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Thermal stresses in a rotating non-homogeneous orthotropic hollow cylinder

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Abstract In this paper the radial deformation and the corresponding stresses in a non-homogeneous hollow elastic cylinder rotating about its axis with a constant angular velocity is investigated. The material of the cylinder is assumed to the non-homogeneous and cylindrically orthotropic. The system of fundamental equations is solved by means of a finite difference method and the numerical calculations are carried out for the temperature, the components of displacement and the components of stress with the time t and through the thickness of the cylinder. The results indicate that the effect of inhomogeneity is very pronounced.

1 Introduction

During the past two decades, wide spread attention has been given to thermoelasticity admit a finite speed for the propagation of thermal signals. In contrast to the conventional theories based on parabolic-type heat equation and are referred to as generalized theories. The problem of rotating disks or cylinders has its application in high-speed cameras, steam and gas turbines, planetary landings and in many other domains. Various authors have formulated these generalized theories on different grounds. Lord and Shulman [1] have developed a theory on the basis of a modified heat conduction law which involves heat-flux rate, and Green and Lindsay [2] have developed a theory by including temperature-rate among the constitutive variables. Lebon [3] has formulated a theory by considering heat-flux as an independent variable. The solutions to the problems of homogeneous isotropic rotating cylinders may be found in Love [4] and Sokolnikoff [5]. The plane thermal stress problem of a thin circular disk of orthotropic material was reported by Parida and Das [6]. Thermal effects on the microstructure level were analysed by Avery and Herakovich [7], by considering an orthotropic fiber in an isotropic matrix under a uniform temperature change. Transient thermal stresses in cylindrically orthotropic tubes were studied by Kardomateas [8].

On the other hand, Mukhopadhyay [9], solved a problem on the effects of non-homogeneity on the stresses in a rotating non-homogeneous aeolotropic cylindrical shell. The rotation of non-homogeneous composite infinite cylinder were investigated by El-Naggar et al. [10], by considering an orthotropic cylinder containing (i) isotropic core and (ii) rigid core. Ahmed and Khan [11] studied thermoelastic plane waves in a rotating isotropic medium.

The present paper deals with the problem of thermal stresses in a hollow elastic cylinder rotating with a constant angular velocity about its normal axis using plane strain assumption. The governing equations for the non-homogeneous circular cylinder in an orthotropic elastic solid are obtained in conservation form. These equations are solved using a numerical method which uses relation from the characteristic theory of finite difference scheme. This scheme is easier to implement than the method of characteristic discussed by Haddow and Mioduchowski [12, 13]. Numerical results are presented for the variation of temperature, displacement and stresses with the time t and through the thickness. The effects of non-homogeneity and the orthotropy on the thermal stress and displacement are investigated.

2 Formulation of the problem

Let us consider a non-homogeneous orthotropic hollow cylinder of inner and outer radii a and b respectively made of cylindrically orthotropic material.

In cylindrical coordinates the stress–strain–temperature relations are given by

$$\sigma_{rr} = c_{11}e_{rr} + c_{12}e_{\theta\theta} + c_{13}e_{zz} - \beta_1 T \tag{2.1a}$$

$$\sigma_{\theta\theta} = c_{12}e_{rr} + c_{22}e_{\theta\theta} + c_{23}e_{zz} - \beta_2 T \tag{2.1b}$$

$$\sigma_{zz} = c_{13}e_{rr} + c_{23}e_{\theta\theta} + c_{33}e_{zz} - \beta_3 T \tag{2.1c}$$

$$\sigma_{\theta z} = c_{44}e_{\theta z}, \quad \sigma_{rz} = c_{55}e_{rz} \tag{2.1d}$$

$$\sigma_{r\theta} = c_{66}e_{r\theta} \tag{2.1e}$$

where

$$\beta_1 = c_{11}\alpha_r + c_{12}\alpha_\theta + c_{13}\alpha_z$$

$$\beta_2 = c_{12}\alpha_r + c_{22}\alpha_\theta + c_{23}\alpha_z$$

$$\beta_3 = c_{13}\alpha_r + c_{23}\alpha_\theta + c_{33}\alpha_z$$

where α_i are the thermal expansion coefficients and c_{ij} are the elastic constants.

