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Thermo-mechanical stresses in functionally graded circular hollow cylinder with linearly increasing boundary temperature

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Abstract

Thermo-mechanical analysis of functionally graded hollow circular cylinders subjected to mechanical loads and linearly increasing boundary temperature is carried out in this study. Thermo-mechanical properties of functionally graded material (FGM) are assumed to be temperature independent and vary continuously in the radial direction of cylinder. Employing Laplace transform techniques and series solving method for ordinary differential equation, solutions for the time-dependent temperature and thermo-mechanical stresses are obtained. As an example, a molybdenum/mullite FGM with material properties follow an exponential law is calculated, and numerical results are graphically presented.

Keywords: Functionally graded material; Circular hollow cylinder; Thermo-mechanical stress; Linear thermal load

1. Introduction

Functionally graded materials (FGMs) are mixtures of two or more different materials. Volume fraction of each material varies continuously along certain direction(s). The gradual change of material properties can be tailored to meet the requirements of different applications and working environments. The FGMs were initially designed as thermal barrier materials for aerospace structures [1,2].

Compared to the fiber-reinforced or laminated composite materials, FGMs are more preferable in many thermal–mechanical applications. It is because continuous change in microstructure of FGMs may not be subjected to a mismatch of mechanical properties across the interface as a reinforced or laminated material does. As a result, the FGMs may not debond even at extremely high thermal loading.

Exploration of the thermo-mechanical performance of FGM structures has been attractions to many researchers. Tanigawa [3] presented an extensive review that covered a wide range of topics from thermoelastic to thermo-inelastic problems. He compiled a comprehensive list of papers on the analytical models of thermoelastic behavior of FGMs. In order to understand the effect of the volumetric ratio of constituents and porosity on thermal stresses, Obata and Noda [4] studied the steady-state thermal stresses in FGM hollow cylinders and FGM hollow spheres. Based on the multi-layered method, Kim and Noda [5] studied the unsteady-state thermal stress of FGM circular hollow cylinders by using of Green function method. Reddy and co-workers [6–11] carried out theoretical as well as finite element analyses of the thermo-mechanical behavior of FGM cylinders, plates and shells. Geometric non-linearity and effect of coupling item was considered for different thermal loading conditions. Batra and co-workers [12–15] studied the thermo-mechanical deformations of thick functionally graded plates. Based on a higher-order shear deformation theory, transient response of a thick plate was derived using a meshless local Petrov–Galerkin method, and analytical solutions was obtained using of Laplace transformation technique and the power series method. Shao and Wang [16–18] studied the thermo-mechanical

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stresses of FGM hollow cylinders and cylindrical panels with the assumption that the material properties of FGM followed simple laws, e.g., exponential law, power law or mixture law in thickness direction. An approximate static solution of FGM hollow cylinders with finite length was obtained by using of multi-layered method; analytical solution of FGM cylindrical panel was carried out by using the Frobinus method; and analytical solution of transient thermo-mechanical stresses of FGM hollow cylinders were derived by using the Laplace transform technique and the power series method, in which effects of material gradient and heat transfer coefficient on time-dependent thermal mechanical stresses were discussed in detail. Similarly, Ootao and Tanigawa [19–21] obtained the analytical solutions of unsteady-state thermal stress of FGM plate and cylindrical panel due to non-uniform heat supply. Using the multi-layered method and through a novel limiting process, Liew and co-workers [22] obtained the analytical solutions of steady-state thermal stress in FGM hollow circular cylinder. Using finite difference method, Awaji and Sivakuman [23] studied the transient thermal stresses of a FGM hollow circular cylinder, which is cooled by surrounding medium. Ching and Yen [24] evaluated the transient thermoelastic deformations of 2-D functionally graded beams under non-uniformly convective heat supply.

Based on the mathematical similarity of the axisymmetric bending and buckling problems of a circular plate between the classical plate theory and Reddy's third-order shear deformation plate theory, Ma and Wang [25] derived the analytical relations of the solutions of bending and buckling of circular FGM plate based on various plate theories, respectively. They obtained the analytical solutions of bending and buckling of circular FGM plate. Ma and Wang [26] also investigated the non-linear bending and post-buckling behavior of FGM circular plate under thermal and mechanical loadings.

In the present work, we consider the unsteady-state thermo-mechanical problem of FGM hollow circular cylinders. The temperature variation from the inner surface to the outer surface is linear. The hollow cylinder is also subjected to mechanical loads on inner and outer surfaces. Material properties of the hollow cylinder are assumed to be temperature independent and vary continuously in the radial direction. The Laplace transform technique and power series method for solving ordinary differential equation(s) are employed to solve the thermo-mechanical problems.

