Computational Finance Exercise Set I

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Exercise 1

Show that

$$\mathbb{E}\left[(X-\mathbb{E}[X])^2\right]=\mathbb{E}\left[X^2\right]-(\mathbb{E}[X])^2 \tag{1}$$

and therefore that

$$\operatorname{Var}[\alpha X]=\alpha^2\operatorname{Var}[X],\ \text{ with } \alpha\in\mathbb{R} \tag{2}$$

Solution:

$$\begin{aligned}\mathbb{E}[(X-\mathbb{E}[X])^2&=\mathbb{E}[X^2-2X\mathbb{E}[X]+(\mathbb{E}[X])^2]]=\mathbb{E}[X]^2-2\mathbb{E}[X\mathbb{E}[X]]+\mathbb{E}[(\mathbb{E}[X])^2]\\&=\mathbb{E}[X^2]-2\mathbb{E}[X]^2+\mathbb{E}[X]^2=\mathbb{E}[X^2]-\mathbb{E}[X]^2\end{aligned}$$

$$\begin{aligned}\operatorname{Var}[X]&=\mathbb{E}[(X-\mathbb{E}[X])^2]\Rightarrow\operatorname{Var}[\alpha X]=\mathbb{E}[(\alpha X-\mathbb{E}[\alpha X])^2]=\mathbb{E}[(\alpha X)^2]-(\mathbb{E}[\alpha X])^2\\&=\alpha^2(\mathbb{E}[X^2]-(\mathbb{E}[X])^2)=\alpha^2\operatorname{Var}[X]\end{aligned}$$

Exercise 2

1. Show theoretically that

$$\int_0^t W(s)\mathrm{d}s=\int_0^t(t-s)\mathrm{d}W(s) \tag{3}$$

2. Also, taking $t=5$ validate the equality above by a numerical experiment.

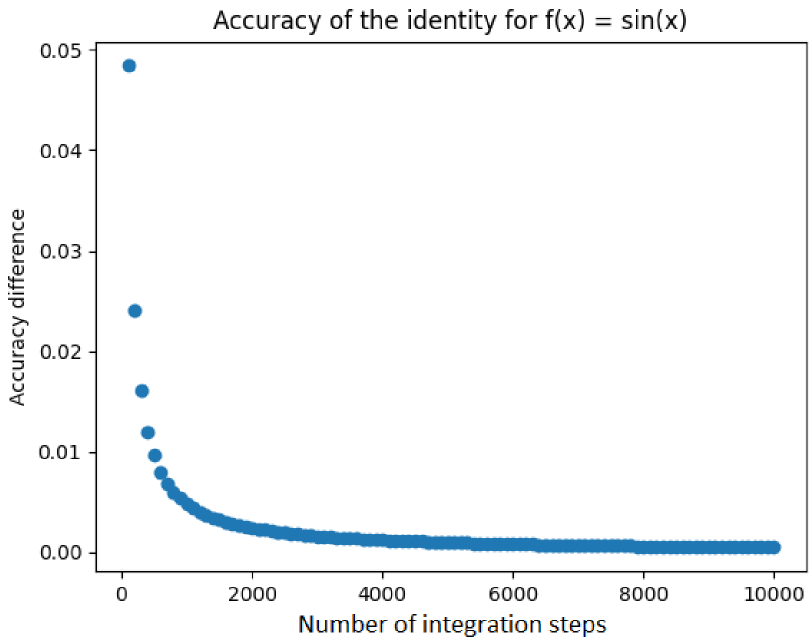
Solution:

If W is a Brownian motion, $W(0)=0$. Then, integrating by parts:

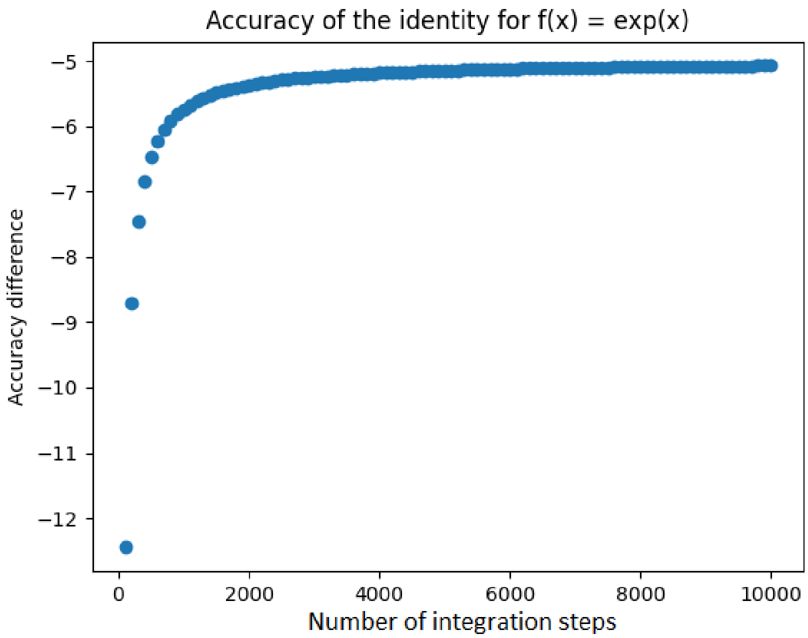
$$\int_0^t W(s)ds=W(s)\cdot s\Big|_0^t-\int_0^t sdW(s)=tW(t)-\int_0^t sdW(s)=\int_0^t(t-s)dW(s)$$

Where we noted that $\int_0^t dW(s)=W(t)-W(0)=W(t)$ in the last equality.

Numerical experiment:
The results above are validated with a numerical experiment, computing the quadrature of the function $W(x)=\sin(x)$. The accuracy between both expressions converges to zero with the size of the integration measure.



Note however that the identity is valid when the function $W(x)$ satisfies $W(0) = 0$. For other functions such as $W(x) = \cos(x)$ or $W(x) = \exp(x)$, the identity is not valid and can be checked numerically.