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The strain components in terms of the displacements u_r, u_θ and u_z are

$$e_{rr} = \frac{\partial u}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right), \quad e_{zz} = \frac{\partial u_z}{\partial z} \tag{2.2a}$$

$$e_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right), \quad e_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \tag{2.2b}$$

$$e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right) \tag{2.2c}$$

Using the preceding assumption which states that, radial displacement is a function of r only, we have conclude that all partial derivatives with respect to θ or z are zero, then Eqs. (2.2) yield

$$e_{rr} = \frac{\partial u}{\partial r}, \quad e_{\theta\theta} = \frac{u}{r} \tag{2.3a}$$

$$e_{zz} = e_{rz} = e_{\theta z} = e_{r\theta} = 0 \tag{2.3b}$$

Substituting from (2.3) in (2.1), we get

$$\begin{aligned} \sigma_{rr} &= c_{11}\frac{\partial u}{\partial r} + c_{12}\frac{u}{r} - \beta_1 T' \\ \sigma_{\theta\theta} &= c_{12}\frac{\partial u}{\partial r} + c_{22}\frac{u}{r} - \beta_2 T' \\ \sigma_{zz} &= c_{13}\frac{\partial u}{\partial r} + c_{23}\frac{u}{r} - \beta_3 T' \\ \sigma_{rz} &= \sigma_{\theta z} = \sigma_{r\theta} = 0 \end{aligned} \tag{2.4}$$

The equation of motion, and taking the rotation term about the z-axis as a body force is

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) - \rho\Omega^2 r = \rho\frac{\partial^2 u}{\partial t'^2} \tag{2.5}$$

where Ω is the uniform angular velocity and ρ is the density of the cylinder material, we assume that the non-homogeneities of the elastic constants c_{ij} are of the form

$$c_{ij} = \beta_{ij} r^{2m}, \quad \rho = \rho_0 r^{2m} \tag{2.6}$$

where β_{ij} and ρ_0 are the values of c_{ij} and ρ in the homogeneous case, respectively, and m is a rational number. The heat conduction equation can be expressed as

$$\frac{\partial^2 T'}{\partial r^2} + \frac{1}{r}\frac{\partial T'}{\partial r} = \frac{1}{k_1}\frac{\partial T'}{\partial t'} \tag{2.7}$$

where k_1 is the thermal diffusivity of the cylinder in the r-direction and T' is the temperature. Substituting from Eqs. (2.4) and (2.6) into Eq. (2.5) we have

$$\begin{aligned} &\frac{\partial^2 u}{\partial r^2} + \frac{(2m+1)}{r}\frac{\partial u}{\partial r} + \frac{(2m\beta_{12} - \beta_{22})}{\beta_{11}}\frac{u}{r^2} \\ &\quad - \frac{\beta_1'}{\beta_{11}}\left(\frac{\partial T'}{\partial r} + \frac{(2m+1)T'}{r} - \frac{\beta_2' T'}{\beta_1' r}\right) \\ &\quad - \frac{\rho_0 \Omega^2 r}{\beta_{11}} = \frac{\rho_0}{\beta_{11}}\frac{\partial^2 u}{\partial t'^2} \end{aligned} \tag{2.8}$$

It is convenient to have the above equations written in non-dimensional form. To this end, we consider the following transformations:

$$b(U, R) = (u, r) \tag{2.9a}$$

$$t' = \frac{b}{v} t \tag{2.9b}$$

$$T = \frac{T'}{T_0} \tag{2.9c}$$

where T_0 is a reference temperature and v is the dimensionless velocity.

