Elementos Finitos y Diferencias Finitas - PEC 1

Diferencias Finitas PEC 1

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1 Problems in Finite Differences

1.a 1D Heat equation

This section consists in finding the solutions for the initial boundary value problem:

$$\begin{aligned} u_t &= 2u_{xx} \\ u(x,0) &= -\sin 3\pi x + \frac{1}{4}\sin 6\pi x \\ u(0,t) &= u(1,t) = 0 \end{aligned} \tag{1.1}$$

which describes the time evolution of a one-dimensional system with a temperature distribution given by $u(x,0)$ and with zero temperature in its extremes. It is known that the analytical solution is given by a Fourier series, however we will solve it numerically applying a couple of Finite Difference Methods. The scripts used will be adaptations of those given in the course.

1.a.1 Explicit method

The explicit method is based on solving the iterative equation relating the solution at the point $(j, n+1)$ of the mesh with the solution at the point (j, n). Substituting the PDE derivatives with their relative difference approximations, we get the relation:

$$u_j^{n+1} = u_j^n + 2\frac{\Delta t}{\Delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) \tag{1.2}$$

where

$$\frac{\Delta t}{\Delta x^2} \equiv \rho \tag{1.3}$$

and ρ has to take values $< \frac{1}{2}$ in order for the method to converge (conditionally stable).

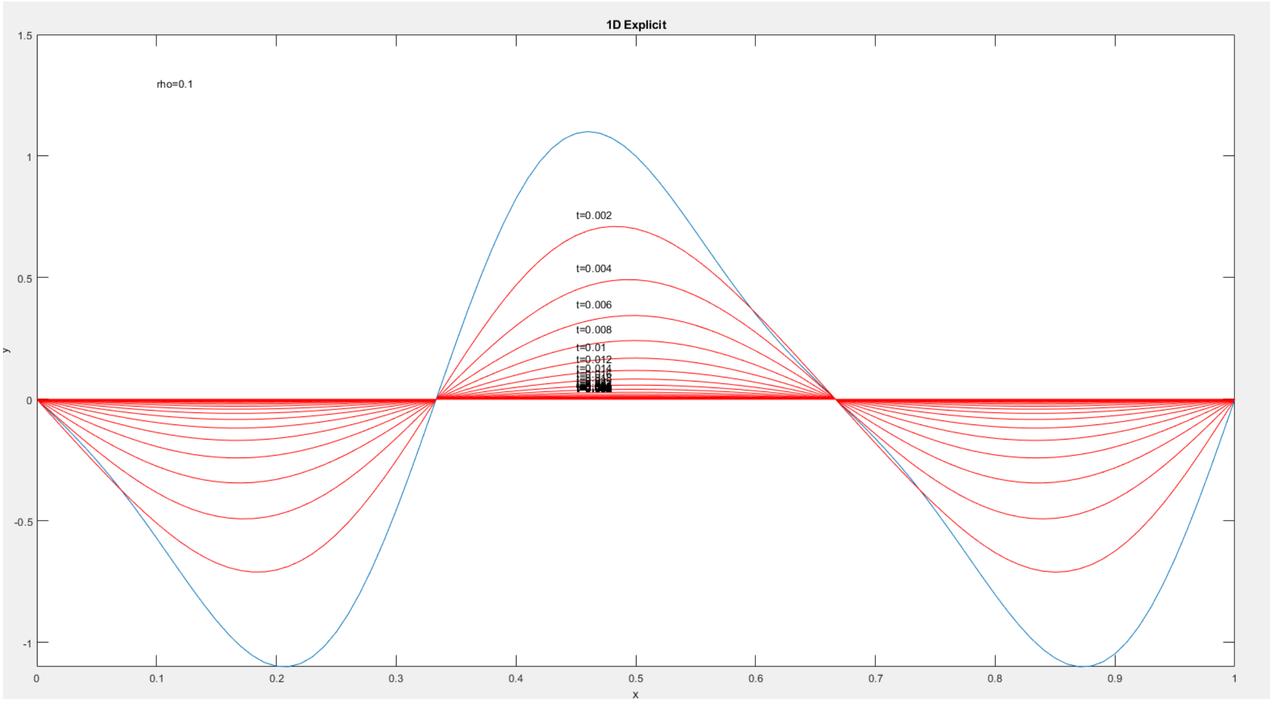

Figure 1.1: *Result for 100 iterations* ($\Delta x = \frac{1}{100}$), $\Delta t = 10\mu s$ *and* $\rho = 0.1$

where the blue line is the initial condition and the red lines are the iterations. If we impose $\rho = 0.52$ we can see that the method does not converge:

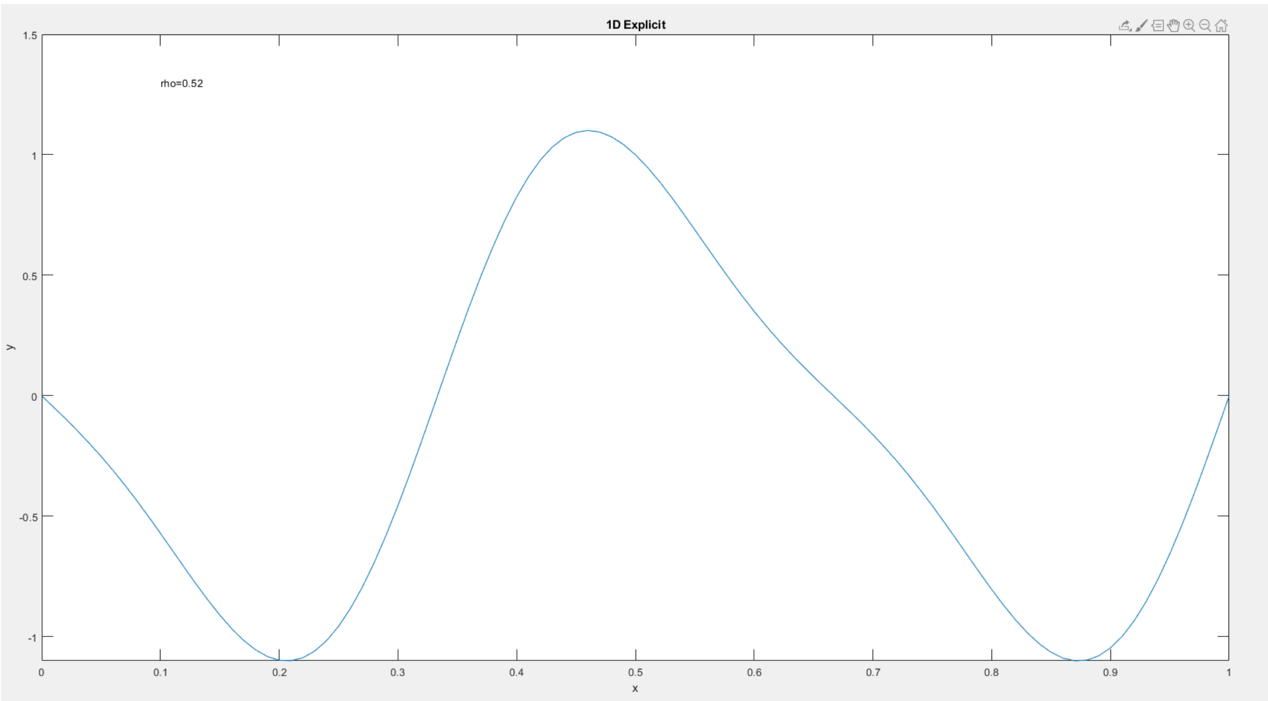

Figure 1.2: *Result for 100 iterations* ($\Delta x = \frac{1}{100}$), $\Delta t = 52\mu s$ *and* $\rho = 0.52$

1.a.2 Implicit method

In the implicit method we relate the solution at the point (j, n) with the solution at three points $(j, n+1)$. The iterative relation is:

$$-\rho u_{j-1}^{n+1} + (1+2\rho)u_j^{n+1} - \rho u_{j+1}^{n+1} = u_j^n \tag{1.4}$$

which gives place to a tridiagonal system of linear equations which can be solve computationally with the Thomas algorithm. The difference between this method and the explicit is that it is **unconditionally stable**, that is, it converges for all values of ρ.

The results for the same configuration as the first explicit result is:

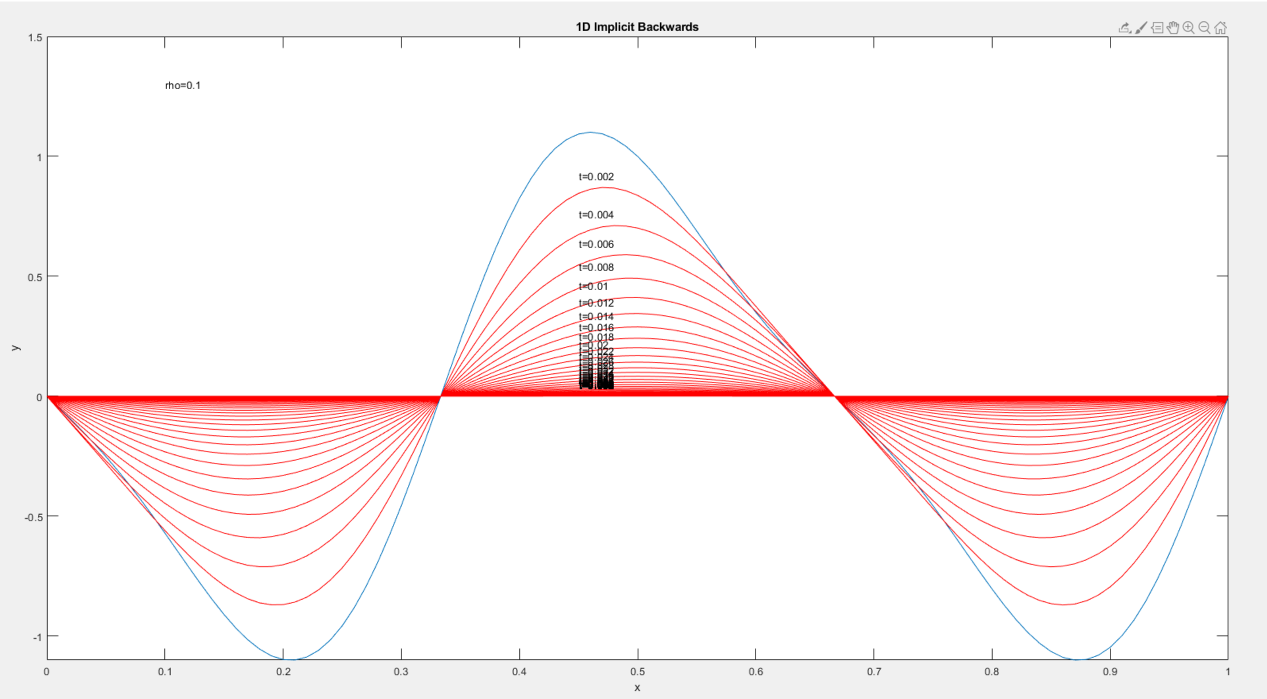

Figure 1.3: *Result for 100 iterations ($\Delta x = \frac{1}{100}$), $\Delta t = 10\mu s$ and $\rho = 0.1$*

However for the second result with $\rho = 0.52$ we see that it now converges:

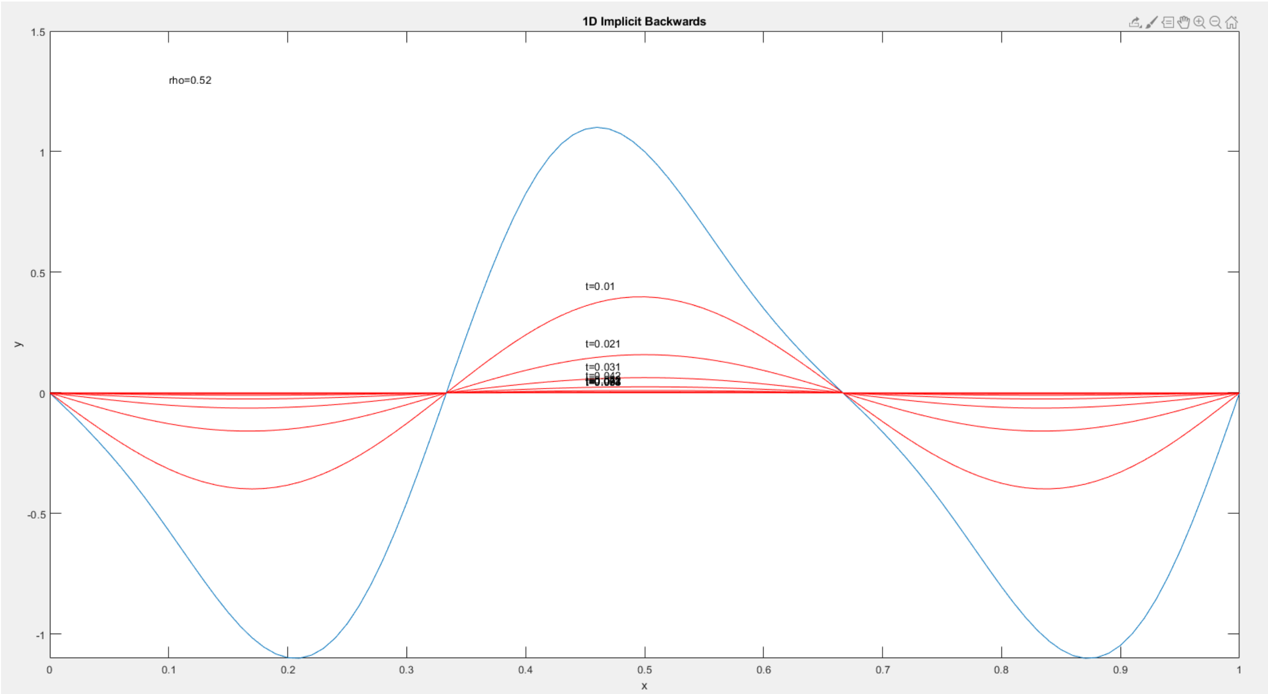

Figure 1.4: Result for 100 iterations ($\Delta x = \frac{1}{100}$), $\Delta t = 52\mu s$ *and* $\rho = 0.52$

this allows us to use larger time intervals and therefore smaller computation times.

1.a.3 DuFort-Frankel method

This method is based on the relation

$$(1 + 2\rho)u_j^{n+1} = 2\rho\left(u_{j+1}^n + u_{j-1}^n\right) + (1 - 2\rho)u_j^{n-1} \tag{1.5}$$

and it is solved using an iterative algorithm (it is an explicit method).

The result is:

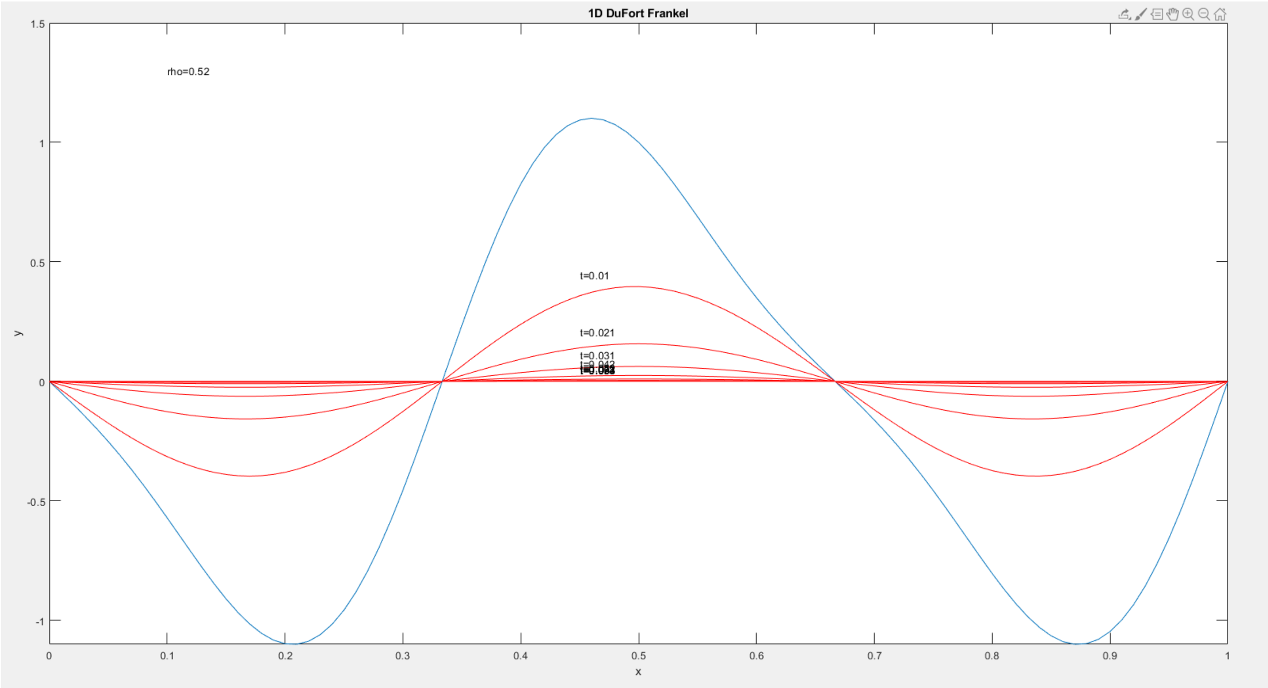

Figure 1.5: Result for 100 iterations ($\Delta x = \frac{1}{100}$), $\Delta t = 52\mu s$ and $\rho = 0.52$

Its amplification factor is:

$$\lambda(k) = \frac{2\rho\cos(k\Delta x) \pm \sqrt{1 - 4\rho^2 \sin^2(k\Delta x)}}{1 + 2\rho} \tag{1.6}$$

it can be shown that $|\lambda(k)| < 1$ for all values of ρ, so the method is unconditionally stable.

1.a.4 Crank-Nicolson implicit method

This method is based on the relation:

$$2(1+\rho)u_j^{n+1} - \rho\left(u_{j+1}^{n+1} + u_{j-1}^{n+1}\right) = 2(1-\rho)u_j^n + \rho\left(u_{j+1}^n + u_{j-1}^n\right) \tag{1.7}$$

which also gives place to a tridiagonal system of equations that can be solved computationally.

The results are:

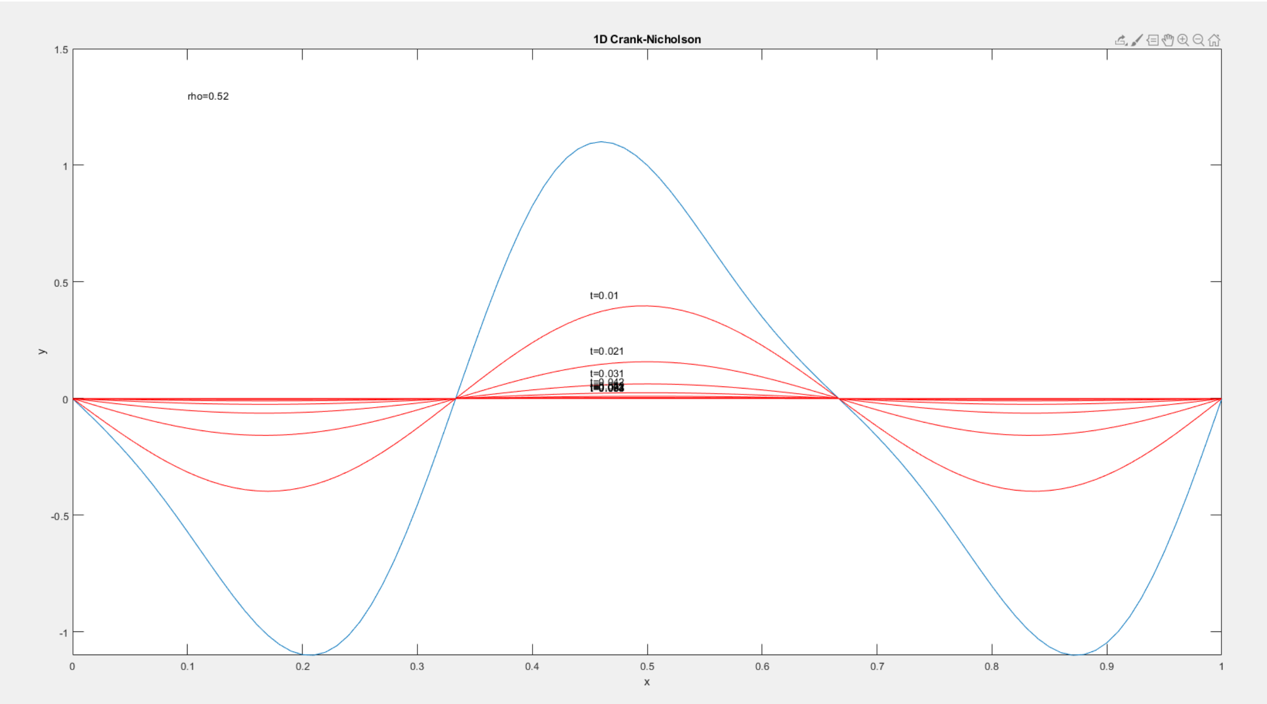

Figure 1.6: Result for 100 iterations ($\Delta x = \frac{1}{100}$), $\Delta t = 52\mu s$ *and* $\rho = 0.52$

This method is also unconditionally stable.

We can see that for all methods the solutions make physical sense: the temperature distributions cool down and flatten as time passes, losing heat only by conduction in the extremes at zero temperature.

1.b 2D Laplace equation

This section consists in finding the solutions for the initial boundary value problem:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \\ u(x,0) &= \sin 3\pi x \\ u(x,1) &= \sin \pi x \\ u(0,y) &= u(1,y) = 0.0 \end{aligned} \tag{1.8}$$

which is an elliptic PDE. It is stationary as it does not depend on time, and substituting the second derivatives by their centered finite differences we can arrive at the 5-point approximation:

$$u_{r,s} = \frac{1}{4}\left(u_{r+1,s} + u_{r-1,s} + u_{r,s+1} + u_{r,s-1}\right) \tag{1.9}$$

As we can find in the bibliography, this system can be expressed in matrix form and solved iteratively using the Gauss-Seidel method. To this end, we use the Matlab script given and adapt the initial conditions and boundary conditions to the problem at hand.

For a grid of size $(50, 50)$ we have the solution:

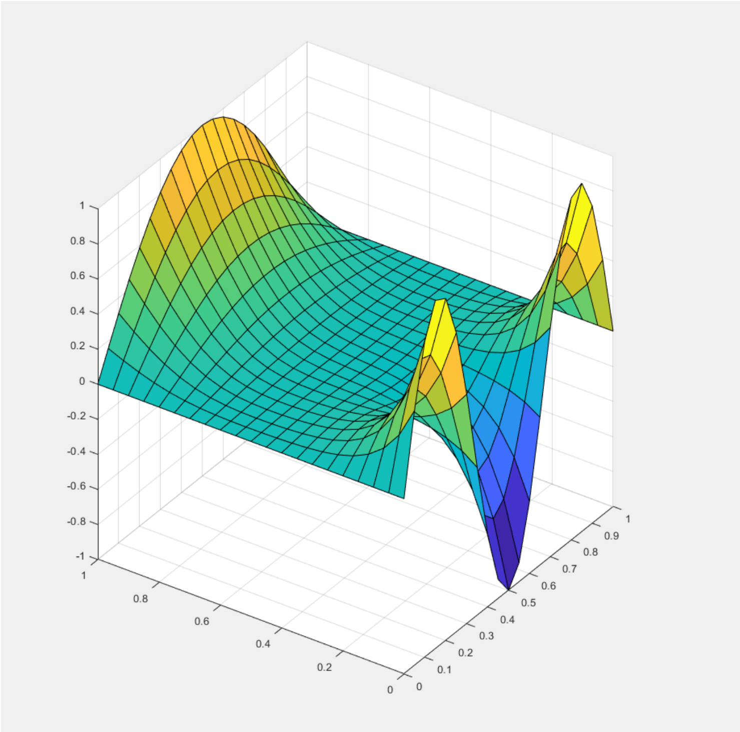

Figure 1.7: *Temperature distribution for the problem with a grid size of 50*

We can also plot the temperature level curves:

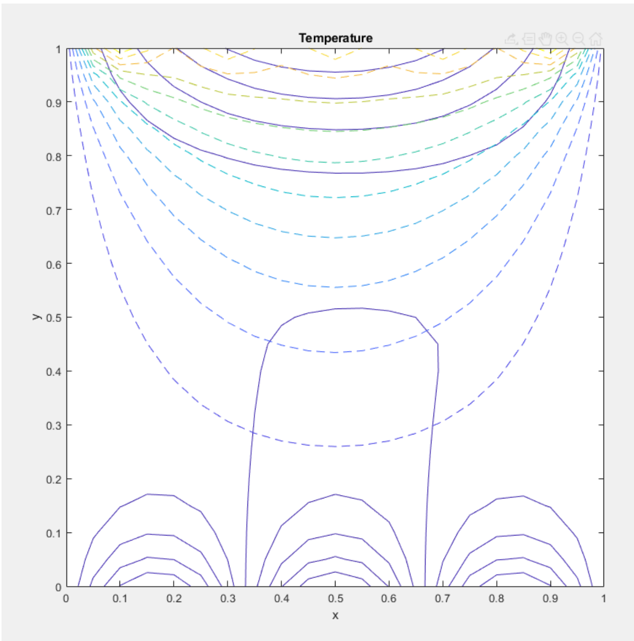

Figure 1.8: *Temperature level curves for the problem with a grid size of 50*

If we increase the grid size we obtain solutions that are closer to the exact analytic one, at the expense of more computational effort. For example, the solution for a grid size of $(100, 100)$ is:

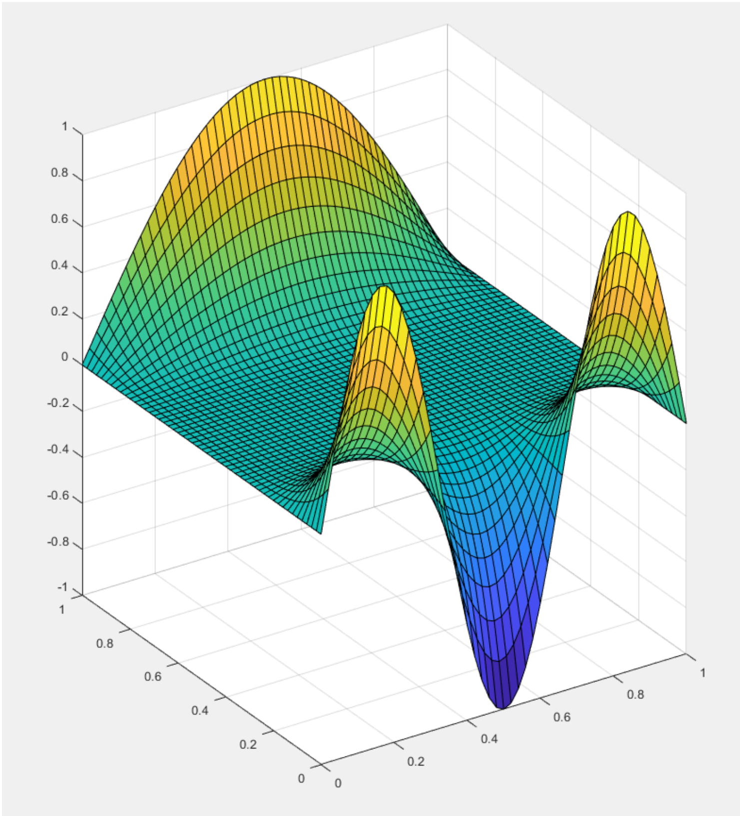

Figure 1.9: *Temperature distribution for the problem with a grid size of 100*

Here the solutions also make physical sense: the temperature distribution matches the boundary conditions at the extremes but flattens in the middle.

1.c Heat problem

Lastly we solve a 2D heat problem, which is a parabollic boundary-value problem, for a rectangular plate with dimensions $L_x = 2$, $L_y = 1$ where the initial temperature is $T_0 = 100°C$. The boundary conditions are of Dirichlet condition, imposing that the upper boundary is completely insulated and all the others are at a constant temperature of $T = 0°C$. Mathematically, if the problem is properly normalized it is expressed as:

$$\begin{aligned} u_t &= u_{xx} + u_{yy} \\ u(x, y, 0) &= 100 \\ u(0, y, t) &= 0 \\ u(2, y, t) &= 0 \\ u(x, 0, t) &= 0 \\ u(x, 1, t) &= 100 \end{aligned} \tag{1.10}$$

where here I have interpreted that the upper border being "completely insulated" means it does not lose heat and therefore its temperature remains constant at its initial value.

Two-dimensional parabollic problems can be solved using explicit and implicit method as their one-dimensional cases. For the explicit method, we can easily calculate the relation between each time step in the same way as in 1D:

$$u_{r,s}^{n+1} = u_{r,s}^{n} + \rho \left(u_{r+1,s}^{n} + u_{r-1,s}^{n} + u_{r,s+1}^{n} + u_{r,s-1}^{n} - 4u_{r,s}^{n} \right) \tag{1.11}$$

this also can be seen as a weighted schema with weights set to zero, $\theta_1 = \theta_2 = 0$. If we set the weights to a number between 0 and 1 we would have a method that is seen as a mix between explicit and implicit, while setting the weights to 1 would mean a purely implicit method. Substiting Fourier modes we can see that $\rho \leq \frac{1}{4}$ for the method to be stable.

The 2D methods analog to the 1D problem (DuFort-Frankel, Crank-Nicolson, etc.) don't work as well as the ADE and ADI methods. The ADE (Alternate Direction Explicit) combines two different finite different schemes for the odd and even steps, which allows for a fast computation time with moderate precision, while the ADI method is an improvement over the Crank-Nicolson that allows for a more precise solution at the expense of a moderate increase in computation time over ADE. Both ADE and ADI methods are unconditionally stable.

The solutions found for the two latter methods are:

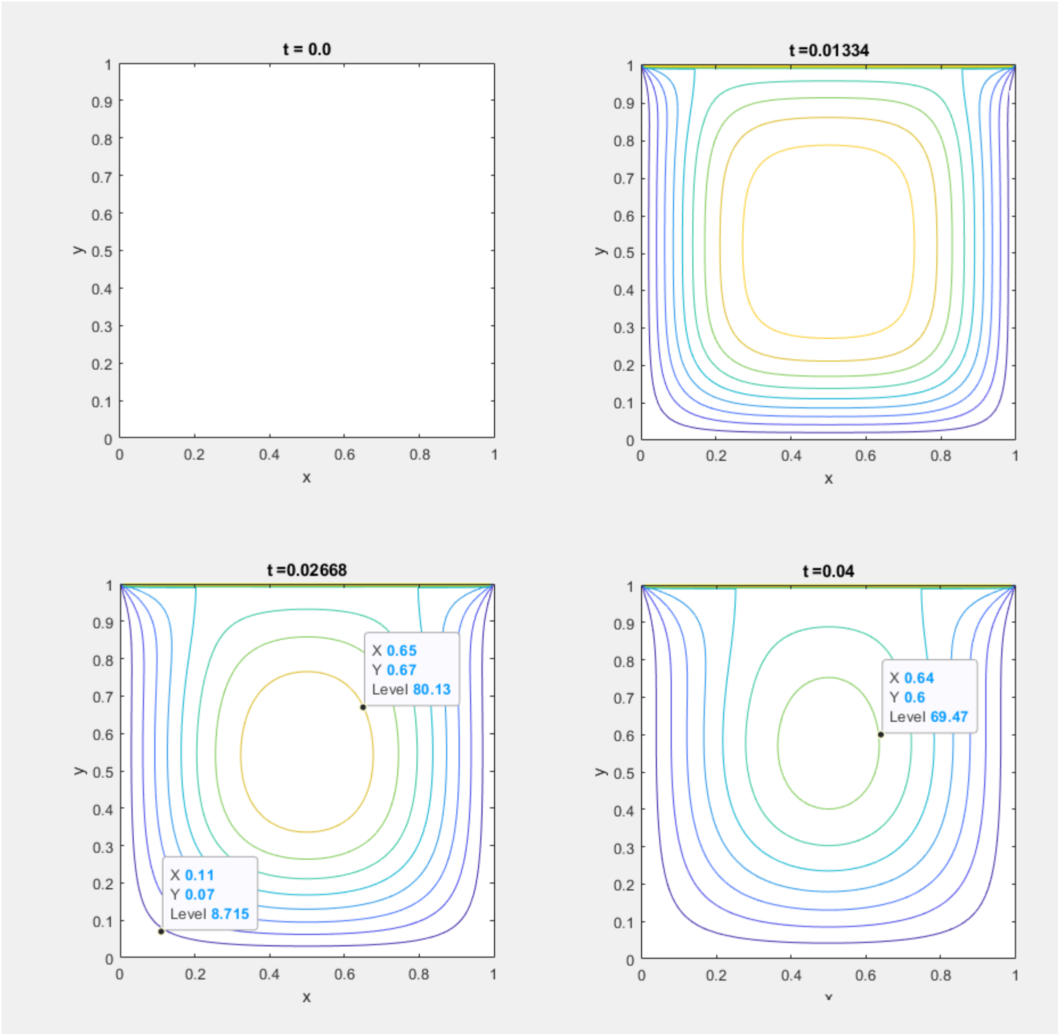

Figure 1.10: Temperature level curves for the ADI method

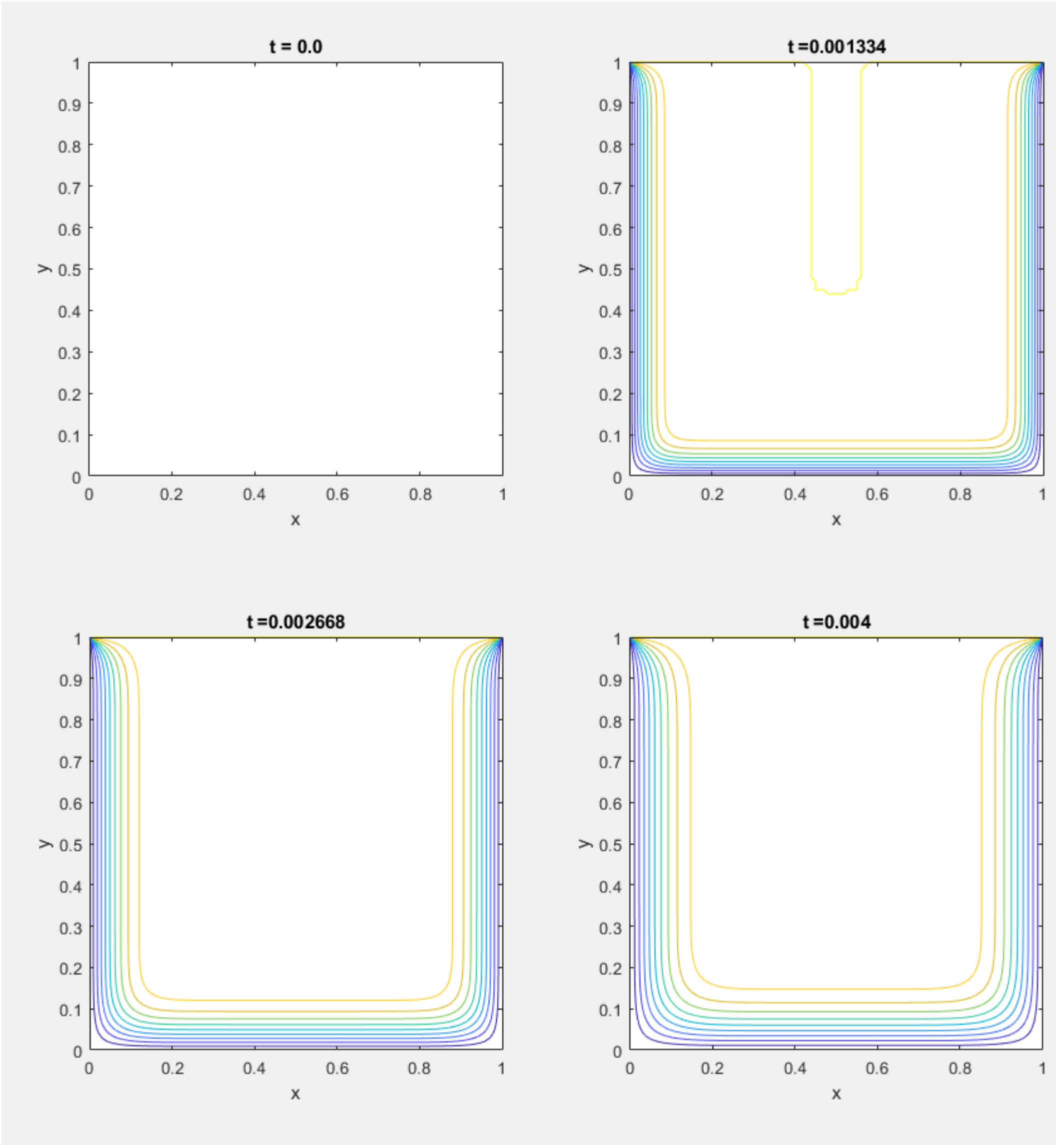

Figure 1.11: *Temperature level curves for the ADE method*

Where the first plot at the top left corner is white because the plate is at constant temperature $u(x, y, 0) = 100$ and therefore the contour plot can't be properly displayed. We can see how the results are physically meaningful: the temperature starts constant but it cools down as time passes. However, the results for the ADE method and the ADI method are discordant.