Problem Set 1 Solutions

1) a) Assume that $y_1(t)$ and $y_2(t)$ are the outputs for the inputs of $x_1(t)$ and $x_2(t)$:

$$y_1(t) = x_1(t) \cdot cos(2\pi f_c t)$$
$$y_2(t) = x_2(t) \cdot cos(2\pi f_c t)$$

Check if the output $y(t)$ satisfies $y(t) = Ay_1(t) + By_2(t)$ for the input $x(t) = Ax_1(t) + Bx_2(t)$:

$$\begin{aligned} y(t) &= (Ax_1(t) + Bx_2(t)) \cdot cos(2\pi f_c t) \\ &= Ax_1(t) \cdot cos(2\pi f_c t) + Bx_2(t) \cdot cos(2\pi f_c t) \\ &= Ay_1(t) + By_2(t) \end{aligned}$$

Then the system is **linear**.

b) Assume that $y_1(t)$ and $y_2(t)$ are the outputs for the inputs of $x_1(t)$ and $x_2(t)$. Check if the outputs satisfies $y_2(t) = y_1(t - t_0)$ for inputs that satisfy $x_1(t) = x_2(t - t_0)$:

$$\begin{aligned} y_1(t) &= x_1(t) \cdot cos(2\pi f_c t) \\ y_1(t - t_0) &= x_1(t - t_0) \cdot cos(2\pi f_c (t - t_0)) \end{aligned}$$

$$\begin{aligned} y_2(t) &= x_2(t) \cdot cos(2\pi f_c t) \\ &= x_2(t) \cdot cos(2\pi f_c t) \\ &= x_1(t - t_0) \cdot cos(2\pi f_c t) \\ &\neq y_1(t - t_0) \end{aligned}$$

Then the system is **not shift-invariant**.

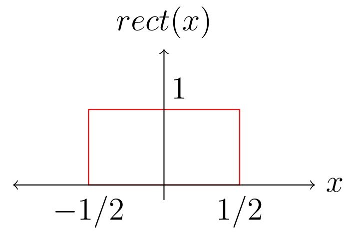
c) Check if the output is always bounded with a constant amplitude $|y(t)| \leq C$ in case the input is bounded $|x(t)| \leq B$:

$$\begin{aligned} |x(t)| &\leq B \\ |cos(2\pi f_c t)| &\leq 1 \end{aligned}$$

$$\begin{aligned} |y(t)| &= |x(t) \cdot cos(2\pi f_c t)| \\ |y(t)| &= |x(t)| \cdot |cos(2\pi f_c t)| \\ &\leq B \cdot 1 \\ |y(t)| &\leq B \end{aligned}$$

Then the system is **BIBO stable**.

2) a) The rectangular function (*rect*) is:



We will show that the Fourier Transform of the *rect* function is the *sinc* function.

$$rect(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

We will use the Fourier Transform formula given in class:

$$X(u) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-i2\pi ux} dx = \frac{1}{-i2\pi u} \left(e^{-i\pi u} - e^{i\pi u}\right) = \frac{2\sin(\pi u)}{2\pi u}$$

Hence the Fourier transform of the rectangular function is:

$$X(u) = \frac{\sin(\pi u)}{\pi u} = \text{sinc}(u)$$

We know that convolution in the Fourier domain is multiplication in the time domain. Therefore:

$$x(t) = tri(x) = rect(x) * rect(x) \Rightarrow X(U) = \mathcal{F}(rect(x)) \cdot \mathcal{F}(rect(x)) = sinc(u) \cdot sinc(u) = sinc^2(u)$$

b) Using Parseval's Theorem

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |\text{sinc}^2(u)|^2 du = \int_{-\infty}^{\infty} |\text{tri}(x)|^2 dx \\ &= \int_{-1}^{1} |1 - |x||^2 dx \\ &= 2 \cdot \int_{0}^{1} (1 - x)^2 dx \\ &= 2 \cdot \left(x - x^2 + \frac{x^3}{3}\right)\Big|_0^1 \\ &= \frac{2}{3} \end{aligned}$$

