - \frac{g}{L}(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots). \tag{197}$$

The conditions of ALS require some equation in the form of

$$\mathbf{A}\begin{Bmatrix} x \\ y \end{Bmatrix} + \mathbf{g} = \mathbf{A}\mathbf{x} + \mathbf{g} = \dot{\mathbf{x}} = \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix}. \tag{198}$$

In Eq. 197 we can partition the part of the equation dependent only on x and y from the nonlinear part. That is,

$$\dot{x} = \begin{Bmatrix} 0 & 1 \end{Bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix} + 0, \tag{199}$$

$$\dot{y} = -\frac{g}{L}x - \frac{c}{m}y + \frac{g}{L}(\frac{x^3}{3!} - \frac{x^5}{5!} + \ldots) \tag{200}$$

$$\to \dot{y} = \begin{Bmatrix} -\frac{g}{L} & -\frac{c}{m} \end{Bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix} + \frac{g}{L}(\frac{x^3}{3!} - \frac{x^5}{5!} + \ldots). \tag{201}$$

Joined together as a system,

$$\underbrace{\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & -\frac{c}{m} \end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{Bmatrix} x \\ y \end{Bmatrix}}_{\mathbf{x}} + \underbrace{\begin{Bmatrix} 0 \\ \frac{g}{L}(\frac{x^3}{3!} - \frac{x^5}{5!} + \ldots) \end{Bmatrix}}_{\mathbf{g}}. \tag{202}$$

To confirm condition (IV), which is $\frac{||\mathbf{g}||}{||\mathbf{x}||} \to 0$ as $\mathbf{x} \to \mathbf{0}$, we graph y against g_y to see that $y \gg g_y$ near $\mathbf{0}$. Therefore the condition is satisfied.

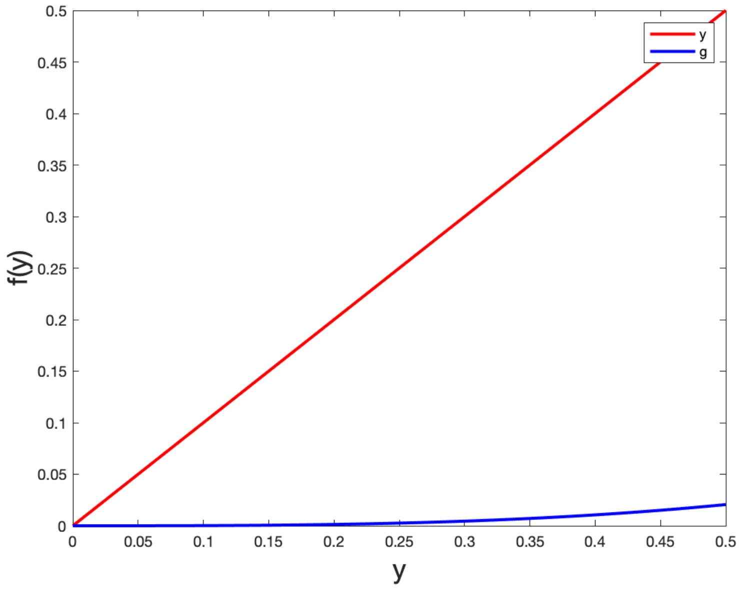

Figure 5: Checking (IV) of ALS: $\frac{||\mathbf{g}||}{||\mathbf{x}||} \to 0$ as $\mathbf{x} \to \mathbf{0}$

```
clear
clc

x = 0:0.01:0.5
g = 0

numiters=10
for iter = 1:numiters
    iter
    g = g + (-1)^(iter -1) * x.^(2*iter+1)/(factorial(2*iter+1))
end

a = plot(x,x)
hold on
b = plot(x,g)
title('')
xlabel('y','FontSize',18)
ylabel('f(y)','FontSize',18)
a.LineWidth = 2
a.Color = 'red'
b.LineWidth = 2
b.Color = 'blue'
legend('y', 'g')
```

Shifting the origin by 2π does not change the behavior of the solution. However, let us shift the origin by pi on x. We let

$$u = x - \pi \leftarrow u + \pi = x, \quad v = y - 0 \leftarrow v + 0 = y. \tag{203}$$

Then

$$\dot{u} = v, \quad \dot{v} = -\frac{c}{m}v - \frac{g}{L}\sin(u+\pi) = -\frac{c}{m}v + \frac{g}{L}\sin(u). \tag{204}$$

This fosters a different behavior. Whereas the origin at $0, 2\pi$ for $\xi < 1$ exhibits a stable spiral, the origin at π for $\xi < 1$ exhibits an unstable saddle. Then the complete phase portrait of the pendulum is a continuous smooth transition between adjacent saddles and spirals.

2.13 Lec 2m Nonlinear ODEs

For many autonomous linear systems a single critical point $\mathbf{x}^* = \mathbf{0}$ is asymptotically stable. That is, the trajectory through any point tends eventually to $\mathbf{x}^*$ as $t \to \infty$. About this we say that the basin of attraction is the entire phase space. The basin of attraction is essentially the set of points which are attracted to the critical point ("the attractor").