2. Solution of time-dependent temperature

A FGM circular hollow cylinder with an inner radius r_1 and an outer radius r_2, as shown in Fig. 1, is considered. It is assumed that the thermal conductivity coefficient λ and the thermal diffusivity κ change continuously through the thickness.

The following dimensionless variables are used in the derivation for easier formulation:

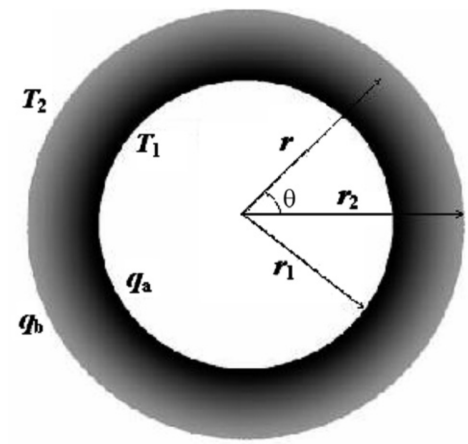

Fig. 1. Dimensions and loading conditions of FGM hollow circular cylinder.

$$
\begin{aligned}
&R = r/r_{\mathrm{m}}, \quad R_1 = r_1/r_{\mathrm{m}}, \quad R_2 = r_2/r_{\mathrm{m}},\\
&Y = E/E_0, \quad \Omega = \alpha/\alpha_0, \quad \Lambda = \lambda/\lambda_0, \quad K = \kappa/\kappa_0,\\
&\Theta = T/T_0, \quad \Theta_1 = T_1/T_0, \quad \Theta_2 = T_2/T_0, \quad \tau = \kappa_0 t/r_{\mathrm{m}}^2,\\
&U = u/\alpha_0 T_0 r_{\mathrm{m}}, \quad \Sigma_r = \sigma_r/\alpha_0 T_0 E_0, \quad \Sigma_\theta = \sigma_\theta/\alpha_0 T_0 E_0,\\
&Q_a = q_a/(\alpha_0 E_0 T_0), \quad Q_b = q_b/(\alpha_0 E_0 T_0),
\end{aligned}
$$

where $r_{\mathrm{m}} = (r_1 + r_2)/2$ is the radius of the mid-surface of the circular hollow cylinder. T_0 is the reference temperature. E_0, α_0, λ_0 and κ_0 are reference values of the elastic modulus, the thermal expansion coefficient, the heat conductivity and the heat diffusivity.

In the cylindrical coordinate system as shown in Fig. 1, the unsteady-state heat conduction equation for the hollow cylinder can be expressed as

$$
\frac{\partial^2 \Theta(R,\tau)}{\partial R^2} + \left(\frac{1}{R} + \frac{\Lambda'(R)}{\Lambda(R)}\right)\frac{\partial \Theta(R,\tau)}{\partial R} = \frac{1}{K(R)}\frac{\partial \Theta(R,\tau)}{\partial \tau}, \qquad (1)
$$

where $\Theta(R,\tau)$ is the dimensionless temperature in the cylinder. In Eq. (1) and following derivations, the primes denote derivatives with respect to the radial or r coordinate.

It is assumed that the initial temperature of the FGM cylinder is zero. The temperature of its inner surface is linearly increased to T_1 within a very short time interval τ_0 and kept constant subsequently (see Fig. 2). Its outer sur-

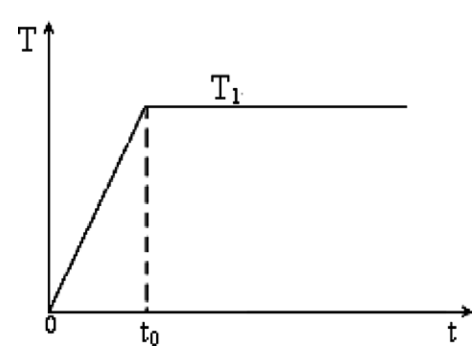

Fig. 2. Linearly increasing temperature on inner surface of the FGM hollow circular cylinder.