3) We have $1 = ||f(x)||^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx$. We will perform integration by parts, by choosing the $\rho = |f(x)|^2$ and $v = x$. Hence $d\rho = 2f(x)f'(x)$ and $dv = dx$. We now have

$$1 = \int_{-\infty}^{\infty} |f(x)|^2 dx = x|f(x)|^2\Big|_{-\infty}^{\infty} - 2\int_{-\infty}^{\infty} xf(x)f'(x)dx$$

Note $x|f(x)|^2$ has to be 0 as $x \to \pm\infty$ since $x|f(x)|^2$ is integrable. Thus, we have

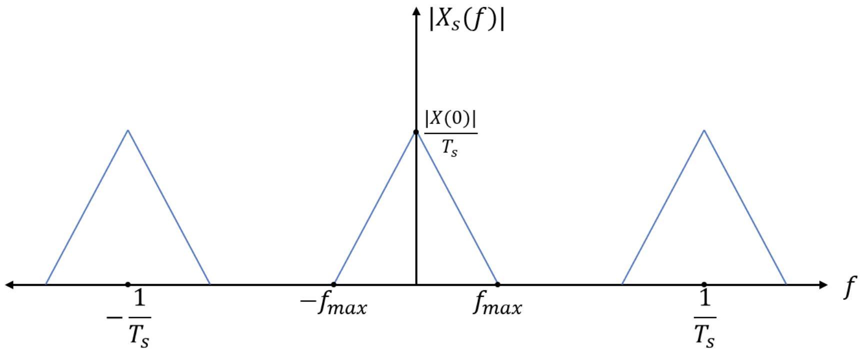
$$\begin{aligned} 1 &= -2\int_{-\infty}^{\infty} xf(x)f'(x)dx \leq 2\left|\int_{-\infty}^{\infty} xf(x)f'(x)dx\right| \\ &\leq 2\Big(\int_{-\infty}^{\infty} |xf(x)|^2 dx\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^{\infty} |f'(x)|^2 dx\Big)^{\frac{1}{2}}, \end{aligned}$$

where the last line follows from Cauchy-Schwarz inequality. Now we use the relationship between the derivative of a function and its Fourier transform, and Parseval's Theorem:

$$\begin{aligned} 1 &\leq 2\Big(\int_{-\infty}^{\infty} x^2|f(x)|^2 dx\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^{\infty} |2\pi i u F(u)|^2 du\Big)^{\frac{1}{2}} \\ &\leq 4\pi\Big(\int_{-\infty}^{\infty} x^2|f(x)|^2 dx\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^{\infty} u^2|F(u)|^2 du\Big)^{\frac{1}{2}} = 4\pi\sigma_x\sigma_u \end{aligned}$$

Thus $\sigma_x\sigma_u \geq \frac{1}{4\pi}$.

4) a)



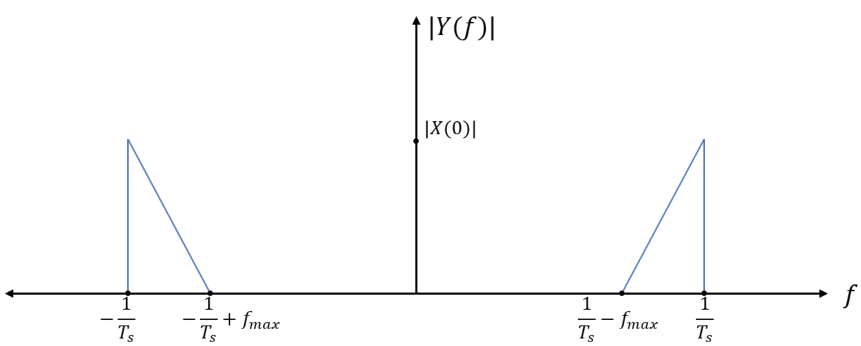
b) We do not want the triangles to intersect so $\frac{1}{T_s} - f_{max}$ for this problem should be larger than f_{max}. Thus, Nyquist criterion for this problem will be:

$$T_s \leq \frac{1}{2 \cdot f_{max}}$$

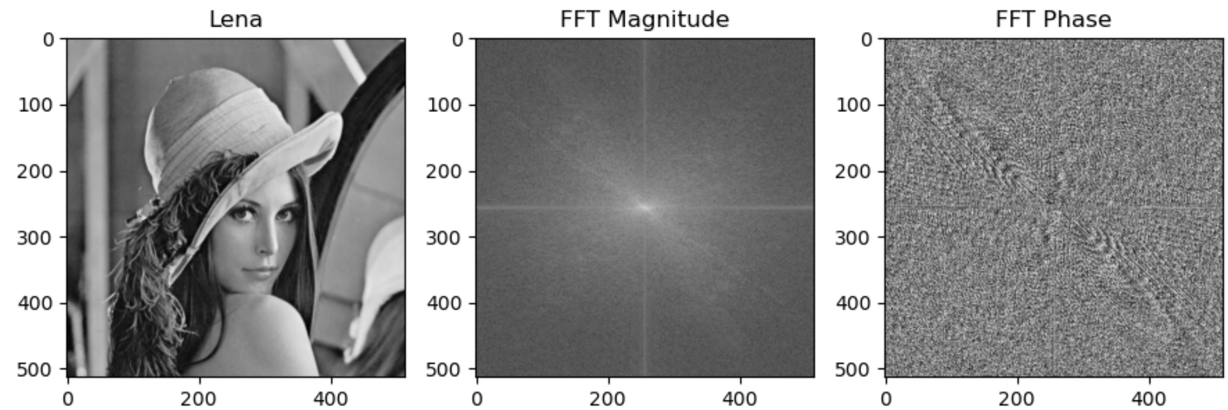
c) The low-pass filter should be larger than f_{max} to recover the full signal but also be smaller than $\frac{1}{T_s} - f_{max}$ so that there will be no aliasing.

$$\frac{1}{T_s} - f_{max} \geq f_h \geq f_{max}$$

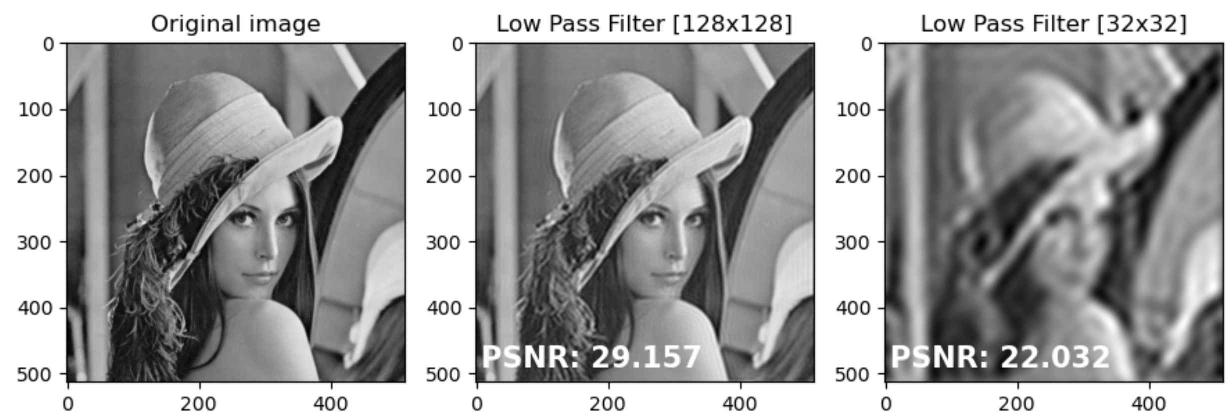
d) This is the Fourier transform of a signal that is different from $x(t)$ but is recovered as the same $x(t)$ using the same procedure



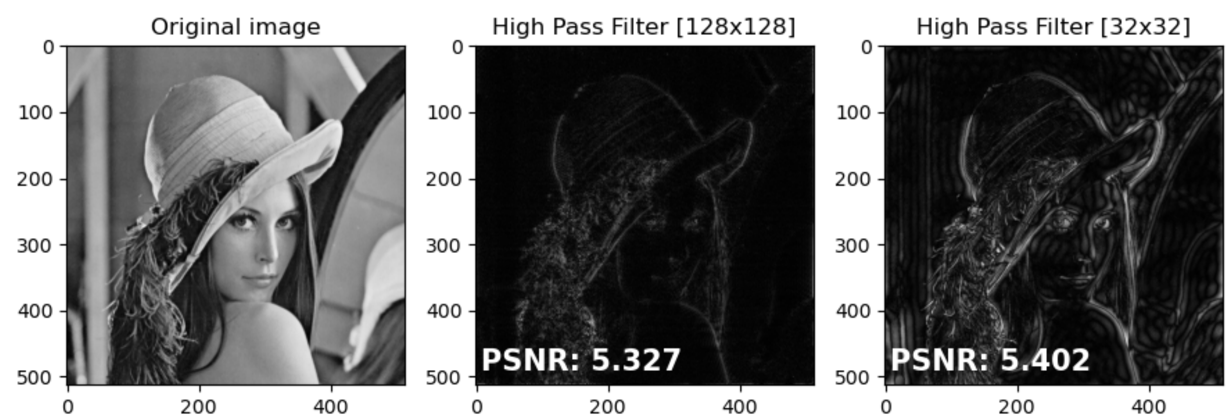
5) Programming Exercise: Code is attached.



