UPE, ESIEE Paris
Laboratory work OUAP-4109
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Design and dSPACE Implementation of Observer-Controller Architecture for Inverted Pendulum Stabilization

1 Introduction

The objective of this lab is to control the Quanser rotary pendulum in its upright position using a controller design method based on state feedback. The pendulum consists of an arm powered by a DC motor with constraints movements around the motor shaft in the horizontal plane. At the end of the rotary arm is articulated a rigid rod we have to position vertically relative to the arm rotation plane. In order to realize the control law we have two angular position sensors that measure respectively the position of the arm motor (θ) angle with respect to its zero position and and the position of the rod (α) angle relative to the vertical position. Both sensors deliver a voltage of $\pm$ **1.33 volt** for an angle θ in the interval **[-pi/8, + pi/8]** and a voltage of $\pm$ **3.18 volt** for an angle α in the interval **[-pi/8, + pi/8]**. The control signal is realized by a DC motor which is controlled by an input voltage which varies in the interval bf [-10v, +10v].

2 Modeling of rotary pendulum

We will consider the rigid rod as a mass concentrated to half of its length. Referring to the plane (x, y) and the rotary motion of the device given by Figure 2, we can write :

$$V_{PenduleCenterOfMass} = Lcos\alpha(\dot{\alpha})\hat{x} + Lsin(\alpha)(\dot{\alpha})\hat{y} \tag{1}$$

We also know that the pendulum moves as the arm rotating at a speed :

$$V_{arm} = r\dot{\theta} \tag{2}$$

Using the equations (1) et (1) and eliminating $\hat{x}$ et $\hat{y}$ we obtain :

$$\begin{aligned} V_x &= Lcos\alpha(\dot{\alpha}) + r\dot{\theta} \\ V_y &= Lsin(\alpha)(\dot{\alpha}) \end{aligned} \tag{3}$$

Equation (2) gives us the projections of the speed of the rigid rod in the plane (x, y).

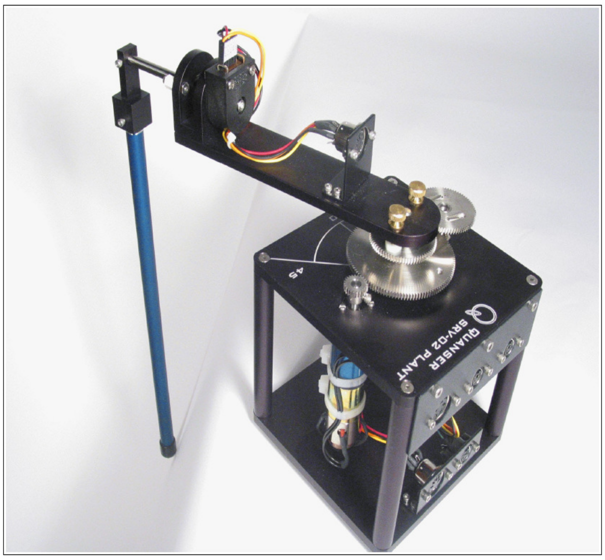

FIGURE 1 – Maquette du Pendule inverse

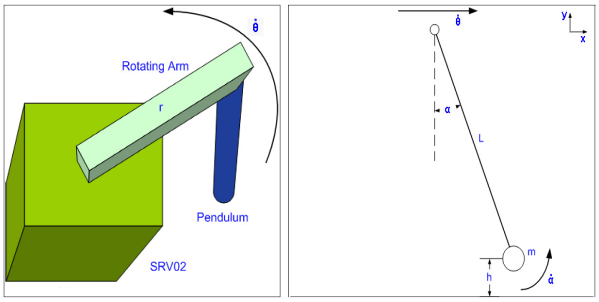

FIGURE 2 – Representation schematique du pendule inverse

2.1 Dynamic equations of the system

From the expression of the speed of the rigid rod we can get the dynamic equations of rotary pendulum using the Euler-Lagrange method.

Potential energy

The only potential energy of the system is related to the force of gravity :

$$V = mgh = mgL(1 - cos\alpha) \tag{4}$$

The kinetic energy

The kinetic energy of the whole is composed of the kinetic energy of the arm motor, kinetic energy of translation of the center of mass of the rod and the kinetic energy of the rotation.

$$T = KE_{bras} + KE_{V_x} + KE_{V_y} + KE_{rot} \tag{5}$$

The moment of inertia of the rod about its center of gravity is given by :

$$J_{cm} = \frac{1}{12}MR^2$$

where R is the rod length equal to $2L$.

Since we defined the length L as $R/\ 2$ then we get the rod moment of inertia given by :

$$J_{cm} = \frac{1}{3}ML^2$$

The total kinetic energy of the system is given by :

$$T = \frac{1}{2}J_{eq}\dot{\theta}^2 + \frac{1}{2}m(Lcos\alpha(\dot{\alpha}) + r\dot{\theta})^2 + \frac{1}{2}m(Lsin(\alpha)(\dot{\alpha}))^2 + \frac{1}{2}J_{cm}(\dot{\alpha})^2 \quad (6)$$

Thus the Lagrangian function is given by :

$$L_a = T - V = \frac{1}{2}J_{eq}\dot{\theta}^2 + \frac{1}{2}m(Lcos\alpha(\dot{\alpha}) + r\dot{\theta})^2 + \frac{1}{2}m(Lsin(\alpha)(\dot{\alpha}))^2 + \frac{1}{2}J_{cm}(\dot{\alpha})^2 - V = mgh = mgL(1 - cos\alpha) \quad (7)$$

To obtain the differential equations of the system we must solve the following Lagrange equations :

$$\begin{aligned} \frac{\delta}{\delta t}\frac{\delta L_a}{\delta \dot{\theta}} - \frac{\delta L_a}{\delta \theta} &= T_{out} - B_{eq}\dot{\theta} \quad (8) \\ \frac{\delta}{\delta t}\frac{\delta L_a}{\delta \dot{\alpha}} - \frac{\delta L_a}{\delta \alpha} &= 0 \end{aligned}$$

(9)

By solving the equations (9) and linearizing around $\alpha = 0$ we get :

$$\begin{aligned} (J_{eq} + mr^2)\ddot{\theta} + mLr\ddot{\alpha} &= T_{out} - B_e q\dot{\theta} \quad (10) \\ \frac{4}{3}mL^2\ddot{\alpha} + mLr\ddot{\theta} + mgL\alpha &= 0 \end{aligned}$$

Knowing that the expression of the DC motor torque is given by :

$$T_{out} = \frac{\eta_m \eta_g K_t K_g (Vm - K_g K_m \dot{\theta})}{R_m} \quad (11)$$

we can get the matrix form of state space equations of the system :

$$\begin{bmatrix} \dot{\theta} \\ \dot{\alpha} \\ \ddot{\theta} \\ \ddot{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{bd}{E} & \frac{-cG}{E} & 0 \\ 0 & \frac{-ad}{E} & \frac{bG}{E} & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \alpha \\ \dot{\theta} \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c\frac{\eta_m \eta_g K_t K_g}{R_m E} \\ -b\frac{\eta_m \eta_g K_t K_g}{R_m E} \end{bmatrix} V_m \quad (12)$$

où :

$$a = J_{eq} + mr^2, b = mLr, c = \frac{4}{3}mL^2, d = mgL, E = ac - b^2, G = \frac{\eta_m \eta_g K_t K_m K_g^2 + B_{eq} R_m}{R_m}$$

Numerical application :

$$\begin{bmatrix} \dot{\theta} \\ \dot{\alpha} \\ \ddot{\theta} \\ \ddot{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 39.32 & -14.52 & 0 \\ 0 & -81.78 & 13.98 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \alpha \\ \dot{\theta} \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 25.54 \\ -24.59 \end{bmatrix} V_m \quad (13)$$

2.2 Objectives of this Lab and work to be done

The objectives of this practical work are multiple. Above all we must study the Matlab and Simulink files provided to familiarize with the open-loop model of the object. The structural analysis of inverted pendulum

model with respect to its stability, reachability and observability is necessary before the controller design. Then we must calculate the feedback gains of the **Observer/Controller** structure in order to achieve closed loop control specification of the rotary inverted pendulum.
Important Note : *Before any test on the real system, the design must be validated by simulation based on Simulink files provided. Once these simulations validated we will proceed with the implementation and test of the designed control architecture on the real rotary inverted pendulum.*
The control input signal will be send to the pendulum via dSPACE card. This card is also used to read the sensor output signals. A Simulink file realizing the input/output signal writing/reading as well as the observer-controller architecture implementation is given.
Evaluation of this laboratory work will be done based on a report and Matlab/Simulink files (zip directory...) to be sent before Monday 26 of October 2019. You have to address the zipped file to a.cela@esiee.fr and noting in the object of your e-mail $Laboratory\_Work\_Signal\_And\_System$ and putting in the copy all the students of the group (2 maximum) who participated in this laboratory work.

3 Annexe

3.1 Observer-Controller design method

— Suppose we have a dynamic system in the following state space form :

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & Ax(t)+Bu(t) \\ y(t) & = & Cx(t)+Du(t) \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} x(k+1) & = & A_d x((k)+B_d u(k) \\ y(k) & = & C_d x(k)+D_d u(k) \end{array}\right. \tag{14}$$

où :

— $x(t), x(k) \in R^n$: is system state in continuous or discrete form

— $y(t), y(k) \in R^p$: is the system output signal in continuous or discrete form

— $u(t), u(k) \in R^m$: is the system control signal in continuous or discrete form

— $\{A, B, C, D\}, \{A_d, B_d, C_d, D_d\}$ are the system matrix with appropriate dimensions in continuous or discrete form .

— **Hypothesis 1** : We suppose that the given dynamic system is reachable and observable one and $D \equiv 0 \wedge D_d \equiv 0$.
— Observer

$$\left\{\begin{array}{rcl} \dot{\hat{x}}(t) & = & A\hat{x}(t)+Bu(t)+L(y(t)-\hat{y(t)}) \\ \hat{y}(t) & = & C\hat{x}(t) \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} \hat{x}(k+1) & = & A_d\hat{x}(k)+B_d u(k)+L_d(y(k)-\hat{y}(k)) \\ \hat{y}(k) & = & C_d\hat{x}(k) \end{array}\right.$$

— System and controller :

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & Ax(t)+Bu(t) \\ y(t) & = & Cx(t) \\ u(t) & = & -K\hat{x}(t)+k_r r(t) \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} x(k+1) & = & A_d x(k)+B_d u(k) \\ y(k) & = & C_d x(k) \\ u(k) & = & -K_d\hat{x}(k)+k_{rd} r(k) \end{array}\right.$$

— Observer/Controller :

$$\left\{\begin{array}{rcl} \dot{\hat{x}}(t) & = & A\hat{x}(t)+Bu(t)+L(y(t)-\hat{y(t)}) \\ \hat{y}(t) & = & C\hat{x}(t) \\ u(t) & = & -K\hat{x}(t)+k_r r(t) \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} \hat{x}(k+1) & = & A_d\hat{x}(k)+B_d u(k)+L_d(y(k)-C_d\hat{x}(k)) \\ y(k) & = & C_d x(k) \\ u(k) & = & -K_d\hat{x}(k)+k_{rd} r(k) \end{array}\right.$$

which can be written as :

$$\left\{\begin{array}{lcl} \dot{\hat{x}}(t) & = & (A-LC)\hat{x}(t)+[B,L][u(t),y(t)]^T \\ \hat{y}(t) & = & C\hat{x}(t) \\ u(t) & = & -K\hat{x}(t)+k_r r(t) \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} \hat{x}(k+1) & = & (A_d-L_dC_d)\hat{x}(k)+[B_d,L_d][u(k),y(k)]^T \\ y(k) & = & C_d x(k) \\ u(k) & = & -K_d\hat{x}(k)+k_{rd}r(k) \end{array}\right. \tag{15}$$

— If we see closely the observer/controller architecture in the discrete case it is the same as that in continuous case with the transformations $\{x(t), \dot{x}(t), u(t), y(t)\} \Rightarrow \{x(k), x(k+1), u(k), y(k)\}$ and $\{A, B, C, D, K, L\} \Rightarrow \{A_d, B_d, C_d, K_d, L_d\}$.