face is heated by surrounding media to temperature T_2. The heat transfer coefficient on outer surface is h_2. Therefore, the initial and boundary temperature conditions of the hollow cylinder can be expressed as

$$\Theta(R,0)=0, \tag{2}$$

$$R=R_1:\Theta=\begin{cases}\Theta_1\frac{\tau}{\tau_0}, & 0\leqslant\tau\leqslant\tau_0,\\ \Theta_1, & \tau>\tau_0,\end{cases} \tag{3a}$$

$$\frac{\partial\Theta(R_2,\tau)}{\partial R}+H_2\Theta(R_2,\tau)=H_2\Theta_2, \tag{3b}$$

where H_2 is dimensionless heat transfer coefficient on the outer surface and defined as

$$H_2=r_{\mathrm{m}}h_2.$$

Firstly, the temperature solution in the phase of $0\leqslant\tau\leqslant\tau_0$ is derived. Considering the initial condition (2), Laplace transformation to Eq. (1) and boundary conditions (3) with respect to variable τ is performed. It derives

$$\frac{\partial^2F(R,s)}{\partial R^2}+\left(\frac{1}{R}+\frac{\Lambda'(R)}{\Lambda(R)}\right)\frac{\partial F(R,s)}{\partial R}=\frac{s}{K(R)}F(R,s), \tag{4}$$

$$R=R_1:F=\frac{\Theta_1}{\tau_0}\cdot\frac{1}{s^2}, \tag{5a}$$

$$R=R_2:\frac{\partial F}{\partial R}+H_2F=\frac{1}{s}H_2\Theta_2. \tag{5b}$$

According to the series solving method for ordinary differential equation, if the coefficients $\Lambda'(R)/\Lambda(R)$ and $1/\mathrm{K}(R)$ are analytical at $R=1$ and can be expressed as the following Taylor's series:

$$f_1(R)=\frac{1}{\Lambda(R)}\frac{\mathrm{d}\Lambda(R)}{\mathrm{d}R}=\sum_{n=0}^{\infty}f_{1n}(r-1)^n, \tag{6a}$$

$$f_2(R)=\frac{1}{K(R)}=\sum_{n=0}^{\infty}f_{2n}(r-1)^n, \tag{6b}$$

where

$$f_{1n}=\frac{1}{n!}f_1^{(n)}(1),\quad f_{2n}=\frac{1}{n!}f_2^{(n)}(1).$$

The solution of Eq. (4) can also be expressed as the following Taylor's series at the point of $R=1$:

$$F(R,s)=\sum_{n=0}^{\infty}A_n(s)\cdot(R-1)^n. \tag{7}$$

Substituting series (6) and (7) into Eq. (4), employing the Abel principle of series multiplication and comparing the coefficients of $(R-1)^n$, the following recurrence relation is then derived

$$\begin{aligned}(n+1)(n+2)A_{n+2}&=s\sum_{k=0}^{n}[A_{k-1}+A_k]f_{2,n-k}\\&\quad-(n+1)^2A_{n+1}-\sum_{k=0}^{n}[kA_k\\&\quad+(k+1)A_{k+1}]f_{1,n-k}.\end{aligned} \tag{8}$$

The coefficients A_n in series (7) are linear combinations of A_0 and A_1 that can be expressed as

$$A_n(s)=P_{1n}(s)\cdot A_0+P_{2n}(s)\cdot A_1, \tag{9}$$

where the coefficient $P_{1n}(s)$ and $P_{2n}(s)$ can be derived from the recurrence Eq. (8). Therefore, the solution of Eq. (4) can be expressed as

$$F(R,s)=\sum_{n=0}^{\infty}(A_0P_{1n}(s)+A_1P_{2n}(s))\cdot(R-1)^n. \tag{10}$$

where A_0 and A_1 are unknown constants. Substituting Eq. (10) into the boundary conditions (5), we obtain the following algebraic equations:

$$\begin{bmatrix}d_{11} & d_{12}\\ d_{21} & d_{22}\end{bmatrix}\begin{Bmatrix}A_0\\ A_1\end{Bmatrix}=\begin{Bmatrix}\frac{\Theta_1}{\tau_0}\cdot\frac{1}{s^2}\\ \frac{H_2\Theta_2}{s}\end{Bmatrix}, \tag{11}$$

where

$$d_{11}=\sum_{n=0}^{\infty}P_{1n}(s)\cdot(R_1-1)^n,$$

$$d_{12}=\sum_{n=0}^{\infty}P_{2n}(s)\cdot(R_1-1)^n,$$

$$d_{21}=\sum_{n=1}^{\infty}nP_{1n}(s)\cdot(R_2-1)^{n-1}+H_2\sum_{n=0}^{\infty}P_{1n}(s)\cdot(R_2-1)^n,$$

$$d_{22}=\sum_{n=1}^{\infty}nP_{2n}(s)\cdot(R_2-1)^{n-1}+H_2\sum_{n=0}^{\infty}P_{2n}(s)\cdot(R_2-1)^n.$$

From Eq. (11), the constants A_0 and A_1 are determined, i.e.