For other, more general nonlinear systems though, there can be other attractors besides just the critical points. At the next level of complexity there is not just one solution but there are periodic solutions at every period T. That is,

$$\mathbf{x}(t+T) = \mathbf{x}(t). \tag{205}$$

An example of such a system is the nonlinear second order Van der Pol oscillator

$$\ddot{u} + \mu(u^2 - 1)\dot{u} + u = 0 \tag{206}$$

where μ is some constant. In an analogy to the pendulum, the $\mu(u^2 - 1)$ term can be thought of as a damp proportional to the velocity $\dot{u}$. The u^2 term implies nonlinearity. If in this system $\mu = 0$ then there is no damp and the equation reduces to

$$\ddot{u} + u = 0 \tag{207}$$

where the solution

$$u(t) = c_1 e^{it} + c_2 e^{-it} \tag{208}$$

has strictly complex roots. In this case therefore the solution trajectory is a circle about the origin. That is to say $\mathbf{x} = \mathbf{0}$ is the only critical point.

Now if $\mu >\neq 0$ then there exists nonlinearity. If (I) $\mu > 0$ then there is damping with coefficient $\mu(u^2 - 1)$. If (Ii) $u > 1$, then $(u^2 - 1)$ is positive and so the entire coefficient is positive. A positive damp will diminish the system by way of an inward spiral. However if (Iii) $u < 1$ then the coefficient is negative and the solution trajectory will grow through an outward spiral.

Converting to phase space form, we let

$$\dot{x} = y, \quad \dot{y} = -\mu(x^2 - 1)y - x. \tag{209}$$

Let us examine the linear portion, which is done by approximating the nonlinear portion x^2 to zero, i.e. $x^2 \approx 0$, and let us do so near the critical point. This is

$$\dot{x} = y, \quad \dot{y} = \mu y - x, \tag{210}$$

or

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \mathbf{A}\mathbf{x}. \tag{211}$$

The eigenvalues of $\mathbf{A}$ are $\lambda = \mu \pm \frac{\sqrt{\mu^2-4}}{2}$. Because the sign of the eigenvalues affects the state space trajectory it is then valuable to partition the system behavior into the cases (I) $\mu^2 \geq 4$ and $\mu^2 < 4$.

Another nonlinear example, one more complicated, is the Lorenz model. It is based on the Rayleigh Bérnard problem where a fluid layer is heated from below such that the temperature at the lower layer T_l is greater than that of the upper layer T_u. So $T_l > T_u$, but if these two are approximately close, then there is simple heat transfer from the hotter end to the colder end. However, if there is a large difference such that $T_l \gg T_u$, then there is a stability issue. If there is a critical temperature difference ΔT_{cr} and if

$$T_l - T_u > \Delta T_{cr}, \tag{212}$$

then fluid motion begins due to buoyancy. Heating the bottom fluid, it attempts to rise. The colder fluid on the top begins to sink. This transfer of position occurs cyclically because as the hotter fluid goes up, it is no longer being heated, becomes cooler, and then goes back down.

The system must satisfy mass balance, which is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{213}$$

the balance of momentum, which is the set of Navier-Stokes equations, or Unfinished

3 Mod3 Fourier analysis and integral transforms

3.1 Lec 3a Fourier series

In Sec 2 we explored power series and Frobenius series. Now we look at trigonometric series to represent periodic functions. A periodic function is defined by

$$f(t+T) = f(t+T+T) = \ldots = f(t+nT) = f(t) \tag{214}$$

with period T for all t. The smallest period is the fundamental period. In this example that is T. That is to say $2T$ is not the fundamental period. Periodic functions include

$$f(t) = C \tag{215}$$

where C is a constant, and where f has no fundamental period;

$$g(t) = \cos\omega t = \cos(\omega t + 2\pi) = \cos(\omega t + 4\pi) = \ldots = \cos(\omega t + 2n\pi) \tag{216}$$

where $n = 1, 2, \ldots$ and where g has fundamental period

$$T = \frac{2\pi}{\omega} \tag{217}$$

as

$$g = \cos\Big(\omega t\Big) = \cos\Big(\omega \underbrace{(t + \frac{2\pi}{\omega})}_{t+T}\Big) = \cos(\omega t + 2\pi). \tag{218}$$

These first two examples are continuous functions, but even functions with discontinuous slopes or completely discontinuous functions can be periodic still. However it is easier to work with smooth functions.

Suppose the function $f(t)$ has period 2π. Let us write this particular periodic function f as a sum of periodic functions, such that

$$f(t) = \frac{a_0}{2} + \sum_n (a_n \cos nt + b_n \sin nt) = \frac{a_0}{2} + a_n \cos t + b_n \sin t + a_n \cos 2t + b_n \sin 2t + \ldots). \tag{219}$$

The fundamental period of each term on the right hand side of Eq. 219 is

$$T = \frac{2\pi}{n} \tag{220}$$

because

$$a_n \cos nt = a_n \cos n(t + \frac{2\pi}{n}) = a_n \cos(nt + 2\pi) \tag{221}$$

and the same is true for $b_n \sin\omega t$. The goal is to find the coefficients $a_n = a_1, a_2, \ldots$, $b_n = b_1, b_2, \ldots$. First of all, integrating Eq. 219 through a single period with bounds $-\pi, \pi$,

$$\int_{-\pi}^{\pi} f(t)dt = \int_{-\pi}^{\pi} \frac{a_0}{2} dt + \int_{-\pi}^{\pi} \Big(\sum_n (a_n \cos nt + b_n \sin nt)\Big) dt. \tag{222}$$