In terms of these non-dimensional variables, the equation of heat conduction (2.7) can be written as

$$\frac{\partial^2 T}{\partial R^2} + \frac{1}{R}\frac{\partial T}{\partial R} = \alpha\frac{\partial T}{\partial t} \tag{2.10}$$

The equation of motion (2.8) has the form

$$\begin{aligned} &\frac{\partial^2 U}{\partial R^2} + \frac{\alpha_1}{R}\frac{\partial U}{\partial R} - \alpha_2\frac{U}{R^2} - \alpha_3\left(\frac{\partial T}{\partial R} + \left(\frac{\alpha_1\beta_1' - \beta_2'}{\beta_1'}\right)\frac{T}{R}\right) \\ &\quad - \alpha_4 R = \alpha_5\frac{\partial^2 U}{\partial t^2} \end{aligned} \tag{2.11}$$

where

$$\alpha = \frac{vb}{k_1}, \quad \alpha_1 = 2m+1, \quad \alpha_2 = \frac{\beta_{22} - 2m\beta_{12}}{\beta_{11}}$$

$$\alpha_3 = \frac{\beta_1 T_0}{\beta_{11}}, \quad \alpha_4 = \frac{\rho_0\Omega^2 b}{\beta_{11}} \quad \text{and} \quad \alpha_5 = \frac{\rho_0 v^2}{\beta_{11}}$$

In terms of these non-dimensional variables, the stresses induced by the temperature T are related to displacement component U by

$$\sigma_{RR} = (bR)^{2m}\left(\beta_{11}\frac{\partial U}{\partial R} + \beta_{12}\frac{U}{R} - \beta_1' T_0 T\right) \tag{2.12a}$$

$$\sigma_{\theta\theta} = (bR)^{2m}\left(\beta_{12}\frac{\partial U}{\partial R} + \beta_{22}\frac{U}{R} - \beta_2' T_0 T\right) \tag{2.12b}$$

$$\sigma_{zz} = (bR)^{2m}\left(\beta_{13}\frac{\partial U}{\partial R} + \beta_{23}\frac{U}{R} - \beta_3' T_0 T\right) \tag{2.12c}$$

where

$$\beta_1' = \beta_{11}\alpha_r + \beta_{12}\alpha_\theta + \beta_{13}\alpha_z$$
$$\beta_2' = \beta_{12}\alpha_r + \beta_{22}\alpha_\theta + \beta_{23}\alpha_z$$
$$\beta_3' = \beta_{13}\alpha_r + \beta_{23}\alpha_\theta + \beta_{33}\alpha_z$$

From the preceding description, the initial conditions may be expressed as:

$$\text{at } t = 0 \quad T = 0 \tag{2.13a}$$

$$\text{at } t = 0 \quad U = \frac{\partial U}{\partial t} = 0 \tag{2.13b}$$

The boundary conditions may be expressed as

$$\text{at } R = \frac{a}{b} \quad T = 1 \tag{2.14a}$$

$$\text{at } R = 1 \quad \frac{\partial T}{\partial R} = 0 \tag{2.14b}$$

$$\text{at } R = \frac{a}{b} \quad U = 0 \tag{2.14c}$$

at $R = 1 \quad \dfrac{\partial U}{\partial R} = 0$ (2.14d)

3
Numerical scheme

A finite difference scheme which is a modification of MacCormack's scheme is described by Wachtman et al. [14]. Where it is used to obtain solutions to problem of thermal stress emanating from cylindrical cavity in a bounded medium. This scheme is a forward-backward predictor corrector scheme.

We take the finite difference grids with spatial intervals h in the direction R and k as the time step, and use the subscripts i and n to denote the ith discrete points in the R direction and nth discrete time.

