Practice 2

Numerical Integration and Nonlinear Equations

Starting date 11/11/2019 Delivery date 6/12/2019

- Your answers to the practice will be in a **pdf file**. The document can be written in Catalan, Spanish or English and using any text processor (or even a scanned document as long as your writing is legible). The name of this pdf file must be **Surname1-Name-Practice2**.
- If you use MATLAB, Octave, Maple or Mathematica you must also deliver the file with the codes ready to run. In this case you must submit a zip (or rar) with this file and the above-mentioned pdf. The file's name will be **Surname1-Name-Practice2**.
- Your document must clearly show the resolution strategy used. You do not need to write down all the calculations but there should be sufficient elements to follow the resolution.
- It will be necessary to justify all the answers properly

1. The equation $x + \ln x = 0$ has a root near $x = 0.5$ and we want to approximate it using a fixed point method. Which one of the following sequences would you use?

(*a*) $x_{n+1} = -\ln(x_n)$,

(*b*) $x_{n+1} = e^{-x_n}$,

(*c*) $x_{n+1} = \frac{x_n + e^{-x_n}}{2}$.

Justify your answer using, if necessary, the graphs of the involved functions. (Here we do not want you to compute the iteration sequence but to argue as in Example 4.1, page 181, of the textbook.)

[3 points]

2. Consider the following table:

x	1.00	1.05	1.10	1.15	1.20	1.25	1.30
e^x	2.71828	2.85765	3.00417	3.15819	3.32012	3.49034	3.66930

Use this information to approximate the value of $\int_{1.00}^{1.30} e^x\,dx$ by means of the composite trapezoidal rule and the composite Simpson's rule. Compare and discuss the results that you obtain with the exact value computed analytically, i.e.,

$$\int_{1.00}^{1.30} e^x\,dx = e^{1.30} - e^{1.00}.$$

[3 points]

3. In this exercise we will study the order of convergence of some iterative methods used to find zeros of functions (see Chapter 4 in the textbook). Our aim is to approximate the order of convergence in four specific examples and to show that it coincides with the expected theoretical value.

We say that a sequence $(x_n)_{n\geqslant 0}$, convergent to a limit α, has *order of convergence* p if the limit

$$\lim_{n\to+\infty}\frac{x_{n+1}-\alpha}{(x_n-\alpha)^p}=L \tag{1}$$

exists and it is different from zero. In this case L is called the *constant of asymptotic error.* This means for instance that if the order of convergence is quadratic (i.e., $p=2$) then the number of correct digits in the approximation doubles with each iterative step.

- The sequence obtained by *Newton's method* is given by

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}.$$

  If the sequence converges to a *simple zero* of f (i.e., such that $f'(\alpha)\neq 0$), then its order of convergence is $p=2$. If the zero is multiple then the convergence is only linear (i.e., $p=1$).
- In order to get quadratic convergence for a zero of multiplicity m we can consider the *modified Newton's method*

$$x_{n+1}=x_n-m\frac{f(x_n)}{f'(x_n)}.$$

  For instance, taking $m=2$, we will get quadratic convergence in case that the zero of f is double (i.e., such that $f'(\alpha)=0$ i $f''(\alpha)\neq 0$).
- If the *secant method* converges to a simple zero then the order of its convergence is $p=\frac{1+\sqrt{5}}{2}\approx 1.618$. The sequence is given by

$$x_{n+1}=x_n-f(x_n)\frac{x_n-x_{n-1}}{f(x_n)-f(x_{n-1})}.$$

In order to ilustrate the order of convergence of these methods we will consider the following "test" functions:

$$\begin{aligned}
f_1(x)&=1-2\,x+2\,x^2-2\,x^3+x^4,\\
f_2(x)&=e^x-x^2+\sqrt{x^2+2}+\cos x,\\
f_3(x)&=x^2\mathrm{e}^{2\,x}+\mathrm{e}^{2\,x}+2\,x^3\mathrm{e}^x+2\,\mathrm{e}^x x+x^4+x^2,\\
f_4(x)&=x^2-x-\cos x.
\end{aligned}$$

To this end we note that if n is large enough then, from (1),

$$\log|x_{n+1}-\alpha|\approx\log L+p\log|x_n-\alpha|,$$

and consequently

$$\log\frac{|x_{n+1}-\alpha|}{|x_n-\alpha|}\approx p\log\frac{|x_n-\alpha|}{|x_{n-1}-\alpha|}.$$

Thus, once we reach k such that $|x_k-\alpha|<\mathrm{tol}$ for some fixed tolerance then we can approximate $p\approx\tilde{p}$ by taking $\alpha=x_k$ and $n=k-2$ in the above expression, i.e.,

$$\tilde{p}=\frac{\log\frac{|x_{k-1}-x_k|}{|x_{k-2}-x_k|}}{\log\frac{|x_{k-2}-x_k|}{|x_{k-3}-x_k|}},$$

(*a*) Using your favourite software (MATLAB, Octave, Maple, Mathematica,...) write a program to implement Newton's method. Taking tol= 10^{-10} and performing the computations with 30 digits, obtain the sequence for the function f_1 with initial value $x_0 = -5.3$. Write in a table the values of x_i for $i = 1, 2, 3$ together with the approximations of the root α and the convergence order p. Comment the results using, if necessary, the graph of the function.

Do the same for f_2 with $x_0 = -7.6$, f_3 with $x_0 = 1.7$ and f_4 with $x_0 = 2.1$.

For example, for the function

$$f_5(x) = 108 - 216\,x + 252\,x^2 - 256\,x^3 + 175\,x^4 - 94\,x^5 + 40\,x^6 - 10\,x^7 + x^8$$

with initial condition $x_0 = -1.1$, you should write the table

$$x_1 = -0.712194813941741033844081495201$$
$$x_2 = -0.316691299996063663874324441470$$
$$x_3 = 0.124394497662126938340273339998$$
$$\tilde{\alpha} = 1.00000000000000000000000000000$$
$$\tilde{p} = 1.99998176697033653107674320518$$

(*b*) Write a program to implement the secant's method. Taking tol= 10^{-10} and performing the computations with 30 digits, obtain the sequence for the function f_1 with initial values $\{x_0 = -3.1, x_1 = -2.3\}$. Write in a table the values of x_i for $i = 2, 3, 4$ together with the approximations of the root α and the convergence order p. Comment the results using, if necessary, the graph of the function.

Do the same for f_2 with $\{x_0 = 3.3, x_1 = 5.7\}$, f_3 with $\{x_0 = -2.3, x_1 = 4.2\}$ and f_4 with $\{x_0 = -5.6, x_1 = -4.7\}$.

(*c*) Use the modified Newton's method to improve the order of convergence for the functions f_1 and f_3 taking the same initial conditions and tolerance as in (*a*).

[4 points]