$$A_0=\frac{1}{s^2}\det\begin{bmatrix}\Theta_1/\tau_0 & d_{12}\\ H_2\Theta_2s & d_{22}\end{bmatrix}\Big/\det\begin{bmatrix}d_{11} & d_{12}\\ d_{21} & d_{22}\end{bmatrix}, \tag{12a}$$

$$A_1=\frac{1}{s^2}\det\begin{bmatrix}d_{11} & \Theta_1/\tau_0\\ d_{21} & H_2\Theta_2s\end{bmatrix}\Big/\det\begin{bmatrix}d_{11} & d_{12}\\ d_{21} & d_{22}\end{bmatrix}. \tag{12b}$$

Substituting the solved constants A_0 and A_1 from Eqs. (12a,b) into the series solution (10), we derive

$$F(R,s)=\sum_{n=0}^{\infty}\frac{N_n(s)}{M(s)}(R-1)^n, \tag{13}$$

where

$$\begin{aligned}N_n(s)&=\det\begin{bmatrix}\frac{\Theta_1}{\tau_0} & d_{12}\\ H_2\Theta_2s & d_{22}\end{bmatrix}P_{1n}(s)\\&\quad+\det\begin{bmatrix}d_{11} & \frac{\Theta_1}{\tau_0}\\ d_{21} & H_2\Theta_2s\end{bmatrix}P_{2n}(S),\end{aligned}$$

$$M(s)=s^2\cdot\det\begin{bmatrix}d_{11} & d_{12}\\ d_{21} & d_{22}\end{bmatrix}.$$

Using of Galerkin method, we then deduce the inverse Laplace transformation for all coefficients of $(R-1)^n$ in Eq. (13). In the case of the transcendental equation $M(s)=0$ has only single root s_k $(k=1,2,\ldots)$, then the inverse transformation of Eq. (13) can be written as

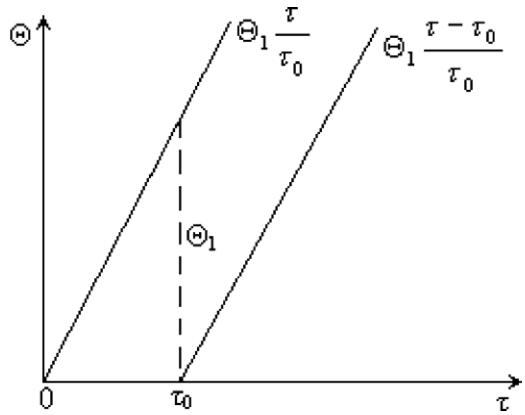

Fig. 3. Sternberg and Chakravorty's analytical method in case of $\tau_0 \leqslant \tau$.

$$\Theta(R,\tau) = \sum_{n=0}^{N}\left[\sum_{k=1}^{K}\frac{N_n(s_k)}{[\mathrm{d}M(s)/\mathrm{d}s]_{s=s_k}}\mathrm{e}^{s_k\tau}\right](R-1)^n. \tag{14}$$

If s_1 is a multiple-root with m orders of function $M(s)$, and $s_{m+1}, s_{m+2}, s_{m+3}\ldots$ are single roots of function $M(s)$, the inverse transform of Eq. (13) can be written as

$$\Theta(R,\tau) = \sum_{n=0}^{N}\left[\frac{1}{(m-1)!}\lim_{s\to s_1}\frac{\mathrm{d}^{m-1}}{\mathrm{d}s^{m-1}}\left[(s-s_1)^m\frac{N_n(s)}{M(s)}\mathrm{e}^{s\tau}\right]\right.$$
$$\left.+\sum_{k=m+1}^{K}\frac{N_n(s_k)}{[\mathrm{d}M(s)/\mathrm{d}s]_{s=s_k}}\mathrm{e}^{s_k\tau}\right](R-1)^n. \tag{15}$$

Based on the temperature solution (14) or (15) for the phase of $0 \leqslant \tau \leqslant \tau_0$, we derive the temperature solution for $\tau_0 \leqslant \tau$. Following the work of Sternberg and Chakravorty [27], as shown in Fig. 3, the temperature change of the inner surface for the phase of $\tau_0 \leqslant \tau$ can be expressed as

$$\Theta(R_1) = \Theta_1\frac{\tau}{\tau_0} - \Theta_1\frac{\tau-\tau_0}{\tau_0}, \quad \tau \succ \tau_0. \tag{16}$$

Thus, the solution of time-dependent temperature of the FGM cylinder can be expressed as

$$\Theta = \Theta(R,\tau) - \Theta(R,\tau-\tau_0), \tag{17}$$

where $\Theta(R,\tau-\tau_0)$ is a function similar to $\Theta(R,\tau)$, in which variable τ is replaced by $\tau-\tau_0$.