Thus the heat conduction equation (2.10) may be expressed in the finite difference as follows (cf. Lapidus and Pinder [15]);

$$T_i^{n+1} = T_i^n + \frac{\rho}{\alpha}\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n + \left(\frac{h}{\frac{a}{b} + ih}\right)\left(T_{i+1}^n - T_i^n\right)\right) \tag{3.1}$$

Also, the equation of motion (2.11) may be expressed in the finite difference as follows:

$$\begin{aligned} U_i^{n+1} = 2U_i^n - U_i^{n-1} + \frac{\rho_1}{\alpha_5}\Bigg[& U_{i+1}^n - 2U_i^n + U_{i-1}^n \\ & + \alpha_1\left(\frac{h}{\frac{a}{b} + ih}\right)(U_{i+1}^n - U_i^n) - \alpha_2\left(\frac{h}{\frac{a}{b} + ih}\right)^2 U_i^n \\ & - \alpha_3 h\left(T_{i+1}^n - T_i^n + \left(\frac{\alpha_1\beta_1' - \beta_2'}{\beta_1'}\right)\left(\frac{h}{\frac{a}{b} + ih}\right)T_i^n\right) \\ & - \alpha_4 h^2\left(\frac{a}{b} + ih\right)\Bigg] \end{aligned} \tag{3.2}$$

where

$$\rho = \frac{k}{h^2}, \quad \rho_1 = \left(\frac{k}{h}\right)^2$$

4
Numerical results and discussion

Let us consider the distribution of the transient temperature, displacement and thermal stresses in the rotating non-homogeneous cylinder. As an illustrative purpose of the foregoing solutions, the cylinder has the following geometric and material constants given by Wachtman et al. [14].

For orthotropic material

$$\beta_{11} = 0.134, \quad \beta_{12} = 0.101, \quad \beta_{22} = 0.674, \quad \beta_{13} = 1.099$$

$$\beta_{33} = 4.981 \times 10^{11}\ \text{dynes/cm}^2, \quad \alpha = 1, \quad \rho_0\Omega^2 = 1.5 \tag{4.1}$$

and

$$\alpha_r = 40 \times 10^{-6}/^\circ\text{C}, \quad \alpha_\theta = 10 \times 10^{-6}/^\circ\text{C},$$

$$\alpha_z = 40 \times 10^{-6}/^\circ\text{C} \tag{4.2}$$

For isotropic material

$$\lambda = 1.525, \quad \mu = 2.642, \quad \alpha = 1, \quad \rho_0\Omega^2 = 1.5 \tag{4.3}$$

Results are presented for cylinder with $a = 0.1$, $b = 1.0$ and $T_0 = 1\,^\circ$C

To study the non-homogeneous case, we assume that $m = 0.5$ and for the homogeneous case we assume that $m = 0.0$. We represented the numerical results graphically.

Figure 1 shows the temperature variation for various non-dimensional time t. It is noticed that, the temperature decreases with increasing R in all the contexts of all three modes and satisfied the boundary conditions.

Figure 2 shows the radial displacement U along the radial direction R at various dimensionless t. From this figure the displacement decreases with increasing R and satisfied the boundary conditions. It is noticed that the displacement component increase with increasing t.

Figure 3 shows the radial stress σ_{RR} along the radial direction R at various time t. It is noticed that, the radial stress decreases with increasing R and t of all three modes and start at $R = 0.9$ increase with increasing R.

Figure 4 shows the radial distribution of the stress σ_{rr}, the deviations from isotropic case are large. It is noticed that, the radial stress is considerably small at inner surface and large at outer surface.

Figure 5 shows the tangential stress $\sigma_{\theta\theta}$, along the radial direction R at various time t. It is noticed that the tangential stress decreases with increasing R and t of all the three modes.

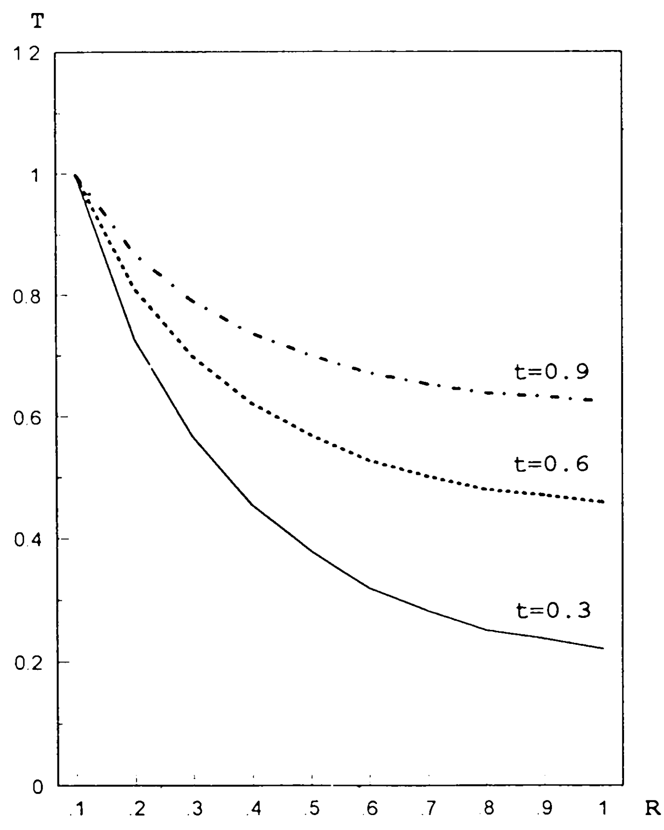

Fig. 1. Temperature distribution

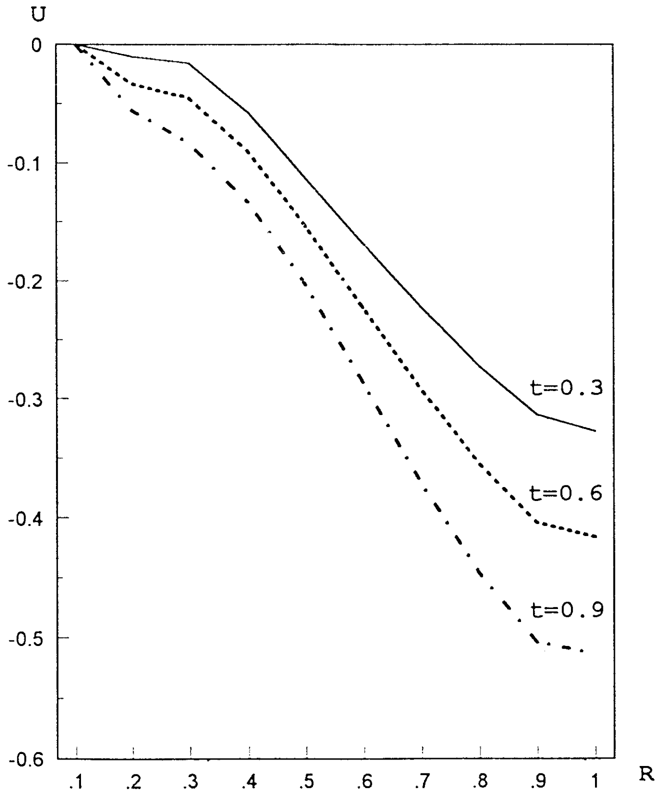

Fig. 2. Radial displacement distribution at different time ($m = 0.5$)

