3 Sequences of Real Numbers

Definition (Limit of a Sequence; Convergence)**.** a is a *limit* of the sequence (x_n) if for every $\epsilon > 0$ the sequence is eventually ϵ-closed to a. We say that (x_n) *converges* to a, and that (x_n) is *convergent*.

Theorem 1. *Limits of sequences are unique.*

Theorem 2. *If* $\lim x_n = a$ *then every subsequence of* (x_n) *converges to* a*.*

Theorem 3. *Every congervent sequence is limited.*

Fact. If a monotone sequence (x_n) has a limited subsequence, then (x_n) itself is limited.

Theorem 4. *Every limited monotone sequence is convergent.*

Corollary (Bolzano–Weierstrass Theorem)**.** *Every limited sequence has a convergent subsequence.*

Theorem 5. *If* $b < \lim x_n$*, then eventually* $b < (x_n)$*. Analogously,* $b > \lim x_n$ *implies that* $b > (x_n)$ *eventually.*

Corollary. *If eventually* $(x_n) \leq (y_n)$*, then* $\lim x_n \leq \lim y_n$*. In particular, if eventually* $(x_n) \leq c$*, then* $\lim x_n \leq c$*.*

Theorem 6 (Sandwich's Theorem)**.** *If* $\lim x_n = \lim z_n = l$ *and eventually* $(x_n) \leq (y_n) \leq (z_n)$*, then* $\lim y_n = l$*.*

Theorem 7. *If* $\lim x_n = 0$ *and* (y_n) *is a limited sequence, then* $\lim x_n y_n = 0$*.*

Fact. $\lim x_n = a \iff \lim(x_n - a) = 0 \iff \lim |x_n - a| = 0$.

Theorem 8. *If* $\lim x_n = a$ *and* $\lim y_n = b$*, then*

1. $\lim(x_n \pm y_n) = a \pm b$.
2. $\lim x_n y_n = ab$.
3. $\lim \frac{x_n}{y_n} = a/b$ *given that* $b \neq 0$.

Fact. If $(x_n) > 0$ and $\lim \frac{x_{n+1}}{x_n} < 1$ then $\lim x_n = 0$.

Fact. If $a > 1$ and $k \in \mathbb{N}$ then

$$\lim_{n\to\infty} \frac{n^k}{a^n} = \lim_{n\to\infty} \frac{a^n}{n!} = \lim_{n\to\infty} \frac{n!}{n^n} = 0$$

Fact. If $a > 0$ then $\lim a^{1/n} = 1$.

Fact. $\lim \left(\frac{n+1}{n}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n} = e$

Fact. $\lim n^{1/n} = 1$

Fact. $\lim \frac{x_n + a/x_n}{2} = \sqrt{a}, a > 0$

Definition (Infinite Limits)**.** We say that $\lim x_n = \infty$ when for each $M > 0$ eventually $(x_n) > M$. Analogously, $\lim x_n = -\infty$ when for each $N < 0$ eventually $(x_n) < N$.

Fact. $\lim x_n = -\infty \iff \lim -x_n = \infty$

Theorem 9.

1. *If* $\lim x_n = \infty$ *and* (y_n) *has a lower bound then* $\lim(x_n + y_n) = \infty$*.*
2. *If* $\lim x_n = \infty$ *and* $(y_n) > c$ *for some* $c > 0$ *then* $\lim x_n y_n = \infty$*.*
3. *If* $(x_n) > c$ *for some* $c > 0$ *and* $(y_n) > 0$ *then* $\lim \frac{x_n}{y_n} = \infty$*.*
4. *If* (x_n) *é limitada e* $\lim y_n = \infty$ *then* $\lim \frac{x_n}{y_n} = 0$*.*

Fact. $\lim \frac{\log n}{n} = 0$

4 Numeric Series

Definition (Series, Partial Sums)**.** A *series* is a limit of the form $\lim(a_1 + \cdots + a_n)$. The numbers $s_n = a_1 + \cdots + a_n$ are the *partial sums* of the series, and the term a_n is its *general term.*

Definition (Congervence)**.** When the limit corresponding to a series exists, the series is *convergent.* Otherwise, it is *divergent.*

Fact. The series $1 + a + a^2 + \ldots$ is the *geometric series.* It converges to $1/1 - a$.

Fact. A series of nonnegative terms converges iff there's a constant that limits each of its partial sums.

Fact. The series $\sum 1/n$ is the *harmonic series.* It diverges.

Theorem 10 (Comparison Criteria)**.** *Let* $\sum a_n$ *and* $\sum b_n$ *be series with nonnegative terms. If there's some* $c > 0$ *such that eventually* $(a_n) \leq c\,(b_n)$*, then the convergence of* $\sum b_n$ *implies that* $\sum a_n$ *converges; and if* $\sum a_n$ *diverges then so does* $\sum b_n$*.*

5 Some Topological Notions

Definition (Interior Point)**.** The point a is *interior* to the set $X \subseteq \mathbb{R}$ when there is some $\epsilon > 0$ st $(a - \epsilon, a + \epsilon) \subseteq \mathbb{R}$. The set of interior points of X, called the *interior* of X, is represented as $\operatorname{int} X$.

Definition (Neighborhood)**.** If $a \in \operatorname{int} X$, then X is a *neighborhood* of a.

Definition (Open Set)**.** $X \subseteq \mathbb{R}$ is open iff $\operatorname{int} X = X$.

Fact (Limit of a Sequence in Terms of Open Sets)**.** $a = \lim x_n$ iff for every open set A containing a, x_n is eventually contained in A.

Theorem 11. *Arbitrary unions of opens are open; finite intersections of opens are open.*

Definition (Adherent Point)**.** a is *adherent* to the set $X \subseteq \mathbb{R}$ when a is the limit of a sequence of points of X.

Definition (Closure)**.** The *closure* of $X \subseteq R$ is the set of points that adhere to X. It is denoted by $\overline{X}$.

Definition (Closed Set)**.** A set X is *closed* when $\overline{X} = X$.

Definition (Dense Set)**.** Let $X \subseteq Y$. X is *dense* on Y if $Y \subseteq \overline{X}$.

Theorem 12. *a adheres to X iff every neighborhood of a intersects X.*

Corollary. *Closures are closed.*

Theorem 13. *A set is closed iff its complement in $\mathbb{R}$ is open.*

Theorem 14. *Arbitrary intersections of closeds are closed; finite unions of closeds are closed.*

Definition (Cision)**.** The sets A and B are a *cision* of $X = A \cup B$ if $A \cap \overline{B} = \overline{A} \cap B = \emptyset$.

Theorem 15. *Intervals of the rect only admit the trivial cision.*

Corollary. *The only clopen subsets of $\mathbb{R}$ are $\emptyset$ and $\mathbb{R}$.*

Definition (Accumulation Point)**.** a is an *accumulation point* of X if every neighborhood of a intersects $X \setminus \{a\}$. The set of accumulation points of X is denoted as X'.

Definition (Isolated Point)**.** $a \in X$ is a point *isolated* from X if a is not an accumulation point of X.

Definition (Discrete Set)**.** A set is *discrete* if all of its elements are isolated points.

Theorem 16. *Given $X \subseteq \mathbb{R}$ and $a \in \mathbb{R}$, the following are equivalent:*

1. $a \in X'$
2. $a \in \overline{X \setminus \{a\}}$
3. *Every interval centered in a has infinite elements of X.*

Theorem 17. *Every bounded infinite set has an accumulation point.*

Definition (Compact Set)**.** A set $X \subseteq \mathbb{R}$ is *compact* if it is limited and closed.

Theorem 18. *A set $X \subseteq \mathbb{R}$ is compact iff every sequence of points of X has a subsequence that converges to a point of X.*

Fact. Every compact set has a mininum and a maximum element.

Theorem 19. *Given a sequence $X_1 \supseteq X_2 \supseteq \dots$ of compact nonempty sets, there exists an element belonging to each X_i.*

Theorem 20 (Borel–Lebesgue)**.** *Every open covering of a compact set has a finite subcovering.*