DETERMINING ZEROS OF QUADRATIC FUNCTIONS USING THE QUADRATIC FORMULA.

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Abstract. This article provides a derivation of the quadratic formula and demonstrates how it may be utilized to obtain solutions to general quadratic equations in standard form.

1. Introduction

Quadratic equations are second degree polynomial equations whose solutions are the zeros of a corresponding quadratic function. Quadratic equations have many applications in fields such as business, statistics, and natural sciences as noted by [4, 1]. In general, determining solutions to quadratic equations can be quite difficult. The goal of this paper is to provide a concise derivation of the quadratic formula. The quadratic formula is a powerful tool used to determine the solutions of quadratic equations.

The structure of this paper is as follows. In Section 2 a formal definition of quadratic functions is provided, as well as some useful facts for determining their solutions. Section 3 describes a method for solving simple forms of quadratic equations that are factorable in terms of integers. We conclude with a step by step derivation of the quadratic formula and demonstrate that the result holds for all general quadratic equations in standard form in Section 4.

2. Preliminaries

If $f : \mathbb{R} \to \mathbb{R}$ defined via $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$, we say f is a polynomial function of degree n. The zeros, or roots, of f are the values of x such that $f(x) = 0$. This relates graphically to the x-intercepts of f, (where the function intersects the x-axis).

2.1. **Quadratic Functions.** We will focus our attention on polynomial functions of degree 2.

Definition 2.1. A *quadratic function* is a function $f : \mathbb{R} \to \mathbb{R}$ defined via

$$f(x) = ax^2 + bx + c,$$

where $a, b, c \in \mathbb{R}$ and $a \neq 0$. Here f is said to be in standard form.

2.2. **Some Useful Properties.** We recall some algebraic facts from [4].

Proposition 2.2. *Zero Product Property.* Assume $n, m \in \mathbb{R}$. If $nm = 0$, then either $n = 0$ or $m = 0$ or both.

Proposition 2.3. *Square Root Property.* Let $x, c \in \mathbb{R}$ such that $c \geq 0$. If $x^2 = c$, then $x = \pm\sqrt{c}$.

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3. Solving Quadratic Equations by factoring

Here we will provide a method for solving quadratic equations that are factorable over the integers. Following [3], we can can decompose the quadratic equation $ax^2 + bx + c = 0$ into a product of linear binomial factors and rewrite the equation as,

$$(x + n)(x + m) = 0,$$

for some $n, m \in \mathbb{Z}$. By Proposition 2.2, we know that this holds if the factors $x + n$ or $x + m$ are equal to 0. Hence,

$$x + n = 0 \text{ or } x + m = 0,$$

implies

$$x = -n \text{ or } x = -m.$$

Example 3.1. Consider the following quadratic function,

$$f(x) = x^2 + 3x - 4.$$

In order to find the roots of f, values of x must be determined such that $f(x) = 0$. Therefore, the resulting quadratic equation is,

$$x^2 + 3x - 4 = 0.$$

Our goal is to rewrite the polynomial as a product of linear factors. Observe that,

$$\begin{aligned}(x - 1)(x + 4) &= x^2 + 4x - x - 4\\ &= x^2 + 3x - 4.\end{aligned}$$

Therefore, the quadratic equation from Example 3.1 can be written as,

$$(x - 1)(x + 4) = 0.$$

Which implies,

$$x = -4 \text{ or } x = 1.$$

Plugging these values into the original function yields,

$$\begin{aligned}f(-4) &= (-4)^2 + 3(-4) - 4\\ &= 16 - 12 - 4\\ &= 16 - 16\\ &= 0,\end{aligned}$$

and,

$$\begin{aligned}f(1) &= (1)^2 + 3(1) - 4\\ &= 1 + 3 - 4\\ &= 4 - 4\\ &= 0.\end{aligned}$$

Thus, $x = -4$ and $x = 1$ are the solutions to the above quadratic equation. The graph of $f = x^2 + 3x - 4$ is given by Figure 3.1. Observe that f intersects the x-axis at $x = -4$ and $x = 1$.

