Homework 1

MIMO Communication Systems (5083)

10th March, 2020

Question 1.

Let $w_1, w_2, \ldots, w_n$ be independent random variables with pdf $w_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$. Show that for $c_1, c_2, \ldots, c_n \in \mathbb{R}$,

$$\sum_{i=1}^{n} c_i w_i \sim \mathcal{N}\left(\sum_{i=1}^{n} c_i \mu_i, \sum_{i=1}^{n} c_i^2 \sigma_i^2\right).$$

Hint: The moment generating function and its properties can be very useful.

Question 2.

Let $w = w_R + j w_I$ be complex Gaussian random variable $\mathcal{CN}(0, \sigma^2)$. Show that

$$\|w\| = \sqrt{w_R^2 + w_I^2} \sim \text{Rayleigh}(\sigma/2),$$

whose pdf is given by

$$f_{\text{Rayleigh}}(r) = \frac{2r}{\sigma^2} \exp\left(-\frac{r^2}{\sigma^2}\right), \quad r \geq 0.$$

Also, show that

$$\|w\|^2 = w_R^2 + w_I^2 \sim \text{Exp}(1/\sigma^2),$$

whose pdf is given by

$$f_{\text{Exp}}(r) = \frac{1}{\sigma^2} \exp\left(-\frac{r}{\sigma^2}\right), \quad r \geq 0.$$

Question 3.

Let $\mathbf{w} \sim \mathcal{N}(0, \mathbf{I}_{2n})$ be an $2n$-dimensional Gaussian random vector. For $n = 1$, **Question 2** has shown that $\|\mathbf{w}\|^2 \sim \text{Exp}(1/2)$. Now show that for general n, we have

$$\|\mathbf{w}\|^2 \sim \chi_{2n}^2,$$

where χ_{2n}^2 is the Chi-square distribution with $2n$ degrees of freedom whose pdf is given by

$$f_{2n}(r) = \frac{1}{2} \frac{r^{n-1}}{2 \cdot 4 \cdot \ldots \cdot (2n-2)} \exp(-r/2), \quad r \geq 0.$$

Hint: Again, the moment generating function and its properties can be very useful.

Question 4.

Let $\mathbf{x} = \mathbf{x}_R + j\mathbf{x}_I$ be a complex Gaussian vector whose mean, covariance matrix, and pseudo-covariance matrix are $\boldsymbol{\mu}, \mathbf{K}$, and $\mathbf{J}$, respectively. Show that its real representation has pdf

$$\begin{bmatrix} \mathbf{x}_R \\ \mathbf{x}_I \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\mu}_R \\ \boldsymbol{\mu}_I \end{bmatrix}, \begin{bmatrix} \frac{1}{2}\mathrm{Re}(\mathbf{K}+\mathbf{J}) & -\frac{1}{2}\mathrm{Im}(\mathbf{K}-\mathbf{J}) \\ \frac{1}{2}\mathrm{Im}(\mathbf{K}+\mathbf{J}) & \frac{1}{2}\mathrm{Re}(\mathbf{K}-\mathbf{J}) \end{bmatrix} \right).$$

Question 5.

Let $\mathbf{x} = \mathbf{x}_R + j\mathbf{x}_I$ be a complex Gaussian vector whose mean, covariance matrix, and pseudo-covariance matrix are $\boldsymbol{\mu}, \mathbf{K}$, and $\mathbf{J}$, respectively. Show that $\mathbf{x}$ is circular symmetric if and only if $\boldsymbol{\mu}$ and $\mathbf{J}$ are 0.

Question 6.

Let x be a complex Gaussian random variable. Show that for x to be circular symmetric, it must have i.i.d. real and imaginary parts.
Hint: From **Question 5**, for x to be circular symmetric, its must have μ and J equal to 0.

Question 7.

(**Programming**) Let $n = 1000$ and $\Delta = 0.25$.

1. Generate i.i.d. complex Gaussian random variable $w_i \sim \mathcal{CN}(0, 1)$ for $i \in \{1, \ldots, n\}$.
2. Partition $[0, 10]$ into bins of length Δ.
3. Generate the empirical distribution function

$$\hat{F}_n(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\|w_i\| \leq t}, \quad t \in \{0, \Delta, 2\Delta, \ldots, 10\}.$$

4. Generate the true cdf of $\|w_i\|$.

Plot and compare the empirical distribution function and the true cdf. You may also try other pairs of (n, Δ).