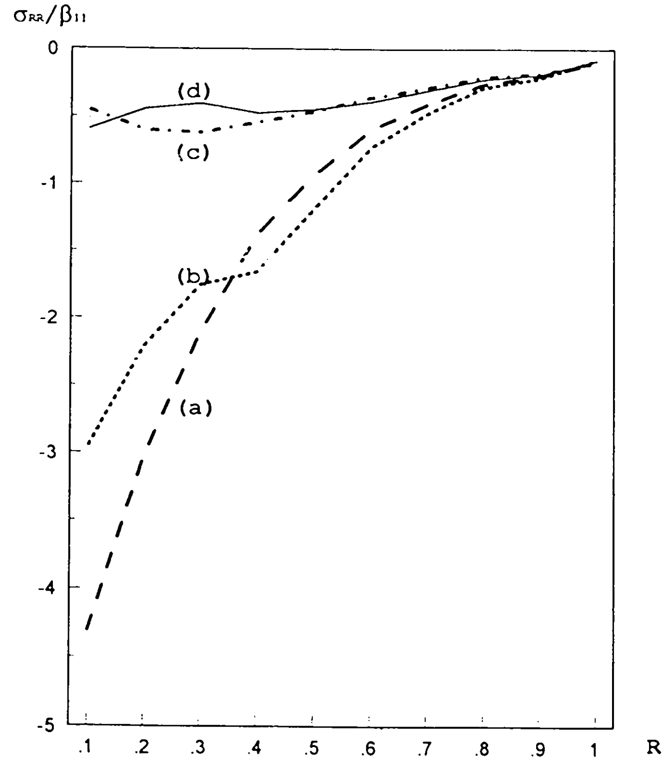

Fig. 4. Radial stress distribution

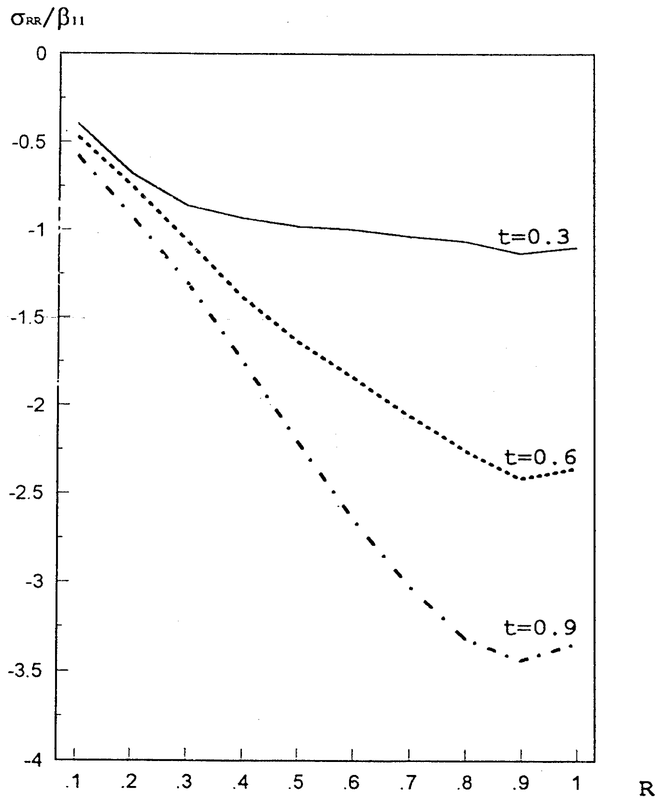

Fig. 3. Radial stress distribution at different time ($m = 0.5$)

