Computational Finance Exercise Set II

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Exercise 1

For a given function $g(t) = W^2(t), f(t) = W^4(t)$ and for $T = \frac{1}{2}$, determine theoretically,

$$\int_0^T g(t)f(t)dW(t).$$

Confirm your answer by a Monte Carlo experiment.

Solution:

In order to compute this integral, we need to apply Ito's lemma to a function in order to solve for the integrand. Starting from a given function $h(t) = \frac{1}{7}W^7(t)$, we have that

$$dh(t) = W^6(t)dW(t) + \frac{5}{2}W^5(t)dt.$$

Integrating both terms from 0 to T,

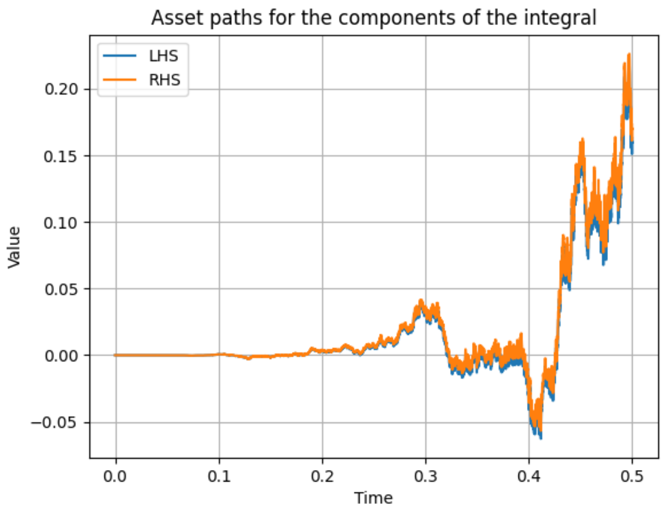
$$\frac{1}{7}W^7(T) = \int_0^T W^6(t)dW(t) + \frac{5}{2}\int_0^T W^5(t)dt$$

so that the desired integral reduces to

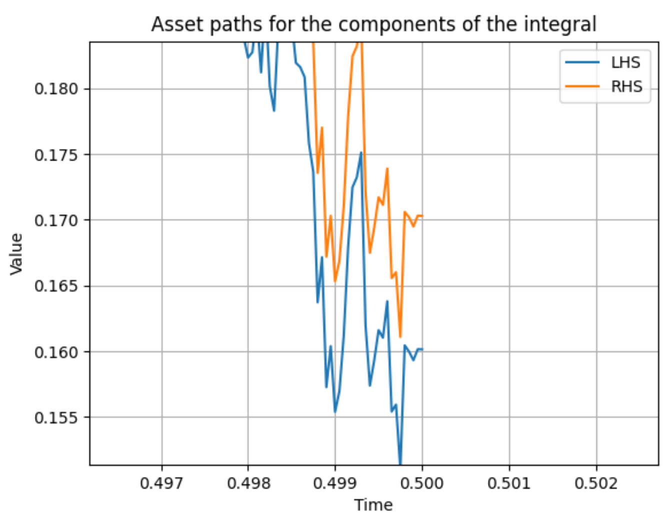
$$\int_0^T W^6(t)dW(t) = \frac{1}{7}W^7(T) - \frac{5}{2}\int_0^T W^5(t)dt.$$

We can check this result numerically by performing the two integrals evaluating the Monte Carlo paths for $W^5(t)$ and $W^6(t)$.

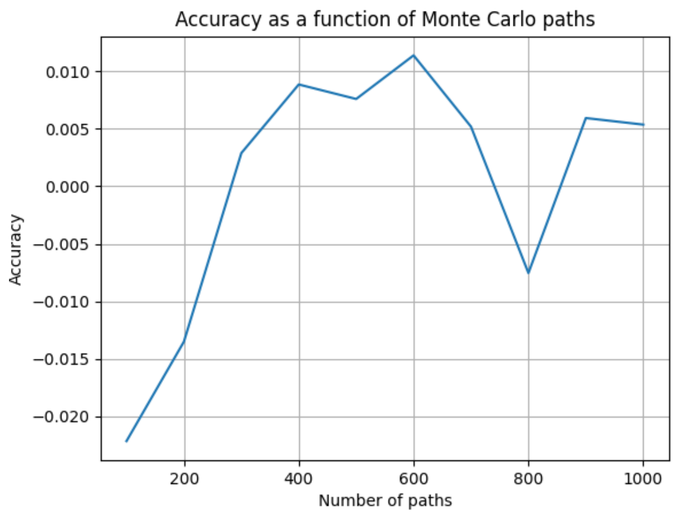
First of all, we check that the asset paths for the left hand side and right hand side of the integral coincide, at least at the endpoint. To this end, we obtain the following paths corresponding to the LHS and RHS of the previous equation, coming from averaging together 1000 Monte Carlo paths, along 10000 steps.



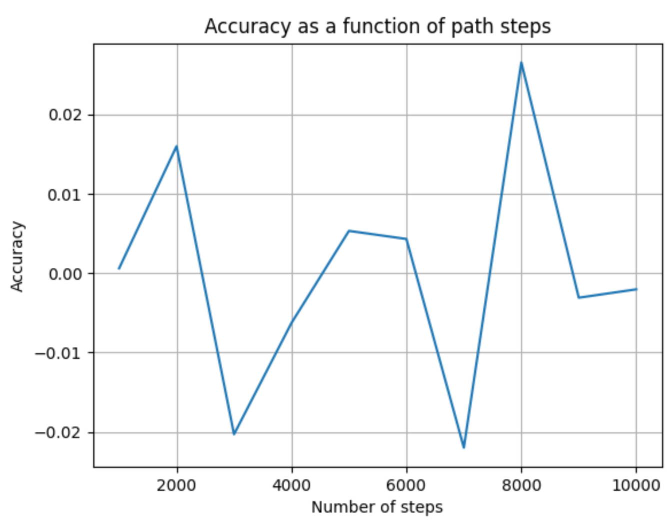
We can see how both components are very close to each other for the whole path. In order to check the accuracy, we must focus on the value at the endpoint. Zooming in, we can see how there is a systematic error between both paths.