Exercise 3

1. Supposing $S(t)$ follows a log-normal Brownian motion given by

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu \mathrm{d}t + \sigma \mathrm{d}W(t) \tag{4}$$

find the dynamics for process $Y(t) = 2S(t)^2$.

2. Apply Ito's formula to $\mathrm{e}^{W(t)}$ where $W(t)$ is a standard Brownian motion. Is it a martingale?

3. Suppose that X(t) and Y(t) satisfy the SDEs

$$\mathrm{d}X(t) = 0.04X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W^{\mathbb{P}}(t) \tag{5}$$

$$\mathrm{d}Y(t) = \beta Y(t)\mathrm{d}t + 0.15Y(t)\mathrm{d}W^{\mathbb{P}}(t) \tag{6}$$

For a given money savings account $\mathrm{d}M(t) = rM(t)\mathrm{d}t$ use Euler discretization to find:

$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T)} \max\left(\frac{1}{2}X(T) - \frac{1}{2}Y(T), K\right) \mid \mathcal{F}(t)\right] \tag{7}$$

Solution:

1. Applying Ito's lemma and using Ito's table:

$$\begin{aligned}
Y(t) = 2S(t)^2 \Rightarrow dY &= \frac{\partial Y}{\partial t}dt + \frac{\partial Y}{\partial S}dS + \frac{1}{2}\frac{\partial^2 Y}{\partial p^2}(dS)^2 \\
&= 4SdS + 2(dS)^2 = 4S(\mu S dt + \sigma S dW) + 2(\sigma^2 S^2 dt) \\
&= (4\mu S^2 + 2\sigma^2 S^2)dt + 4\sigma S^2 dW
\end{aligned}$$

2. Analogously,

$$\begin{aligned}
Y(t) = e^{W(t)} \Rightarrow dY &= \frac{\partial Y}{\partial t}dt + \frac{\partial Y}{\partial W}dW + \frac{1}{2}\frac{\partial^2 Y}{\partial W^2}(dW)^2 \\
&= e^W dW + \frac{1}{2}e^2(dW)^2 = e^W dW + \frac{1}{2}e^W dt
\end{aligned}$$

We can see how this process is **not** a martingale by seing how it has a drift term $\frac{1}{2}e^W dt$. More rigurously:

$$\begin{aligned}
\mathbb{E}[e^{W(t)}|\mathcal{F}(s)] &= \mathbb{E}[e^{W(t)-W(s)+W(s)}|\mathcal{F}(s)] = \mathbb{E}[e^{W(s)} \cdot e^{W(t)-W(s)}|\mathcal{F}(s)] \\
&= e^W(s)\mathbb{E}[e^{W(t)-W(s)}|\mathcal{F}(s)]
\end{aligned}$$

Because of the property of independent increments, and that $W(t) - W(s) \approx \mathcal{N}(\mu, \sigma)$ with mean $\mu = 0$ and standard deviation $\sigma^2 = t - s$:

$$\begin{aligned}
&\mathbb{E}[e^{W(t)-W(s)}|\mathcal{F}(s)] = \mathbb{E}[e^{W(t)-W(s)}] = e^{\mu+\sigma^2/2} = e^{\frac{t-s}{2}} \\
&\Rightarrow \mathbb{E}[e^W(t)|\mathcal{F}(s)] = e^{W(s)} \cdot e^{\frac{t-s}{2}} \neq e^{W(s)} \text{ not a martingale.}
\end{aligned}$$

3. Given a money-savings account $M(t)$ and an asset process $S(t)$ with equations

$$\begin{aligned}
dM(t) &= rM(t)dt \\
dS(t) &= \frac{1}{2}dX(t) - \frac{1}{2}dY(t)
\end{aligned}$$

where the underlying processes satisfy two stochastic differential equations $dX(t)$ and $dY(t)$, suppose there is a contingent claim of the form

$$\chi = V(S(T), T) = \max\left(\frac{1}{2}X(t) - \frac{1}{2}Y(t), K\right) \tag{8}$$

that is, a call option on the process $S(t)$ with a given strike K.

We would like to find the discounted payoff under risk-neutral measure. For that, we must check that the process $\frac{S(t)}{M(t)}$ is a martingale. Its dynamics are given by the equation

$$d\frac{S}{M} = \frac{S}{M}(\mu - r)dt + \frac{1}{M}\sigma S dW^{\mathbb{P}} \tag{9}$$

and we need to have $\mu = r$ in order to have a driftless process, which happens when setting the parameters values correctly to $\beta = 0.1$ and $r = 0.06$. Then, we know we must look for option prices in the form

$$V(S(t),t) = M(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T)}V(S(T),T) \mid \mathcal{F}(t)\right]. \tag{10}$$

Assuming the money-savings account stays constant at time t, $M(t) = 1$ and considering the call option payout at time T, we arrive at the desired identity:

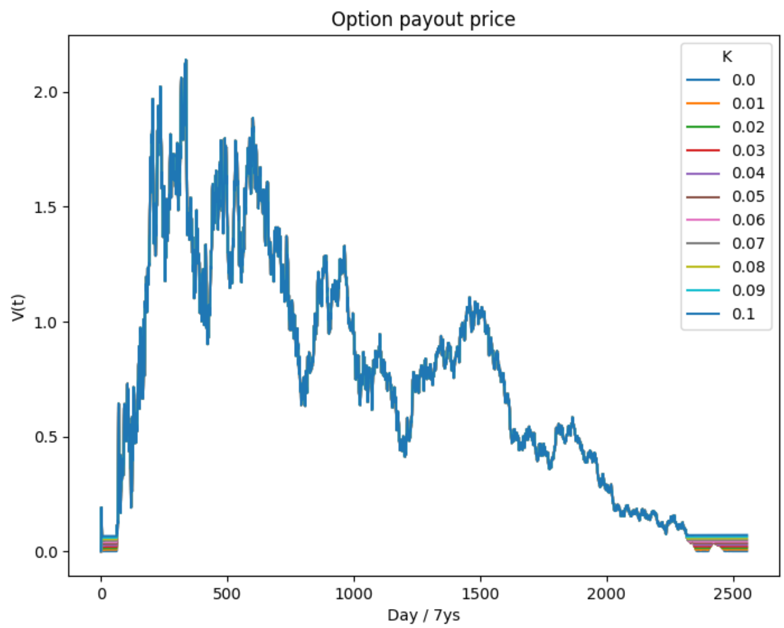
$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T)}\max\left(\frac{1}{2}X(T) - \frac{1}{2}Y(T), K\right) \mid \mathcal{F}(t)\right] \tag{11}$$