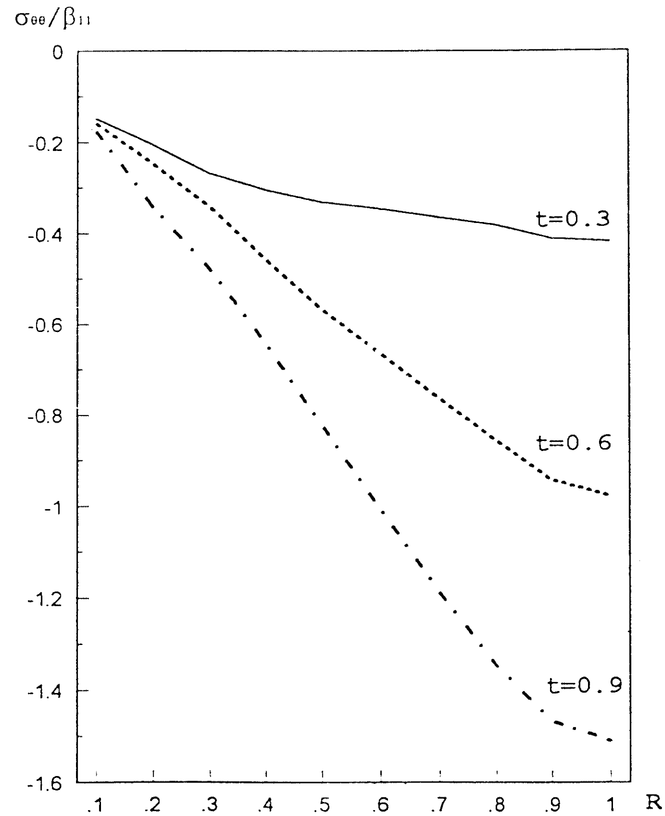

Fig. 5. Tangential stress distribution at different time ($m = 0.5$)

Figure 6 shows the tangential stress $\sigma_{\theta\theta}$, the deviations from isotropic case are considerably large.

Also figures 4 and 6 show the influence of the orthotropy and non-homogeneity of the material constants on the displacement distribution, radial stress, tangential stress. Also, they show the difference among the four theories namely homogeneous orthotropic (a), homo-

