An Extension of Hölder's Inequality with an Application to Economics

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Abstract

We extend a well known fact in functional analysis, Hölder's Inequality, and use it to get simple derivations of certain economic formulas involving CES, Arminton, or n-stage Armington functions.

1 Introduction

Solow introduced CES functions in (Solow, 1956), see also (Arrow et al., 1961), to develop a growth model without the Harrod-Domar model assumption, which leads to model instability, that capital and labor are not substitutes. Since its appearance in the seminal papers, CES functions have become an essential economic modeling tool. We recall some of their basic properties here to fix notation. By definition, a CES function with parameter $r \leq 1$, $r \neq 0$ assigns to a vector $x = (x_1, \ldots, x_n)$ with positive entries the number:

$$\left(\sum_{i=1}^{n} x_i^r \right)^{\frac{1}{r}} . \tag{1}$$

When $r > 0$, the CES function describes inputs that are substitutes, or perfect substitutes when $r = 1$. When $r < 0$, the inputs of the CES function are complements, or perfect complements in the limiting case $r = -\infty$, which is a Leontief function. If θ is a weight vector, we can replace x with $\theta x = (\theta_1 x_1, \ldots, \theta_n x_n)$ in Formula 1 and take the limit $r \to 0$ to get a Cobb-Douglas function. We can also change Formula 1 by multiplying it by a factor or raising it to a power to change homogeneity. The CES functional form, when including all extra parameters and limiting cases, is the only function with the CES between any pair of inputs (see Arrow et al. (1961)).

Given their versatility, CES functions are the building blocks of several models. For example, many Computable General Equilibrium models consist of families of nested CES functions. A drawback of CES functions, however, is solving optimization problems involving them. As stated in (Rutherford, 2002), the usual Lagrangian calculus approach to solving such optimization problems can lead to long derivations. We claim that, by recasting CES functions as a type of norm of a vector spaces, and by extending Hölder's inequality to that setting, it is easy to solve these optimization problems. Similarly, the (n-stage) Armington functions of (Armington, 1969) can be viewed as the a type of norm on a direct sum of "normed" vector spaces. We can still extend Hölder's inequality to that setting and get simple solutions methods of optimization problems involving these functions.

2 L^p-spaces and Hölder's inequality

A norm on a vector space V is a real valued function $\| \cdot \|$ satisfying: (1) $\| \cdot \|$ is finite and convex, (2) $\|\alpha v\| = |\alpha| \, \|v\|$ for any vector v of V and scalar α, and (3) $\| \cdot \|$ is zero only at the zero vector. The convexity

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assumption is usually replaced by the triangular inequality. If we allow $\|\cdot\|$ to be zero away from zero, then it is called a pseudo-norm. If $\|\cdot\|$ only satisfies the triangular inequality up to a constant multiple[1], then it is called a quasinorm. L^p-spaces, where p is a number larger than 1, are a particular rich family of normed vector spaces that have been the subject of intense study (see, for example, (Rudin, 1991)). We will denote these spaces by $L^{p\geq 1}$. Their norms are defined as:

$$\|f\|_p = \left(\int |f|^p\right)^{\frac{1}{p}}.$$

Where f is, roughly, a real valued function on a measurable space. $L^{p\geq 1}$-spaces have a wide range of applications. For example, the state spaces in quantum mechanics are L^2-spaces. If the measure is a probability measure, then f is a random variable and $\|f\|_q$ is its q-moment. To simplify our arguments, we will restrict to $L^p(\mathbb{R}^n)$, the set of vectors $x = (x_1, \ldots, x_n)$ of $\mathbb{R}^n$ with norm:

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}. \tag{2}$$

The main fact on $L^{p\geq 1}$-spaces that we want to generalize to the case when $p < 1$ is Hölder's inequality (see Rudin (1991)):

Lemma 1 (Hölder's inequality)**.** *For $p \geq 1$ and q satisfying*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

it follows that

$$\|x\,y\|_1 \leq \|x\|_p \|y\|_q$$

for any two vectors x in L^p and y in L^q, where $x\,y$ is the point-wise multiplication. (In the finite dimensional case, we can replace $\|x\,y\|_1$ by $x \cdot y$ since $x \cdot y \leq \|x\,y\|_1$.)

