University of Minnesota
Driven to Discover℠

EE 5561: Image Processing and Applications

Lecture 2

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Recap and Outline

- Last lecture
  - Basics of image processing
  - Examples of image processing tasks & state-of-the-art

- Today
  - Image transformations
  - 2D signals and systems

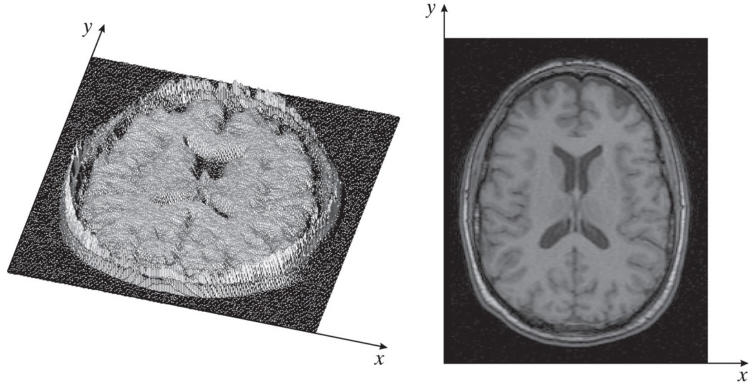
Basics

- Recall that we view images as functions

$$f : \mathbb{R}^2 \to \mathbb{R}$$

or

$$f : [a, b] \times [c, d] \to \mathbb{R}$$



(a) (b)

- Continuous images?
  - Where are they?
- In practice, images can be
  - Discrete-valued
  - Binary
  - Real-valued (or complex-valued)

Image Transformations

- Several ways to transform images
  - Acting on intensities/amplitude
  - Acting on the image spatial coordinates

  - Affine amplitude transformation

$$g(x, y) = a \cdot f(x, y) + b$$

  - Non-linear amplitude transformation
    - e.g. clipping

$$g(x, y) = \begin{cases} 255 & \text{if} \quad a \cdot f(x, y) + b > 255 \\ a \cdot f(x, y) + b & \text{if} \quad 0 \leq a \cdot f(x, y) + b \leq 255 \\ 0 & \text{if} \quad a \cdot f(x, y) + b < 0 \end{cases}$$

Image Transformations

- Several ways to transform images
  - Translation

Camera panning

$$g(x,y) = f(x - x_0, y - y_0)$$

  - Mirroring

$$g(x,y) = f(-x, -y)$$

  - Zooming

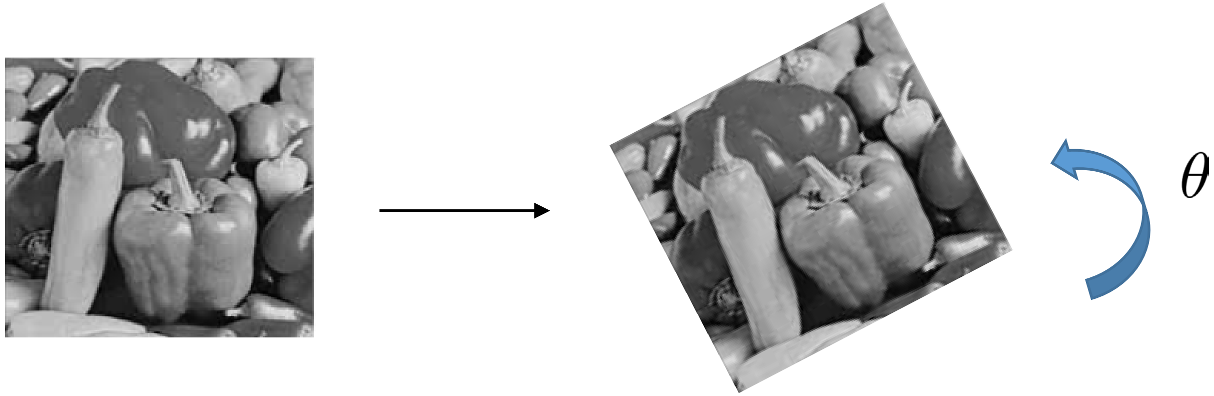
$$g(x,y) = f(ax, ay) \text{ with } a > 0$$

