Continuous Assessment Test Number 1

Starting date: 30/09/2019. Delivery date: 18/10/2019.

- This activity is a theoretical study of propagation of the error and the convergence of iterative methods for linear systems. It can be done without the use of MATLAB, although it can be used on specific occasions to perform some computations.
- You can deliver the resolution in any language (Catalan, Spanish, English), using any text processor or even an scanned document if this writing is legible. In any case, the filename must be stated as Surname1-Name-PEC1T19.pdf

  If you use a MATLAB file, you must also deliver it ready to be execute quickly.
- Your document must clearly show the resolution strategy used, that is, you do not need to write down all the calculations but there should be the sufficient elements to complete the resolution.
- It will be necessary to justify all the answers properly.

Exercise 1. Consider the sequence of real numbers defined by

$$\begin{aligned} x_1 &= 1, \\ x_2 &= 0.\hat{3}, \\ x_n &= 4.\hat{3}\, x_{n-1} - 1.\hat{3}\, x_{n-2}, \qquad \text{for} \quad n > 2 \end{aligned}$$

a) Using only 8 significant digits, compute up to the fifth term of the sequence.
b) Using the formulas of the propagation of errors in the input data, give an upper bound of the absolute error for this fifth term.
c) How many right figures do you expect to have in the term that occupies the position 100?

Exercise 2. Assume that the exact value of number e is $e = 2.7182818$, but we are not able to calculate any power of e. We want to know the value of $f(x) = e^{x^2}$ when x is any value close to 1, that we can write as $x_0 = 1 + \epsilon$ for a given constant ϵ.

a) Give the absolute error and the relative error for the approximate value $x_0^* = 1$.
b) Using the formulas of the propagation of errors in the input data, give an upper bound for the absolute error and another one for the relative error of $f(x_0)$ when we use the approximate value $x_0^* = 1$.
c) Using the formulas of the propagation of errors in the input data, deduce how far from 1 we can take x_0 in order to assure that the approximation $f(x_0) \simeq f(x_0^*)$ will have an error smaller than $0.5 \cdot 10^{-6}$
d) If the exact value of e were $e = 3$, is there any significance difference in the answer of section c)? What do you think is the reason?

Exercise 3. Considerar the problem of solving the linear system:

$$\begin{aligned} 51x + 82y &= 235 \\ 50.33x + 81y &= 232 \end{aligned}$$

a) For a system written in matrix form as $\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{b}$, give the matrix $\mathbf{A}$ and vector $\mathbf{b}$ for the current case.
b) Is the matrix $\mathbf{A}$ strictly diagonally dominant?
c) Give the Jacobi's and Gauss-Seidel's iterative matrices for this system.
d) Compute the spectral radius of these Jacobi's and Gauss-Seidel's iterative matrices (if you want, you can use MATLAB to do it).
e) Discuss, without solving the system, the convergence of both methods in this case.
f) Make a prediction about the number of iterations necessary to obtain a solution with a relative error less than $0.5 \cdot 10^{-3}$, without computing any iteration. Discuss the result: the value is very low/high, what do you think is the reason.

Exercise 4. Let be the square matrix $\mathbf{A} = (a_{ij})$ of dimension n defined as

$$\begin{cases} a_{ij} = n + j - i + 1 & \text{if} \quad j < i \\ a_{ii} = 1 + p & \\ a_{ij} = 1 + j - i & \text{if} \quad j > i \end{cases}$$

where i is the row of the matrix, j is the column and p is a constant. And let be the n-dimensional vector $\mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ \vdots \\ 2 \end{pmatrix}$. Our goal is to solve the linear system $\mathbf{A}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{b}$ using an iterative method.

First, we use $n = 4$, $p = 19$ and the initial approximation $\mathbf{x}^{(0)} = \begin{pmatrix} x_1^{(0)} \\ \vdots \\ x_4^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$; the system will be:

$$\begin{pmatrix} 20 & 2 & 3 & 4 \\ 4 & 20 & 2 & 3 \\ 3 & 4 & 20 & 2 \\ 2 & 3 & 4 & 20 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

Provide the answer to the next questions for Jacobi's and Gauss-Seidel's iterative methods

a) Give the iterative matrix and array.
b) Fill in the following tables with the first four approximations of the solution, $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$ and $\mathbf{x}^{(4)}$, their residuals $\mathbf{r}^{(k)} = \mathbf{b} - \mathbf{A} \cdot \mathbf{x}^{(k)}$ and the euclidian norms $E^{(k)} = \|\mathbf{r}^{(k)}\|_2$

	$\mathbf{x}^{(0)}$	$\mathbf{x}^{(1)}$	$\mathbf{x}^{(2)}$	...
x_1	0	...	...	...
x_2	0	...	...	...
x_3	0	...	...	...
x_4	0	...	...	...

	$\mathbf{r}^{(0)}$	$\mathbf{r}^{(1)}$	$\mathbf{r}^{(2)}$	...
r_1	2	...	...	...
r_2	0	...	...	...
r_3	0	...	...	...
r_4	2	...	...	...
$E^{(k)}$	2.828427	...	...	...

c) Fill in the following table giving the minimum number, k_{min}, of iterations such that

$$E^{(k)} \leq 10^{-8}$$

and the value of $E^{(k)}$ in the iteration number $k_{min} - 1$ and k_{min}.

Tolerance	k_{min}	$E^{(k_{min}-1)}$	$E^{(k_{min})}$
10^{-8}	...	...	...

d) In the case of convergence, is the order of convergence the same for both methods? Justify your answer.
e) Fill in the table of section c) with the results for $p = 69$ and $n = 9$. What happens in this case?
f) Fill in the table for $\mathbf{r}^{(k)}$ of section b) with the results for $p = 0$ and $n = 4$. What happens in this case? Which do you believe is the reason for this behavior?