Now, assuming the series converges uniformly, the summation and integration in Eq. 222 can be interchanged so that

$$
\begin{aligned}
\int_{-\pi}^{\pi} f(t)dt &= \int_{-\pi}^{\pi} \frac{a_0}{2}dt + \sum_n \int_{-\pi}^{\pi} (a_n \cos nt + b_n \sin nt)dt \\
&= \frac{a_0}{2}(\pi + \pi) + \sum_n \int_{-\pi}^{\pi} (a_n \cos nt + b_n \sin nt)dt \\
&= a_0\pi + \sum_n (a_n \frac{1}{n} \sin nt\Big|_{-\pi}^{\pi} - b_n \frac{1}{n} \cos nt\Big|_{-\pi}^{\pi}) \\
&= a_0\pi = \int_{-\pi}^{\pi} f(t)dt \Longrightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt. \qquad (223)
\end{aligned}
$$

Here because of the nature of the trigonometric functions, there exists just as much in the positive regime as in the negative regime, so the entire second term cancels.

a_0 is the first term in the series. To obtain the remaining terms, multiply Eq. 222 by $\cos mt$ so that

$$
\int_{-\pi}^{\pi} f(t)\cos mtdt = \int_{-\pi}^{\pi} \frac{a_0}{2}\cos mtdt + \int_{-\pi}^{\pi} \left(\sum_n (a_n \cos nt + b_n \sin nt)\right)\cos mtdt. \qquad (224)
$$

Again assuming uniform convergence,

$$
\int_{-\pi}^{\pi} f(t)\cos mtdt = \int_{-\pi}^{\pi} \frac{a_0}{2}\cos mtdt + \sum_n \int_{-\pi}^{\pi} (a_n \cos nt \cos mt + b_n \sin nt \cos mt)dt \qquad (225)
$$

Trigonometric identities state

$$
\begin{aligned}
\cos nt \cos mt &= \frac{1}{2}(\cos(n+m)t + \cos(n-m)t) \\
\sin nt \cos mt &= \frac{1}{2}(\sin(n+m)t + \sin(n-m)t). \qquad (226)
\end{aligned}
$$

Substituting Eq. 226 into the right hand side of Eq. 225,

$$
\begin{aligned}
\int_{-\pi}^{\pi} f(t)\cos mtdt &= \int_{-\pi}^{\pi} \frac{a_0}{2}\cos mtdt \\
&+ \sum_n \Bigg(\underbrace{\int_{-\pi}^{\pi} \frac{a_n}{2}\cos((n+m)t)dt}_{\textbf{I.}} + \underbrace{\int_{-\pi}^{\pi} \frac{a_n}{2}\cos((n-m)t)dt}_{\textbf{II.}} \\
&+ \underbrace{\int_{-\pi}^{\pi} \frac{b_n}{2}\sin((n+m)t)dt}_{\textbf{III.}} + \underbrace{\int_{-\pi}^{\pi} \frac{b_n}{2}\sin((n-m)t)dt}_{\textbf{IV.}} \Bigg). \qquad (227)
\end{aligned}
$$

Considering Eq. 227: if $n \neq m$ so that $n+m=\xi, n-m=\eta$,

$$\int_{-\pi}^{\pi} f(t)\cos mt dt = \int_{-\pi}^{\pi} \frac{a_0}{2}\cos mt dt$$

$$+\sum_n \Bigg(\underbrace{\int_{-\pi}^{\pi} \frac{a_n}{2}\cos(\xi t)dt}_{\textbf{I.}} + \underbrace{\int_{-\pi}^{\pi} \frac{a_n}{2}\cos(\eta t)dt}_{\textbf{II.}}$$

$$+\underbrace{\int_{-\pi}^{\pi} \frac{b_n}{2}\sin(\xi t)dt}_{\textbf{III.}} + \underbrace{\int_{-\pi}^{\pi} \frac{b_n}{2}\sin(\eta t)dt}_{\textbf{IV.}} \Bigg) = 0+0+0+0+0 \tag{228}$$

again because the curves oscillate evenly between the positive and negative number space. Now, if $n=m$,

$$\int_{-\pi}^{\pi} f(t)\cos mt dt = \int_{-\pi}^{\pi} \frac{a_0}{2}\cos mt dt$$

$$+\sum_m \Bigg(\underbrace{\int_{-\pi}^{\pi} \frac{a_m}{2}\cos(2mt)dt}_{\textbf{I.}} + \underbrace{\int_{-\pi}^{\pi} \frac{a_m}{2}\cos(0t)dt}_{\textbf{II.}}$$

$$+\underbrace{\int_{-\pi}^{\pi} \frac{b_m}{2}\sin(2mt)dt}_{\textbf{III.}} + \underbrace{\int_{-\pi}^{\pi} \frac{b_m}{2}\sin(0t)dt}_{\textbf{IV.}} \Bigg) = 0+0+a_m\pi+0+0 \tag{229}$$

since in **II**, $\cos 0 = 1$, so $\int_{-\pi}^{\pi} dt = 2\pi$. Therefore,

$$\int_{-\pi}^{\pi} f(t)\cos mt dt = a_m\pi$$

$$\implies a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos mt dt. \tag{230}$$

To find b_n, the same procedure that developed Eq. 224 could be done for $\sin mt$. That is, multiply 222 by $\sin mt$. Then,

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin mt dt. \tag{231}$$

Eqs. 230 and 231 are so called Euler formulas. These are all the coefficients in Eq. 219 ($f(t) = \frac{a_0}{2} + \sum_n (a_n \cos nt + b_n \sin nt)$), which is called the Fourier series corresponding to f. a_m, b_m are called the Fourier coefficients of f.