2.1 $L^{p<1}$-spaces

It is clear that the definition of CES functions, Equation 1, and the definition of the L^p-norms, Equation 2, have the same functional form. The only difference is that CES functions depend on a parameter $r \leq 1$ and for L^p-spaces one usually assumes that $p \geq 1$. For now on we use p for price and use r instead of p for L^p-spaces, even when $p > 1$. From the viewpoint of functional analysis, the case when $r < 1$ has a number of undesirable properties. For example, for the infinite dimensional case, the dual space of $L^{0<r<1}$ is trivial[2]. Furthermore, $\|\cdot\|_r$ is not a norm but a quasinorm for $0 < r < 1$ (see Conrad (2002)) and not even a pseudo-quasinorm[3] when $r < 0$. Nevertheless, properties of $L^{r\geq 1}$-spaces may have $L^{r\leq 1}$ counterparts. For the purposes of this note, we state the $L^{r<1}$ counterpart of Hölder's inequality. Its proof is a straightforward adaptation of the original one and the main part of the argument has a simple geometric interpretation.

Lemma 2 (Hölder's inequality for $L^{r\leq 1,\, r\neq 0}$-spaces)**.** *For $r < 1$, $r \neq 0$, and s satisfying*

$$\frac{1}{r} + \frac{1}{s} = 1,$$

[1] i.e., $\|x + y\| \leq K(\|x\| + \|y\|)$ for some $K > 0$.

[2] Namely, under these assumptions, if ϕ is a continuous linear function on L^r then $\phi = 0$. For a proof, see (Conrad, 2002). In particular, if $x : [0,1] \to \mathbb{R}$ is a continuum of goods with prices $p : [0,1] \to \mathbb{R}$ then the linear functional $p(x) = \int p\,x$ is not continuous with respect to a utility function defined by $\|\cdot\|_r$. For example, if we set prices to be equal to 1 for every good and consider the sequence of consumption bundles $x_n = n\,\chi_{[0,\frac{1}{n}]}$, where χ is an indicator function, then $\|x_n\|_r = n^{r-1} \to 0$ as $n \to \infty$, (x_n approaches the zero bundle as n increases) but $p(x_n) = 1$ for all n whereas $p(0) = 0$.

[3] When $r < 0$, it is natural to assume that $\|x\|_r = 0$ whenever x has a zero entry. We can write any vector v of $L^{r<0}(\mathbb{R}^n)$ as a sum of two vectors v_1, v_2 each of which with a zero entry. If $\|\cdot\|_r$ was a pseudo-quasinorm, then $\|v\|_r \leq K(\|v_1\|_r + \|v_2\|_r) = 0$ for any vector v, a contradiction.

it follows that for any two non-negative vectors y and x:

$$y \cdot x \geq \|x\|_r \|y\|_s.$$

Proof: We begin by showing that if a and b are positive numbers, and for r and s satisfying the lemma's assumptions, then

$$ab \geq \frac{a^r}{r} + \frac{b^s}{s}. \tag{3}$$

(This is an extension of Young's inequality, see Rudin (1991)). To check this, consider the function $y = x^{r-1}$ and its inverse $x = y^{s-1}$. Both functions are decreasing since $r, s < 1$. By evaluating the right hand side of the equation below, it is easy to show that:

$$\frac{a^r}{r} + \frac{b^s}{s} = \frac{1}{2}\left(\int_{b^{s-1}}^{a} x^{r-1}dx\right) + \frac{1}{2}\left(\int_{a^{r-1}}^{b} y^{s-1}dy\right) + \frac{1}{2}b^{s-1}(b - a^{r-1}) + \frac{1}{2}a^{r-1}(a - b^{s-1}) + a^{r-1}b^{s-1}. \tag{4}$$