The resulting accuracy is close to zero, although it does not seem to converge with respect to the number of Monte Carlo paths or steps after a given amount. To show this, we plot the accuracy as a function of the Monte Carlo paths and number of steps. First, we set the number of steps to 10000 and range the paths between 100 and 1000 in steps of 100, giving the following result



Then, we do the opposite fixing the amount of Monte Carlo paths to 1000 and ranging the amount of steps between 1000 and 10000 in steps of 1000, giving the following plot



We can see how the amount of error is reasonably close to zero although the convergence is not perfect and there are noticeable oscillations.

Exercise 2

For a Wiener process $W(t)$ consider

$$X(t) = W(t) - \frac{1}{3}\frac{t}{T}W(T-t), \text{ for } 0 \leq t \leq T.$$

For $T = 15$, find analytically $\mathrm{Var}[X(t)]$ and perform a numerical simulation to confirm your result. Is the accuracy sensitive to T ? Also show a few paths for process $X(t)$.

Solution:

Using the known formula for the variance of the sum of two random variables,

$$\mathrm{Var}(aX - bY) = a^2 \,\mathrm{Var}(X) + b^2 \,\mathrm{Var}(Y) - 2ab \,\mathrm{Cov}(X, Y)$$

we find for $X(t)$ that

$$\mathrm{Var}\, X(t) = \mathrm{Var}\, W(t) + \left(\frac{t}{3T}\right)^2 \mathrm{Var}\, W(T-t) - \frac{2t}{3T} \,\mathrm{Cov}[W(t), W(T-t)]$$

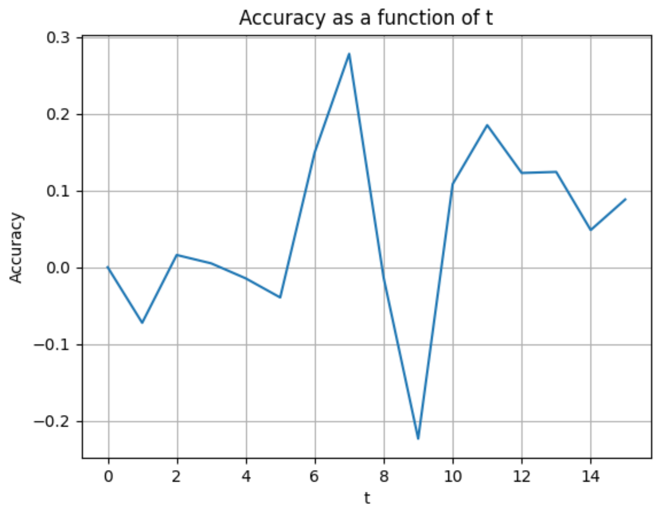
replacing the know values for the variance and covariance of brownian motions, namely $\mathrm{Var}\, W(t) = t$ and $\mathrm{Cov}[W(t_1), W(t_2)] = \min(t_1, t_2)$

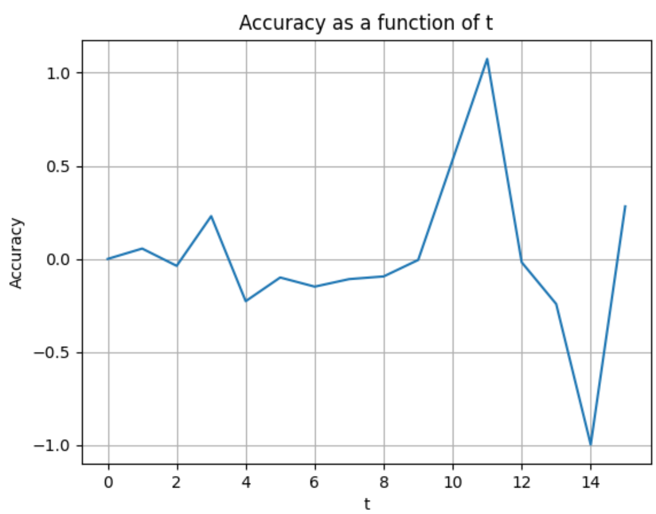
$$\mathrm{Var}\, X(t) = t + \left(\frac{t}{3T}\right)^2 (T-t) - \frac{2t}{3T} \min(t, T-t).$$

The last term in the equation can be expressed with an absolute value as $\min(t, T-t) = -|t - \frac{T}{2}| + \frac{T}{2}$. Therefore, replacing the value of $T = 15$ we arrive at the expression

$$\mathrm{Var}\, X(t) = \frac{t^2}{45^2}(15 - t) + \frac{2t}{45}\left|t - \frac{15}{2}\right| + \frac{2}{3}t.$$

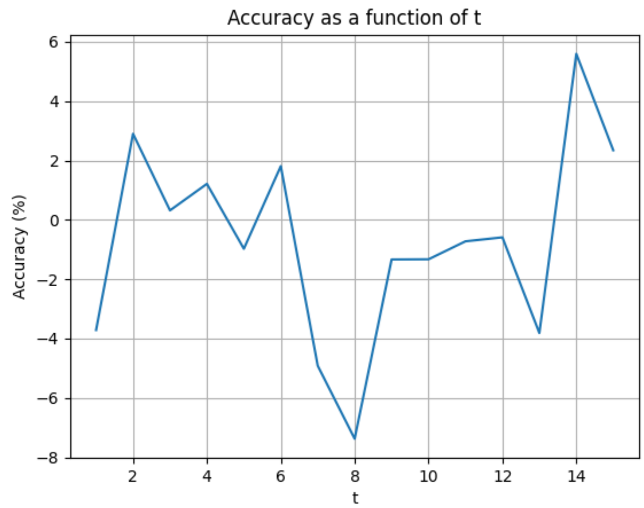
We now check this result numerically by computing 1000 Monte Carlo paths for $X(t)$ with 10000 steps each and comparing the resulting variance to the analytical formula at several values for $t \leq T$, resulting in the following plots:



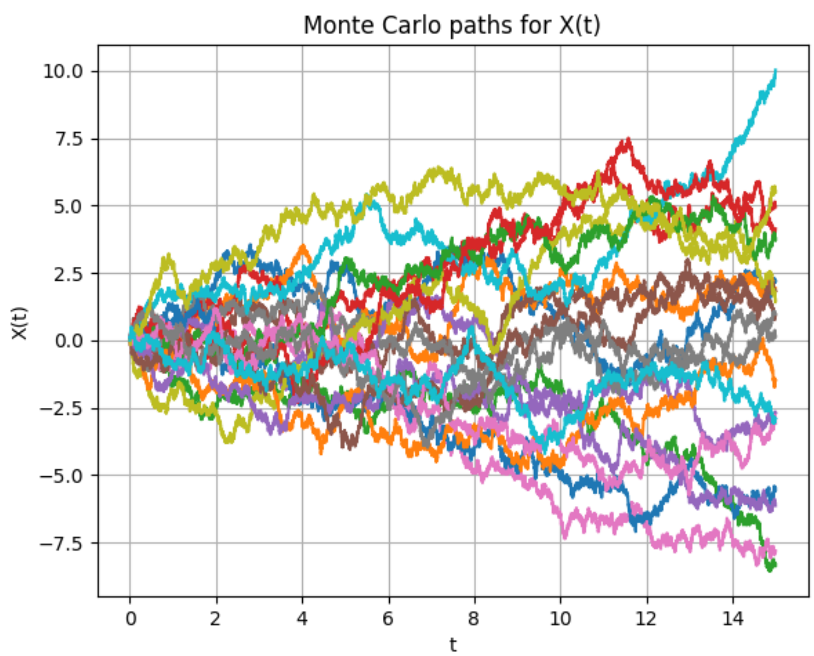


It seems that the error between the variances increases after crossing the mid-point $t = \frac{15}{2}$, so it looks like it is slightly sensitive to t.

As we can see in the following plot, the relative error sits at around 5% or less for all the values of t.



The process $X(t)$ looks as follows:



Exercise 3

In GBM, with $S(t_0) = 1, r = 0.05$ consider the following time-dependent volatility function,

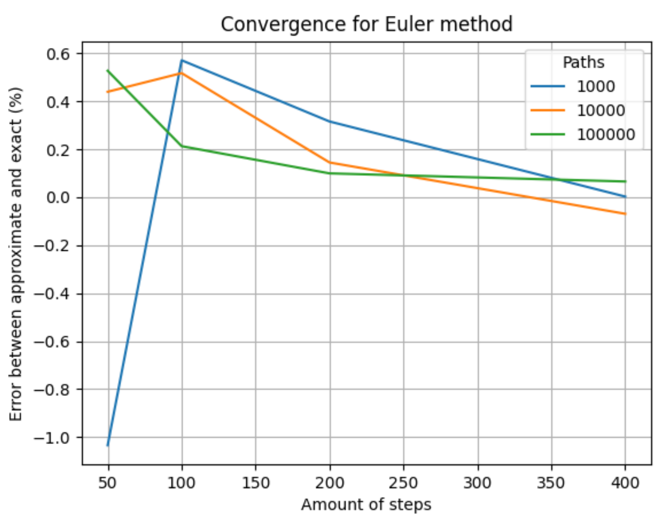
$$\sigma(t) = 0.6 - 0.2e^{-1.5 \cdot t}$$

Implement a Monte Carlo Euler and also a Milstein simulation to price a European call option with $K = 1.6$ for maturity time $T = 4$, with $N = 1000, 10000$ and 100.000 paths and $m = 50, 100, 200, 400$ time-steps. Make a statement about the observed numerical convergence.

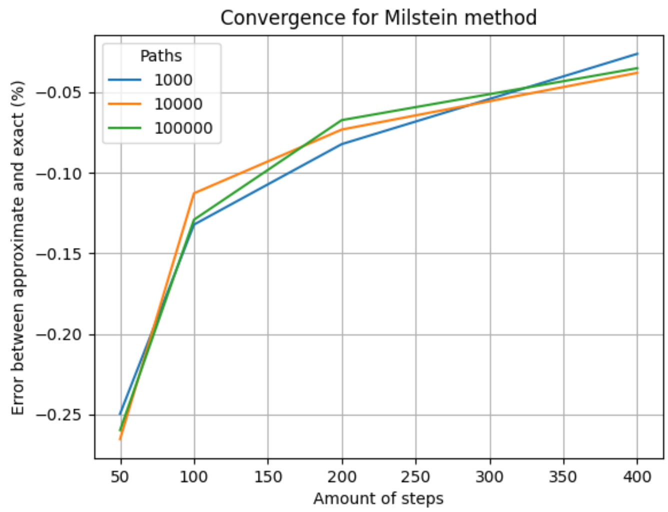
Choose barrier $B = 1.5$ and determine the price of an up-and-out barrier option, and an up-and-in barrier option. Can you make statements about the accuracy?

Solution:

We implement the Euler and Milstein discretization methods to price an European call option with the given parameters. The resulting plots show the convergence of the solutions, that is, the relative difference between the exact and approximate (Euler, Milstein) solutions. First of all, for Euler discretization we have the following



While for Milstein we have the following

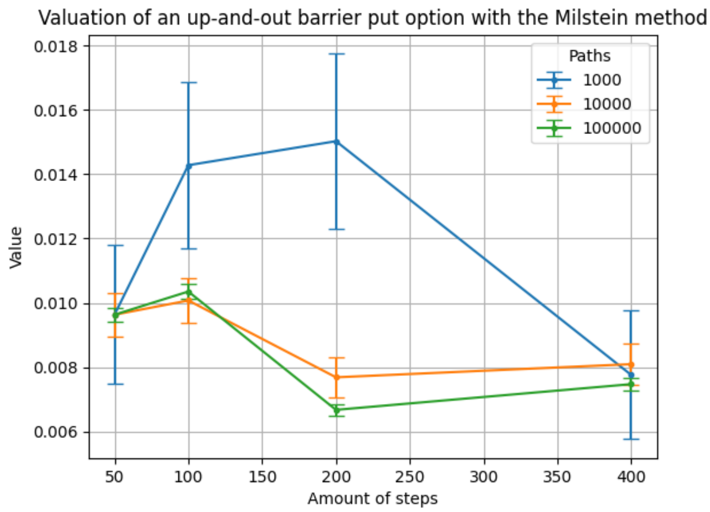


We can see how the Milstein method shows a much better convergence even for a small number of paths, while the Euler method converges more slowly.