Now, recall that f has period 2π. However, we want to generalize the Fourier series for any T. So, instead of integrating from $-\pi$ to π and multiplying by

$$\cos mt = \cos m\frac{\pi}{\pi}t,\ \sin mt = \sin m\frac{\pi}{\pi}t,$$

we integrate from $-p$ to p and multiply by

$$\cos m\frac{\pi}{p}t,\ \sin m\frac{\pi}{p}t.$$

Then, The Fourier series representation of any f with period $2p$ is

$$f(t) = f(t + nT) = \frac{a_0}{2} + \sum_n (a_n \cos\frac{n\pi t}{p} + b_n \sin\frac{n\pi t}{p}), \tag{232}$$

where

$$a_n = \frac{1}{p}\int_{-p}^{p} f(t)\cos\frac{n\pi t}{p} dt, \tag{233}$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(t)\sin\frac{n\pi t}{p} dt, \tag{234}$$

$$T = 2p. \tag{235}$$

3.2 Lec 3b Orthogonality

An important part of the proof in Lec 3.1 was that trigonometric functions sin and cos were symmetric over the period $-\pi$ to π. This is called the property of orthogonality. Formally, in continuous form, if a set of functions $\phi_i = \phi_i(x)$ has the property

$$\int_a^b \phi_m \phi_n dx \quad \begin{cases} = 0, & m \neq n \\ \neq 0, & m = n, \end{cases} \tag{236}$$

then we say these functions form an orthogonal set. Moreover, if

$$\int_a^b \phi_m^2 dx = 1, \tag{237}$$

then this set is orthonormal. Then we can generalize and say

$$\int_a^b \phi_m \phi_n dx = \delta_{mn}. \tag{238}$$

It is easy to convert any orthogonal set into an orthonormal one by normalizing each ϕ_i.

In discrete form, an orthogonal set

$$\boldsymbol{\phi}_m^T \boldsymbol{\phi}_n \quad \begin{cases} = 0, & m \neq n \\ \neq 0, & m = n. \end{cases} \tag{239}$$

An orthonormal set

$$\boldsymbol{\phi}_m^T \boldsymbol{\phi}_n = 1. \tag{240}$$

This means

$$\boldsymbol{\phi}_m \cdot \boldsymbol{\phi}_n = \delta_{mn}. \tag{241}$$

Take for example the Cartesian coordinate system $\mathbf{e}_i$, which is orthonormal. We know

$$\mathbf{e}_1^T \mathbf{e}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 + 0 + 0 = 1; \quad \mathbf{e}_3^T \mathbf{e}_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0. \tag{242}$$

A set of vectors is orthogonal with respect to the matrix $\mathbf{A}$ if

$$\boldsymbol{\phi}_m^T \mathbf{A} \boldsymbol{\phi}_n \quad \begin{cases} = 0, & m \neq n \\ \neq 0, & m = n. \end{cases} \tag{243}$$

Now, recall that to find Fourier coefficients a_n or b_n of f with period $T = (-p, p)$, we integrate Eq. 232, which is

$$f(t) = f(t + nT) = \frac{a_0}{2} + \sum_n (a_n \cos \frac{n\pi t}{p} + b_n \sin \frac{n\pi t}{p}),$$

over $(-p, p)$ and multiply by $\cos(m\pi t/p)$ to find a_m or $\sin(m\pi t/p)$ to find b_m. We then discussed that the trigonometric functions are symmetrical (orthogonal) over that period because they oscillate back and forth between the positive and negative regimes. However, we discussed that if $n = m$, then the trigonometric identities Eq. 226 transform one of the $\int_{-p}^{p} \cos(n-m)dt$ terms into $\frac{1}{2}\int_{-p}^{p} 1dt = p$. Altogether,

$$\int_{-p}^{p} \cos\left(\frac{m\pi t}{p}\right) \cos\left(\frac{n\pi t}{p}\right) dt = \begin{cases} 0, & m \neq n \\ p, & m = n. \end{cases} \tag{244}$$

In a completely analogous way, we also find this to be true of $\sin(*)$, in that

$$\int_{-p}^{p} \sin\left(\frac{m\pi t}{p}\right) \sin\left(\frac{n\pi t}{p}\right) dt = \begin{cases} 0, & m \neq n \\ p, & m = n. \end{cases} \tag{245}$$

However, for the product $\sin(*)\cos(*)$,

$$\int_{-p}^{p} \sin\left(\frac{m\pi t}{p}\right) \cos\left(\frac{n\pi t}{p}\right) dt = 0. \tag{246}$$

Again this is because of the trigonometric identity Eq. 226 ($\sin nt \cos mt = \frac{1}{2}(\sin(n+m)t + \sin(n-m)t)$). Then integration is possible. Again, these ideas helped us form the Euler formulas for the coefficients of the Fourier series of f, i.e., Eqs. 232-234.