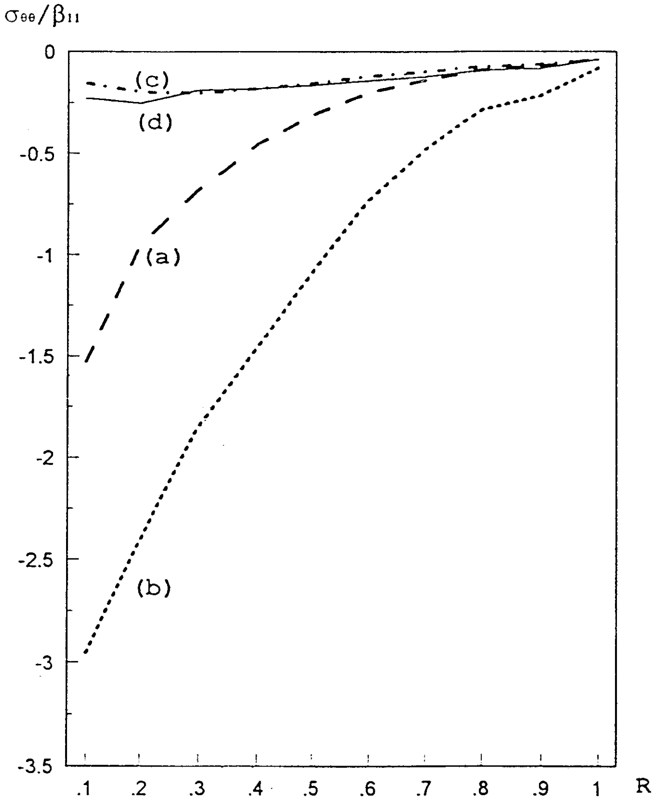

Fig. 6. Tangential stress distribution

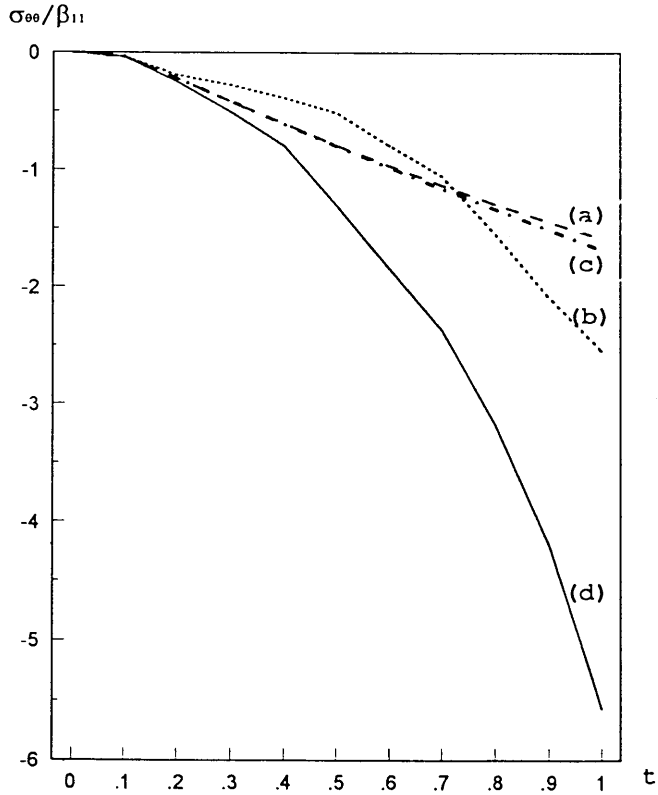

Fig. 8. Variation of tangential stress with time ($R = 1$)

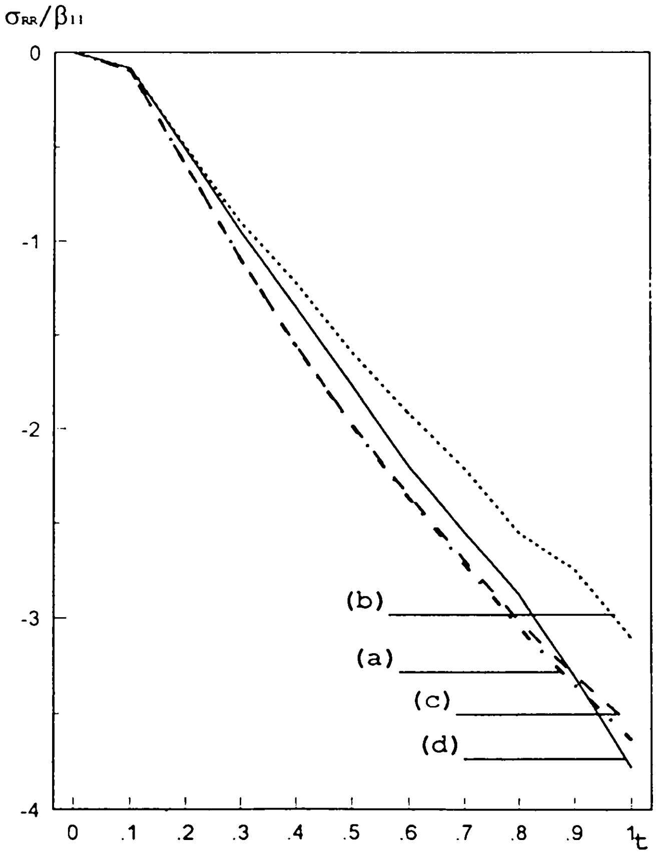

Fig. 7. Variation of radial stress with time ($R = 1$)

geneous isotropic (b), non-homogeneous orthotropic (c) and non-homogeneous isotropic (d).

Figures 7 and 8 show the radial stress σ_{rr} and tangential stress $\sigma_{\theta\theta}$. From these figures the radial and tangential stress decrease with increasing time t in four cases, homogeneous orthotropic, homogeneous isotropic, non-homogeneous orthotropic and non-homogeneous isotropic.

The variation of stresses and displacement $\sigma_{RR}, \sigma_{\theta\theta}$ and U due to the effect of inertia and rotation. It is evident that orthotropy has a significant influence on stresses. Also, the influence of the non-homogeneity on displacement and stresses is very pronounced. These results are specific for the example considered, but the other examples may have different trends because of the dependence of the results on the mechanical and thermal constants of the material.

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