Either $a \geq b^{s-1}$ or $a \leq b^{s-1}$ in the domain of the first integral of the above equation. If we assume the former case, then the right-hand side of Equation 4 calculates the area of shaded region of Figure 1. Since this region is contained in a rectangle with area equal to ab, the statement follows. We can give an algebraic proof of the statement by starting with $a \geq x \geq b^{s-1}$, and raising these numbers by $r-1$ to get $b \geq x^{r-1} \geq a^{r-1}$. Setting $y = x^{r-1}$, we can raise the last inequality by $s-1$ to get $a \geq y^{s-1} \geq b^{s-1}$. Hence,

$$\int_{b^{s-1}}^{a} x^{r-1}dx \leq \int_{b^{s-1}}^{a} b\,dx = b(a - b^{s-1}), \tag{5}$$

and

$$\int_{a^{r-1}}^{b} y^{s-1}dy \leq \int_{a^{r-1}}^{b} a\,dy = a(b - a^{r-1}). \tag{6}$$

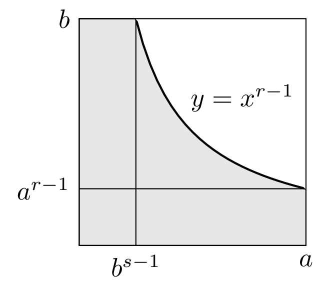

Figure 1: When $a > b^{s-1}$, Equation 4 computes the shaded area in a rectangle with area ab.

By plugging in Equations 5, 6 into Equation 4 we get that $\frac{a^r}{r} + \frac{b^s}{s} \leq ab$.

Notice that the same inequalities hold true if $a \leq b^{s-1}$ (and hence $b \leq a^{r-1}$) since now the integrals of Equation 4 become negative numbers. Visually, as depicted in Figure 2, are now subtracting from the are of a rectangle an amount larger than the area of a region whose complement has area ab. Hence, we still have that $\frac{a^r}{r} + \frac{b^s}{s} \leq ab$.

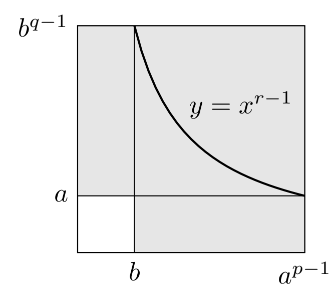

Figure 2: When $a < b^{s-1}$, Equation 4 subtracts an from the area of the depicted rectangle a number larger than the shaded are shaded.

With the extended Young's inequality (Equation 3) hand, we can now finish the proof of the lemma. Suppose $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are two vectors with positive entries. Then:

$$\begin{aligned}
\frac{x}{\|x\|_r} \cdot \frac{y}{\|y\|_s} &:= \sum_{i=1}^{n} \frac{x_i}{\|x\|_r} \cdot \frac{y_i}{\|y\|_s} \\
&\geq \frac{1}{r} \sum \frac{x_i^r}{\|x\|_r^r} + \frac{1}{s} \sum \frac{y_i^s}{\|y\|_s^s} \\
&= \frac{1}{r} + \frac{1}{s} = 1.
\end{aligned}$$

Where the inequality in the above equation follows from Equation 3. Hence $x \cdot y \geq \|x\|_r \|y\|_s$. $\square$

Corollary 1 (Hölder's inequality for L^0-spaces)**.** *Suppose $\theta = (\theta_1, \ldots, \theta_n)$ is a weight vector, that is, a vector with positive entries where $\sum_{i=1}^n \theta_i = 1$. For any two vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ with positive entries, we have that:*

$$x \cdot y \geq \prod_{i=1}^n x_i^{\theta_i} \prod_{i=1}^n (\theta_i^{-1} y_i)^{\theta_i} = \|x\|_{0,\,\theta}\, \|\theta^{-1} y\|_{0,\,\theta}.$$

Where $\|x\|_{0,\theta} = \prod_{i=1}^n x_i^{\theta_i}$ and $\theta^{-1}y = (\theta_i^{-1} y_1, \ldots, \theta_n^{-1} y_n)$.

Remark 1 (Versions of L^0-spaces)**.** To clarify, we are considering $L^0_\theta(\mathbb{R}^n)$ as $\mathbb{R}^n$ together with the Cobb-Douglas function $\prod |x_i|^{\theta_i}$. There are other natural candidates for L^0-spaces such as the F-norm of (Banach, 1987), the counting norm of (Donoho and Elad, 2002), and the example in (Kalton et al., 1984) of functions with the convergence in measure topology.