3.3 Lec 3c Dirichlet conditions

For a function f to satisfy the Dirichlet (DEER-ish-lay) conditions,

- f must be bounded;
- f must be periodic;
- f cannot have infinite local minima and maxima at one period; and
- f cannot be discontinuous at infinite points within one period.

If f satisfies the Dirichlet conditions, then

- The Fourier series of f converges to f wherever f is continuous; and
- Wherever f is discontinuous, its Fourier series converges to the average of its right and left hand limits.

This is all true for Fourier series representation of f with period $T = 2p$ (Eqs. 232-234),

$$f(t) = f(t + nT) = \frac{a_0}{2} + \sum_n (a_n \cos\frac{n\pi t}{p} + b_n \sin\frac{n\pi t}{p}), \tag{247}$$

where

$$a_n = \frac{1}{p}\int_{-p}^{p} f(t)\cos\frac{n\pi t}{p}dt,$$
$$b_n = \frac{1}{p}\int_{-p}^{p} f(t)\sin\frac{n\pi t}{p}dt.$$

As an example, consider a function with period $T = 2\pi$

$$f(x) = \begin{cases} -k, & -\pi < x < 0, \\ k, & 0 < x < \pi. \end{cases} \tag{248}$$

Notice that the function is discontinuous at $x = n\pi$. However, the function is only discontinuous at a finite number of points within a single period. So, the function satisfies the Dirichlet conditions. Of course, this is also a periodic function. To solve, we split f into its two intervals, making

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\frac{n\pi x}{\pi}dx \\ &= \frac{1}{\pi}\left(\int_{-\pi}^{0} -k\cos(nx)dx + \int_{0}^{\pi} k\cos(nx)dx\right) \\ &= \frac{-k}{n\pi}\sin(nx)\Big|_{-\pi}^{0} + \frac{k}{n\pi}\sin(nx)\Big|_{0}^{\pi} = 0 + 0 = 0 \end{aligned} \tag{249}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi}\left(\int_{-\pi}^{0} -k\sin(nx)dx + \int_{0}^{\pi} k\sin(nx)dx\right) \\ &= \frac{k}{n\pi}\cos(nx)\Big|_{-\pi}^{0} - \frac{k}{n\pi}\cos(nx)\Big|_{0}^{\pi} \\ &= \frac{k}{n\pi}(\cos 0 - \underbrace{\cos(-n\pi) - \cos(n\pi)} + \cos(0)) = \frac{2k}{n\pi}(1 - \underbrace{\cos n\pi}_{\cos n\pi = \cos -n\pi}) = b_n. \end{aligned} \tag{250}$$

For b_n, if n is even so that $(1 - \cos 0\pi) = (1 - \cos 2\pi) = \ldots = 0$, then $b_n = 0$. But, if n is odd so that $(1 - \cos\pi) = (1 - \cos 3\pi) = \ldots = 2$, then $b_n = 2(2k/n\pi)$. Altogether,

$$b_n = \begin{cases} 4k/n\pi, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases} \tag{251}$$

Then, from 247, the Fourier series

$$f(x) = \frac{4k}{\pi} \sum_{n \text{ odd}} \frac{\sin nx}{n}. \tag{252}$$

Because of the Dirichlet theorem, we know that at points of discontinuity, the Fourier series converges to the average between the left and right limits, which is zero, as those limits are k and $-k$. This is shown in Fig. 6.

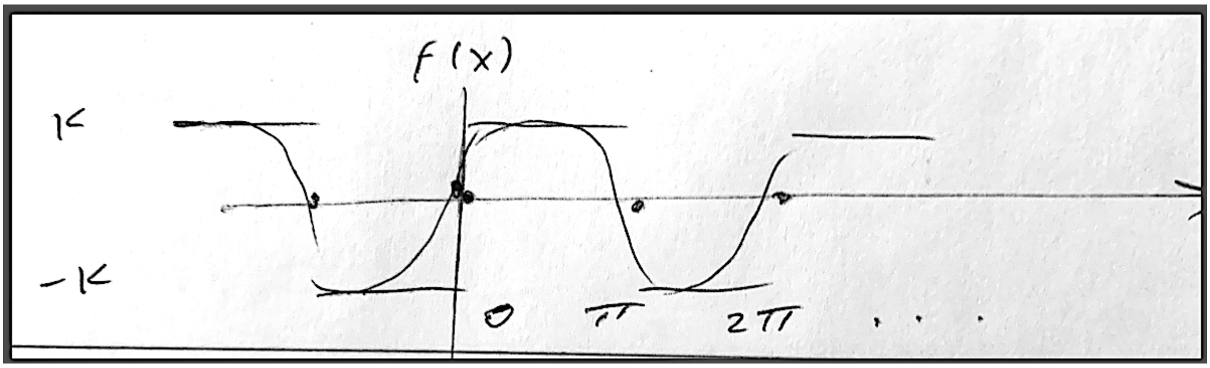

Figure 6: $f(x) = \frac{4k}{\pi} \sum_{n \text{ odd}} \frac{\sin nx}{n}$.

