Mathematics of Computer Science. - M.I.T. opencourseware

Abstract:
Random walks can reveal whether possibility of game or any field will derive player to doom or win.

For the English:
get through it, martingale, are gonna, numerator, formula has nothing to do with, outweight, take over, drift,

Lec 25. RANDOM WALKS:

- Gambler ruin problem
start with \$n / each bet : win \$1 w/ prob P <-> 1-p
play until + \$m or lose \$n (broke)
EX: Roulette p = 18/38 = 9/19 = 0.473
m = \$100 (goal) // n = \$1,000
no matter how much money you bring -> you're likely to go to broke
Pr(win \$100) =< 1/37,648

1-dimensional random walk prob of up = p <-> 1-p
: mut indep of post moves => martingale
● graph swing but drift down
If p != half, random walk is biased <-> unbiased
- Boundary condition
DEF: w* = event hit \$T = ntm before it hits 0 (broke)
D = # \$ at start
Xn = Pr(w*|D=n)
Claim: Xn = { 0 if n=0| 1 if n=T | pXn+T + (1-p)Xn-1 if 0<n<T }
PF: X0 = Pr(w*|D=0) = start with broke = no chance to start = 0
XT = Pr(w*|D=T) = 1 = already m money = hit the top boundary
For 0<n<T DEF: E = event win 1st bet <-> !E = lose 1st bet
Xn = Pr(w*|D=n)
= Pr(w*∩E|D=n) + Pr(w*∩!E|D=n)
= Pr(E|D=n)Pr(w*|E∩D=n) + Pr(!E|D=n)Pr(w*|!E∩D)
= P*Pr(w*|D=n+1) + (1-p)Pr(w*|D=n-1) = pXn+1 + (1-p)Xn-1
martingale
pXn+1 - Xn + (1-p)Xn-1 = 0 X0 = 0, XT = 1
Char-eqn: pr^2 - r + (1-p) = 0 => 1+-√1-4p+4p^2 / 2p = 2-2p/2p or 2p/2p = 1-p/p, 1
if p != 1/2, then Xn = A(1-P/P)^N + b(1)^n = A(1-p/p)^n + B
Bdry conds: 0 = X0 = A+B => B = -A
1 = XT = A(1-p/p)^T -A => A = 1/(1-p/p)^T - 1
=> Xn = (1-p/p)^n-1/(1-p/p)^T - 1 if p < 1/2, 1-p/p > 1 -> Lim => 0
=> =< (1-p/p)^n-T = (p/1-p)^T-n = (p/1-p)^m

THM: If $p < 1/2$, then Pr(win \$m before lose \$n) $\leq (p/1-p)^m$
EX: $p = 9/19$, $p/1-p = 9/10$, $m = 100$, $n = 1000$, Pr(win \$100) $\leq (9/10)^{100} < 1/37{,}648$
formula has nothing to do with n, ever $m = 10 \Rightarrow 0.35$ and it's meaningless

$p = 1/2$, $X_n = (An+B)(1)^n = An+B$
Bdry conds: $0 = X_0 = B = 0$, $1 = X_T = AT = 1 \Rightarrow A = 1/T$
$X_n = n/T = n/n+m$
THM: If $p = 1/2$, then Pr(win \$m before lose \$n) $\leq n/n+m \Rightarrow \text{Lim} \Rightarrow 1 = 100\%$
EX: $m = 100$, $n = 1000$ pr(win) $= 10/11$

check the origin two graph that unbiased random walk and biased random walk
later one is dirft outweight swings = take over
drift $= (1-2p)$

After x steps: drifted $(1-2p)x$ = downward linear graph
Ex(lose) $= 1(1-p) - 1(p) = 1-2p$
swing is $\Theta(\sqrt{x})$ => square root => so drift outweight swing

DEF: S = # steps until we hit bdry, $E_n = Ex(S|D=n)$
Claim: $E_n = \{0 \text{ if } n = 0 | 0 \text{ if } n = T | 1+pE_{n+1} + (1-p)E_{n-1} \text{ if } 0<n<T \}$
$\Rightarrow E_0 = 0$, $E_T = 0$, $pE_{n+1} - E_n + (1-p)E_{n-1} = -1$
1. Homo sol: $E_n = A(1-p/p)^n + B$ for $p \neq 1/2$
2. Particular sol: Guess $E_n = a$ -> fails => $E_n = an+b$ -> ok // $a = 1/1-2p$, $b = 0$
3. Gen sol: $E_n = A(1-p/p)^n + B + n/1-2p$
Bdry cond: $E_n = n/1-2p - T/1-2p * (1-p/p)^n - 1 / (1-p/p)^T - 1$
EX: $m = 100$, $n = 1000$, $T = 1100$, $p = 9/19 \Rightarrow$ Exp(#bets) $= 19000 - 0.56 = 19000 \sim$
play foever until lose everyting -> AsM -> infinite $E_n \sim n/1-2p$
If $p = 1/2$,
1. Homo sol: $E_n = An+B$
2. Part sol: Try $E_n = a$, fails, $an+b$, fails, $an^2 + bn + c$ => work! $= a = -1$, $b,c = 0$
$\Rightarrow E_n = an+B - n^2$
bdry cond: $E_0 = 0 = B$, $E_T = 0$ 0 $AT - T^2 \Rightarrow A = T \Rightarrow E_n = Tn - n^2$
$= (n+m)n - n^2 = nm$

last THM: (Quit while you're ahead!) - If you start w/ \$n & $p = 1/2$, & you play til you go broke
then Pr(go broke) $= 1$
PF: by contradiction
Assume it's not true $\exists n$ & $\varepsilon>0 \Rightarrow$ Pr(broke) $\leq 1-\varepsilon$
$\Rightarrow \forall m$ Pr(lose \$n before win \$m) $\leq 1-\varepsilon \Rightarrow \forall m * m/n+m \leq 1-\varepsilon \Rightarrow m \leq (1-\varepsilon/\varepsilon)n$
-> fixed value

the end.