Public key cryptology, RSA, ElGamal, Elliptic Curve

Jason Pearson and Colin MacCreery

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Key Terms

Plain text: typically a simple text such as this line
Cipher text: a message after it has been encrypted

One-way Function: A function that is easy to compute on every input but hard to invert given the output

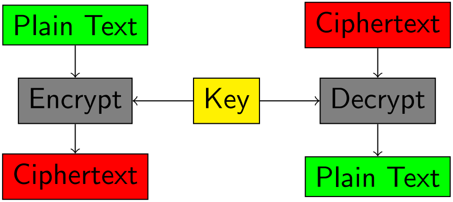
Trap Door One-way Function: A function that is easy to compute but hard to invert given the output unless you know the special information of the trap door

Types of Encryption

Symmetric Encryption
Asymmetric Encryption
Hybrid Encryption

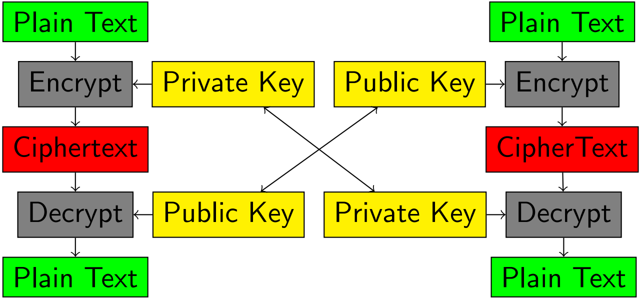
Symmetric Encryption

Encryption and Decryption use the same key
Symmetric systems include the Advanced Encryption Standard (AES), Blowfish and Serpent



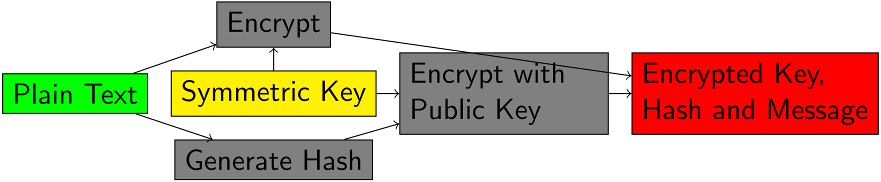
Asymmmetric Encryption

Public key and private key pair
Creating the key tends to be computationally expensive
Examples of asymmetric systems are ElGamal, RSA, DSS



Hybrid Encryption

Uses ideas from symmetric and asymmetric encryption methods
An asymmetric cryptosystem is used for key encapsulation and an symmetric system is used for data encapsulation
An example of a hybrid system is OpenPGP



RSA Cryptosystem

First designed in 1973 and declassified in 1997.

Named after its rediscoverers Ron Rivest, Adi Shamir and Leonar Adleman

Uses large prime numbers to create a private and public key

Security arises from the presumed difficulty of factoring large prime numbers

RSA Generating Public and Private Keys

Generate two large prime numbers and use to calculate n
Compute Euler's Totient Function
Create Public Key and Private Key

Generating Large Prime Numbers

Generate two large prime numbers.
Typically uses AKS testing and/or the Miller-Rabin test for prime numbers
These two values we will call p and q
$p = 991$
$q = 821$
We next calculate $n = p * q$
So we have $n = 991 * 821 = 813611$

Public Information	Secret Information
$n = 813611$	$q = 821$
	$p = 991$

Euler's Totient

The $\phi(n)$ value counts the number of positive numbers relatively prime to n
$\phi(7)$ would be 6 because 1,2,3,4,5,6 are all relatively prime to 7
We then can determine Euler's totient value by using the following equation.
$\phi(n) = \phi(p)\phi(q) = (p-1)(q-1)$
And for $\phi(n) = \phi(813611) = 811800$

Public Information	Secret Information
n = 813611	q = 821
	p = 991
	$\phi(n) = 811800$

Create Public and Private Keys

To create a public key we pick an arbitrary number e between $1 < e < \phi(n)$ for example we will use e $= 7423$
To create a private key we need to find the modular multiplicative inverse of e.
$d \equiv e^{-1} \pmod{\phi(n)}$ or
$d * 7423 \equiv 1 \pmod{\phi(n)}$
$788287 * 7423 \equiv 1 \pmod{\phi(n)}$
$5851454401 \equiv 1 \pmod{\phi(n)}$
$1 \equiv 1 \pmod{\phi(n)}$
This is commonly done using the Extended Euclidean Algorithm.
This makes our value of d $= 788287$

Public Information	Secret Information
n $= 813611$	q $= 821$
e $= 7423$	p $= 991$
	$\phi(n) = 811800$
	d $= 788287$

Working Example

Using these values we can create a cipher text c and decrypt it using the following equations
$c \equiv m^e \,(mod\,(n))$ and $m \equiv c^d \,(mod\,(n))$
Finishing up our example we will encrypt "Hi" using our new values
plain text m = 72105, e = 7423, d = 788287, n = 813611

Public Encrypt	Private Encrypt
$c \equiv m^e \,(mod\,(n))$	$c \equiv m^d \,(mod\,(n))$
$c \equiv 72105^{7423} mod(813611)$	$c \equiv 72105^{788287} mod(813611)$
$c = 707473$	$c = 616895$

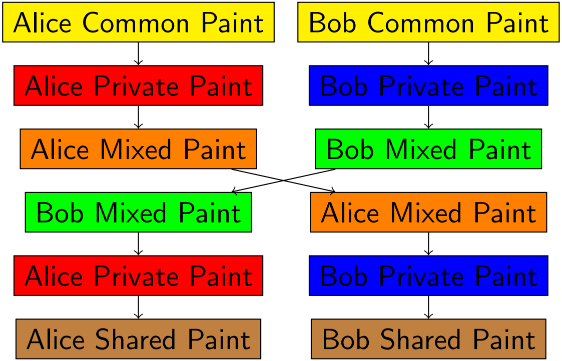
Private Decrypt	Public Decrypt
$m \equiv c^d \,(mod\,(n))$	$m \equiv c^e \,(mod\,(n))$
$m \equiv 707473^{788287} mod(813611)$	$m \equiv 616895^{7423} mod(813611)$
$m = 72105$	$m = 72105$

RSA Practical Considerations

Used in many secure applications escpecially on the internet
Keys need to be very large and continue to lengthen as computers become more powerful
A quantum computer can factor in polynomial time breaking RSA

Diffie-Hellman Key Exchange

This key exchange allows for secret communication over a public network.



ElGamal Cryptosystem

First described by Taher Elgamal in 1985

ElGamal can be represented over any cyclic group

This method for cryptogphy uses discrete logarithms with a large prime modulus

Generating Large Prime Numbers

First we generate a large prime number p
For us p = 17
We then create a generator g of multiplicative group $\mathbb{Z}_p^*$ of integers modulo p
A generator when raised to a power x is evenly distributed over $\mathbb{Z}_p^*$
For this example g = 6
So $< g >= \{6^1, 6^2, 6^3, 6^4, 6^5, 6^6, 6^7, 6^8, 6^9, 6^{10}, ...\}$
Or $\{6, 36, 216, 1296, 7776, 46656, 279936, 1679616, 10077696, ...\}$
Thus $\{6, 2, 12, 4, 7, 8, 14, 16, 11, 15, ...\}$

Public Information	Private Information
p = 17	
g = 6	

ElGamal Creating Public and Private Keys

We then select a private key a where $1 \leq a \leq p - 2$
For this example a = 5
We can then use this to generate the last part of the public key
$g^a \mathrm{mod} p = 6^5 \mathrm{mod} 17 = 7$

Public Information	Private Information
p = 17	a = 5
g = 6	
$g^a \mathrm{mod} p = 7$	

Encrypting a Message

We will have our message m = 13
A public sender to send a message to the private key holder picks a random value k for this example k = 10
We then compute $c_1 = g^k modp = 15$
Now $c_2 = m * g^k \text{mod} p = 13 * 6^{10} mod 17 = 8$
Cipher text sent through c_1 and c_2 to private key holder

Public Information	Private Information
p = 17	a = 5
g = 6	
$g^a \text{mod} p = 7$	

Decrypting a Message

First we must calculate the shared secret
$s = (c_1{}^a) * c_2 \mathrm{mod} p = \left(15^5\right) * 8 \mathrm{mod} 17 = 16$
We then take the modular inverse of s and multiply it by c_2
Finding the modular inverse is commonly done with the Extended Euclidean Algorithm
$m = \left(c_{2*} s^{-1}\right) \mathrm{mod} p = (8 * 8) \mathrm{mod} 17 = 64 \mathrm{mod} 17 = 13$

Public Information	Private Information
p = 17	a = 5
g = 6	
$g^a \mathrm{mod} p = 7$	

ElGamal Practical Usage

Cipher text double the size in bits than the message
Commonly used in hybrid Cryptosystems
El Gamal is probablistic meaning that the same message encrypted won't always give the same ciphertext

Elliptic Curve Cryptography

Elliptic Curves are a set of points defined by

$$E = \{(x, y) | y^2 = x^3 + ax + b\}$$

where

$$a, b \in K$$

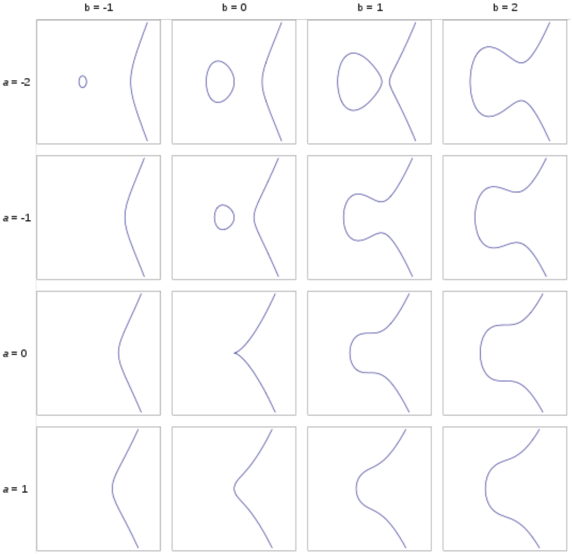
point at infinity: $\mathcal{O}$
Also the following inequality must be true

$$4a^3 + 27b^2 \neq 0$$

The field K can be defined as

$$\mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{Z}/p\mathbb{Z}$$

What Do Elliptic Curves Look Like?



Why Are They Used?

Encryption keys are shorter and use fewer memory and CPU resources

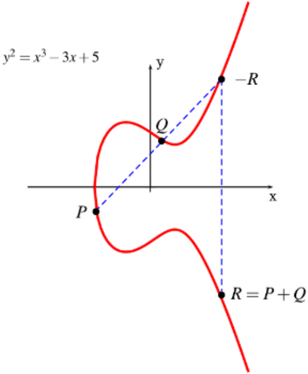
Operations on points are quickly calculated and are difficult to reverse

Group Operations: Addition

Given two points in the set

$$E = \{(x, y) | y^2 = x^3 + ax + b\} \cup \mathcal{O}$$

how can we calculate $P + Q$?



Group Operations: Addition

Compute the slope

$$s = \frac{y_P - y_Q}{x_P - x_Q}$$

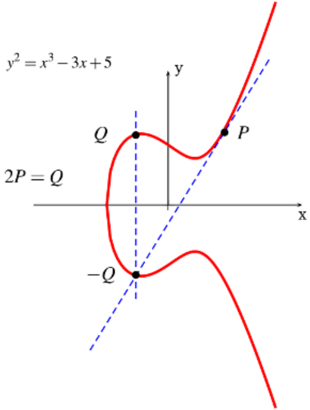
Now compute (x, y) for point R

$$x_R = s^2 - (x_P + x_Q)$$

$$y_R = s(x_P - x_R) - y_P$$

Group Operations: Point Doubling

How can we calculate $P + P = R = 2P$?



Group Operations: Point Doubling

Compute the slope

$$s = \frac{3x_P^2 + a}{2y_p}$$

Now compute (x, y) for point $R = 2P$

$$x_R = s^2 - 2x_P$$

$$y_R = s(x_P - x_R) - y_P$$

What If Points Are Vertical?

For point addition

$$P + Q = \mathcal{O} \text{ if } x_P = x_Q$$

or for point doubling

$$P + P = \mathcal{O} \text{ if } y_P = 0$$

Group Operation: Scalar Multiplication

If a point Q is defined as

$$Q = kP \text{ where } k \in \mathbb{Z}$$

then it is calculated through repeated addition

$$Q = P + P + \ldots + P$$

K times

Elliptic Curve Discrete Logarithm Problem

Scalar Multiplication is effectively a one-way function
Given

$$Q, P \in E(\mathbb{Z}/p\mathbb{Z})$$

finding

$$k \text{ such that } Q = kP$$

is infeasible!

Base Point or Generator

Some point

$$G \in E(\mathbb{Z}/p\mathbb{Z})$$

that generates a cyclic group where

$$\text{ord}(G) = n \text{ and } kG = \mathcal{O}$$

The cofactor is defined as

$$h = \frac{|E(\mathbb{Z}/p\mathbb{Z})|}{n}$$

and ideally $h = 1$
Cofactors over 4 are more susceptible to attacks

Domain Parameters

Public parameters p, a, b, G, n, h:

p: finite field

a, b: curve parameters

G: generator

n: ord(G)

h: cofactor