As another example, which currently needs further explanation, consider

$$f(x) = \begin{cases} x, & -2 < x \le 0, \\ x, & 0 < x \le 2, \end{cases} \qquad T = 4. \tag{253}$$

This is continuous everywhere, and is periodic, so, certainly it satisfies the Dirichlet conditions. Then

$$a_n = \frac{1}{2} \int_{-2}^{0} -x \cos \frac{n\pi x}{2} dx + \frac{1}{2} \int_{0}^{2} x \cos \frac{n\pi x}{2} dx. \tag{254}$$

Using integration by parts

$$\int u dv = uv - \int v du; \quad u = x, \quad dv = \cos \frac{n\pi}{2} x dx,$$

we obtain

$$a_n = \frac{4}{n^2 \pi} (\cos n\pi - 1) = \begin{cases} -\frac{8}{n^2 \pi^2}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases} \tag{255}$$

Using the analogous integration by parts technique on b_n, but for $\sin(*)$, we find

$$b_n = 0 \quad \forall \quad n. \tag{256}$$

Then

$$f(x) = \sum_{n \text{ odd}} (-\frac{8}{n^2 \pi^2} \cos \frac{n\pi x}{2}). \tag{257}$$

Notice that in Eq. 257 there are only cos terms. However, in the result derived from the first example Eq. 252, there are only sin terms. This is predictable based on inspection: The function $\cos\omega t$ is itself even, as it is symmetrical along the y axis, and the function $\sin\omega t$ is odd, as it is symmetrical along the line $y = x$. This extrapolates to more complex functions: if f is even, then the cos terms will prevail; if f is odd, then the sin terms will prevail. This is obvious thinking about it geometrically. If you are taking the integral along the interval $(-p, p)$, and the function is even, that is, symmetric along the y axis, then the area under the curve of the left half is exactly that of the right half because they are mirror images. That is,

$$\int_{-p}^{p} g(x)dx = 2\int_{0}^{p} g(x)dx, \quad g \text{ even.} \tag{258}$$

On the other hand, taking the same integral but of an odd function, the area under the curve of the left half will be the exact inverse (negative) of that of the right half, because of the axis along which the two halves are symmetrical. That is,

$$\int_{-p}^{p} h(x)dx = 0, \quad h \text{ odd.} \tag{259}$$

Product rules for even g and odd h are

$$g_1 g_2 \text{ is even;} \;\; h_1 h_2 \text{ is even;} \;\; g_1 h_2 \text{ is odd.} \tag{260}$$

We can apply Eqs. 258- 260 to the Fourier series representation of f, Eqs. 232-234. If f is even,

$$f(t) = \frac{a_0}{2} + \sum_n (a_n \cos\frac{n\pi t}{p} + b_n \sin\frac{n\pi t}{p}),$$

$$a_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(t)}_{\text{even}} \underbrace{\cos\frac{n\pi t}{p}}_{\text{even}} dt = \frac{2}{p}\int_{0}^{p} f(t)\cos\frac{n\pi t}{p}dt,$$

$$b_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(t)}_{\text{even}} \underbrace{\sin\frac{n\pi t}{p}}_{\text{odd}} dt = 0$$

$$\implies f(t) = \frac{a_0}{2} + \sum_n a_n \cos\frac{n\pi t}{p}, \quad a_n = \frac{2}{p}\int_{0}^{p} f(t)\cos\frac{n\pi t}{p}dt, \quad f \text{ even;} \tag{261}$$

if f is odd,

$$a_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(t)}_{\text{odd}} \underbrace{\cos\frac{n\pi t}{p}}_{\text{even}} dt = 0,$$

$$b_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(t)}_{\text{odd}} \underbrace{\sin\frac{n\pi t}{p}}_{\text{odd}} dt = \frac{2}{p}\int_{0}^{p} \underbrace{f(t)}_{\text{odd}} \underbrace{\sin\frac{n\pi t}{p}}_{\text{odd}} dt$$

$$\implies f(t) = \sum_n b_n \sin\frac{n\pi t}{p}, \quad b_n = \frac{2}{p}\int_0^p f(t)\sin\frac{n\pi t}{p}dt, \quad f \text{ odd.} \tag{262}$$

These are called the Fourier cosine and Fourier sine functions.

Now, suppose f is not periodic but is instead defined only over one interval. If this is the case, we can apply either a Fourier sine or Fourier cosine extension to this function (depending on if it is even or odd; if neither, then either extension works). Here we only evaluate the interval in which the function exists. These extensions are called half range expansions.

3.4 Lec 3d Fourier integrals

The standard form of the Fourier series of f with period $T = 2p$, once again (Eqs. 232-234),

$$f(t) = f(t+nT) = \frac{a_0}{2} + \sum_n (a_n \cos\frac{n\pi t}{p} + b_n \sin\frac{n\pi t}{p}),$$

where

$$a_n = \frac{1}{p}\int_{-p}^{p} f(t)\cos\frac{n\pi t}{p}dt,$$
$$b_n = \frac{1}{p}\int_{-p}^{p} f(t)\sin\frac{n\pi t}{p}dt.$$

Let us convert the standard form of this series into a complex exponential form. Specifically, let

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}. \tag{263}$$

Applying Eq. 263 to Eq. 232, with $\theta = 2\pi t/p$,

$$f(t) = c_n e^{in\pi t/p} \tag{264}$$

where

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(t)e^{-in\pi t/p}dt = \frac{1}{2p}\int_{-p}^{p} f(t)e^{-i\omega_n t}dt. \tag{265}$$

The correspondence between the real and complex coefficients are

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2}. \tag{266}$$

Here n in c_n is an index that travels negatively as well as positively. It is not practical but it is a good way to generalize the Fourier series. The derivation is in Greenberg.