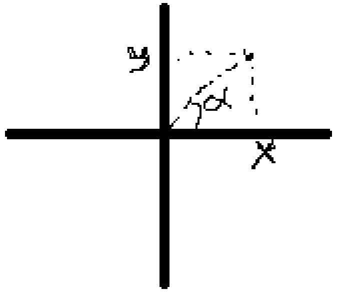
$$a < 1 \rightarrow \text{zoom-in}$$

Image Transformations

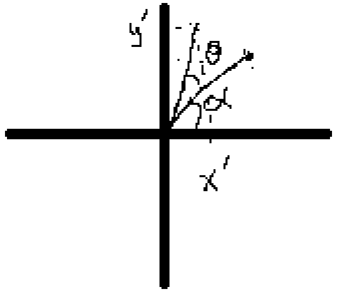
- Several ways to transform images
  - Rotation

$$g(x,y) = f(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$$





$$x = r\cos\alpha$$
$$y = r\sin\alpha$$

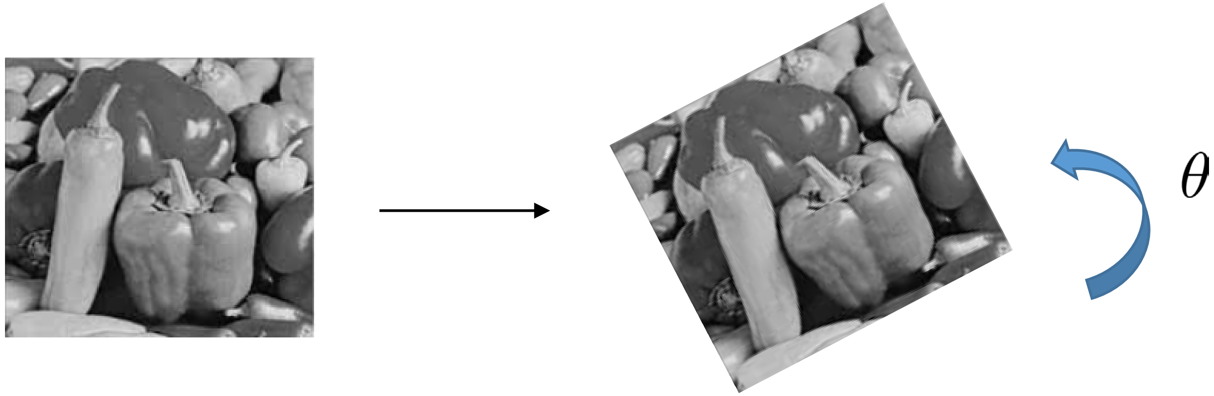


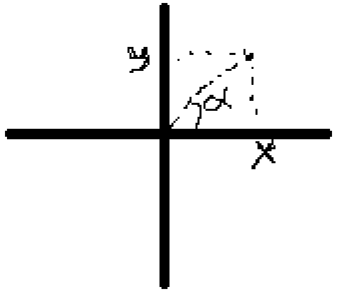
$$x' = r\cos(\alpha + \theta)$$
$$= r\cos\alpha\cos\theta - r\ sin\alpha\sin\theta$$

Image Transformations

- Several ways to transform images
  - Rotation

$$g(x,y) = f(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$$





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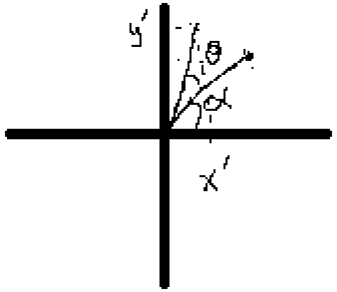


$$x' = r\cos(\alpha+\theta)$$
$$= \underbrace{r\cos\alpha}_{x}\cos\theta - \underbrace{r\ sin\alpha}_{y}\sin\theta$$
$$y' = r\cos\alpha\sin\theta + r\ sin\alpha\cos\theta$$

Image Transformations

- Several ways to transform images
  - Warping

$$g(x, y) = f(T_x(x, y), T_y(x, y))$$

$$T_x, T_y : \mathbb{R}^2 \to \mathbb{R}$$

  - Component of concepts like image morphing and registration

Ref: https://ccrma.stanford.edu/~jacobliu

2D Signals and Systems

- Outline
  - First extend the basic concepts (signals, systems, Fourier analysis) to 2D for continuous space images
  - Then review sampling
  - Then move to discrete space images
  - Finally move to discrete space images with finite extent

2D Signals and Systems

- Dirac impulse (e.g. from EE 3015)
  - Now in 2D
  - $\delta(x, y)$ generalized function/distribution defined by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x-x_0,y-y_0)dxdy = f(x_0,y_0)$$ (sifting)

$$f(x,y)\delta(x-x_0,y-y_0) = f(x_0,y_0)\delta(x-x_0,y-y_0)$$ (sampling)

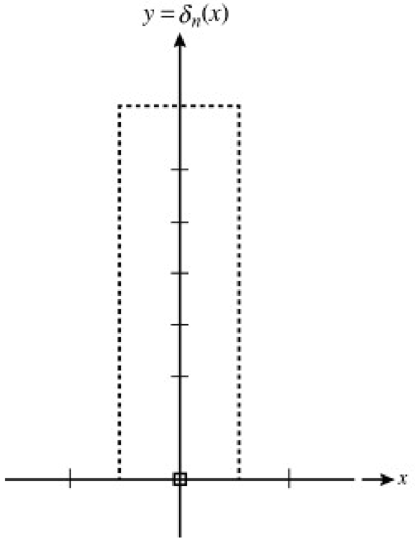
[technical note: assumes *f* is continuous at (x_0,y_0)]

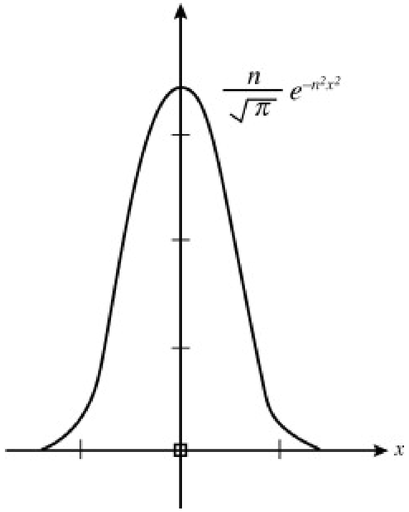
  - Easy to deduce

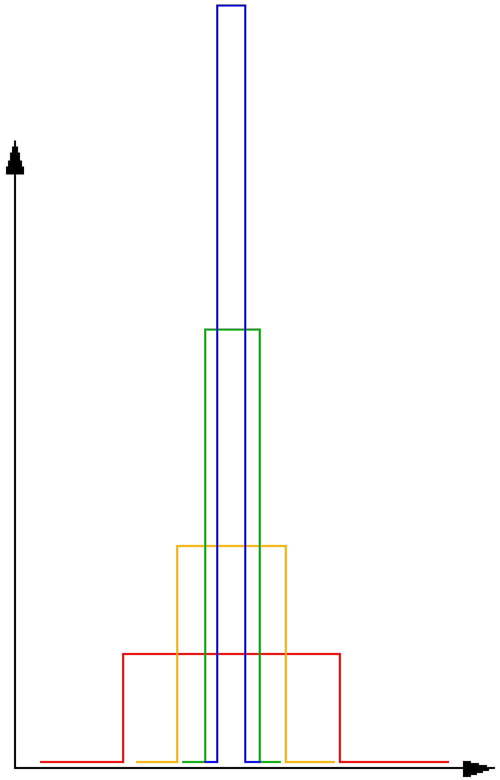
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)dxdy = 1$$ (unit area)

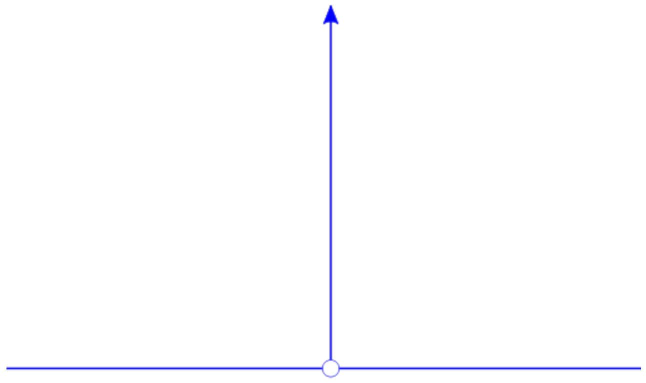
2D Signals and Systems

- Again from your earlier signals and systems courses









2D Signals and Systems

- Other properties

$$\delta(x, y) = \delta(-x, -y)$$ (symmetry)

$$\delta(x - x_0, y - y_0) = 0 \qquad \text{if} \qquad x \neq x_0, y \neq y_0$$

- Ordinary functions do not satisfy these properties (hence “generalized”)

- Approximations exist (examples on the previous slide)
  - e.g., Gaussian

$$\delta(x, y) = \lim_{\alpha \to \infty} f(x, y; \alpha)$$

$$f(x, y; \alpha) = \alpha e^{-\pi\alpha^2(x^2 + y^2)}$$

2D Systems

- Now in 2D

Input image

$f(x, y)$

true image

<u>or</u>

true spatial distribution of some physical quantity

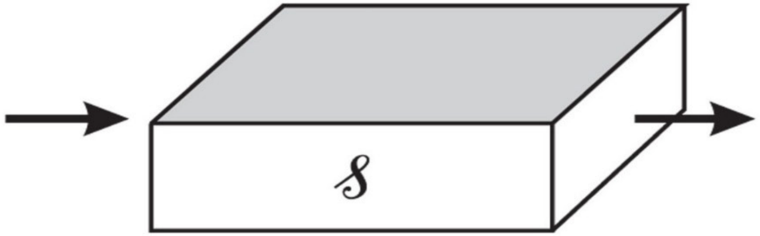
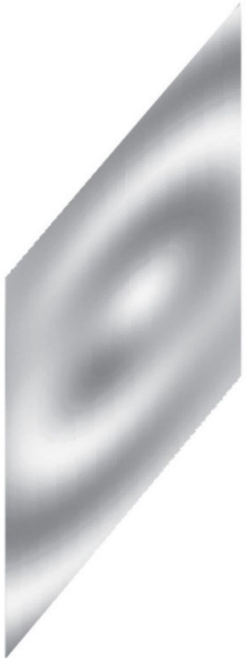


Image <u>formation</u> system

(design parameters)

e.g., camera
CT scanner
MRI scanner



Output image

$g(x, y)$

observed image

<u>or</u>

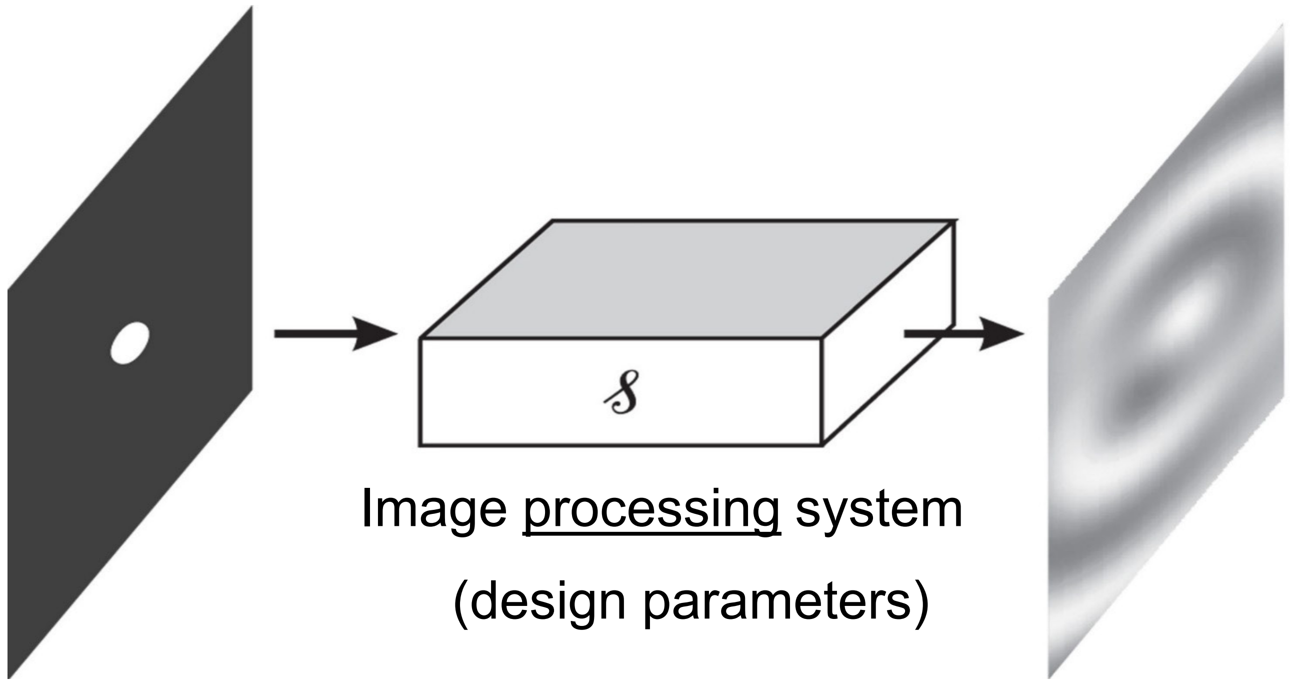
recorded "raw data"

  - System design: Make g and f as close as possible

2D Systems

$f(x, y)$

image produced by some formation system
(e.g. blurry, noisy)



$g(x, y)$

enhanced version
or
analyzed version
or
….

- e.g.
  - continuous: zoom lens, analog photocopiers
  - discrete: scanner/digitizer, compressor, edge detector

2D Systems

$$f(x,y) \longrightarrow \boxed{\mathbf{S}} \longrightarrow g(x,y)$$

- Properties
  - Amplitude-based
    - linearity, stability, invertibility, etc
  - Spatial-based
    - Causality, seperability, memory, shift-invariance, rotation-invariance

2D Systems

$$f(x,y) \longrightarrow \boxed{\mathbf{S}} \longrightarrow g(x,y)$$

- Linearity

$$\mathcal{S}(\alpha f_1(x,y) + \beta f_2(x,y)) = \alpha\mathcal{S}(f_1(x,y)) + \beta\mathcal{S}(f_2(x,y))$$

(proper notation)

$$\mathcal{S}(\alpha f_1 + \beta f_2) = \alpha\mathcal{S}(f_1) + \beta\mathcal{S}(f_2)$$

(since spatial dependence is obvious)

2D Systems

- Stability
  - Bounded-input bounded-output (BIBO) stable
  - If and only if (iff) every bounded input produces a bounded output

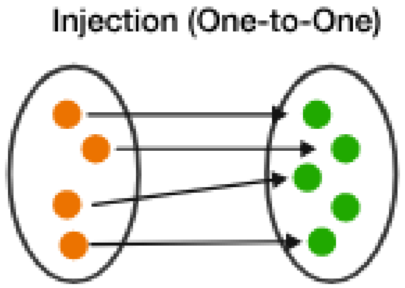
$$\text{if} \quad \exists B_f \quad \text{such that} \quad |f(x,y)| \leq B_f < \infty \quad \forall x, y$$

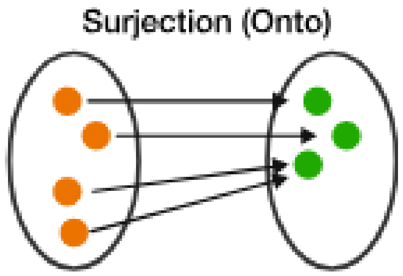
$$\text{then} \quad \exists B_g \quad \text{such that} \quad |g(x,y)| \leq B_g < \infty \quad \forall x, y$$

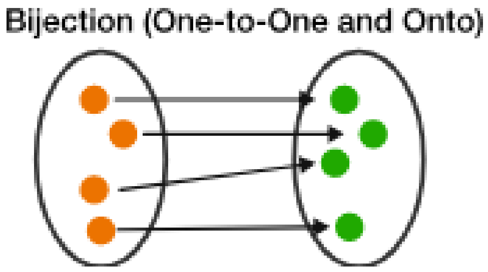
  - Otherwise the system is unstable

2D Systems

- Invertible
  - Iff each (possible) output signal corresponds to only one input signal







invertible, but it
inverse need not
be invertible

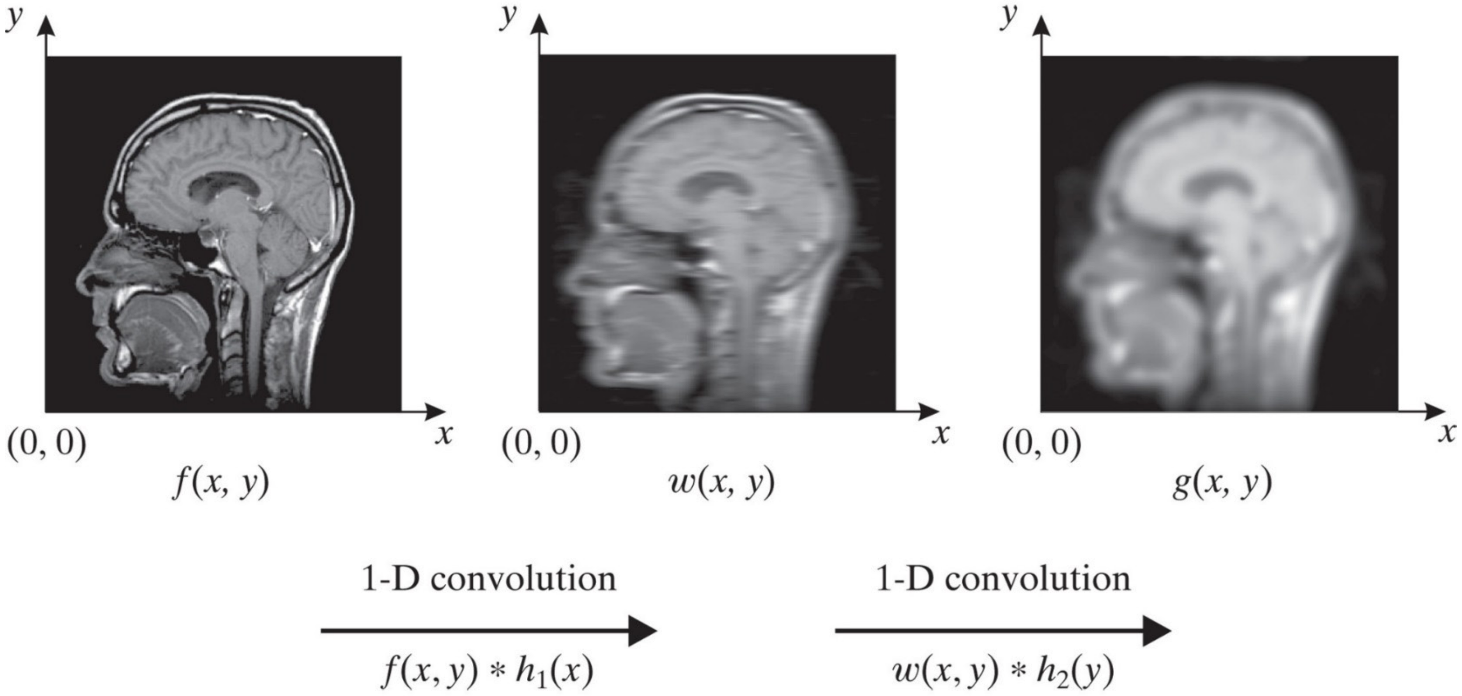
$$f(x,y) \longrightarrow \boxed{\mathbf{S}} \longrightarrow g(x,y) \longrightarrow \boxed{\mathbf{S}^{-1}} \longrightarrow f(x,y)$$

2D Systems

- Separability

$$S = S_h \cdot S_v$$

  - Enables us to process rows and columns of the image independently



2D Systems

- Shift-invariance

$$f(x, y) \to \mathcal{S} \to g(x, y)$$

$$f(x - x_0, y - y_0) \to \mathcal{S} \to g(x - x_0, y - y_0)$$

  - Zooming?

$$\mathcal{S}(f(x, y)) = f\left(\frac{x}{2}, \frac{y}{2}\right)$$

2D Systems

- Rotational-invariance

$$f(x, y) \to \mathcal{S} \to g(x, y)$$

$$f_\theta(x, y) \to \mathcal{S} \to g_\theta(x, y)$$

rotated version

  - e.g. Fourier transform

Point Spread Function

- Impulse response (linear systems) → point spread function (PSF)
  - By definition of linear systems

$$\mathcal{S}\Big(\sum_k \alpha_k f_k\Big) = \sum_k \alpha_k \mathcal{S}(f_k)$$

  - Strategy
    - Decompose the input into some “elementary” functions
    - Compute response to each elementary function
    - Determine total response
  - How to decompose?
    - Dirac impulse

Point Spread Function

- Impulse response (linear systems) → point spread function (PSF)
  - Recall the sifting property

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\delta(x_0 - x, y_0 - y)\,dx\,dy = f(x_0, y_0)$$

  - Using Dirac impulses

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')\,\underbrace{\delta(x - x', y - y')}_{\triangleq\,\delta(x,y;x',y')}\,dx'\,dy'$$

Point Spread Function

- Impulse response (linear systems) → point spread function (PSF)
  - Input an impulse centered at (*x'*,*y'*)

$$\delta(x, y; x', y') \rightarrow \mathcal{S} \rightarrow h(x, y; x', y')$$

  - This is the PSF (impulse response function)
  - Then

$$g(x,y) = \mathcal{S}(f(x,y)) = \mathcal{S}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')\delta(x,y;x',y')dx'dy'\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')\mathcal{S}\Big(\delta(x,y;x',y')\Big)dx'dy'$$ (linearity)

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')h(x,y;x',y')dx'dy'$$ superposition integral
(*S* fully-characterized by PSF)

Point Spread Function

- Example: Moving average filter

$$g(x,y) = \mathcal{S}(f(x,y)) \triangleq \frac{1}{\Delta^2}\int_{y-\frac{\Delta}{2}}^{y+\frac{\Delta}{2}}\int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}} f(x',y')dx'dy'$$

$$= \frac{1}{\Delta^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')\mathrm{rect}\left(\frac{x-x'}{\Delta}, \frac{y-y'}{\Delta}\right)dx'dy'$$

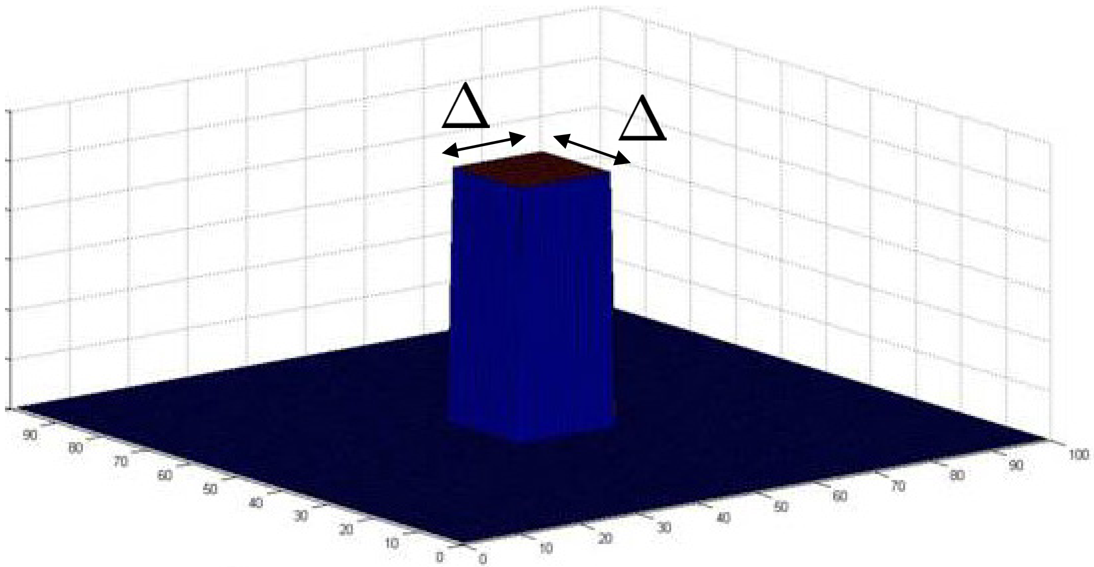
  - Now input a Dirac delta

$$f(x',y') = \delta(x',y';x'',y'') \triangleq \delta(x'-x'',y'-y'')$$

Point Spread Function

- Example: Moving average filter

$$h(x, y; x', y') = \frac{1}{\Delta^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x' - x'', y' - y'') \mathrm{rect}\left(\frac{x - x''}{\Delta}, \frac{y - y''}{\Delta}\right) dx'' dy''$$
$$= \frac{1}{\Delta^2} \mathrm{rect}\left(\frac{x - x'}{\Delta}, \frac{y - y'}{\Delta}\right)$$



Linear Shift-Invariant Systems

- Recall shift-invariance

$$f(x,y) \rightarrow \mathcal{S} \rightarrow g(x,y)$$

$$f(x-x_0,y-y_0) \rightarrow \mathcal{S} \rightarrow g(x-x_0,y-y_0)$$

- Additionally linear system

$$h(x,y;x',y') = h(x-x',y-y';0,0) \qquad \forall x,y,x',y'$$

- If *S* is linear & shift-invariant
  - System is characterized by the response for a single impulse!

Linear Shift-Invariant Systems

- