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EE5561 Image Processing

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HW1

1) Consider the 1D system $y(t) = x(t)\cos(2\pi f_c t) = S$

a) linear? b) shift-invariant? c) BIBO stable?

A system is linear if $S(\alpha x_1(t) + \beta x_2(t)) = \alpha S(x_1(t)) + \beta S(x_2(t))$

LHS: $S(\alpha x_1(t) + \beta x_2(t)) = [\alpha x_1(t) + \beta x_2(t)]\cos(2\pi f_c t)$

RHS: $\alpha S(x_1(t)) + \beta S(x_2(t)) = \alpha[x_1(t)\cos(2\pi f_c t)] + \beta[x_2(t)\cos(2\pi f_c t)]$

$= [\alpha x_1(t) + \beta x_2(t)]\cos(2\pi f_c t)$

$\boxed{\text{LHS} = \text{RHS then the system is linear}}$

A system is shift invariant if $\left\{ \begin{array}{l} \text{if } x(t) \to S \to y(t) \\ \text{then } x(t-t_0) \to S \to y(t-t_0) \end{array} \right\}$

$x(t) \to S \to x(t)\cos(2\pi f_c t) = y(t)$

$x(t-t_0) \to S \to x(t-t_0)\cos(2\pi f_c t)$ not the same for all t_0 not $\in \mathbb{Z}$

$y(t-t_0) = x(t-t_0)\cos(2\pi f_c(t-t_0)) = x(t-t_0)\cos[2\pi f_c t - 2\pi f_c t_0]$

$\boxed{\text{LHS} \neq \text{RHS then system is not shift invariant}}$

A system is BIBO stable if $\left\{ \begin{array}{ll} \text{if } \exists B_x & |x(t)| \le B_x < \infty \;\forall t \\ \text{then } \exists B_y & |y(t)| \le B_y < \infty \;\forall t \end{array} \right\}$

assume $x(t)$ is a bounded input then output $= x(t)\cos(2\pi f_c t)$

bounded

$\cos(2\pi f_c t) \in [-1, 1]$ which is also bounded

then $y(t) = \text{bounded} * \text{bounded} = \text{bounded output}$

$\boxed{\text{since bounded input produces a bounded output then system is BIBO stable}}$

2) a) Show that the 1D Fourier transform of the tri function is a sinc^2

$$\text{tri}(x): \Lambda(x) \equiv \max(1-|x|, 0) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & \text{otherwise} \end{cases}$$

b) Determine energy of sinc^2 function

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based from the hint

$\Rightarrow \text{tri}(x) = \text{rect}(x) * \text{rect}(x)$

& convolution property of the Fourier transform $\mathcal{F}\{f(x) * g(x)\} = F(w)G(w)$

$$\mathcal{F}(\text{tri}(x)) = \mathcal{F}(\text{rect}(x)) \cdot \mathcal{F}(\text{rect}(x))$$

$$= \text{sinc}(w) \cdot \text{sinc}(w)$$

then

$$\boxed{\mathcal{F}(\text{tri}(x)) = \text{sinc}^2(w)}$$

The energy of a function is the same even after a Fourier transform

then $\text{Energy}(\text{sinc}^2(w)) = \text{Energy}(\text{tri}(x))$ Parseval's Relation

tri function is

$$E = \int_{-1}^{0} (1+x)^2 dx + \int_0^1 (1-x)^2 dx$$

$$= \int_{-1}^{0} 1+2x+x^2 \, dx + \int_0^1 1-2x+x^2 \, dx = \left[x + x^2 + \frac{x^3}{3}\right]_{-1}^{0} + \left[x - x^2 + \frac{x^3}{3}\right]_0^1$$

$$= -(\cancel{-1}\cancel{+1} - \tfrac{1}{3}) + (\cancel{1}\cancel{-1} + \tfrac{1}{3}) = \tfrac{1}{3} + \tfrac{1}{3} = \tfrac{2}{3}$$

$$\boxed{\text{Energy of } \text{sinc}^2(w) = \tfrac{2}{3}}$$

3) Fourier uncertainty principle

Assume $\|f(x)\|^2 = 1$ & $f(x), xf(x)$ & $uF(u)$ are square integrable

$$\int_{-\infty}^{\infty} x|f(x)|^2 dx = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} u|F(u)|^2 du = 0$$

$$\sigma_x = \sqrt{\int_{-\infty}^{\infty} x^2|f(x)|^2 dx} \quad \& \quad \sigma_u = \sqrt{\int_{-\infty}^{\infty} u^2|F(u)|^2 du} \quad \text{prove } \sigma_x\sigma_u \geq \frac{1}{4\pi}$$

$$\sigma_x\sigma_u = \left(\int_{-\infty}^{\infty} x^2|f(x)|^2 dx\right)^{1/2}\left(\int_{-\infty}^{\infty} u^2|F(u)|^2 du\right)^{1/2}$$

$$= \left(\int_{-\infty}^{\infty} |xf(x)|^2 dx\right)^{1/2}\left(\int_{-\infty}^{\infty} |uF(u)|^2 dx\right)^{1/2} \Bigg\} \text{ these are both definitions of norm } = \|xf(x)\| \, \|uF(u)\|$$