Eq. 265 is a representation of f in the time domain. On the other hand, Eq. 266 can be made into a plot of $\mathrm{Re}(c_n)$ vs. ω_n, where

$$\omega_n = \frac{n\pi}{p}. \tag{267}$$

Then c_n represents the spectrum of f. Explanation needs work

Now, suppose f is not periodic. Then we do not have a frequency domain representation that involves only discrete frequencies. This is because we cannot obtain a Fourier series representation in the first place. Therefore, we must generalize this function f and allow the frequency representation to assume continuous values. Generalizing Eqs. 264 ($f(t) = c_n e^{in\pi t/p}$) and Eq. 265 ($c_n = \frac{1}{2p}\int_{-p}^{p} f(t)e^{-i\omega_n t}$), with frequency $\omega = n\pi/p$, we let

$$f(t) = \int_{-\infty}^{\infty} \mathcal{C}(\omega)e^{i\omega t}d\omega, \tag{268}$$

where complex coefficients

$$\mathcal{C}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}dt. \tag{269}$$

This is called the Fourier integral representation. The Fourier integral is basically a natural extension of the idea of a complex esponential Fourier series into functions f that are not periodic. In physical problems, time t corresponds to circular frequency ω. On the other hand, space x corresponds to wave number k. Basically, the analog of t is x and the analog of ω is k.

In general, if the Dirichlet conditions are satisfied and if the integral $\int_{-\infty}^{\infty} f(t)dt$ exists in the first place, then the Fourier integral that is Eq. 268 with $\mathcal{C}$ given in Eq. 269 actually gives the value of f wherever the function is continuous. Also, the Fourier integral converges to the average of the left and right hand limits of f where the function is discontinuous. And this is completly analoguos to the Dirichlet theorem itself In Lec 3.3.

Now, many functions satisfy the Dirichlet conditions. However, the condition that $\int_{-\infty}^{\infty} f(t)dt$ must exist is more exclusive. Basically, for this to be true, the function must decay as $t \to \pm\infty$. If the function does not decay sufficiently fast, then the integral is not bounded, and so the area under the curve does not exist, and so a Fourier representation is not possible.

However, while these conditions are sufficient, they are not necessary. There do exist some functions that possess a Fourier representation despite not being periodic. For example, consider

$$f(t) = \begin{cases} 0, & t < 0 \\ e^{-\alpha t}, & t > 0 \end{cases} \quad \text{for} a > 0. \tag{270}$$

The Dirichlet conditions are satisfied. However, this function is not periodic. Still, using Eq. 269, we can write the Fourier integral representation of f

$$\begin{aligned}\mathcal{C}(\omega) = \frac{1}{2\pi}\int_0^{\infty} e^{-\alpha t}e^{-i\omega t}dt = \frac{1}{2\pi}\int_0^{\infty} e^{-(\alpha+i\omega)t}dt \\ = \frac{1}{-2\pi(a+i\omega)}\lim_{T\to\infty}[e^{-(\alpha+i\omega)t}]\Big|_0^T \\ = \frac{1}{-2\pi(a+i\omega)}(\lim_{T\to\infty}[e^{-(\alpha+i\omega)T}] - \lim_{T\to\infty}[e^0])\end{aligned}$$

$$= \frac{1}{2\pi(a+i\omega)}(1 - \lim_{T\to\infty}[e^{-\alpha t}e^{-i\omega T}]) = \frac{1}{2\pi(a+i\omega)} = \mathcal{C}(\omega). \tag{271}$$

With $\mathcal{C}$, we can now use Eq. 268 to obtain

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2\pi(a+i\omega)} d\omega. \tag{272}$$

Lastly, recall the Fourier-cosine and Fourier-sine integral representations in the real form Eqs. 261 and 262,

$$f(t) = \sum_n b_n \sin\frac{n\pi t}{p}, \quad b_n = \frac{2}{p}\int_0^p f(t)\sin\frac{n\pi t}{p}dt, \quad f \text{ odd.}$$

$$f(t) = \frac{a_0}{2} + \sum_n a_n \cos\frac{n\pi t}{p}, \quad a_n = \frac{2}{p}\int_0^p f(t)\cos\frac{n\pi t}{p}dt, \quad f \text{ even.}$$

The representation is completely analogous in the frequency domain. If $\omega = n\pi/p$ and

$$f(t) = \int_0^{\infty} [\mathcal{A}(\omega)\cos\omega t + \mathcal{B}(\omega)\sin\omega t]d\omega, \tag{273}$$

where

$$\mathcal{A}(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t dt, \quad \mathcal{B}(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t dt, \tag{274}$$

then the corresponding Fourier cosine integral for f even is

$$f(t) = \int_0^{\infty} \mathcal{A}(\omega)\cos\omega t d\omega, \quad \mathcal{A}(\omega) = \frac{2}{\pi}\int_0^{\infty} f(t)\cos\omega t dt; \tag{275}$$

the corresponding Fourier sine integral for f odd is

$$f(t) = \int_0^{\infty} \mathcal{B}(\omega)\sin\omega t d\omega, \quad \mathcal{B}(\omega) = \frac{2}{\pi}\int_0^{\infty} f(t)\sin\omega t dt. \tag{276}$$