Now, we can apply this pricing procedure to price barrier up-and-in and up-and-out options. The only change needed is to adapt the payoff function to be path dependent and consider the whole asset path instead of just the end of it.

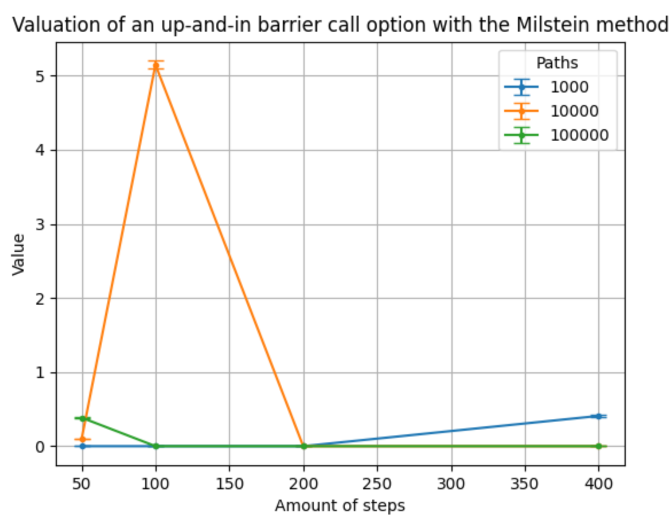
While for the vanilla European options we only considered calls, in this case we also need to consider put options because the barrier is smaller than the strike. More specifically, an up-and-out call option with a barrier smaller than the strike will always be worth zero, as the option gets knocked out before it gets in the money.

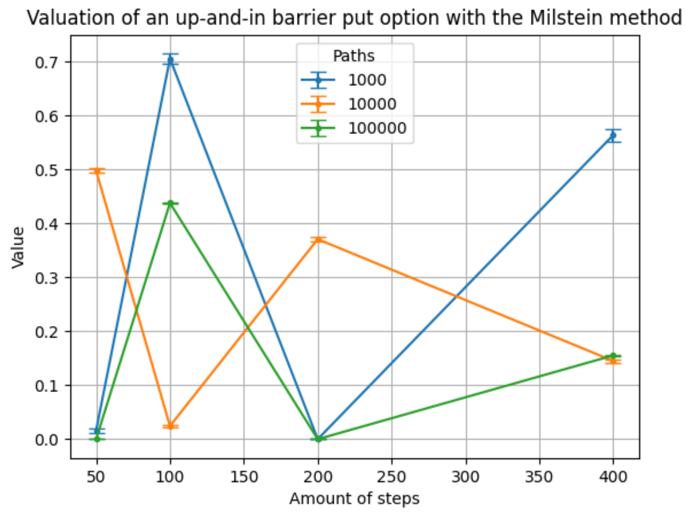
The prices for an up-and-out put option with the Milstein method, ranging over paths and steps, is given by the following plot:



We can see how increasing the number of paths diminishes the amount of error. The Monte Carlo error is considered as the standard deviation of the asset divided by the square root of the number of paths.

While for up-and-in options, both calls and puts are sensible because as soon as the asset price reaches the barrier value, the option behaves as if it was European. The corresponding plots for the up-and-in barrier option are the following.





We can see how the error again diminishes with the number of paths, however the valuation is unclear because there is no clear convergence to a given value. It looks like the call option expires worthless very often because it is unlikely for the path to reach the strike.

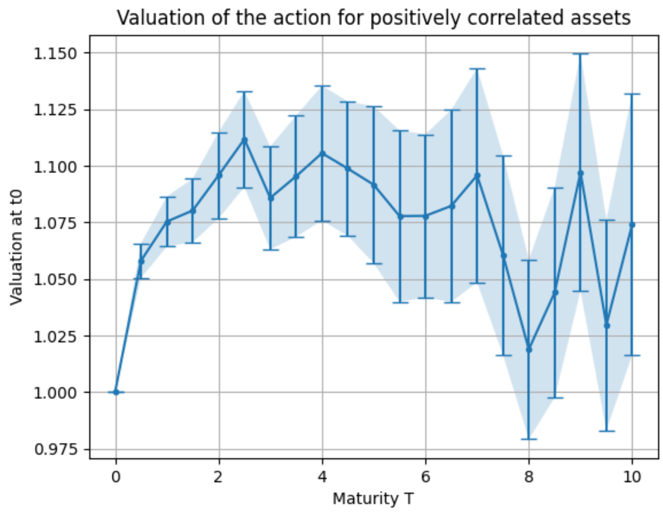
Exercise 4

Perform a Monte Carlo simulation for a two-asset option with payoff $\max(S_1(T), S_2(T))$, where the asset prices are governed by correlated Brownian motion, $\sigma_1 = 0.4, \sigma_2 = 0.15, r = 0.01, S_1(0) = S_2(0) = 1$. Consider in two numerical experiments $\rho = -0.9$ and $\rho = 0.9$, respectively. Give confidence intervals for the results, and describe the observations regarding positive and negative correlation.

