Computational Finance WISM467
Exercise set 2

deadline: 1 day before examination

Exercise 1 (1p.)

For a given function $g(t) = W^2(t), f(t) = W^4(t)$ and for $T = \frac{1}{2}$, determine theoretically,

$$\int_0^T g(t)f(t)\mathrm{d}W(t).$$

Confirm your answer by a Monte Carlo experiment.

Exercise 2 (2p.)

For a Wiener process $W(t)$ consider

$$X(t) = W(t) - \frac{1}{3}\frac{t}{T}W(T-t), \text{ for } 0 \leq t \leq T.$$

For $T = 15$, find analytically $\mathbb{V}\mathrm{ar}[X(t)]$ and perform a numerical simulation to confirm your result. Is the accuracy sensitive to T? Also show a few paths for process $X(t)$.

Exercise 3 (4p.)

In GBM, with $S(t_0) = 1$, $r = 0.05$ consider the following time-dependent volatility function,

$$\sigma(t) = 0.6 - 0.2e^{-1.5\cdot t}.$$

Implement a Monte Carlo Euler and also a Milstein simulation to price a European call option with $K = 1.6$ for maturity time $T = 4$, with $N = 1000, 10000$ and 100.000 paths and $m = 50, 100, 200, 400$ time-steps. Make a statement about the observed numerical convergence.

Choose barrier $B = 1.5$ and determine the price of an *up-and-out barrier option*, and an *up-and-in barrier option.* Can you make statements about the accuracy?

Exercise 4 (3p.)

Perform a Monte Carlo simulation for a two-asset option with payoff $\max(S_1(T), S_2(T))$, where the asset prices are governed by correlated Brownian motion, $\sigma_1 = 0.4, \sigma_2 = 0.15$, $r = 0.01$, $S_1(0) = S_2(0) = 1$. Consider in two numerical experiments $\rho = -0.9$ and $\rho = 0.9$, respectively. Give confidence intervals for the results, and describe the observations regarding positive and negative correlation.

Exercise 5 (5p.)

Euler or Milstein schemes are not well-suited for simulating stochastic processes that are, by definition, positive and have a probability mass around zero. An example is the CIR process with the following dynamics:

$$\mathrm{d}v(t) = \kappa(\bar{v} - v(t))\mathrm{d}t + \gamma\sqrt{v(t)}\mathrm{d}W(t), \quad v(t_0) > 0. \tag{1}$$

When the Feller condition, $2\kappa\bar{v} > \gamma^2$, is satisfied, $v(t)$ can never reach zero, however, if this condition does not hold, the origin is accessible. In both cases, $v(t)$ cannot become negative.

a. (2p.) Choose a parameter set yourself and show that an Euler discretization can get to negative $v(t)$ values.

b. (3p.) Perform "exact simulation of the variance process" (meaning sampling from the non-central chi-squared distribution) and confirm that the asset paths are not negative when using this technique for the same parameter values as above. Perform two tests in which the time step is varied. Give expectations and variances.

Exercise 6 (10p.)

For a state vector $X(t) = [S(t), r(t)]^T$ and fixed probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with a filtration $\mathcal{F} = \{\mathcal{F}(t) : t \geq 0\}$, which satisfies the usual conditions, consider the bivariate so-called Black-Scholes Vasicek (BSV) process given by:

$$\begin{cases} \mathrm{d}S(t) = & r(t)S(t)\mathrm{d}t + & \sigma S(t)\mathrm{d}W^S(t), & S_0 > 0, \\ \mathrm{d}r(t) = & \kappa\left(\theta - r(t)\right)\mathrm{d}t + & \gamma \mathrm{d}W^r(t), & r_0 > 0, \end{cases} \tag{2}$$

with correlation $\mathrm{d}W^S(t)\mathrm{d}W^r(t) = \rho \mathrm{d}t$, $t > 0$, $\theta > 0$, $\sigma > 0$, $\gamma > 0$ and $\kappa > 0$. Choose $\rho = -0.5, \theta = 0.2, \gamma = 0.7, \sigma = 0.5, S_0 = 1, r_0 = \theta, T = 1$, and kappa is either $\kappa = 1.0$ or $\kappa = 0.2$.

a) (1p.) Show that the solution of $S(t), t > 0$ can be expressed as:

$$S(t) = S_0 \exp\left(\int_0^t (r(s) - \frac{1}{2}\sigma^2)\mathrm{d}s + \sigma W^S(t)\right)$$

b) (2p.) Construct a self-financing portfolio and derive pricing PDE.
Hint: The construction of the pricing PDE is similar to the one for the Heston model, however, with zero-coupon bond.

c) (2p.) Perform a Monte-Carlo simulation to price a put option with $K = 1.1$, with the Euler discretization method, for the two κ-values, and discuss the results obtained.

d) (2p.) For this system of SDEs under the log-transform for $S(T)$, derive the characteristic function for $\log(S(T))$.

e) (3p.) Employ the COS method, to recover the density function for the BSV system given here.