3. Solution of transient thermo-mechanical stresses

Generally, the Poisson's ratio μ of FGM varies in a small range. Its effects on thermo-mechanical stresses are insignificant. For simplicity, we assume μ to be a constant. On the other hand, the elastic modulus E and thermal expansion coefficient α change continuously through the thickness of the hollow cylinder. Using the cylindrical coordinate system shown in Fig. 1, the equilibrium equation of the FGM hollow cylinder can be expressed as

$$\frac{\partial^2 U}{\partial R^2} + \left(\frac{1}{R}+\frac{Y'(R)}{Y(R)}\right)\frac{\partial U}{\partial R} + \left(\frac{\mu}{1-\mu}\frac{Y'(R)}{Y(R)}-\frac{1}{R}\right)\frac{U}{R}$$
$$-\frac{1+\mu}{1-\mu}\frac{[\Omega(R)\cdot Y(R)]'}{Y(R)}\Theta - \frac{1+\mu}{1-\mu}\Omega(R)\frac{\partial\Theta}{\partial R} = 0. \tag{18}$$

We assume that the FGM cylinder is subjected to axisymmetric pressures q_a and q_b on its inner and outer surfaces, respectively. Therefore, considering the aforementioned dimensionless variables, the boundary condition can be expressed as

$$R = R_1 : (1-\mu)\frac{\partial U}{\partial R} + \frac{\mu}{R_1}U - (1+\mu)\Omega\cdot\Theta = Q_a, \tag{19a}$$

$$R = R_2 : (1-\mu)\frac{\partial U}{\partial R} + \frac{\mu}{R_2}U - (1+\mu)\Omega\cdot\Theta = Q_b. \tag{19b}$$

According to the theory of series solving method for ordinary differential equation, if the coefficients $Y'(R)/Y(R), [\Omega(R)\cdot Y(R)]'/Y(R)$ and $\Omega(R)$ are analytical at $R=1$ and can be expressed as the following Taylor's series in terms of $(R-1)$:

$$f_3(R) = \frac{1}{Y(R)}\frac{\mathrm{d}Y(R)}{\mathrm{d}R} = \sum_{n=0}^{\infty} f_{3n}\cdot(R-1)^n, \tag{20a}$$

$$f_4(R) = \frac{1}{Y(R)}\frac{\mathrm{d}[\Omega(R)\cdot Y(R)]}{\mathrm{d}R} = \sum_{n=0}^{\infty} f_{4n}\cdot(R-1)^n, \tag{20b}$$

$$f_5(R) = \Omega(R) = \sum_{n=0}^{\infty} f_{5n}\cdot(R-1)^n, \tag{20c}$$

where

$$f_{3n} = \frac{1}{n!}f_3^{(n)}(1),$$
$$f_{4n} = \frac{1}{n!}f_4^{(n)}(1),$$
$$f_{5n} = \frac{1}{n!}f_5^{(n)}(1),$$

then the solution of Eq. (18) can also be expressed as the following Taylor's series at the point of $R=1$:

$$U(R,\tau) = \sum_{n=0}^{\infty} B_n(\tau)\cdot(R-1)^n. \tag{21}$$

Additionally, we rewrite the time-dependent temperature (14) or (15) as

$$\Theta(R,\tau) = \sum_{n=0}^{\infty}\Theta_n(\tau)\cdot(R-1)^n, \tag{22}$$

where

$$\Theta_n(\tau) = \sum_{k=1}^{K}\frac{N_n(s_k)}{[\mathrm{d}M(s)/\mathrm{d}s]_{s=s_k}}\mathrm{e}^{s_k\tau} \quad \text{or}$$
$$\Theta_n(\tau) = \frac{1}{(m-1)!}\lim_{s\to s_1}\frac{\mathrm{d}^{m-1}}{\mathrm{d}s^{m-1}}\left[(s-s_1)^m\frac{N_n(s)}{M(s)}\mathrm{e}^{s\tau}\right]$$
$$+\sum_{k=m+1}^{K}\frac{N_n(s_k)}{[\mathrm{d}M(s)/\mathrm{d}s]_{s=s_k}}\mathrm{e}^{s_k\tau}.$$

Substituting Eqs. (20), (21) and (22) into Eq. (18), employing the Abel principle of series multiplication and comparing the coefficient of $(R-1)^n$, we obtain the following recurrence relation,

$$
\begin{aligned}
&-(n+1)(n+2)B_{n+2}\\
&\quad=(n+1)(2n+1)B_{n+1}+(n^2-1)B_n\\
&\qquad-\frac{\mu}{1-\mu}\sum_{k=0}^{n}[B_{k-1}+B_k]f_{3,n-k}+\sum_{k=0}^{n}[(k-1)B_{k-1}\\
&\qquad+2kB_k+(k+1)B_{k+1}]f_{3,n-k}\\
&\qquad-\frac{1+\mu}{1-\mu}\left\{\sum_{k=0}^{n}(\Theta_{k-2}+2\Theta_{k-1}+\Theta_k)f_{2,n-k}\right.\\
&\qquad\left.+\sum_{k=0}^{n}[(k-1)\Theta_{k-2}+2k\Theta_{k-1}+(k+1)\Theta_k]f_{3,n-k}\right\}.
\end{aligned}
\tag{23}
$$