3.5 Lec 3e Fourier transforms

The purpose of Fourier analysis in the complex representation (Eq. 264-265) is to judge f in the domain of frequency and in the domain of time. We have essentially imposed transformations between these two domains, so that

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega = \mathcal{F}^{-1}(F(\omega)), \tag{277}$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \mathcal{F}(f(t)). \tag{278}$$

Script $\mathcal{F}$ is called the Fourier transform. The Fourier transform of the time domain takes you to the frequency domain. The inverse Fourier transform takes you from the frequency domain back to the time domain.

Some useful examples are

$$f(t) = \begin{cases} 0, & t < 0 \\ e^{-\alpha t}, & t > 0 \end{cases} \Longrightarrow F(\omega) = \mathcal{F}(f(t)) = \frac{1}{\alpha + i\omega}, \tag{279}$$

$$f(t) = \begin{cases} e^{\alpha t}, & t \leq 0 \\ e^{-\alpha t}, & t > 0 \end{cases} \Longrightarrow F(\omega) = \mathcal{F}(f(t)) = \frac{2\alpha}{\alpha + i\omega}. \tag{280}$$

These are from tables in Erdelyi (1954) and Greenberg (1998).

Besides for tabulating various F, $\mathcal{F}$ has useful mathematical properties. Assuming

$$F(\omega) = \mathcal{F}(f(t)), \quad f(t) = \mathcal{F}^{-1}(F(\omega)), \tag{281}$$

linearity holds, which means

$$\mathcal{F}(a_1 f_1 + a_2 f_2) = a_1 \mathcal{F}(f_1(t)) + a_2 \mathcal{F}(f_2(t)). \tag{282}$$

Symmetry holds, which means

$$\mathcal{F}(F(t)) = -2\pi f(\omega). \tag{283}$$

Here, while normally $\mathcal{F}$ transforms $f(t)$ into $F = F(\omega)$, it is also possible for a function to be given in terms of its frequency ω such that $f = f(\omega)$ and then to impose on it $\mathcal{F}$ to obtain $F(t)$. Eq. 283 describes that reverse relationship. As an extension of linearity, time scaling holds:

$$\mathcal{F}(F(\alpha t)) = \frac{1}{\alpha} F(\frac{\omega}{\alpha}); \tag{284}$$

time shifting holds:

$$\mathcal{F}(f(t - t_0)) = e^{-i\omega t} F(\omega); \tag{285}$$

and frequency shifting holds:

$$\mathcal{F}^{-1}(F(\omega - \omega_0)) = e^{i\omega_0 t} f(t). \tag{286}$$

Next, suppose we wish to impose Fourier transform $\mathcal{F}$ on f. This is possible if

- f is continuous everywhere;
- f' is piece wise continuous everywhere; and
- $\int_{-\infty}^{\infty} f(t)dt, \int_{-\infty}^{\infty} f'(t)dt$ exist.

Then the Fourier transform of the time derivative (Eq. 278)

$$\mathcal{F}(f'(t)) = \int_{-\infty}^{\infty} f'(t) e^{-i\omega t} dt; \tag{287}$$

integrating by parts with $u = e^{-i\omega t}, dv = f'(t)dt, \int u dv = uv - \int v du$, we obtain

$$\mathcal{F}(f'(t)) = i\omega F(\omega). \tag{288}$$

If this is repeated for higher order derivatives,

$$\mathcal{F}(f^{(n)}(t)) = (i\omega)^n F(\omega). \tag{289}$$

This rule is significant becuse it allows us to convert differential equations with constant coefficients in the time domain to algebraic equations in the frequency domain. The reverse operation from the frequency domain to the time domain is

$$\mathcal{F}^{-1}(F'(\omega)) = -itf(t), \tag{290}$$

$$\mathcal{F}^{-1}(F^{(n)}(\omega)) = -(it)^n f(t). \tag{291}$$

From here we introduce the convolution operation. Given functions $f(t)$ and $g(t)$, then

$$\text{convolution}(f, g) = (f * g)(t) := \int_a^{t-a} f(\tau)g(t-\tau)d\tau. \tag{292}$$

In the special case of $a = 0$, which is called unilateral convolution,

$$(f * g)(t) := \int_0^t \overbrace{f(\tau)}^{\boldsymbol{\Psi(\tau)}} \quad \underbrace{g(t-\tau)}_{\text{fundamental solution}} \quad d\tau. \tag{293}$$

For $a = -\infty$, which is called bilateral convolution,

$$(f * g)(t) := \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau. \tag{294}$$

Time convolution is related to the product of Fourier transforms. Given $F(\omega)$ and $G(\omega)$, the product

$$F(\omega)G(\omega) = \mathcal{F}(\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau) \tag{295}$$

which is the exact form of a bilateral convolution in Eq. 294

3.6 Lec 3f Generalized functions

3.7 Lec 3g Laplace transforms

3.8 Lec 3h Integral transform summary

3.9 Lec 3i Boundary value problems

4 Mod4 PDEs