However, many quadratic equations cannot be factored over the integers. Take for example,

$$f(x) = x^2 + 10x + 18.$$

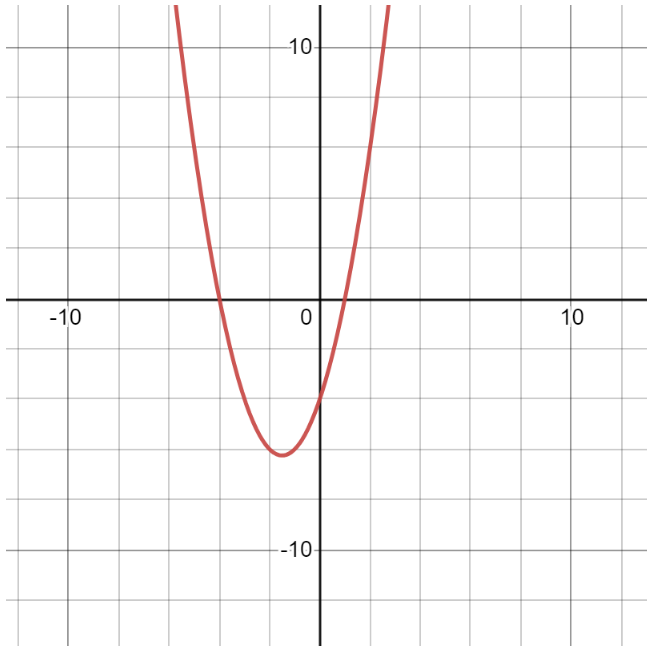

FIGURE 3.1. Graph of $f(x) = x^2 + 3x - 4$. Note that f has zeros at $x = 1$ and $x = -4$ where the graph intersects the x-axis.

Since f does not have a nice factorization in terms of integers, we cannot solve this equations using the above method. We must therefore rely on a more robust method to tackle such an equation.

4. Derivation of The Quadratic Formula

4.1. **Derivation.** In this section we will derive a result known as the *quadratic formula*. Consider a general quadratic function in standard form as given by Definition 2.1,

$$f(x) = ax^2 + bx + c.$$

In order to determine the zeros of this function we set $f(x) = 0$ and obtain the following quadratic equation,

$$ax^2 + bx + c = 0.$$

Our goal is to create a perfect square binomial on the left hand side of the equation. First, isolate the variable terms by moving c to the right hand side of the equation, then divide through by a to achieve a leading coefficient of 1. which yields,

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Next, consider the desired form for the left hand side of our equation,

$$\begin{aligned}(x+k)^2 &= (x+k)(x+k)\\ &= x^2 + kx + kx + k^2\\ &= x^2 + 2kx + k^2.\end{aligned}$$

This implies,

$$k = \frac{b}{2a}.$$

Therefore, adding $k^2 = \frac{b^2}{4a^2}$ to both sides of the equations yields,

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}.$$

By construction, we now have a perfect square binomial in expanded form on the left hand side of the equation, which can now be written in factored form as,

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Finally, using Proposition 2.3, we can take the square root of both sides of the equation and solve for x to obtain the quadratic formula,

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}.$$

For an alternative derivation of the quadratic formula see [2]. We must now show that this result holds for all quadratic equations in standard form

4.2. **Verifying The Result.** Here we will demonstrate that the quadratic formula provides solutions to any quadratic equation in standard from. Assume $f : \mathbb{R} \to \mathbb{R}$ defined via,

$$f(x) = ax^2 + bx + c \text{ where } a, b, c \in \mathbb{R} \text{ and } a \neq 0.$$

In order the show that the quadratic formula holds for all quadratic equations of this form, the function value must be equal to 0 when evaluated at the above values of x. Notice that,

$$\begin{aligned}
f\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) &= a\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)^2 + b\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) + c \\
&= \frac{ab^2 - 2ab\sqrt{b^2 - 4ac} + ab^2 - 4a^2c}{4a^2} + \frac{-b^2 + b\sqrt{b^2 - 4ac}}{2a} + c \\
&= \frac{ab^2 - 2ab\sqrt{b^2 - 4ac} + ab^2 - 4a^2c}{4a^2} + \frac{-2ab^2 + 2a\sqrt{b^2 - 4ac}}{4a^2} + \frac{4a^2c}{4a^2} \\
&= \frac{2ab^2 - 2ab\sqrt{b^2 - 4ac} - 2ab^2 - 4a^2c + 2ab\sqrt{b^2 - 4ac} + 4a^2c}{4a^2} \\
&= 0.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
f\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right) &= a\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)^2 + b\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right) + c \\
&= \frac{ab^2+2ab\sqrt{b^2-4ac}+ab^2-4a^2c}{4a^2} + \frac{-b^2-b\sqrt{b^2-4ac}}{2a} + c \\
&= \frac{ab^2+2ab\sqrt{b^2-4ac}+ab^2-4a^2c}{4a^2} + \frac{-2ab^2-2ab\sqrt{b^2-4ac}}{4a^2} + \frac{4a^2c}{4a^2} \\
&= \frac{2ab^2+2ab\sqrt{b^2-4ac}-4a^2c-2ab^2-2ab\sqrt{b^2-4ac}+4a^2c}{4a^2} \\
&= 0.
\end{aligned}
$$

Therefore, the roots of any quadratic function of the form $f(x) = ax^2 + bx + c$ can be obtained using the quadratic formula.

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