using Cauchy-Schwarz inequality $|\langle f,g\rangle| \leq \|f\|\|g\|$

where our $f = xf(x)$ & $g = uF(u)$ these are two different integrating variables

so need to express g in terms of x

looking at $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)e^{iux} du \Rightarrow f'(x) = \frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)e^{iux} du\right)$

$$\Rightarrow f'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} iuF(u)e^{iux} du \Rightarrow \mathcal{F}^{-1}(f'(x)) = iuF(u)$$

using Parseval-Plancherel relation $\langle f,g\rangle = \frac{1}{2\pi}\langle \hat{f},\hat{g}\rangle$

$$|\langle f,g\rangle| = \frac{1}{2\pi}\int_{-\infty}^{\infty} xf(x)f'(x)dx = \frac{1}{2\pi}\cdot\frac{1}{2}\int_{-\infty}^{\infty} x\frac{d|f(x)|^2}{dx}dx$$

using integration by parts $\quad u = x \quad du = 1$

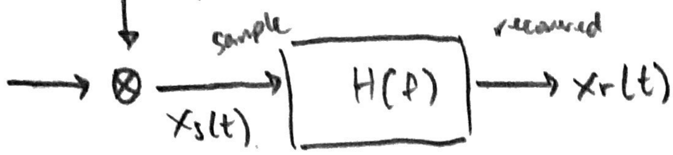
$dv = \frac{d|f(x)|^2}{dx}dx \quad v = |f(x)|^2$

$$= \frac{1}{4\pi}\left[\underbrace{x|f(x)|^2\Big|_{-\infty}^{\infty}}_{=0} - \underbrace{\int_{-\infty}^{\infty}|f(x)|^2 dx}_{\text{from definition of norm } \|f(x)\|^2 = 1 \text{ this is equal to } 1}\right]$$

$= \frac{1}{4\pi}$ then $|\langle f,g\rangle| \leq \|f\|\|g\|$

$$\boxed{\sigma_x\sigma_u \geq \frac{1}{4\pi}}$$

4) $x_s(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$

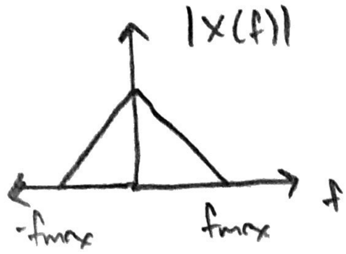


Find a) $\mathcal{F}\{x_s(t)\}$

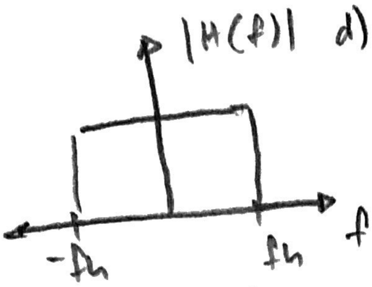
b) Nyquist criterion of T_s

c) f_h recovering $x(t)$ from $x_s(t)$

d) $\mathcal{F}\{y(t)\}$



$\mathcal{F}\{x(t)\}$



freq response of low-pass filter

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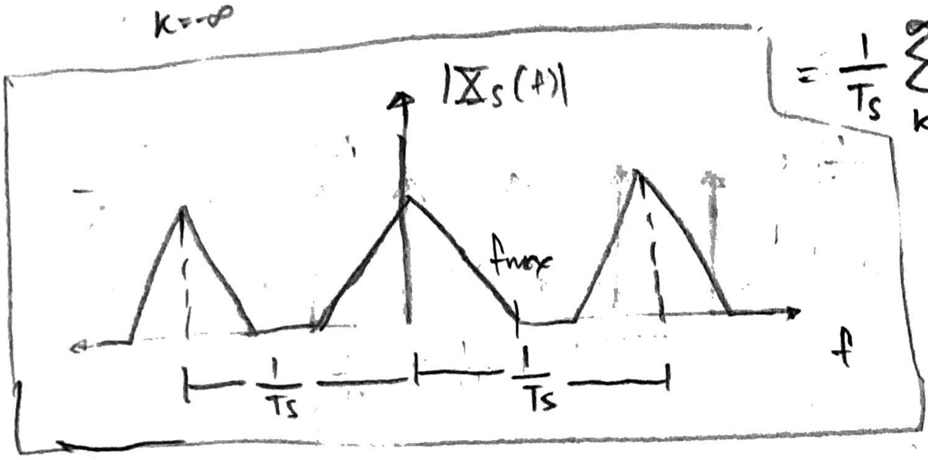
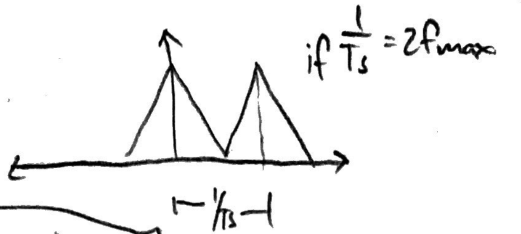
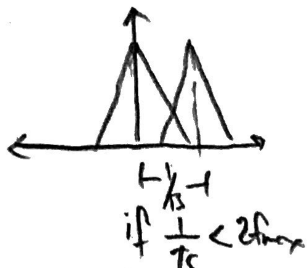
a) $x_s(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s) x(t) \Rightarrow \mathcal{F}\{x_s(t) * x(t)\} = \mathcal{F}\{x_s(t)\}\, \mathcal{F}\{x(t)\}$