Solution:

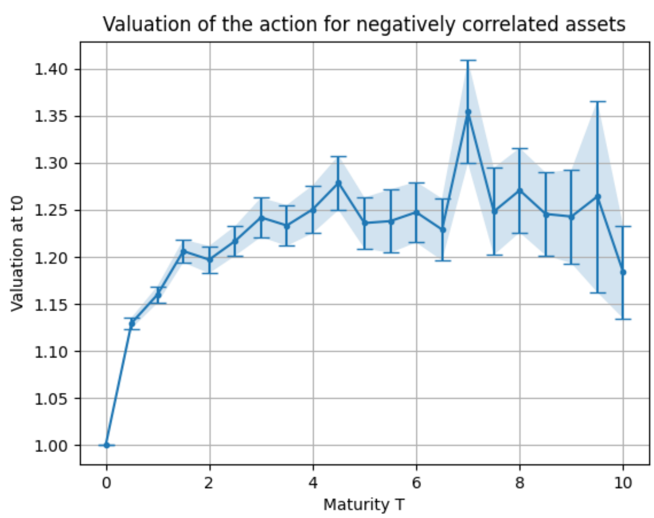
We compute the pricing of the two-asset option with the given payoff by computing the correlated brownian motions and evaluating the two-asset payoff function on them. We evaluate the prices for several equispaced maturities between 0 and 10, for 1000 Monte Carlo paths of 2000 time steps each. For each price, we compute a Monte Carlo error that is given by the standard deviation of the prices divided by the square root of the number of paths.

For positively correlated assets, with a correlation $\rho = 0.9$, we obtain the following valuations:



We can see how the errors increase with the maturity as should be expected, and become very large for long maturities.

On the other hand, for negatively correlated assets with a correlation $\rho = -0.9$, we obtain the following valuations:



We can see how in this case the errors are noticeably smaller.

Exercise 5

Euler or Milstein schemes are not well-suited for simulating stochastic processes that are, by definition, positive and have a probability mass around zero. An example is the CIR process with the following dynamics:

$$\mathrm{d}v(t) = \kappa(\bar{v} - v(t))\mathrm{d}t + \gamma\sqrt{v(t)}\mathrm{d}W(t), \quad v(t_0) > 0.$$

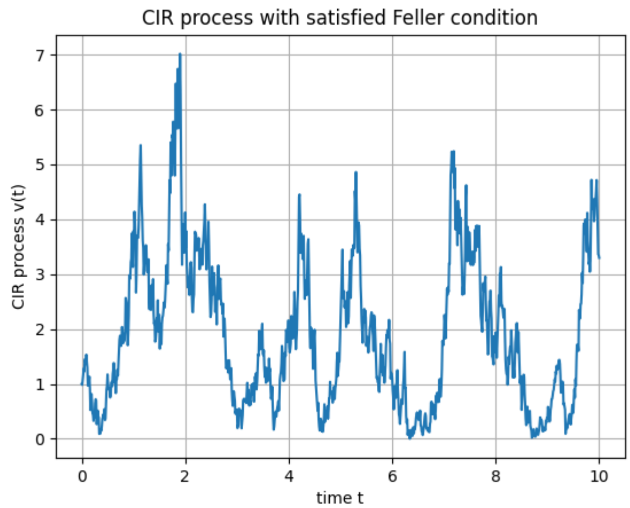
When the Feller condition, $2\kappa\bar{v} > \gamma^2$, is satisfied, $v(t)$ can never reach zero, however, if this condition does not hold, the origin is accessible. In both cases, $v(t)$ cannot become negative.

1. (2p.) Choose a parameter set yourself and show that an Euler discretization can get to negative $v(t)$ values.
2. (3p.) Perform "exact simulation of the variance process" (meaning sampling from the non-central chisquared distribution) and confirm that the asset paths are not negative when using this technique for the same parameter values as above. Perform two tests in which the time step is varied. Give expectations and variances.

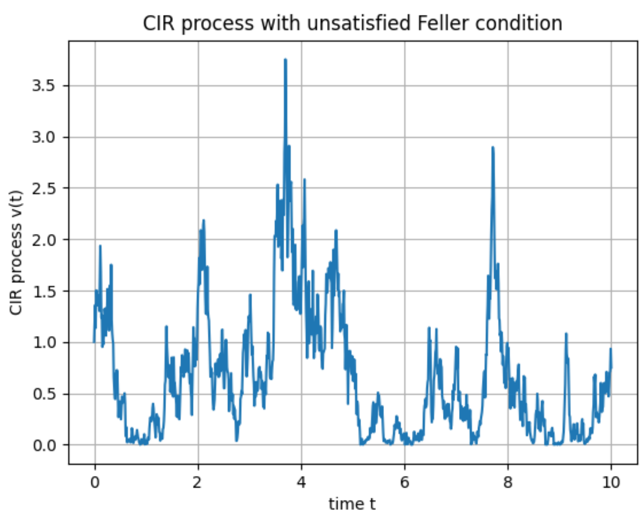
Solution:

In this exercise we compute asset paths for the CIR process defined above using the Euler and Milstein discretizations. For all the simulations we set the values $T = 10$, $v_0 = 1$ and we use 1000 time steps, while we change the parameters $\bar{v}$, κ and γ in order to observe the different regimes.

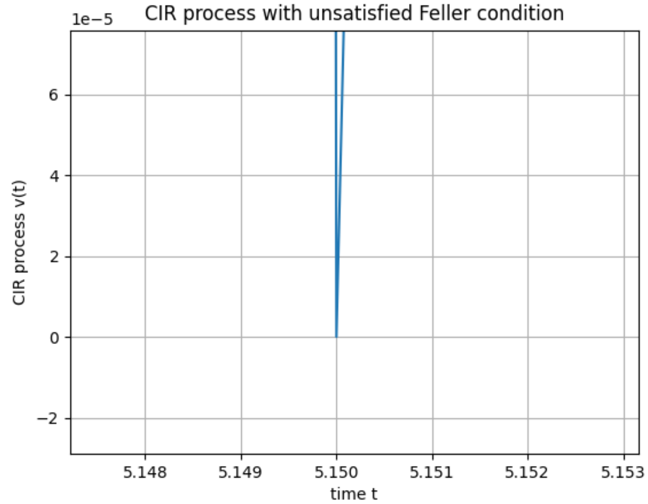
First of all, starting with Euler discretization we select a set of parameters that satisfies the Feller condition. This ensures that the asset paths for $v(t)$ do not reach the origin. For the parameter values $\bar{v} = 2$, $\kappa = 2$ and $\gamma = 2$ we get the following plot:



We can see how $v(t)$ never reaches the origin. On the other hand, for the parameter set $\bar{v} = 1$, $\kappa = 2$ and $\gamma = 1$ the Feller condition is not satisfied and therefore it is possible for $v(t)$ to reach the origin. The corresponding path is given by:



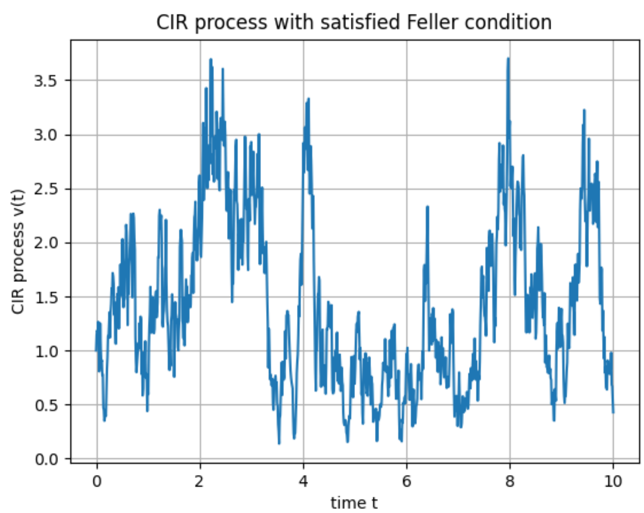
We can see how it looks like $v(t)$ touches the origin. To be sure of this, we zoom in to one part in 10^{-5} around one of the peaks to see if it actually does.



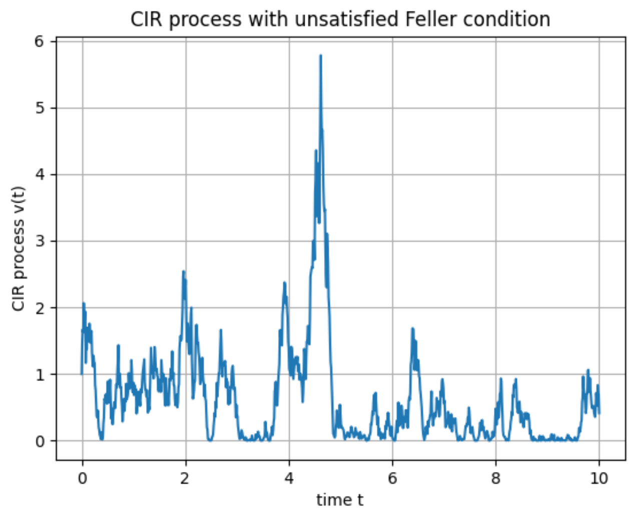
We can see how it indeed reaches the origin.

After this, we repeat the same procedure but this time employing an exact simulation by sampling directly from the non-central chisquared distribution. This time, the paths should not reach the origin in any case even if the Feller condition is not satisfied. To check this, we repeat the previous plots for the same parameter vales.

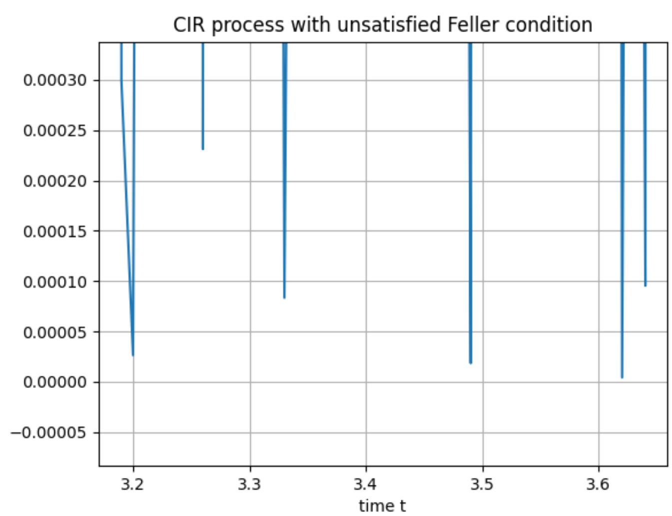
For the satisfied Feller condition, we see how $v(t)$ keeps far from the origin



However, when it is not satisfied, it looks like it reaches the origin.

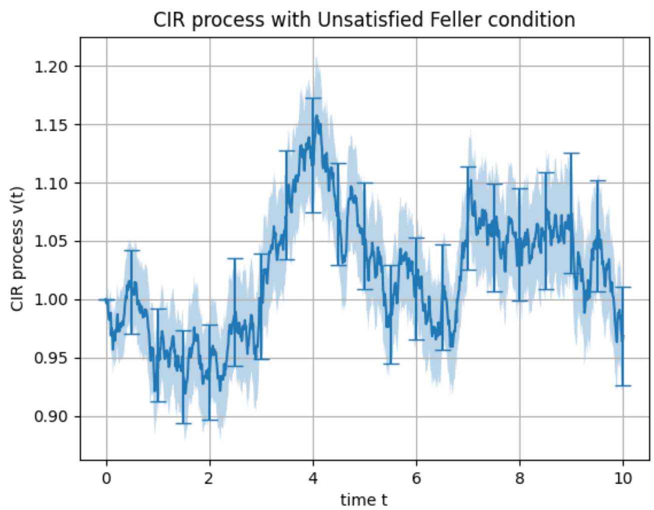


Zooming in around some of the peaks to one part in 10^{-5} we see how actually it does not reach the origin but gets arbitrarily close to it.

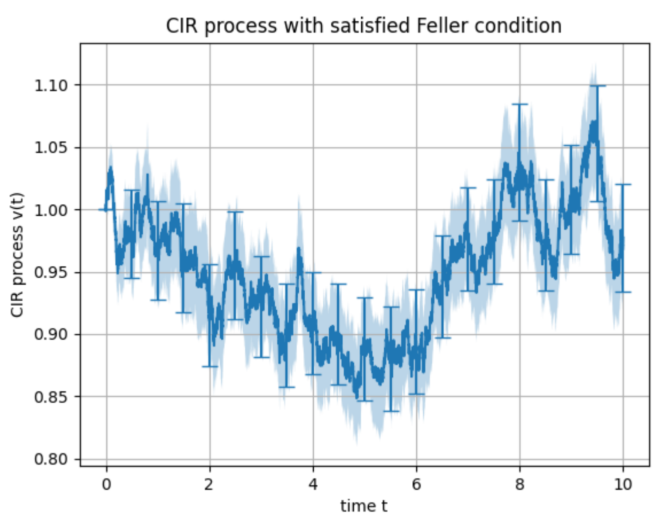


Finally, we perform two tests with expectations and variances by plotting the path for $v(t)$ for a larger amount of time steps, 1000 and 10000 steps respectively, and for 1000 Monte Carlo paths, taking the average between them. We also include the errors with respect to the paths. In both cases, the Feller condition is not satisfied and the second parameter set is used.

1000 steps:



10000 steps:



In both cases we see how the Monte Carlo average for $v(t)$ keeps far from the origin. This is most likely due to the larger amount of paths used and the fact that on average each one of them keeps far away from the origin.

Exercise 6

For a state vector $X(t) = [S(t), r(t)]^T$ and fixed probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with a filtration $\mathcal{F} = \{\mathcal{F}(t) : t \geq 0\}$, which satisfies the usual conditions, consider the bivariate so-called Black-Scholes Vasicek (BSV) process given by:

$$\begin{aligned} \mathrm{d}S(t) &= \quad r(t)S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^S(t), \quad S_0 > 0, \\ \mathrm{d}r(t) &= \kappa(\theta - r(t))\mathrm{d}t + \quad \gamma \mathrm{d}W^r(t), \quad r_0 > 0, \end{aligned}$$

with correlation $\mathrm{d}W^S(t)\mathrm{d}W^r(t) = \rho \mathrm{d}t, t > 0, \theta > 0, \sigma > 0, \gamma > 0$ and $\kappa > 0$. Choose $\rho = -0.5, \theta = 0.2, \gamma = 0.7, \sigma = 0.5, S_0 = 1, r_0 = \theta, T = 1$, and kappa is either $\kappa = 1.0$ or $\kappa = 0.2$.

1. (1p.) Show that the solution of $S(t), t > 0$ can be expressed as:

$$S(t) = S_0 \exp\left(\int_0^t \left(r(s) - \frac{1}{2}\sigma^2\right) \mathrm{d}s + \sigma W^S(t)\right)$$

2. (2p.) Construct a self-financing portfolio and derive pricing PDE. Hint: The construction of the pricing PDE is similar to the one for the Heston model, however, with zero-coupon bond.
3. (2p.) Perform a Monte-Carlo simulation to price a put option with $K = 1.1$, with the Euler discretization method, for the two κ-values, and discuss the results obtained.
4. (2p.) For this system of SDEs under the log-transform for $S(T)$, derive the characteristic function for $\log(S(T))$.
5. (3p.) Employ the COS method, to recover the density function for the BSV system given here.

Solution:

1. The given solution for $S(t)$ comes from integrating the expression for $dS(t)$. Note however how Ito's lemma is needed in order to perform the integration.

   First of all, notice how

$$\frac{dS(t)}{S(t)} = r(t)dt + \sigma dW^S(t) = d(\log S(t)).$$

   Applying Ito's lemma on $d(\log S(t))$,

$$d(\log S(t)) = \left(r(t) - \frac{1}{2}\sigma^2\right) dt + \sigma dW^s(t).$$

Finally, integrating the expression between 0 and t we reach the desired expression:

$$\int_0^t d(\log S(s)) = \int_0^t \left(r(s) - \frac{1}{2}\sigma^2\right) ds + \sigma(W^S(t) - W^S(0))$$

which leads to

$$S(t) = S_0 \exp\left(\int_0^t \left(r(s) - \frac{1}{2}\sigma^2\right) \mathrm{d}s + \sigma W^S(t)\right).$$

2. We can derive the pricing PDE by imposing the martingale property, that is, by constructing a self-financing portfolio consisting of a money account $M(t)$ together with an option of value $V(S,r,t)$ and imposing that $\frac{V(S,r,t)}{M(t)}$ is a martingale under the filtration $\mathcal{F}(t)$.

   We know from the course that this is equivalent to imposing a vanishing drift on the dynamics of the new portfolio, and that turns out to be easier. Therefore, all we need to do is find the dynamics of $\frac{V(S,r,t)}{M(t)}$.

   $$d\left(\frac{V(S,r,t)}{M(t)}\right) = \frac{1}{M(t)} dV(S,r,t) - V(S,r,t) d\left(\frac{1}{M(t)}\right)$$

   For simplicity, from now on we drop the arguments from the functions. The dynamics of M are given by a simple application of Ito's lemma:

   $$d(\frac{1}{M}) = -\frac{r}{M}$$

   while the dynamics for V are more involved. Expanding dV:

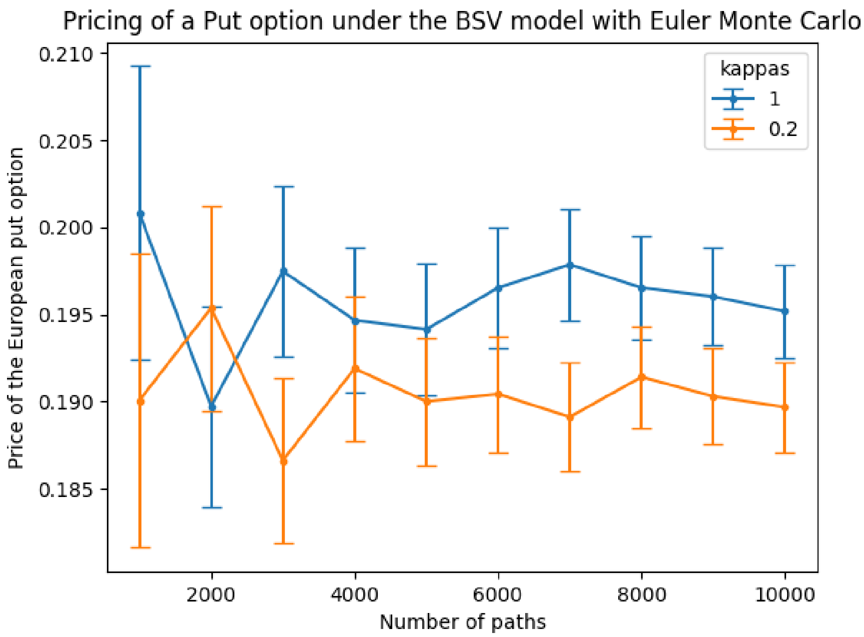
   $$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial r} dr + \frac{1}{2}\frac{\partial^2 V}{\partial r^2} dr^2 + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} dS^2 + \frac{\partial^2 V}{\partial S \partial r} dS dr$$

   Replacing the expressions for dS and dr on the equation above and grouping all the terms multiplying dt, we can reach the pricing PDE by equating them to zero. In order to do that, we need to apply Ito's table on the differentials, and consider the correlations between the brownian motions, $dW^S dW^r = \rho dt$. The result is:

   $$\frac{1}{M}\left\{\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \kappa(\theta - r)\frac{\partial V}{\partial r} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\gamma^2 \frac{\partial^2 V}{\partial r^2} + \sigma\gamma\rho S \frac{\partial^2 V}{\partial S \partial r}\right\} - \frac{rV}{M} = 0$$

3. In this section we perform a Monte Carlo simulation to price a put option under the BSV model using the Euler discretization method. In order to do that, we implement the asset paths for $r(t)$ and $S(t)$ including their corresponding correlation, and evaluate a given payoff function.

The prices are evaluated for 1000 Monte Carlo steps and for several Monte Carlo paths in order to appreciate the convergence of the method with respect to the number of paths. The results are as follows:



We can see how the valuations of the put option for the two values of $\kappa = 1, 0.2$ are clearly distinguished at around 0.2 and 0.19 respectively, and how the Monte Carlo error diminishes with the amount of paths albeit slowly.

4. For this section we employ the Vasicek characteristic function that can be found in the course's book. However, it only takes into consideration the evolution of the interest rate $r(t)$ and should be corrected in order to also incorporate the effects of the asset $S(t)$.

$$\phi_{\text{Vas}}(u;t,T) = \exp\left(-\int_t^T \psi(z)\mathrm{d}z + iu\psi(T)\right) \cdot \phi_{\overline{\text{HW}}}(u;t,T)$$

where

$$\exp\left(-\int_t^T \psi(z)\mathrm{d}z + iu\psi(T)\right) = \exp\left[\frac{1}{\kappa}(r_0 - \theta)\left(\mathrm{e}^{-\kappa T} - \mathrm{e}^{-\kappa t}\right) - (T-t)\theta\right]$$
$$\times \exp\left[iu\left(\theta + (r_0 - \theta)\,\mathrm{e}^{-\kappa T}\right)\right]$$

and

$$\phi_{\overline{\mathrm{HW}}}(u;t,T) = \exp(\bar{A}(u,\tau) + \bar{B}(u,\tau)\widetilde{r}(t))$$

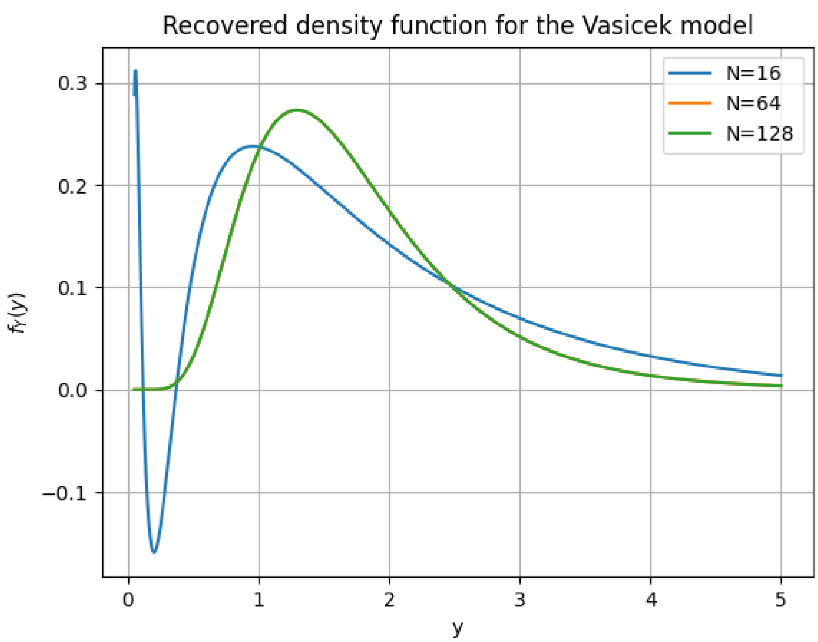
with coefficients given by

$$\bar{A}(u,\tau) = \frac{\gamma^2}{2\kappa^3}\left(\kappa\tau - 2\left(1-\mathrm{e}^{-\kappa\tau}\right) + \frac{1}{2}\left(1-\mathrm{e}^{-2\kappa\tau}\right)\right) - iu\frac{\gamma^2}{2\kappa^2}\left(1-\mathrm{e}^{-\kappa\tau}\right)^2 - \frac{1}{2}u^2\frac{\gamma^2}{2\kappa}\left(1-\mathrm{e}^{-2\kappa\tau}\right),$$

$$\bar{B}(u,\tau) = iu\mathrm{e}^{-\kappa\tau} - \frac{1}{\kappa}\left(1-\mathrm{e}^{-\kappa\tau}\right)$$

where $\tau = T - t_0$.

5. We can recover the density function for the Vasicek model by employing the COS method over the characteristic function defined above. The resulting density function is given by:



We can see how increasing the value of N results in a better density function. Note how the density function for $N = 64$ converges closely to the value of $N = 128$ so that it cannot be appreciated in the plot.