Proof: For any $r \leq 1$, $r \neq 0$, and given θ, x and y satisfying the Corollary's assumptions, we have, by Lemma 2:

$$x \cdot y = (\theta^{1/r} x) \cdot (\theta^{-1/r} y) \geq \|\theta^{1/r} x\|_r . \|\theta^{1/s} (\theta^{-1} y)\|_s.$$

In the limit:

$$\begin{aligned} x \cdot y &\geq \lim_{r \to 0} \|\theta^{1/r} x\|_r \, \|\theta^{1/s} (\theta^{-1} y)\|_s. \\ &= \prod x_i^{\theta_i} \prod \left(\theta_i^{-1} y_i\right)^{\theta_i}. \end{aligned}$$

□

Below, we extend Hölder's inequality to the direct sum of $L^{r \leq 1}$-spaces. When $r \geq 1$, there is no obvious best candidate for a norm on a direct sum of L^r-spaces. Banach considered the following norms for such spaces (see, Banach (1987)). Let $L^{r_1}(\mathbb{R}^{n_1}), \ldots, L^{r_m}(\mathbb{R}^{n_m})$ be spaces with $r_i \geq 1$, $i = 1, \ldots, m$. We can use the given norms to define a function on their direct sum:

$$\begin{aligned} N : \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m} &\longrightarrow \mathbb{R}^m, \\ (x_1, \ldots, x_m) &\mapsto (\|x_1\|_{r_1}, \ldots, \|x_m\|_{r_m}). \end{aligned}$$

We can endow $\mathbb{R}^m$, the target space of N, with an L^r norm, for any $r \geq 1$. Composing the function N with any such norm defines a new function on the direct sum. The new function is still a norm and the space is still a Banach space.

If in the previous construction we assume instead that $r_i \leq 1$ for all cases i, then the constructed "norm" is an Armington function (usually the norm picked over $\mathbb{R}^m$ is the L^0 one). The following corollary extends Hölder's inequality to this setting.

Corollary 2 (Hölder's inequality for direct sums of $L^{r \leq 1}$-spaces)**.** *Fix a finite set $L^{r_1}(\mathbb{R}^{n_1}), \ldots, L^{r_m}(\mathbb{R}^{n_m})$ where $r_i \leq 1$, $i = 1, \ldots, m$. For $r \leq 1$ and s satisfying*

$$\frac{1}{r} + \frac{1}{s} = 1,$$

and for $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ vectors in $\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m}$ where the vectors x_i and y_i have non-negative entries, we have

$$\begin{aligned} x \cdot y &\geq \|X\|_r \|Y\|_s, \quad \text{if } r \neq 0, \\ x \cdot y &\geq \|X\|_{0,\,\theta}\, \|\theta^{-1} Y\|_{0,\,\theta}, \quad \text{if } r = 0 \text{ and any weight vector } \theta. \end{aligned}$$

Where $X = (\|x_1\|_{r_1}, \ldots, \|x_m\|_{r_m})$, $Y = (\|y_1\|_{s_1}, \ldots, \|y_m\|_{s_m})$, and $s_i = \frac{r_i}{r_i - 1}$. (Notice the slight abuse of notation: whenever $r_i = 0$, we assume a weight vector θ_i has being fixed and we replace $\|x_i\|_{r_i}$ and $\|y_i\|_{s_i}$ by $\|x_i\|_{0,\,\theta_i}$ and $\|\theta_i^{-1} y_i\|_{0,\,\theta_i}$, respectively.)

Proof: by Lemma 2 or Corollary 1, we have that:

$$
\begin{aligned}
X \cdot Y &\geq \|X\|_r \|Y\|_s, \quad \text{if } r \neq 0 \\
X \cdot Y &\geq \|X\|_{0,\,\theta}\, \|\theta^{-1} Y\|_{0,\,\theta} \quad \text{if } r = 0 \text{ and any weight vector } \theta.
\end{aligned}
$$

By definition, $X \cdot Y = \sum_{i=1}^m X_i Y_i$. We can apply again Lemma 2 or Corollary 1, to $X_i Y_i$ to get:

$$x_i \cdot y_i \geq X_i Y_i,$$

for $i = 1, \ldots, m$. Since $x \cdot y = \sum_{i=1}^m x_i \cdot y_i$, the proof is complete. $\square$

3 Calculations

In this section we go over some concrete calculations.

Example 1 (CES Utility Function)**.** Let $x = (x_1, \ldots, x_n)$ be a bundle of goods (a vector in $\mathbb{R}^n$ with positive entries), and U an utility function over such bundles. Assuming that U is a CES function then, from the previous section, we think of the bundle x as a point in a $L^r(\mathbb{R}^n)$-space for some $r \leq 1$, $r \neq 0$. Similarly, from the perspective of the previous section, it is natural to consider a price vector p (a vector of positive entries in $\mathbb{R}^n$) of the bundle x as a point in $L^s(\mathbb{R}^n)$ where

$$\frac{1}{r} + \frac{1}{s} = 1.$$

(In other words, a "r-norm" $\|x\|_r$ on the space of goods induces a "s-norm" $\|p\|_s$ on the space of prices.) Let us start with the problem of minimizing expenditure given utility and price levels. Suppose that $U(x) = u$, from Lemma 2 (extended Hölder's inequality), we always have that:

$$x \cdot p = \sum_{i=1}^n p_i x_i \geq \|x\|_r \|p\|_s = u\|p\|_s.$$

Since the above inequality is true for any bundle with utility level u, we get a lower bound to the expenditure function: $e(u, p) \geq u\|p\|_s$. To show that this lower bound is sharp, we show that there is a feasible consumption bundle x (a vector with positive entries) satisfying the lower bound. If we had that $e(u, p) = u\|p\|_s$, then, since the utility function is locally nonsatiated and strictly convex, the Hicksian demand function equals to:

$$x(u, p) = \nabla_p e(u, p) = u\|p\|_s^\sigma p^{-\sigma}.$$

Where $\sigma = 1 - s = \frac{1}{1-r}$ and $p^{-\sigma} = (p_1^{-\sigma}, \ldots, p_n^{-\sigma})$. Since all terms on the left-hand side of the above equation are positive, $x(u, p)$ is a feasible bundle and the lower bound under consideration is sharp. Furthermore, since the lower bound is sharp, we can view $\|p\|_s$ as the cost of a unit of utility for a given price p, and hence the price index[4] is:

$$P^K(p_1, p_0, u) = \frac{e(u, p_1)}{e(u, p_0)} = \frac{\|p_1\|_s}{\|p_0\|_s}.$$

We now consider the dual problem of maximizing utility given price and wealth constraints. This can be solved by noting that if the expenditure function is $e(u, p) = u\|p\|_s$ then the indirect utility function is equal to:

$$\nu(m, p) = m\|p\|_s^{-1}.$$

By Roy's identity, the Marshallian demand function equals to:

$$x(m, p) = -\frac{\nabla_p \nu(p, m)}{\nabla_m \nu(p, m)} = m\frac{\|p\|_s^\sigma p^{-\sigma}}{\|p\|_s} = m\frac{p^{s-1}}{\|p\|_s^s}.$$

Hence $x_i\, p_i = m\frac{p_i^s}{\|p\|_s^s}$ and $p_i^s/\|p\|_s$ is the share of the budget spent on sector i.

[4]This is true cost-of-living index (or Konüs expenditure-based cost-of-living index). It measures the necessary compensation to fix a consumer's utility level after a movement in prices.

Remark 2. The above example avoided solving expenditure minimization problems via Lagrange multiplier methods by getting a lower bound to the expenditure function using ideas from functional analysis and showing that the bound is sharp. Related utility maximization problems were then solved by using results that are applicable to general settings. When including the arguments of Section 2.1, this example is derived from first principles.

Remark 3 (CES with weights and Cobb-Douglas)**.** We can adapt Example 1 to the CES with weights or the Cobb-Douglas case by starting from the inequalities (see Corollary 1):

$$\begin{aligned} x \cdot p &= (\theta^{1/r}x) \cdot (\theta^{-1/r}p) \geq \|\theta^{1/r}x\|_r . \|\theta^{1/s}(\theta^{-1}p)\|_s, \\ x \cdot p &\geq \prod x_i^{\theta_i} \prod \left(\theta_i^{-1}p_i\right)^{\theta_i} = \|x\|_{0,\theta}\|\theta^{-1}p\|_{0,\theta}. \end{aligned}$$

The key calculations for these two cases are:

$$\begin{aligned} \nabla_p \|\theta^{1/s}(\theta^{-1}p)\|_s &= \|\theta^{1/s}(\theta^{-1}p)\|_s^{\sigma}(\theta^{-1}p)^{-\sigma}, \\ \nabla_p \|\theta^{-1}p\|_{0,\theta} &= \|\theta^{-1}p\|_{0,\theta}(\theta^{-1}p)^{-1}. \end{aligned}$$

Example 2 (Armington functions)**.** Suppose x_i is a bundle of n_i goods for $i = 1, \ldots, m$ and let $x = (x_1, \ldots, x_m)$. For each x_i fix an CES utility function $\| \cdot \|_{r_i}$ and let $\| \cdot \|_{s_i}$ be the corresponding CES function on prices p_i where $s_i = \frac{r_i}{1-r_i}$. Let

$$\begin{aligned} X &= \left(\|x_1\|_{r_1}, \ldots, \|x_m\|_{r_m}\right), \\ P &= \left(\|p_1\|_{s_1}, \ldots, \|p_m\|_{s_m}\right). \end{aligned}$$

For a given $r < 1,\ r \neq 0$ and $s = \frac{r}{r-1}$ let: $\|x\| = \|X\|_r$, and $\|p\|^{\bullet} = \|P\|_s$. These are Armington functions. By Corollary 2, we have $x \cdot p \geq \|X\|_r \|P\|_s$. Hence, $e(u,p) \geq u \,\|P\|_s$. By the chain rule, $\nabla_p u \,\|P\|_s$ is a product of positive numbers, hence the lower bound is sharp. In particular, the Hicksian demand functions are:

$$x_i(u,p) = \nabla_{p_i} e(u,p) = u\|P\|_s^{\sigma} P_i^{-\sigma} \|p_i\|_{s_i}^{\sigma_i} p_i^{-\sigma_i}.$$

Where $\sigma = 1 - s$, and $\sigma_i = 1 - s_i$ are the elasticities. As in Example 1, $\| \cdot \|^{\bullet}$ is the cost of a unit of utility and the price index is:

$$P^K(p', p, u) = \frac{\|p'\|^{\bullet}}{\|p\|^{\bullet}} = \frac{\left(\sum_{i=1}^m \|p_i'\|_{s_i}^s\right)^{1/s}}{\left(\sum_{i=1}^m \|p_i\|_{s_i}^s\right)^{1/s}}.$$

Since the indirect utility function is $\nu(m,p) = m\|P\|_s^{-1}$, by Roy's lemma the Marshallian demand functions are:

$$\begin{aligned} x_i(m,p) &= -\frac{\nabla_{p_i}\nu(p,m)}{\nabla_m \nu(p,m)} = m\frac{\|P\|_s^{\sigma}\, P_i^{-\sigma}\, \|p_i\|_{s_i}^{\sigma_i} p_i^{-\sigma_i}}{\|P\|_s} = m\frac{\|P\|_s^{\sigma}\, P_i^{-\sigma}(P_i P_i^{-1})\, \|p_i\|_{s_i}^{\sigma_i} p_i^{-\sigma_i}}{\|P\|_s} \\ &= m\frac{P_i^s}{\|P\|_s^s} \frac{p_i^{s_i - 1}}{\|p_i\|_{s_i}^{s_i}}. \end{aligned}$$

As in Example 1, we can interpret the fractions of the last formula in terms of shares of allocated budget.

Remark 4. As in Remark 3, we can introduce weights θ to the previous Armington function example and take the limit $r \to 0$ to an Armington utility function where the aggregate function is a Cobb-Douglas function. In this case, we have: $e(u,p) = u \prod \left({\theta_i}^{-1}\|p_i\|_{s_i}\right)^{\theta_i} = u\|\theta^{-1}P\|_{0,\theta}$. The price index is then:

$$P^K(p', p, u) = \prod \left(\frac{\|{p'}_i\|_{s_i}}{\|p_i\|_{s_i}}\right)^{\theta_i}.$$

The indirect utility function is $\nu(m,p) = m\|P\|_{0,\theta}^{-1}$. By Roy's identity,

$$x_i(m,p) = m\frac{\theta_i}{P_i}\nabla_{p_i} P_i = m\theta_i = m\theta_i \frac{p_i^{s_i - 1}}{\|p_i\|_{s_i}^{s_i}}.$$

As before, we can interpret the last formula in terms of budget shares.

Remark 5 (Armington with several stages)**.** We can introduce more stages to the Armington function by aggregating finite sets of Armington functions using Cobb-Douglas or CES functions. These process can be repeated n times. Hölder's inequality, as stated in Corollary 2, can be extended to these cases by induction. Example 2 can be adapted to these cases. In particular, the Marshallian demand function is again a product of factors that can be understood as budget shares.

4 Conclusion

In this paper, we recast CES, Cobb-Douglas and Armington functions as $L^{r\leq 1}(\mathbb{R}^n)$-spaces, or direct sums of these spaces. We extend a central result of $L^{r\geq 1}$-spaces, Hölder's inequality, to these settings and use it to get simple derivations of main economic formulas such as Hicksian and Marshallian demand functions. Going in the other direction, economic theory leads to a new family of L^0-spaces. To aid intuition, see Table 1 for a dictionary of guiding concepts and Figure 3 for a depiction of sample solution sets $\|x\|_r = 1$, for $-\infty \leq r \leq \infty$.

L^r-spaces, $r \geq 1$	L^r-spaces $r \leq 1$
upper bounds	lower bounds
L^1, L^∞ pairing	L^1, $L^{-\infty}$ pairing
L^2, Hilbert Spaces	L^0, Cobb-Douglas spaces
L^r-spaces, $1 \leq r < 2$	complementary goods
L^r-spaces, $2 < r$	substitute goods
$\|\cdot\|_r$ and $\|\cdot\|_s$ where $r^{-1} + s^{-1} = 1$	Utility functions on space of goods ($\|\cdot\|_r$), cost of a an unit of utility on the space of prices ($\|\cdot\|_s$). For a vector of prices p, the share $p_i^s/\|p\|_s^s$ is the share of the budget allocated to item i.
Banach's norms on direct sums of L^r-spaces	Armington functions

Table 1: Dictionary between $L^{r\geq 1}$-spaces and $L^{r\leq 1}$-spaces.

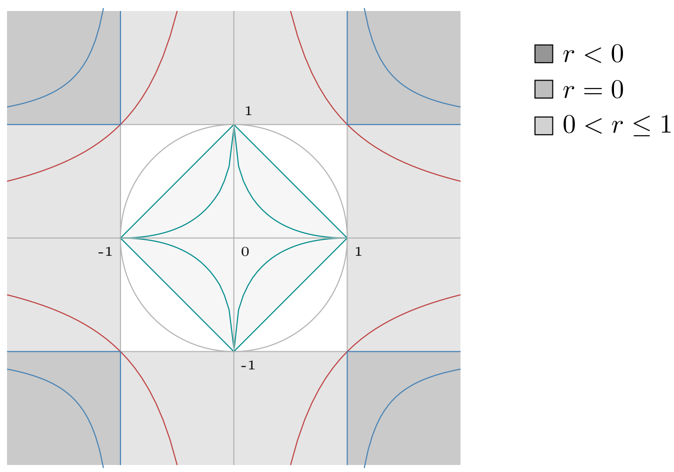

Figure 3: Regions of solutions sets of $\|x\|_r = 1$. For $r < 0$ the solution set lies in the outer corner squares; the special case when $r = -\infty$ is traced by the boundaries of these squares that lie inside the picture. For $r = 0$ the solution set lies inside the outer rectangles. For $0 < r \leq 1$ the solution set lies inside the inner diamond; the solution set is the edge of the diamond when $r = 1$. For $r > 1$ the solution set lies in the non-shaded region; the solution set is the boundary of square containing the non-shaded region when $r = \infty$.

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