It is observed that all the coefficients of Taylor's series (21) can be expressed as

$$B_n(\tau)=B_0Q_{1n}+B_1Q_{2n}+Q_{3n}, \tag{24}$$

where Q_{1n}, Q_{2n} and Q_{3n} are derived from the recurrence relation (23). Furthermore, the solution of the radial displacement of the FGM hollow cylinder is expressed as

$$U(R,\tau)=\sum_{n=0}^{\infty}[B_0Q_{1n}+B_1Q_{2n}+Q_{3n}](R-1)^n, \tag{25}$$

where B_0 and B_1 are unknown constants and can be determined by substituting solutions (25) into boundary conditions (19).

Substituting Eq. (25) into the geometric equation and then into the constitutive equation, we obtain the transient thermo-mechanical stresses of the FGM circular hollow cylinder.

4. Numerical results and discussion

Before any calculation of the thermal stresses of FGM circular hollow cylinder, we firstly check the convergence of the series solution (14). Carslaw and Jaeger [28] studied the heat conduction problem in a homogeneous circular hollow with heat transfer boundary conditions. Using Laplace transformation and Bessel functions, they carried out the analytical solution of this problem, i.e.

$$
\begin{aligned}
\Theta(R,\tau)=&\frac{R_1H_1\Theta_1[1-R_2H_2\ln(R/R_2)]+R_2H_2\Theta_2[1+R_1H_1\ln(R/R_1)]}{R_1H_1+R_2H_2+R_1R_2H_1H_2\ln(R_2/R_1)}\\
&-\pi\sum_{k=1}^{\infty}\frac{e^{-\kappa\alpha_n^2\tau}}{F(\alpha_n)}[\alpha_nJ_1(R_2\alpha_n)-H_2J_0(R_2\alpha_n)]\cdot C_0(R,\alpha_n)\\
&\times\{-H_1\Theta_1[\alpha_nJ_1(R_2\alpha_n)-H_2J_0(R_2\alpha_n)]\\
&-H_2\Theta_2[\alpha_nJ_1(R_1\alpha_n)+H_1J_0(R_1\alpha_n)]\},
\end{aligned}
\tag{26}
$$

where

$$
\begin{aligned}
F(\alpha_n)=&(\alpha_n^2+H_2^2)[\alpha_nJ_1(R_1\alpha_n)+H_1J_0(R_1\alpha_n)]^2\\
&-(\alpha_n^2+H_1^2)[\alpha_nJ_1(R_2\alpha_n)-H_2J_0(R_2\alpha_n)]^2,\\
C_0(R,\alpha_n)=&J_0(R\alpha_n)[\alpha_nY_1(R_1\alpha_n)+H_1Y_0(R_1\alpha_n)]\\
&-Y_0(R\alpha_n)[\alpha_nJ_1(R_1\alpha_n)+H_1J_0(R_1\alpha_n)],
\end{aligned}
$$

and α_n are the roots of the following equation:

$$
\begin{aligned}
&[\alpha J_1(R_1\alpha)+H_1J_0(R_1\alpha)][\alpha Y_1(R_2\alpha)-H_2Y_0(R_2\alpha)]\\
&\quad-[\alpha Y_1(R_1\alpha)+H_1Y_0(R_1\alpha)][\alpha J_1(R_2\alpha)-H_2J_0(R_2\alpha)]=0.
\end{aligned}
$$

We can obtain the temperature solution of this heat conduction problem by using of the present analytical study. For the sake of brevity, we will not repeat the process of derivation. For homogeneous circular hollow cylinder, the coefficients of Taylor's series (6) and the recurrence relation (8) can be rewritten as

$$
\begin{aligned}
&f_{1n}=0,\quad n=0,1,2,\ldots,\\
&f_{20}=1/K,\quad f_{2n}=0,\ \ n=1,2,3,\ldots,\\
&-(n+1)(n+2)A_{n+2}=(n+1)^2A_{n+1}-\frac{s}{K_d}(A_n+A_{n-1}).
\end{aligned}
\tag{27}
$$

The relevant coefficient items $P_{1n}(s)$ and $P_{2n}(s)$ can be obtained from the recurrence relation (8). Then the relevant coefficients d_{11}, d_{12}, d_{21} and d_{22} can be carried out by using the corresponding temperature boundary conditions. Submitting these coefficient items into Eq. (14), the temperature solution of this problem can be finally obtained.

