Notes for the course
- – REAL ANALYSIS

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January 23, 2020

Contents

Part I

Real Analysis I, by Elon L. Lima

Chapter 6

Some Topological Notions

Definition (Interior Point)**.** The point a is *interior* to the set $X \subseteq \mathbb{R}$ when there is some $\epsilon > 0$ st $(a - \epsilon, a + \epsilon) \subseteq \mathbb{R}$. The set of interior points of X, called the *interior* of X, is represented as $\operatorname{int} X$.

Definition (Neighborhood)**.** If $a \in \operatorname{int} X$, then X is a *neighborhood* of a.

Definition (Open Set)**.** $X \subseteq \mathbb{R}$ is open iff $\operatorname{int} X = X$.

Fact (Limit of a Sequence in Terms of Open Sets)**.** $a = \lim x_n$ iff for every open set A containing a, x_n is eventually contained in A.

Theorem 1. *Arbitrary unions of opens are open; finite intersections of opens are open.*

Definition (Adherent Point)**.** a is *adherent* to the set $X \subseteq \mathbb{R}$ when a is the limit of a sequence of points of X.

Definition (Closure)**.** The *closure* of $X \subseteq R$ is the set of points that adhere to X. It is denoted by $\overline{X}$.

Definition (Closed Set)**.** A set X is *closed* when $\overline{X} = X$.

Definition (Dense Set)**.** Let $X \subseteq Y$. X is *dense* on Y if $Y \subseteq \overline{X}$.

Theorem 2. *a adheres to X iff every neighborhood of a intersects X.*

Corollary. *Closures are closed.*

Theorem 3. *A set is closed iff its complement in $\mathbb{R}$ is open.*

Theorem 4. *Arbitrary intersections of closeds are closed; finite unions of closeds are closed.*

Definition (Cision)**.** The sets A and B are a *cision* of $X = A \cup B$ if $A \cap \overline{B} = \overline{A} \cap B = \emptyset$.

Theorem 5. *Intervals of the rect only admit the trivial cision.*

Corollary. *The only clopen subsets of* $\mathbb{R}$ *are* $\emptyset$ *and* $\mathbb{R}$.

Definition (Accumulation Point)**.** a is an *accumulation point* of X if every neighborhood of a intersects $X \setminus \{a\}$. The set of accumulation points of X is denoted as X'.

Definition (Isolated Point)**.** $a \in X$ is a point *isolated* from X if a is not an accumulation point of X.

Definition (Discrete Set)**.** A set is *discrete* if all of its elements are isolated points.

Theorem 6. *Given* $X \subseteq \mathbb{R}$ *and* $a \in \mathbb{R}$*, the following are equivalent:*

1. $a \in X'$
2. $a \in \overline{X \setminus \{a\}}$
3. *Every interval centered in* a *has infinite elements of* X.

Theorem 7. *Every bounded infinite set has an accumulation point.*

Definition (Compact Set)**.** A set $X \subseteq \mathbb{R}$ is *compact* if it is limited and closed.

Theorem 8. *A set* $X \subseteq \mathbb{R}$ *is compact iff every sequence of points of* X *has a subsequence that converges to a point of* X.

Fact. Every compact set has a mininum and a maximum element.

Theorem 9. *Given a sequence* $X_1 \supseteq X_2 \supseteq \dots$ *of compact nonempty sets, there exists an element belonging to each* X_i.

Theorem 10 (Borel–Lebesgue)**.** *Every open covering of a compact set has a finite subcovering.*