$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T_s}\right) X(f)$





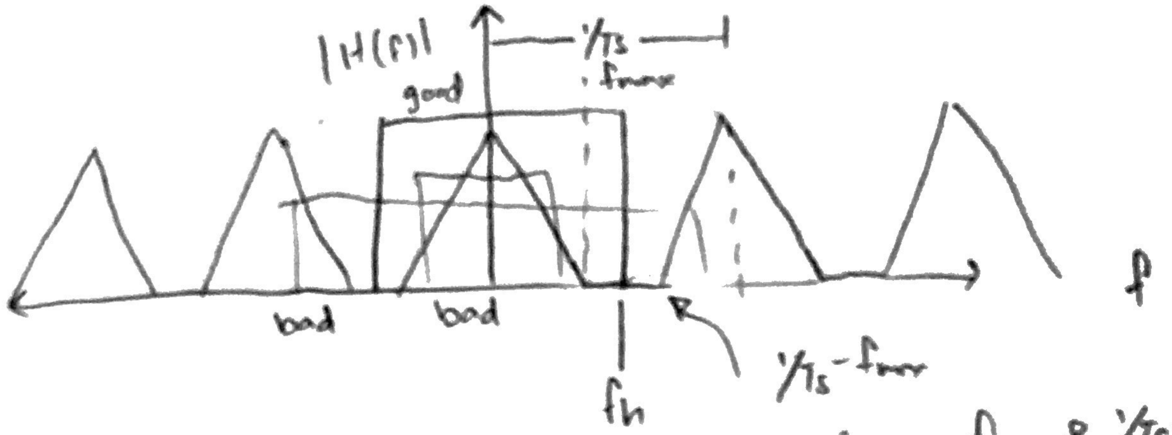
b) Recoverable if $\frac{1}{T_s} \geq 2 f_{max}$ then $\boxed{T_s \leq \frac{1}{2 f_{max}}}$



c) need to apply a low-pass filter to recover by using a rect(f_h)

$\boxed{\text{want } f_h \geq f_{max}}$

$|X_s(f)|$ assuming $\frac{1}{T_s} \geq 2f_{max}$



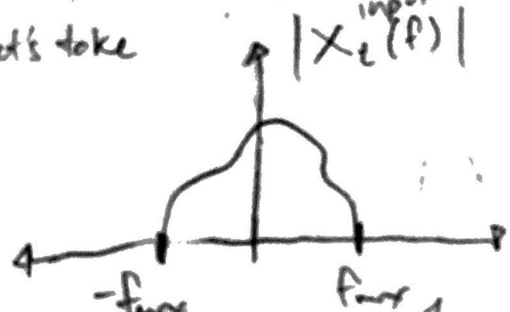
overall want f_n to be between f_{max} & $1/T_s - f_{max}$

overall

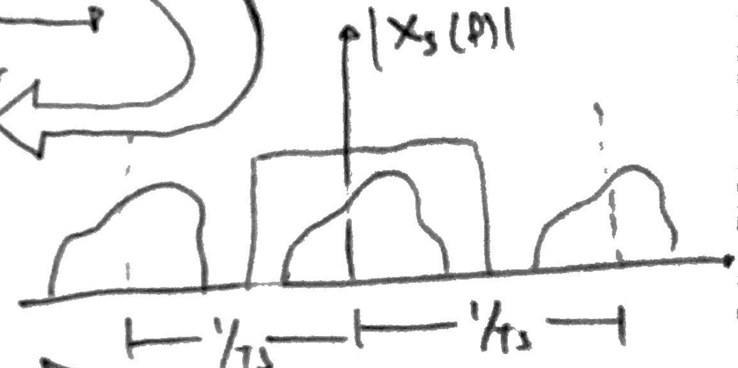
$$\boxed{f_{max} \leq f_n \leq \frac{1}{T_s} - f_{max}}$$

d) To fully recover a different signal, $x(t)$ it has to be <u>band limited</u>

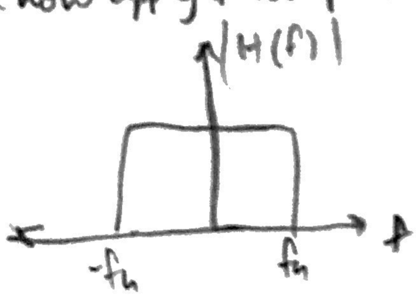
so assure let's take $|X_i(f)|$ (input)



the sampled signal assuming $\frac{1}{T_s} \geq 2f_{max}$ becomes



and now apply a low pass filter



such that

$$f_{max} \leq f_n \leq \frac{1}{T_s} - f_{max}$$

to fully recover $|X_r(f)| = |X_i(f)| \Rightarrow x_r(t) = x_i(t)$