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Problem Set 2
EE5561 Image Processing & Applications

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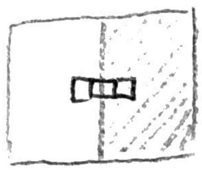
1) Design a 3×3 Convolution kernel for edge detection
   a) focuses on detecting horizontal edges while suppressing vertical edges
   b) get equivalent filter $H(u,v)$
   c) Determine if low pass/high pass ← bandpass?

1D kernel for horizontal edge



can use central differencing
$x_{i+1} - x_{i-1}$
$\Rightarrow [-1 \;\; 0 \;\; 1]$
↳ more details on next page

for suppressing the vertical edges
we can use a Gaussian kernel to blur the edges
$[1 \;\; 2 \;\; 1]$

$$h[m,n] = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} [-1 \;\; 0 \;\; 1] = \boxed{\begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}}$$

$$|H(w_x, w_y)| = |H(w_x)||H(w_y)| = \Big|\underbrace{\sum_n h[n] e^{-iwn}}_{1}\Big| \Big|\underbrace{\sum_n h[n] e^{-iwn}}_{2}\Big|$$

1 $= -1e^{-iw(0)} + e^{-iw(2)} \Rightarrow e^{-iw}\left[-e^{iw} + e^{-iw}\right] = e^{-iw}\left[-\cancel{\cos w_x} - i\sin w_x + \cancel{\cos w_x} - i\sin w_x\right]$

$= e^{-iw}\left[-2i\sin w_x\right] \Rightarrow |H(w_x)| = 2i\sin(w_x)$

2 $= 1e^{-iw(0)} + 2e^{-iw(1)} + 1e^{-iw(2)} \Rightarrow e^{-iw}\left[e^{iw} + 2 + e^{-iw}\right]$

$= e^{-iw}\left[\cos w_y + \cancel{i\sin w_y} + 2 + \cos w_y - \cancel{i\sin w_y}\right] = e^{-iw}\left[2 + 2\cos w_y\right]$

$\Rightarrow |H(w_y)| = 2(1+\cos(w_y))$

$$\boxed{H(u,v) = e^{-iw_x}(2i\sin(w_x))\, e^{-iw_y}(2(1+\cos(w_y)))}$$

$|H(w_x, w_y)| = 2i\sin(w_x) \cdot 2(1+\cos(w_y)) = |4i\sin(w_x)(1+\cos(w_y)) = |H(u,v)|$

x-kernel use differencing

$\frac{df}{dx} \approx_{\text{continuos}} \frac{f(x+h)-f(x)}{h}$ & discrete $f[m+1]-f[m]$

or a kernel $h[n] = [\,-1 \quad 0 \quad 1\,]$

y-kernel use Gaussian to reduce noise, suppress vertical lines, smoothing

discrete Gaussian filter $g_\sigma(u) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(\frac{-(u)^2}{2\sigma^2}\right) \Rightarrow$

or a kernel $h_m = [1,2,1]$

From part B we have

$$|H(w_x, w_y)| = 4i\sin(w_x)(1+\cos(w_y))$$

let's fix $w_x = 0$ & vary w_y

$$|H(0, w_y)| = 4i\,(0)\,(1+\cos(w_y)) = 0$$

let's fix $w_y = \pm\pi$ & vary $w_x \geq 0$

$$|H(w_x, \pm\pi)| = 4i\sin(w_x)(1-1) = 0$$

$|H(0,0)| = 0$

$|H(\pm\pi, w_y)| = 4i(0)(1+\cos(w_y)) = 0$

$|H(w_x, 0)| = 4i\sin(w_x)(1+1) = 8i\sin(w_x)$

not sure but seems like a high-pass/band pass for x-direction

& low-pass for y-direction

2) mean & median filters — neighborhood operations $|N(i)| = \text{odd}$

I_j = intensity

a) Show

$$N(i) \xrightarrow[\text{filter}]{\text{mean}} \min_{I_i^{new}} \sum_{j \in N(i)} |I_i^{new} - I_j|^2$$

b) Show

$$N(i) \xrightarrow[\text{filter}]{\text{median}} \min_{I_i^{new}} \sum_{j \in N(i)} |I_i^{new} - I_j|$$

c) suppose solving $\min_{I_i^{new}} \sum_{j \in N(i)} w_j |I_i^{new} - I_j|$ for $\{w_j\}$, how calculate I_i^{new} from I_j, $j \in N(i)$

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a) $\min_{I_i^{new}} \sum_{j \in N(i)} |I_i^{new} - I_j|^2$ to get minimum need to take $\frac{\partial}{\partial I_i^{new}}(\ ) = 0$

$$\frac{\partial}{\partial I_i^{new}}\left(\sum_{j \in N(i)} |I_i^{new} - I_j|^2\right) = 2\sum_{j \in N(i)} |I_i^{new} - I_j| \, \text{sign}(I_i^{new} - I_j) = 2\sum_{j \in N(i)} (I_i^{new} - I_j) = 0$$

combine to just remove | |

let $m = |N(i)|$ is odd

$$\sum_{j \in N(i)} (I_i^{new} - I_j) = 0 \Rightarrow m I_i^{new} = \sum_{j \in N(i)} I_j \Rightarrow \boxed{I_i^{new} = \frac{\sum_{j \in N(i)} I_j}{|N(i)|}}$$

This is the mean

b) $\min_{I_i^{new}} \sum_{j \in N(i)} |I_i^{new} - I_j|$ same process $\Rightarrow \frac{\partial}{\partial I_i^{new}}\left(\sum_{j \in N(i)} |I_i^{new} - I_j|\right) = 0$

$$\sum_{j \in N(i)} \text{sign}(I_i^{new} - I_j) = 0 \Rightarrow \sum_{j \in N(i)} (I_i^{new} - I_j)^+ + \sum_{j \in N(i)} (I_i^{new} - I_j)^- = 0$$

+ means positive

$$\sum_{j \in N(i)} (I_i^{new} - I_j)^+ = \sum_{j \in N(i)} (I_j - I_i^{new})^-$$

equal amount of $I_j > I_i^{new}$ & $I_j < I_i^{new}$ but since $|N(i)|$ is odd then there should exist $I_j = I_i^{new}$

$$\boxed{I_i^{new} = \text{median } N(i)}$$

c) get $\{w_j\}$ from $\min_{I_{new}} \sum_{j \in N(i)} w_j \left| I_i^{new} - I_j \right|$ take $\frac{d}{dI_{new}}(\ ) = 0$

$$\sum_{j \in N(i)} w_j \, \text{sign}(I_i^{new} - I_j) = 0$$

In order to calculate this I_i^{new}, we should first

sort all the I_j, $j \in N(i)$.

All of the I_j s are associated with a certain w_j

assume $I_1, \dots I_n$ are the I_j s sorted

$$\begin{array}{cccc} I_1 & I_2 & \cdots\cdots & I_{n-1}, & I_n \\ w_1 & w_2 & \cdots\cdots & W_{n-1} & w_j \end{array}$$

for $I_k < I_i^{new} \Rightarrow \text{sign}(I_i^{new} - I_j) = +$

for $I_k > I_i^{new} \Rightarrow \text{sign}(I_i^{new} - I_j) = -$

choose I_i^{new} such that

$$\sum_{1}^{k} w_j \simeq \sum_{k+1}^{n} w_j \qquad \text{so that} \qquad \sum_{j \in N(i)} w_j \, \text{sign}(I_i^{new} - I_j) \approx 0$$

it is possible to be $\neq 0$ but still want to minimize as much as possible

<u>quick possibility</u>

take $\frac{\sum w_j}{2}$ & we aim to get as close to this as possible ①

take cumulative w_j array : ②

find index of ② closest to ① : ③

chose value I_i^{new} based on $I_{③-1}$ & $I_{③+1}$ & $I_{③}$

3) Consider Gaussian random vectors $x_1 \sim N(\mu_1, V_1)$, $x_2 \sim N(\mu_2, V_2)$

pdf $p_1(x)$ $\quad$ $p_2(x)$

a) show $p_1(x)p_2(x)$ = Gaussian pdf

b) Generalize to $\prod_k p_k(x)$ when $x_k \sim N(\mu_k, V_k)$

$x_1 \sim N(\mu_1, V_1) \Rightarrow p_1(x) = \frac{1}{\sqrt{2\pi V_1}} e^{-\frac{(x-\mu_1)^2}{2V_1^2}}$

μ = mean

V = Variance $= \sqrt{\sigma}$

$x_2 \sim N(\mu_2, V_2) \Rightarrow p_2(x) = \frac{1}{\sqrt{2\pi V_2}} e^{-\frac{(x-\mu_2)^2}{2V_2^2}}$

$$p_1(x)p_2(x) = \left[\frac{1}{\sqrt{2\pi V_1}} e^{-\frac{(x-\mu_1)^2}{2V_1}}\right]\left[\frac{1}{\sqrt{2\pi V_2}} e^{-\frac{(x-\mu_2)^2}{2V_2}}\right] = \frac{1}{\sqrt{2\pi(2\pi V_1 V_2)}} e^{\frac{-(x-\mu_1)^2}{2V_1} + \frac{-(x-\mu_2)^2}{2V_2}} \quad \text{Ⓐ}$$

let $V_☆$ be new variance & seen from here $\sqrt{2\pi V_☆} = \sqrt{4\pi^2 V_1 V_2} \Rightarrow V_☆ = 2\pi V_1 V_2$

look at exponent of e Ⓐ

$$\frac{-(x-\mu_1)^2 V_2 - (x-\mu_2)^2 V_1}{2V_1V_2} \Rightarrow \underbrace{\frac{-[x^2V_2 + \mu_1^2 V_2 - 2x\mu_1 V_2 + x^2 V_1 + \mu_2^2 V_2 - 2x\mu_2 V_1]}{2V_1V_2}}$$

$2V_1V_2 \rightarrow \frac{2V_☆}{2\pi} = V_☆/\pi$

this is a somewhat failed attempt

next attempt at next page

$\underline{x}_1 \sim N(\underline{\mu}_1, \underline{\underline{V}}_1)$ & $\underline{x}_2 \sim N(\underline{\mu}_2, \underline{\underline{V}}_2)$

using Lecture 8 Gaussian random vector, can express

$$p_1(\underline{x}) = \frac{1}{\sqrt{(2\pi)^n |\det(\underline{\underline{V}}_1)|}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu}_1)^T \underline{\underline{V}}_1^{-1} (\underline{x}-\underline{\mu}_1)} \quad \&$$

$$p_2(\underline{x}) = \frac{1}{\sqrt{(2\pi)^n |\det(\underline{\underline{V}}_2)|}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu}_2)^T \underline{\underline{V}}_2^{-1} (\underline{x}-\underline{\mu}_2)}$$

now taking $p_1(\underline{x})p_2(\underline{x})$, we get

$$p_1(\underline{x})p_2(\underline{x}) = \underbrace{\frac{1}{(2\pi)^n \sqrt{|\det(\underline{\underline{V}}_1)|\,|\det(\underline{\underline{V}}_2)|}}}_{\substack{\text{no issue here} \\ \downarrow \\ |\det(\underline{\underline{V}}_1 \underline{\underline{V}}_2)|}} e^{\overbrace{-\frac{1}{2}(\underline{x}-\underline{\mu}_1)^T \underline{\underline{V}}_1^{-1} (\underline{x}-\underline{\mu}_1) - \frac{1}{2}(\underline{x}-\underline{\mu}_2)^T \underline{\underline{V}}_2^{-1} (\underline{x}-\underline{\mu}_2)}}$$

need to show that it can be converted to a form of

$$e^{-\frac{1}{2}(\underline{x}-\underline{\mu}_{12})^T \underline{\underline{V}}_{12}^{-1} (\underline{x}-\underline{\mu}_{12})} = \text{A}$$

looking at just the exponent of A & let's expand this & ignore $-1/2$

$$(\underline{x}-\underline{\mu}_{12})^T \underline{\underline{V}}_{12}^{-1} (\underline{x}-\underline{\mu}_{12}) = \left(\underline{x}^T \underline{\underline{V}}_{12}^{-1} - \underline{\mu}_{12}^T \underline{\underline{V}}_{12}^{-1}\right)(\underline{x}-\underline{\mu}_{12}) = \underline{x}^T \underline{\underline{V}}_{12}^{-1} \underline{x} - \underline{x}^T \underline{\underline{V}}_{12}^{-1} \underline{\mu}_{12} - \underline{\mu}_{12}^T \underline{\underline{V}}_{12}^{-1} \underline{x} + \underline{\mu}_{12}^T \underline{\underline{V}}_{12}^{-1} \underline{\mu}_{12} \quad \heartsuit$$

now let's look at exponent of $p_1(\underline{x})p_2(\underline{x})$ & ignore $-1/2$ & expand as well

$$= \underline{x}^T \underline{\underline{V}}_1^{-1} \underline{x} - \underline{x}^T \underline{\underline{V}}_1^{-1} \underline{\mu}_1 - \underline{\mu}_1^T \underline{\underline{V}}_1^{-1} \underline{x} + \underline{\mu}_1^T \underline{\underline{V}}_1^{-1} \underline{\mu}_1 + \underline{x}^T \underline{\underline{V}}_2^{-1} \underline{x} - \underline{x}^T \underline{\underline{V}}_2^{-1} \underline{\mu}_2 - \underline{\mu}_2^T \underline{\underline{V}}_2^{-1} \underline{x} + \underline{\mu}_2^T \underline{\underline{V}}_2^{-1} \underline{\mu}_2$$

rearrange terms

$$= \underbrace{\left[\underline{x}^T \underline{\underline{V}}_1^{-1} \underline{x} + \underline{x}^T \underline{\underline{V}}_2^{-1} \underline{x}\right]}_{=\underline{x}^T\left(\underline{\underline{V}}_1^{-1}\underline{x} + \underline{\underline{V}}_2^{-1}\underline{x}\right)} + \underbrace{\left[-\underline{x}^T \underline{\underline{V}}_1^{-1} \underline{\mu}_1 - \underline{x}^T \underline{\underline{V}}_2^{-1} \underline{\mu}_2\right]}_{-\underline{x}^T\left(\underline{\underline{V}}_1^{-1}\underline{\mu}_1 + \underline{\underline{V}}_2^{-1}\underline{\mu}_2\right)} + \underbrace{\left[-\underline{\mu}_1^T \underline{\underline{V}}_1^{-1} \underline{x} - \underline{\mu}_2^T \underline{\underline{V}}_2^{-1} \underline{x}\right]}_{-\left(\underline{\mu}_1^T \underline{\underline{V}}_1^{-1} + \underline{\mu}_2^T \underline{\underline{V}}_2^{-1}\right)\underline{x}} + [\ldots\ldots]$$

$$= \underline{x}^T\left(\underline{\underline{V}}_1^{-1} + \underline{\underline{V}}_2^{-1}\right)\underline{x}$$

Do the same for others

$+ [\ldots]$

$\heartsuit\heartsuit$

looking at ♡ let $\underline{\underline{V}}_{12}^{-1} = \underline{\underline{A}}$ & $\underline{\underline{V}}_{12}^{-1}\underline{M}_{12} = \underline{\underline{B}}$

then $\underline{\underline{B}}^T = M_{12}^T \underline{\underline{V}}_{12}^{-1}$ $\quad$ $\underline{\underline{V}}$ is positive definite so $\underline{\underline{V}}^{-T} = \underline{\underline{V}}^{-1}$

Now looking at ♡♡, compare w/ ♡ & notice that

$$\underline{\underline{A}} = \underline{\underline{V}}_1^{-1} + \underline{\underline{V}}_2^{-1} \quad \& \quad \underline{\underline{B}} = \underline{\underline{V}}_1^{-1}\underline{M}_1 + \underline{\underline{V}}_2^{-1}\underline{M}_2$$

using these relations then it is possible to form $P_1(\underline{x})P_2(x)$ into another gaussian PDF. to get the new $\underline{\underline{V}}_{12}$ & $\underline{M}_{12}$ let's solve it

$$\underline{\underline{V}}_{12}^{-1} = \underline{\underline{V}}_1^{-1} + \underline{\underline{V}}_2^{-1} \Rightarrow \underline{\underline{V}}_{12} = (\underline{\underline{V}}_1^{-1} + \underline{\underline{V}}_2^{-1})^{-1}$$

$$\underline{\underline{V}}_{12}^{-1}\underline{M}_{12} = \underline{\underline{V}}_1^{-1}\underline{M}_1 + \underline{\underline{V}}_2^{-1}\underline{M}_2 \Rightarrow \underline{M}_{12} = \underline{\underline{V}}_{12}(\underline{\underline{V}}_1^{-1}\underline{M}_1 + \underline{\underline{V}}_2^{-1}\underline{M}_2)$$

$$\underline{M}_{12} = (\underline{\underline{V}}_1^{-1} + \underline{\underline{V}}_2^{-1})^{-1}(\underline{\underline{V}}_1^{-1}\underline{M}_1 + \underline{\underline{V}}_2^{-1}\underline{M}_2)$$

$\boxed{P_1(\underline{x})P_2(x) \text{ forms a gaussian PDF with } \underline{M}_{new} = (\underline{\underline{V}}_1^{-1} + \underline{\underline{V}}_2^{-1})^{-1}(\underline{\underline{V}}_1^{-1}\underline{M}_1 + \underline{\underline{V}}_2^{-1}\underline{M}_2) \text{ \& } \underline{\underline{V}}_{new} = (\underline{\underline{V}}_1^{-1} + \underline{\underline{V}}_2^{-1})^{-1}}$

b) To generalize to $\prod_{i=1}^{n} P_i(x)$ we can do the same approach.

& our ♡♡ terms just turn into

$$= \underline{x}^T \underbrace{\left(\sum_{i=1}^{n} \underline{\underline{V}}_i^{-1}\right)}_{\underline{\underline{A}} = \underline{\underline{V}}_{[1,n]}^{-1}} \underline{x} - \underline{x}^T \underbrace{\left(\sum_{i=1}^{n} \underline{\underline{V}}_i^{-1}\underline{M}_i\right)}_{\underline{\underline{B}} = \underline{\underline{V}}_{[1,n]}^{-1}\underline{M}_{[1,n]}} - \underbrace{\left(\sum_{i=1}^{n} \underline{M}_i^T \underline{\underline{V}}_i^{-1}\right)}_{\underline{\underline{B}}^T} \underline{x} + [\dots]$$

From This can be seen that

$$\underline{\underline{V}}_{[1,n]}^{-1} = \sum_{i=1}^{n} \underline{\underline{V}}_i^{-1} \implies \underline{\underline{V}}_{[1,n]} = \left(\sum_{i=1}^{n} \underline{\underline{V}}_i^{-1}\right)^{-1}$$

$$\& \quad \underline{\underline{V}}_{[1,n]}^{-1} \underline{\mu}_{[1,n]} = \sum_{i=1}^{n} \underline{\underline{V}}_i^{-1} \underline{\mu}_i \implies \underline{\mu}_{[1,n]} = \underline{\underline{V}}_{[1,n]} \left(\sum_{i=1}^{n} \underline{\underline{V}}_i^{-1} \underline{\mu}_i\right)$$

$$\underline{\mu}_{[1,n]} = \left(\sum_{i=1}^{n} \underline{\underline{V}}_i^{-1}\right)^{-1} \left(\sum_{i=1}^{n} \underline{\underline{V}}_i^{-1} \underline{\mu}_i\right)$$

For k random vectors $\underline{x}$ the $\prod_{i=1}^{k} p_k(\underline{x})$

produces another gaussian pdf

with $\underline{\mu}_{new} = \left(\sum_{i=1}^{k} \underline{\underline{V}}_i^{-1}\right)^{-1} \left(\sum_{i=1}^{k} \underline{\underline{V}}_i^{-1} \underline{\mu}_i\right)$

$\&$ $\underline{\underline{V}}_{new} = \left(\sum_{i=1}^{k} \underline{\underline{V}}_i^{-1}\right)^{-1}$