Computational Finance WISM467
Exercise set 1
deadline 12.05.2022

Exercise 1 (2p.)

Exercise 1.2 from the book "*Mathematical Modeling and Computation in Finance*".

Exercise 2 (3p.)

1. Show theoretically that
$$\int_0^t W(s)\mathrm{d}s = \int_0^t (t-s)\mathrm{d}W(s).$$
2. Take $t = 5$ and validate by a numerical experiment the equality above.

Exercise 3 (6p.)

a. Suppose $S(t)$ follows a log-normal Brownian motion given by:
$$\frac{\mathrm{d}S(t)}{S(t)} = \mu\mathrm{d}t + \sigma\mathrm{d}W(t),$$
with constant parameters μ, σ and Wiener process $W(t)$. Find the dynamics for process $Y(t) = 2S(t)^2$.

b. Apply Itô's formula to $\mathrm{e}^{W(t)}$, where $W(t)$ is a standard Brownian motion. Is this a martingale?

c. Suppose that $X(t)$ satisfies the following Stochastic Differential Equation (SDE):
$$\mathrm{d}X(t) = 0.04X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W^{\mathbb{P}}(t),$$
and $Y(t)$ satisfies:
$$\mathrm{d}Y(t) = \beta Y(t)\mathrm{d}t + 0.15Y(t)\mathrm{d}W^{\mathbb{P}}(t).$$
For a given money-savings account,
$$\mathrm{d}M(t) = rM(t)\mathrm{d}t,$$
take $\beta = 0.1$, $\sigma = 0.38$, with risk-free rate $r = 6\%$, maturity $T = 7y$, and for strike values $K = [0 : 0.1 : 10]$ and use the Euler discretization to find:
$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T)}\max(\frac{1}{2}X(T) - \frac{1}{2}Y(T), K)|\mathcal{F}(t)\right],$$
and plot the results (K vs. $V(t)$)

.

Exercise 4 (3p.)

For a Wiener process $W(t)$ consider

$$X(t) = W(t) - \frac{t}{T}W(T-t), \text{ for } 0 \leq t \leq s \leq T.$$

For $T = 10$, find analytically $\text{Var}(X(t))$ and perform a numerical simulation to confirm your result. Is the accuracy sensitive to t?

Exercise 5 (3p.)

a. Use a stochastic representation result (Feynman-Kac theorem) to solve the following boundary value problem in the domain $[0,T] \times \mathbb{R}$.

$$\begin{aligned} \frac{\partial V}{\partial t} + \mu x \frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} &= 0 \\ V(T,x) &= \log(x^3), \end{aligned}$$

where μ and σ are known constants.

b. Prove that if we are given the final condition problem:

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \\ V(S,T) = \text{given} \end{cases}$$

with the sum of the first derivatives of the option square integrable,
then the value, $V(S(t),t)$, is the unique solution of

$$V(S,t) = \mathrm{e}^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[V(S(T),T)|S(t)\right]$$

with $S(t)$ governed by the stochastic differential equation:

$$dS(t) = rS(t)dt + \sigma S(t)\mathrm{d}W^{\mathbb{Q}}(t),$$

Exercise 6 (2p.)

Exercise 1.4 from the book "*Mathematical Modeling and Computation in Finance*".

Exercise 7 (2p.)

Exercise 2.4 from the book "*Mathematical Modeling and Computation in Finance*".

Exercise 8 (2p.)

Exercise 2.6 from the book "*Mathematical Modeling and Computation in Finance*".

Exercise 9 (4p.)

Exercise 3.12 from the book "*Mathematical Modeling and Computation in Finance*".