Numerical experiment:
We can plot the option price $V(T)$ against the option strikes K for the parameters asked by solving numerically $V(t)$. To that end, we reuse the function *GeneratePathsGBMABM* to generate the asset paths $X(t)$, $Y(t)$ as well as the money-saving account $M(t)$.

Evaluating numerically the call option payout to:

$$\begin{aligned} V(t) &= \frac{e^{-r(T-t)}}{M(t)}\mathbb{E}^{\mathbb{Q}}\left\{\max\left[\frac{1}{2}X(t)e^{\alpha(T-t)+\sigma[W(T)-W(t)]} - \frac{1}{2}Y(t)e^{\beta(T-t)+\tilde{\sigma}[W(T)-W(t)]}, K\right]\middle|\mathcal{F}(t)\right\} \\ &= \frac{e^{-r(T-t)}}{M(t)}\max\left[\frac{1}{2}X(t)e^{\alpha(T-t)+\sigma[T-t]} - \frac{1}{2}Y(t)e^{\beta(T-t)+\tilde{\sigma}[T-t]}, K\right] \end{aligned}$$

the corresponding plot of $V(t)$ against K is:



The implementation seems faulty because most of the option prices lie above the strike which is not realistic.

Exercise 4

For a Wiener process $W(t)$, consider

$$X(t) = W(t) - \frac{t}{T}W(T-t), \text{ for } 0 \leq t \leq s \leq T \tag{12}$$

For $T = 10$ find analitically $\mathrm{Var}(X(t))$ and perform a numerical simulation to confirm yout result. Is the accuracy sensitive to t?

Solution:

Starting from

$$X(t) = W(t) - \frac{t}{T}W(10-T)$$

we know that $\mathrm{Var}(aX+bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) + 2ab\mathrm{Cov}(X,Y)$. Then,

$$\mathrm{Var}[W(t) - \frac{t}{T}W(T-t)] = \mathrm{Var}[W(t)] + \frac{t^2}{T^2}Var[W(T-t)] - 2\frac{t}{T}\mathrm{Cov}[W(t), W(T-t)]$$

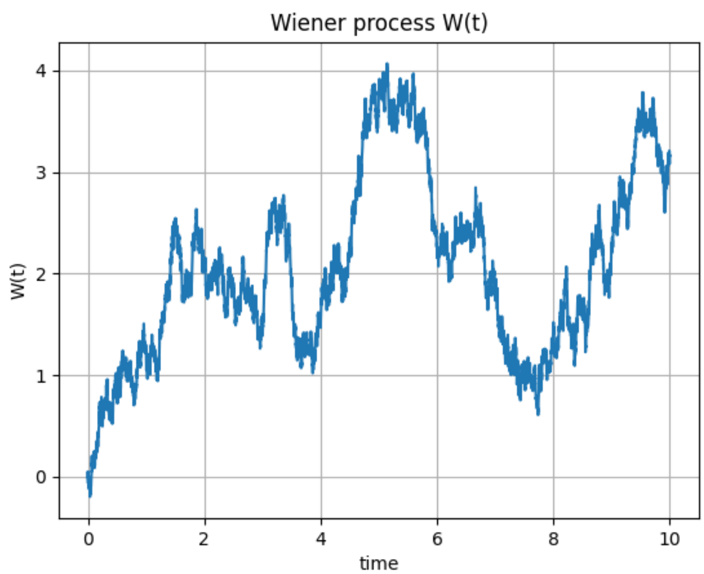
Now, $\mathrm{Cov}[W(s), W(t)] = \min(s,t)$.

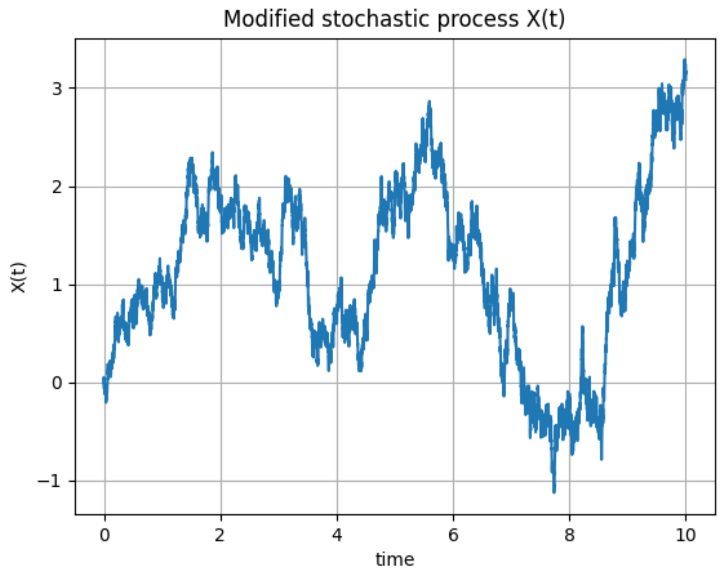
- If $t < T - t \Rightarrow t < \frac{T}{2}$, $\text{Cov}[W(t), W(T-t)] = t$.
- If $t > T - t \Rightarrow t > \frac{T}{2}$, $\text{Cov}[W(t), W(T-t)] = T - t$.

