

Notes for the course
- – REAL ANALYSIS

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Part I

Real Analysis I, by Elon L.
Lima

Chapter 3

Sequences of Real Numbers

Definition (Limit of a Sequence; Convergence). a is a *limit* of the sequence (x_n) if for every $\epsilon > 0$ the sequence is eventually ϵ -closed to a . We say that (x_n) *converges* to a , and that (x_n) is *convergent*.

Theorem 1. *Limits of sequences are unique.*

Theorem 2. *If $\lim x_n = a$ then every subsequence of (x_n) converges to a .*

Theorem 3. *Every convergent sequence is limited.*

Fact. If a monotone sequence (x_n) has a limited subsequence, then (x_n) itself is limited.

Theorem 4. *Every limited monotone sequence is convergent.*

Corollary (Bolzano–Weierstrass Theorem). *Every limited sequence has a convergent subsequence.*

Theorem 5. *If $b < \lim x_n$, then eventually $b < (x_n)$. Analogously, $b > \lim x_n$ implies that $b > (x_n)$ eventually.*

Corollary. *If eventually $(x_n) \leq (y_n)$, then $\lim x_n \leq \lim y_n$. In particular, if eventually $(x_n) \leq c$, then $\lim x_n \leq c$.*

Theorem 6 (Sandwich's Theorem). *If $\lim x_n = \lim z_n = l$ and eventually $(x_n) \leq (y_n) \leq (z_n)$, then $\lim y_n = l$.*

Theorem 7. *If $\lim x_n = 0$ and (y_n) is a limited sequence, then $\lim x_n y_n = 0$.*

Fact. $\lim x_n = a \iff \lim(x_n - a) = 0 \iff \lim |x_n - a| = 0$.

Theorem 8. *If $\lim x_n = a$ and $\lim y_n = b$, then*

1. $\lim(x_n \pm y_n) = a \pm b$.
2. $\lim x_n y_n = ab$.

3. $\lim \frac{x_n}{y_n} = a/b$ given that $b \neq 0$.

Fact. If $(x_n) > 0$ and $\lim \frac{x_{n+1}}{x_n} < 1$ then $\lim x_n = 0$.

Fact. If $a > 1$ and $k \in \mathbb{N}$ then

$$\lim_{n \rightarrow \infty} \frac{n^k}{a^n} = \lim_{n \rightarrow \infty} \frac{a^n}{n!} = \lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0$$

Fact. If $a > 0$ then $\lim a^{1/n} = 1$.

Fact. $\lim \left(\frac{n+1}{n}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} = e$

Fact. $\lim n^{1/n} = 1$

Fact. $\lim \frac{x_n + a/x_n}{2} = \sqrt{a}, a > 0$

Chapter 5

Some Topological Notions

Definition (Interior Point). The point a is *interior* to the set $X \subseteq \mathbb{R}$ when there is some $\epsilon > 0$ st $(a - \epsilon, a + \epsilon) \subseteq X$. The set of interior points of X , called the *interior* of X , is represented as $\text{int } X$.

Definition (Neighborhood). If $a \in \text{int } X$, then X is a *neighborhood* of a .

Definition (Open Set). $X \subseteq \mathbb{R}$ is open iff $\text{int } X = X$.

Fact (Limit of a Sequence in Terms of Open Sets). $a = \lim x_n$ iff for every open set A containing a , x_n is eventually contained in A .

Theorem 9. *Arbitrary unions of opens are open; finite intersections of opens are open.*

Definition (Adherent Point). a is *adherent* to the set $X \subseteq \mathbb{R}$ when a is the limit of a sequence of points of X .

Definition (Closure). The *closure* of $X \subseteq \mathbb{R}$ is the set of points that adhere to X . It is denoted by \overline{X} .

Definition (Closed Set). A set X is *closed* when $\overline{X} = X$.

Definition (Dense Set). Let $X \subseteq Y$. X is *dense* on Y if $Y \subseteq \overline{X}$.

Theorem 10. *a adheres to X iff every neighborhood of a intersects X .*

Corollary. *Closures are closed.*

Theorem 11. *A set is closed iff its complement in \mathbb{R} is open.*

Theorem 12. *Arbitrary intersections of closed sets are closed; finite unions of closed sets are closed.*

Definition (Cision). The sets A and B are a *cision* of $X = A \cup B$ if $A \cap \overline{B} = \overline{A} \cap B = \emptyset$.

Theorem 13. *Intervals of the real line only admit the trivial clopen sets.*

Corollary. *The only clopen subsets of \mathbb{R} are \emptyset and \mathbb{R} .*

Definition (Accumulation Point). a is an *accumulation point* of X if every neighborhood of a intersects $X \setminus \{a\}$. The set of accumulation points of X is denoted as X' .

Definition (Isolated Point). $a \in X$ is a point *isolated* from X if a is not an accumulation point of X .

Definition (Discrete Set). A set is *discrete* if all of its elements are isolated points.

Theorem 14. *Given $X \subseteq \mathbb{R}$ and $a \in \mathbb{R}$, the following are equivalent:*

1. $a \in X'$
2. $a \in \overline{X \setminus \{a\}}$
3. *Every interval centered at a has infinite elements of X .*

Theorem 15. *Every bounded infinite set has an accumulation point.*

Definition (Compact Set). A set $X \subseteq \mathbb{R}$ is *compact* if it is limited and closed.

Theorem 16. *A set $X \subseteq \mathbb{R}$ is compact iff every sequence of points of X has a subsequence that converges to a point of X .*

Fact. Every compact set has a minimum and a maximum element.

Theorem 17. *Given a sequence $X_1 \supseteq X_2 \supseteq \dots$ of compact nonempty sets, there exists an element belonging to each X_i .*

Theorem 18 (Borel–Lebesgue). *Every open covering of a compact set has a finite subcovering.*