We consider a homogeneous circular hollow cylinder with inner radius $R_1=0.9$ and outer radius $R_2=1.1$. Its initial temperature is zero and suddenly heated by surrounding media on the inner and outer surfaces. The temperatures of internal and external surrounding medium are $\Theta_1=1$ and $\Theta_2=0$. The heat transfer coefficients on the inner and outer surfaces are $H_1=H_2=30$. Results obtained from Carslaw and Jaeger's solution and the present method are graphically presented in Fig. 4. Results with three, five, and seven series items of the present method are all summarized in this figure. It is seen that differences of numerical results between these two methods decrease as the time proceeds. The numerical results of the present method converge rapidly with the increase of the series items considered. Excellent agreement can be obtained when seven terms in the series solution (14) are considered.

Now we consider a FGM circular hollow cylinder with radius $R_1=0.9$ and $R_2=1.1$, as shown in Fig. 1. The

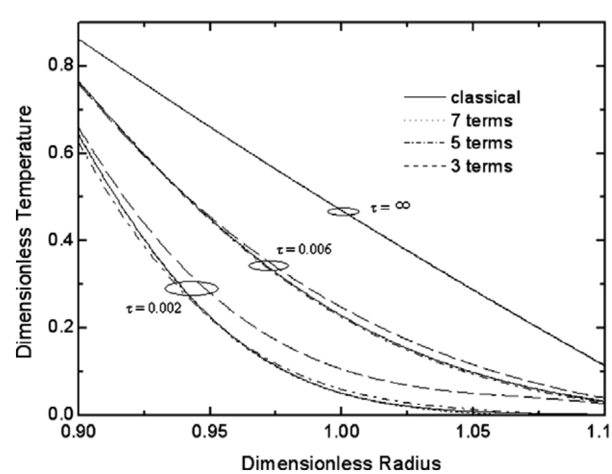

Fig. 4. Time-dependent temperature of a homogeneous circular hollow cylinder.

dimensionless temperature of the inner surface is heated up to 1.0 in a short time interval $\tau_0 = 0.1$. Its outer surface is cooled by surrounding medium. The heat transfer coefficient on the outer surface is 30. The FGM circular hollow cylinder is made from molybdenum and mullite. Its inner surface is pure mullite, and its outer surface is composite of molybdenum/mullite. Both molybdenum/mullite vary continuously from the inner to outer surfaces of the cylinder. E_0, α_0, λ_0, κ_0 and μ of mullite are 225 GPa, $4.8 \times 10^{-6}\,\mathrm{K}^{-1}$, $5.9\,\mathrm{W(m\,K)}^{-1}$, $2.8 \times 10^{-6}\,\mathrm{m}^2\,\mathrm{s}^{-1}$ and 0.3, respectively [23]. It is assumed that variations of the material properties through the thickness of the hollow circular cylinder follows the following exponential laws:

$$Y(R) = \mathrm{Exp}[m_1(R - R_1)], \tag{28a}$$

$$\Omega(R) = \mathrm{Exp}[m_2(R - R_1)], \tag{28b}$$

$$\Lambda(R) = \mathrm{Exp}[m_3(R - R_1)], \tag{28c}$$

$$K(R) = \mathrm{Exp}[m_4(R - R_1)], \tag{28d}$$

where m_1, m_2, m_3 and m_4 are material constants, and $m_1 = 2.0$, $m_2 = 0.3$, $m_3 = 3.0$, $m_4 = 2.0$.

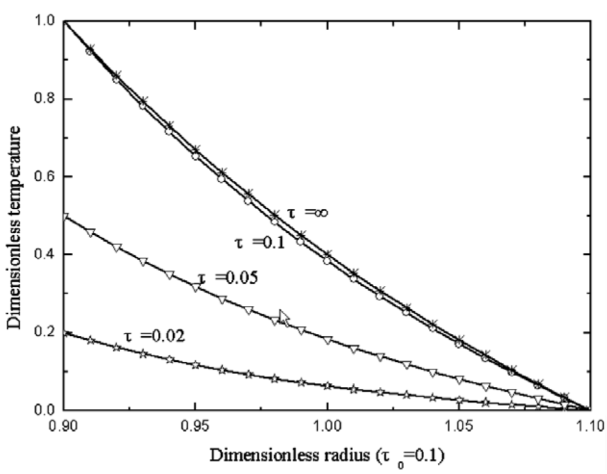

Fig. 5. Time-dependent temperature in the FGM hollow cylinder, ($\tau_0 = 0.1$).

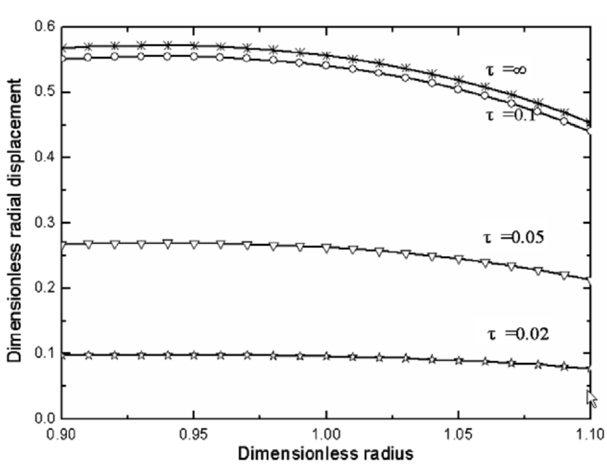

Fig. 6. Transient radial displacement of the FGM hollow cylinder, ($\tau_0 = 0.1$).

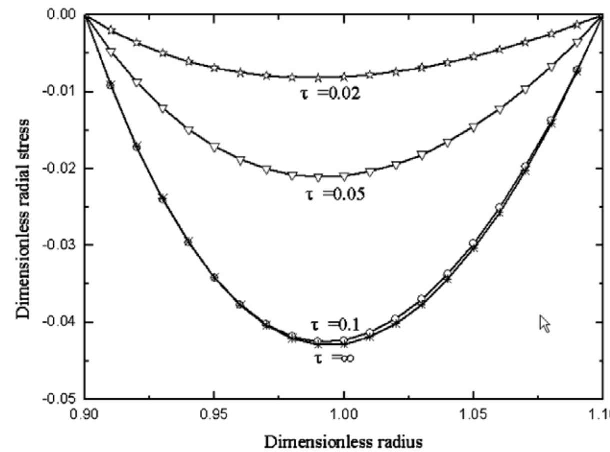

Fig. 7. Transient radial stress of molybdenum/mullite FGM hollow cylinder, ($\tau_0 = 0.1$).

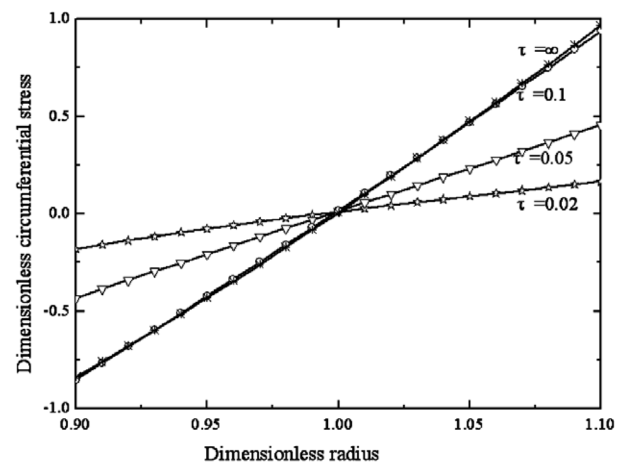

Fig. 8. Transient circumferential stress of the FGM hollow cylinder, ($\tau_0 = 0.1$).

Fig. 5 shows the numerical results of the time-dependent temperature in the FGM cylinder. It is seen that temperature increases with time in the phase of $0 \leqslant \tau \leqslant \tau_0$ and rapidly approaches to the steady-state value in the phase of $\tau_0 \leqslant \tau$.

Fig. 6 shows the numerical results of the dimensionless radial displacement. The radial displacement increases with time in the phase of $0 \leqslant \tau \leqslant \tau_0$ and approaches rapidly to the maximum value at steady state in the subsequent phase of $\tau_0 \leqslant \tau$. Figs. 7 and 8 show the numerical results of the transient radial and circumferential thermal stresses, respectively. It is seen that absolute values of the radial and circumferential thermal stresses increase with time in the phase of $0 \leqslant \tau \leqslant \tau_0$and rapidly approaches to the maximum values at steady state in the following phase of $\tau_0 \leqslant \tau$.

5. Conclusions

Thermo-mechanical analysis of FGM circular hollow cylinders subjected to mechanical loads and linear-type heat supply is presented in this paper. Using Laplace transform technique and series solving method for ordinary differential equation, solutions of time-dependent temperature and unsteady thermo-mechanical stresses of FGM hollow cylinders are derived. For exemplification, using the derived temperature solution and employing the exponential material law for FGM, we analyzed a molybdenum/mullite FGM hollow circular cylinder.

The virtue of the present method lies on that the variation continuity of material properties in one direction can be fully taken into account, and it is applicable to any material model suggested for FGMs.

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