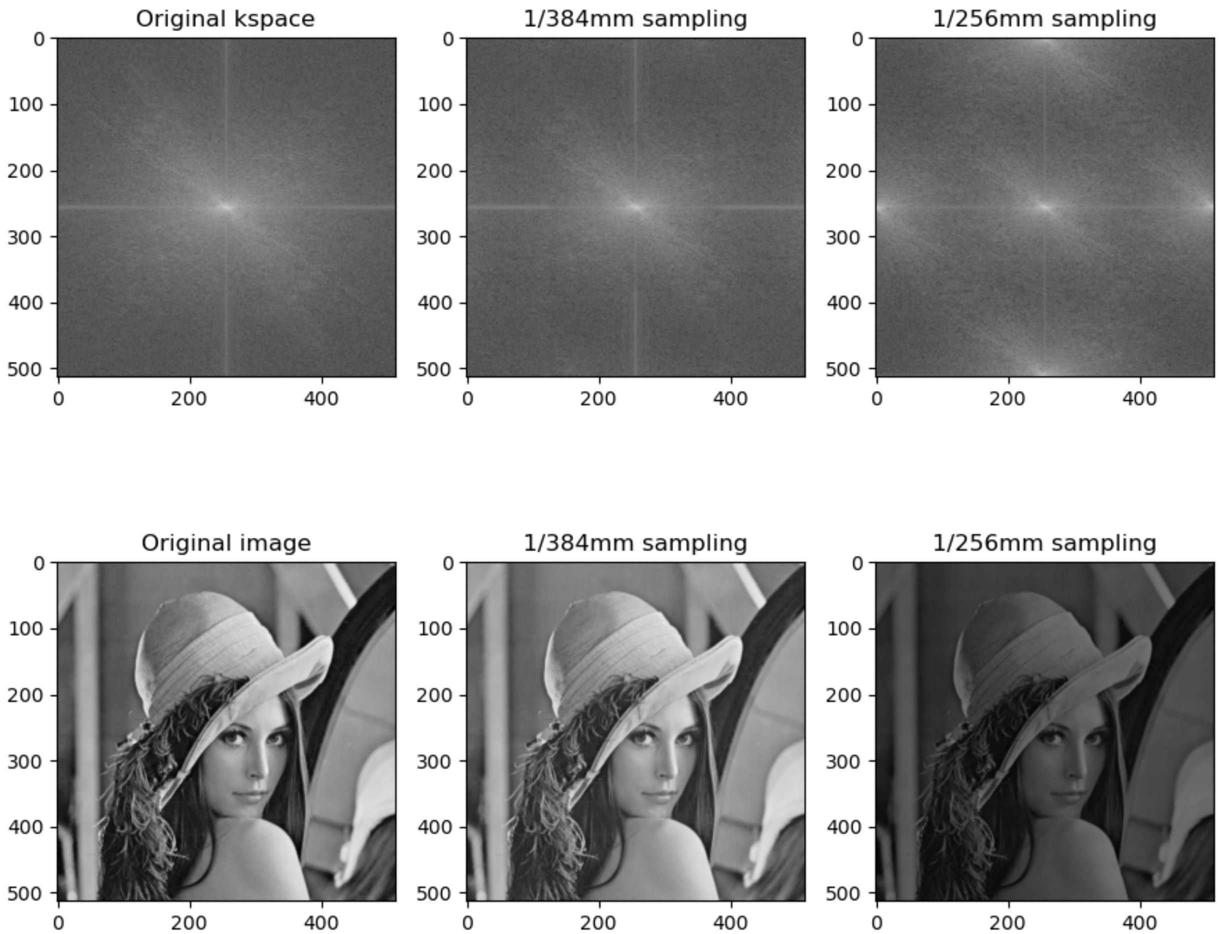
(a)



(b)



(c)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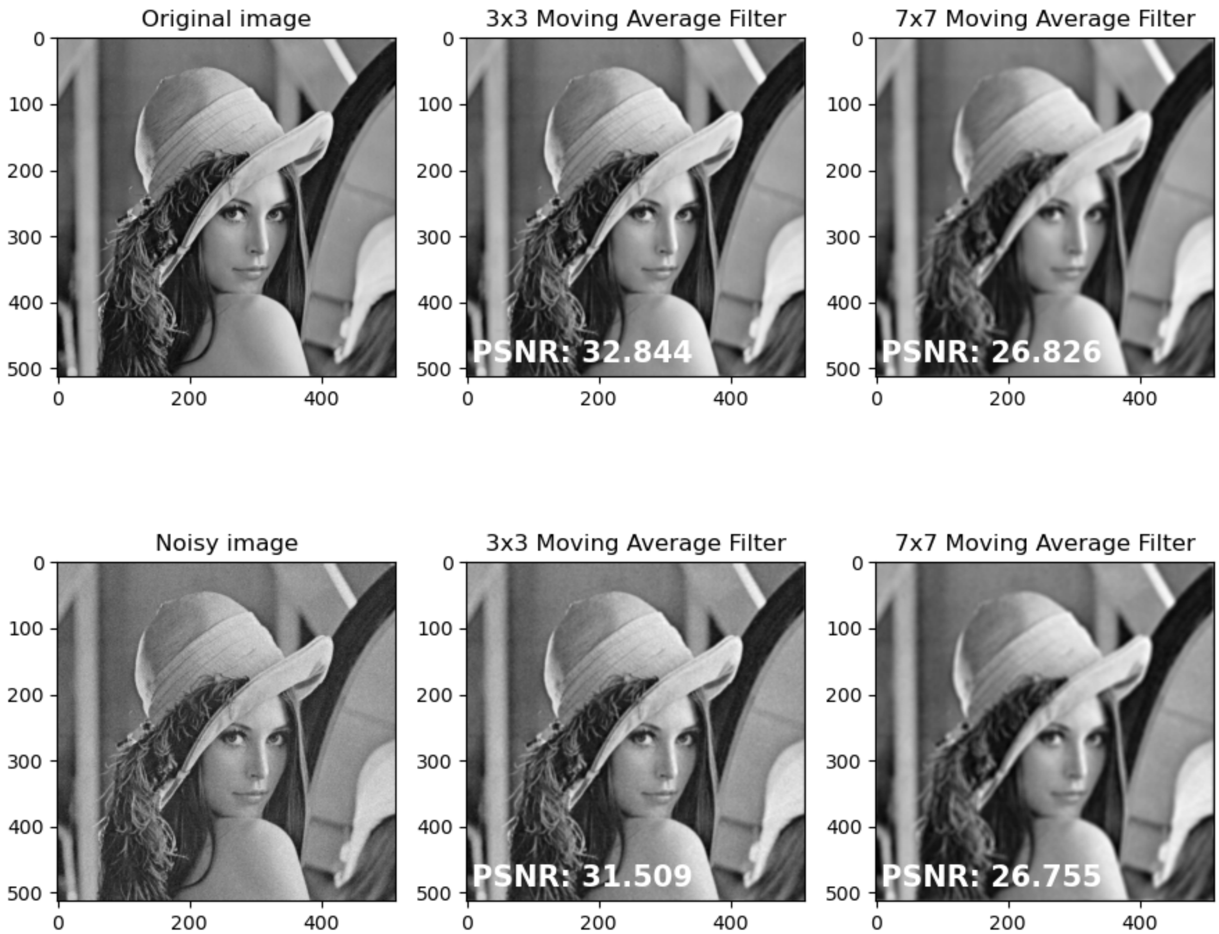
(d)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7x7 Moving Average Filter
PSNR: 26.826
Noisy image
3x3 Moving Average Filter
PSNR: 31.509
7x7 Moving Average Filter
PSNR: 26.755

(e)