Putting everything together and substituting $T = 10$:

$$\begin{aligned}\text{Var}[X(t)] &= t + \frac{t^2}{100}(10-t) - \frac{2t}{10}\left(-|t-5| + 5\right) \\ &= \frac{t^2}{100}(10-t) + \frac{t|t-5|}{5}\end{aligned}$$

Numerical experiment:
First of all, we compute the paths for the Wiener process as well as the stochastic process $X(t)$.

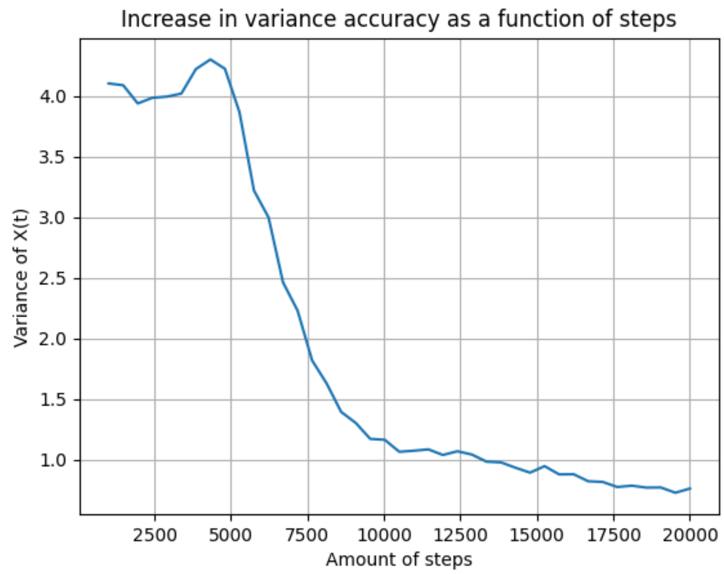




For $t = 5$, $T = 10$, the analytical formula gives a variance value of $\text{Var}[X(t)] = 1.25$. Unfortunately, the numerical results seem to be off by a factor of two. Taking into consideration this missing factor of two, the results get closer with an increasing number of paths and steps.

For 500 paths and 5000 steps each path, the numerical value for the variance is $\text{Var}[X(t)] = 1.194$ and increasing for larger amounts of steps.

However, further analysis reveals that the variance does not converge to the analytical value, probably due to a fault in the implementation.



Exercise 5

1. Use a stochastic representation result (Feynman-Kac theorem) to solve the following boundary value problem in the domain $[0, T] \times \mathbb{R}$

$$\begin{aligned} \frac{\partial V}{\partial t} + \mu x \frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} &= 0 \\ V(T, x) &= \log\left(x^3\right) \end{aligned} \tag{13}$$

2. Prove that if we are given the final condition problem:

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \\ V(S, T) = \text{ given} \end{cases} \tag{14}$$

with the sum of the first derivatives of the option square integrable, then the value, $V(S(t); t)$, is the unique solution of

$$V(S, t) = \mathrm{e}^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[V(S(T), T) \mid S(t)] \tag{15}$$

with $S(t)$ governed by the stochastic differential equation:

$$dS(t) = rS(t)dt + \sigma S(t) \mathrm{d}W^{\mathbb{Q}}(t) \tag{16}$$

Solution:

1. By means of the Feynman-Kac formula, we know that the solution is given by a process

$$V(x,t) = \mathbb{E}(\log[X(T)]^3|\mathcal{F}(t)) \tag{17}$$

where $X(T)$ is an Ito process driven by the equation

$$dX(s) = \mu x dt + \sigma x^2 dW(t) \tag{18}$$

and an initial condition $X(t) = x$. Therefore, we are left with the evaluation of the expectation value:

$$V(x,t) = \mathbb{E}\left\{\log\left[x + \mu x(T-t) + \sigma^2 x^2 (W(T) - W(t))\right]^3 \Big| \mathcal{F}(t)\right\} \tag{19}$$

Because the only random variable is the Wiener process $W(t)$ and we know that it follows a normal distribution with expected value t, the evaluation gives place to:

$$V(x,t) = \log\left[x + \mu x(T-t) + \sigma^2 x^2 (T-t)\right]^3 \tag{20}$$

2. To prove this theorem, we need to find the dynamics of $\frac{V(t,S)}{M(t)}$, which is known to be a martingale.

$$\begin{aligned} d\frac{V(t,S)}{M(t)} &= d\left(e^{-r(T-s)}V(t,S)\right) \\ &= V(t,S)d(e^{-r(T-s)} + e^{-r(T-s)}dV(t,S)) \end{aligned}$$

Now, the dynamics of the process $V(S,t)$ can be derived by Ito's lemma, giving

$$\mathrm{d}V = \left(\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)\mathrm{d}t + \sigma^2 S^2 \frac{\partial V}{\partial S}\,\mathrm{d}W^{\mathbb{Q}} \tag{21}$$

Multiplying the previous equation by $e^{r(T-s)}$ in order to solve for $dV(t,S)$ and replacing our expression for it, we get

$$\mathrm{e}^{r(T-s)}\mathrm{d}\left(\mathrm{e}^{-r(T-s)}V\right) = \left\{\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV\right\}\mathrm{d}t + \sigma^2 S^2 \frac{\partial V}{\partial S}\,\mathrm{d}W^{\mathbb{Q}} \tag{22}$$

The term between keys is zero by the own definition of the problem. The second to last step is to integrate this expression,

$$\begin{aligned} \int_t^T \mathrm{d}\left(\mathrm{e}^{-r(T-s)}V(t,S)\right) &= \mathrm{e}^{-r(T-t)}V(T,S) - V(t,S) \\ &= \int_t^T \mathrm{e}^{-r(T-s)}\bar{\sigma}\frac{\partial V}{\partial S}\,\mathrm{d}W^{\mathbb{Q}}(t) \end{aligned}$$

Finally, reordering the expression and taking an expectation with respect to the $\mathbb{Q}$ measure, we arrive at the desired expression for $V(t,S)$:

$$
\begin{aligned}
V(t,S) &= \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-t)}V(T,S) \mid \mathcal{F}(t)\right] - \mathbb{E}^{\mathbb{Q}}\left[\int_t^T \mathrm{e}^{-r(T-s)}\bar{\sigma}\frac{\partial V}{\partial S}\,\mathrm{d}W^{\mathbb{Q}}(t) \mid \mathcal{F}(t)\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-t)}V(T,S) \mid \mathcal{F}(t)\right]
\end{aligned}
$$

Exercise 6

Show that

$$
F_{\mathcal{N}(0,1)}(x) = \frac{1+\mathrm{erf}(\frac{x}{\sqrt{2}})}{2}, \qquad \text{where} \quad \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}ds \tag{23}
$$

Solution:
From the book (1.2), we know that:

$$
F_{\mathcal{N}(\mu,\sigma^2)}(x) = \frac{1}{\sigma\sqrt{2\pi}}\int_{\infty}^x \exp\left(\frac{-(z-\mu)^2}{2\sigma^2}\right)dz \tag{24}
$$

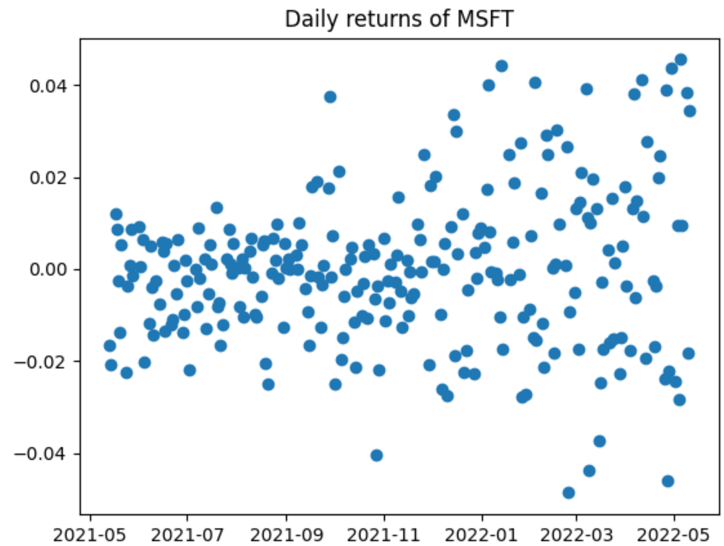
Then,

$$
\begin{aligned}
\Rightarrow F_{\mathcal{N}(0,1)}(x) &= \frac{1}{\sqrt{2\pi}}\int_{\infty}^x \exp\left(\frac{-z^2}{2}\right)dz \\
&= \frac{1}{\sqrt{2\pi}}\left\{\int_{\infty}^0 e^{\frac{-z^2}{2}}dz + \int_0^x e^{\frac{-z^2}{2}}dz\right\} \\
&= \frac{1}{\sqrt{2\pi}}\left\{\sqrt{\frac{\pi}{2}} + \int_0^x e^{\frac{-z^2}{2}}dz\right\} \\
&= \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\pi}{2}}\mathrm{erf}(x) \\
&= \frac{1+\mathrm{erf}(x)}{2}
\end{aligned}
$$

Exercise 7

1. Find two data sets with daily stock prices of two different stocks S_1 and S_2, that are "as independent as possible". Check the independence by means of a scatter plot of daily returns.

   Comprobation:
   We check an scatter plot of the daily returns of the Microsoft stock (MSFT) and the prices of wheat, represented by the ETF Teucrium Wheat Fund (WEAT), expecting no correlation between the two. Then, in order to check the independence, we compute an scatter plot of the 7-day rolling window correlations of their daily returns.


Daily returns of MSFT
0.04
0.02
0.00
−0.02
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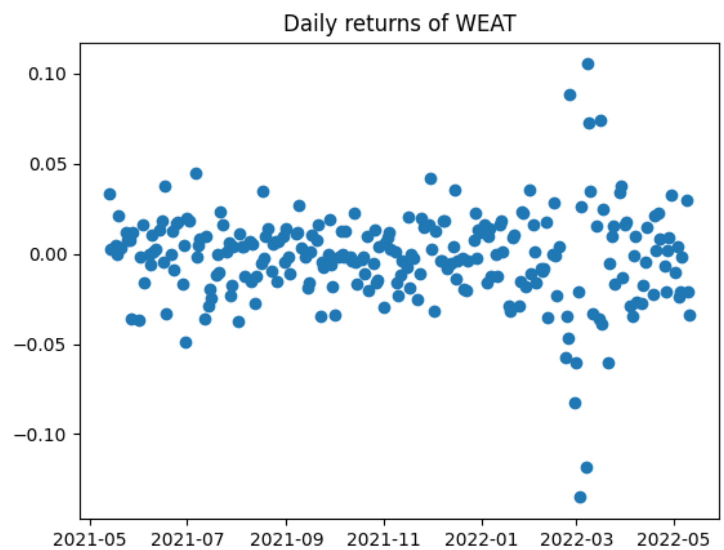
2021-05
2021-07
2021-09
2021-11
2022-01
2022-03
2022-05


Daily returns of WEAT
0.10
0.05
0.00
−0.05
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2021-05
2021-07
2021-09
2021-11
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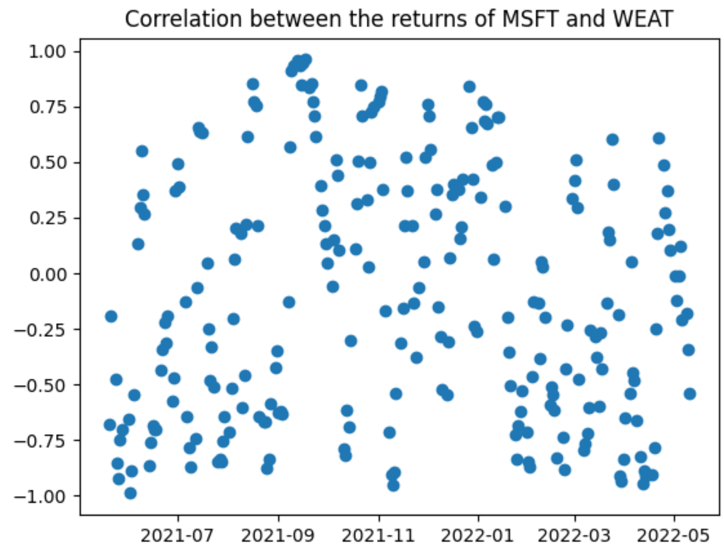
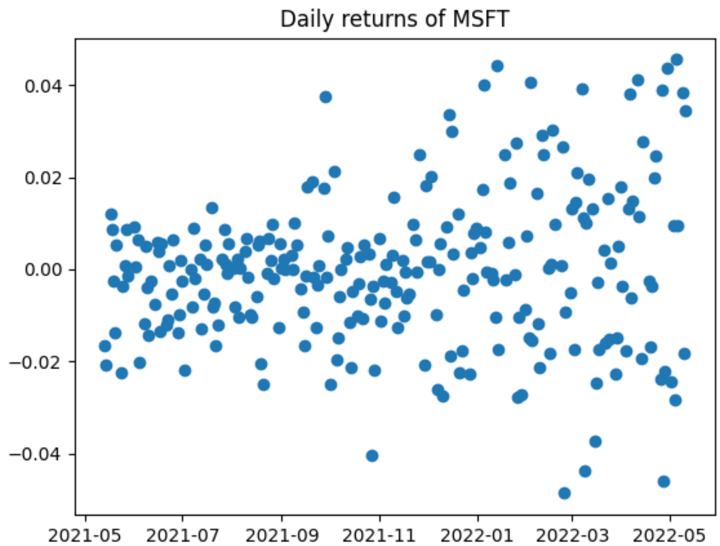
2022-03
2022-05

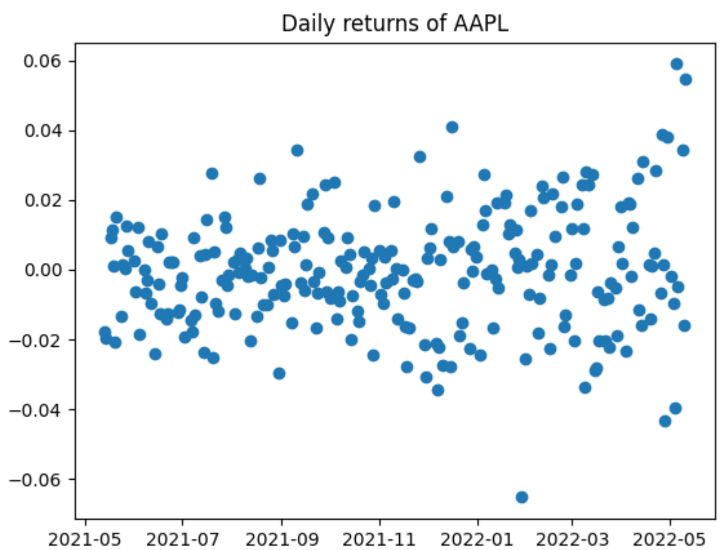


We can see how the return 7-day rolling correlations are spread over all possible values between -1 and 1, showing no clear correlation between the prices. Note that there is a bias introduced in selecting the number of days for the rolling window.

2. Find yourself two data sets with daily stock prices of two different stocks S_3 and S_4, that are "as dependent as possible". Check the dependence by means of a scatter plot of daily returns.

   This time we do the same but instead with the stock prices of Microsoft (MSFT) and Apple (AAPL), hoping that this time there is a correlation between the stocks.


Daily returns of MSFT
0.04
0.02
0.00
−0.02
−0.04
2021-05
2021-07
2021-09
2021-11
2022-01
2022-03
2022-05


Daily returns of AAPL
0.06
0.04
0.02
0.00
−0.02
−0.04
−0.06
2021-05
2021-07
2021-09
2021-11
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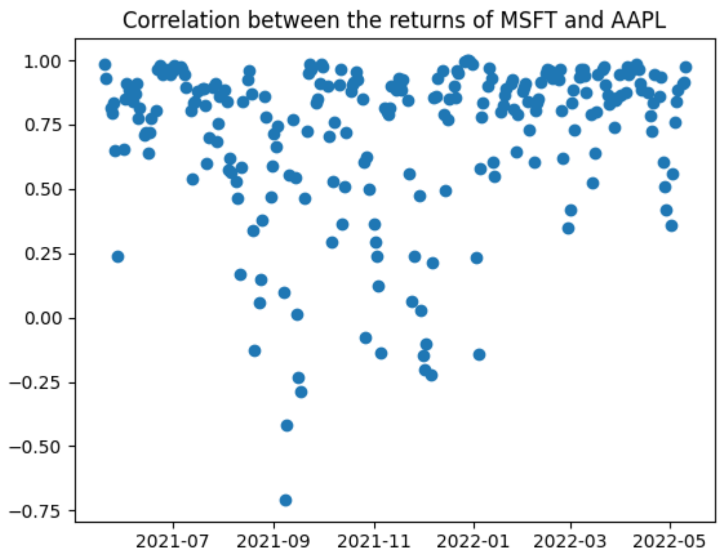
2022-03
2022-05



This time, there is a clear positive correlation between two stocks.

Exercise 8

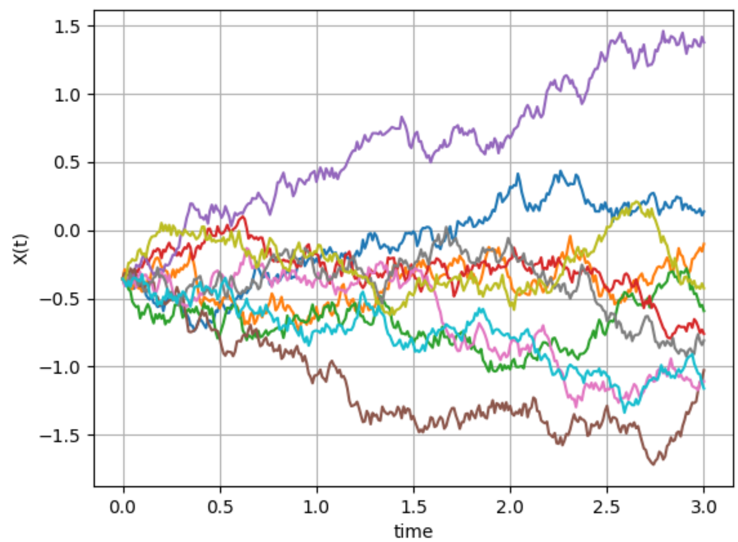
Choose two realistic values: $0.1 \leq \sigma \leq 0.75, 0.01 \leq \mu \leq 0.1$

1. With $T = 3$, $S_0 = 0.7$, $\Delta t = 10^{-2}$, generate 10 asset paths that are driven by a geometric Brownian motion and the parameters above.
2. Plot for these paths the "running sum of square increments", i.e.

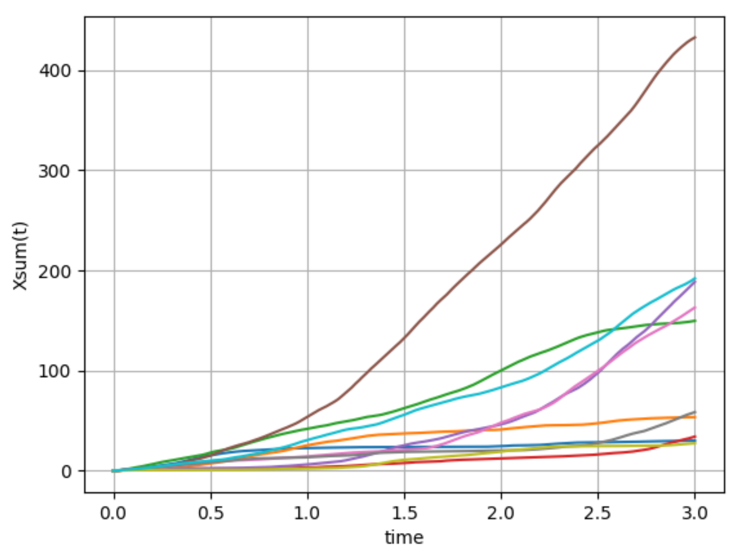
$$\sum_{k=1}^{m} \left(\Delta S\left(t_k\right)\right)^2 \tag{25}$$

3. Use asset price market data (those from Exercise 7) and plot the asset return path and the running sum of square increments.

Solution:
For this exercise, we reuse the function *GeneratePathsGBMABM* shown in the lecture. The resulting paths for this parameters are the following:

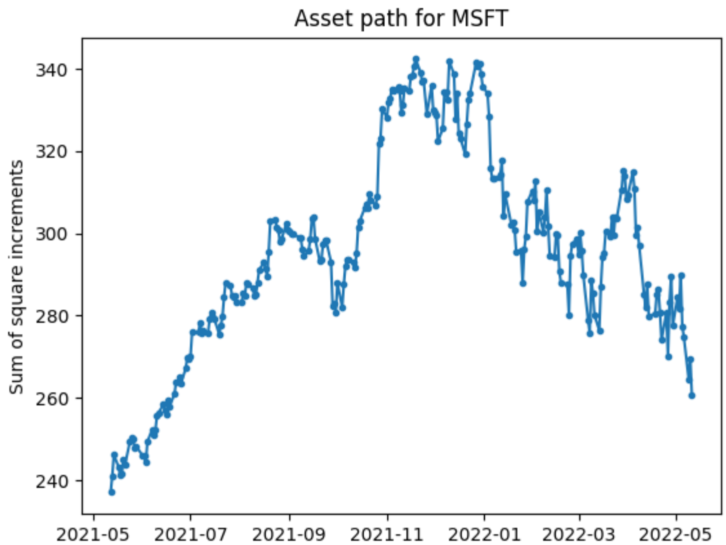


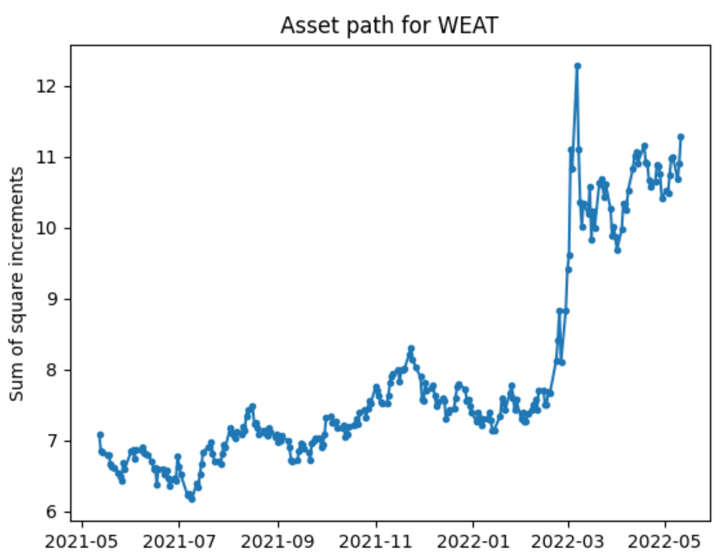
The corresponding running sum of square increments can be easily implemented from the paths generated by the function and looks as:

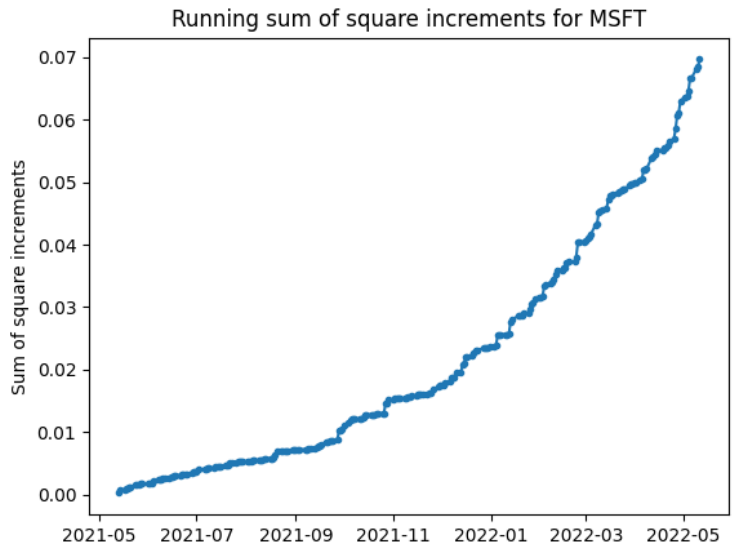


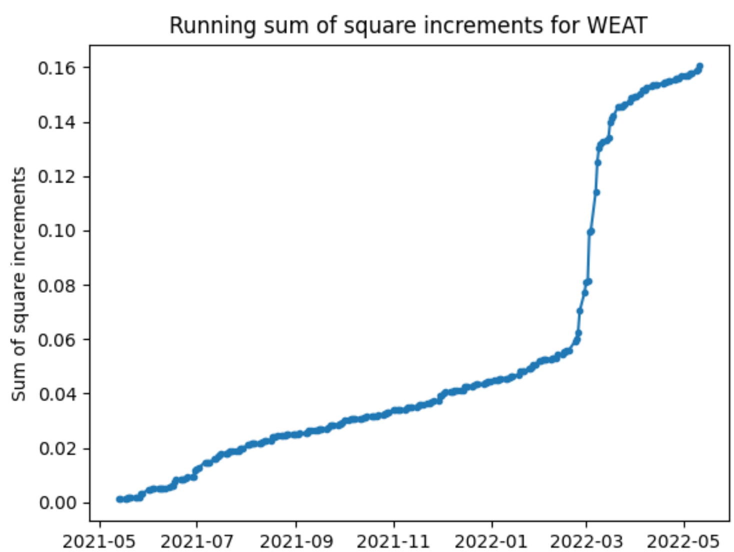
We can adapt this function to use the stock data shown in the previous exercise instead of synthetic data. In particular, for the MSFT stock and the

WEAT ETF, the asset paths and running sum of square increments are:






Running sum of square increments for MSFT
Sum of square increments
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.07
2021-05
2021-07
2021-09
2021-11
2022-01
2022-03
2022-05


Running sum of square increments for WEAT
Sum of square increments
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
2021-05
2021-07
2021-09
2021-11
2022-01
2022-03
2022-05

Exercise 9

Consider the Black-Scholes equation for a cash-or-nothing call option, with the "in-the-money" value, $V^{\text{cash}}(T, S(T)) = A$, for $t = T$.

1. What are suitable boundary conditions, at $S(t) = 0$ and for $S(t)$ "large", for this option, where $0 \le t \le T$?

2. The analytic solution is given by $V_c^{\text{cash}}(t, S) = Ae^{-r(T-t)} F_{\mathcal{N}(0,1)}(d_2)$. Derive a putcall parity relation for the cash-or-nothing option, for all $t \le T$, and determine the value of a cash-or-nothing put option (same parameter settings).

3. Give the value of the delta for the cash-or-nothing call, and draw in a picture the delta value just before expiry time, at $t = T^-$.

Solution:

1. The boundary conditions for a cash-or-nothing call option are, as the name implies, bivaluated, with a return $V^{\text{cash}}(T, S(T)) = A$ when $S(T)$ is above a given strike K; and $V^{\text{cash}}(T, S(T)) = 0$ when it is below the threshold.

2. A put-call parity for the cash-or-nothing call option is given by:

$$C + P = e^{-r(T-t)} \tag{26}$$

because buying a put and call with the same strike gives the same payoff than putting $e^{-r(T-t)}$ in the money market in order to not show arbitrage opportunities.

3. The delta of a cash-or-nothing call option is given by the sensitivity of its value to the underlying. Checking out the analytical expression for the value,

$$V(t, S) = K\mathrm{e}^{-r(T-t)} F_{\mathcal{N}(0,1)}(d_2) \tag{27}$$

then

$$\begin{aligned}
\Delta(t, S) &= \frac{\partial V(t, S)}{\partial S} \\
&= \frac{\partial}{\partial S}\left(K\mathrm{e}^{-r(T-t)} F_{\mathcal{N}(0,1)}(d_2)\right) \\
&= Ke^{-r(T-t)} \frac{\partial}{\partial S} \int_{-\infty}^{d_2} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx \\
&= Ke^{-r(T-t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \frac{\partial}{\partial S} d_2 \\
&= Ke^{-r(T-t)} \frac{F'_{\mathcal{N}(0,1)}(d_2)}{\sigma S(t)\sqrt{T-t}}
\end{aligned}$$

where we have used the derivative of the function d_2, defined as:

$$d_2 = \frac{\log \frac{S(t)}{K} + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{28}$$

We can see how this function has a pole at $t = T$, meaning that the sensitivity of delta to time diverges close to expiry time. Abstracting all of the terms except the dominant $\frac{1}{\sqrt{T-t}}$ close to expiry time, and considering $T = 1$, the shape of this function is: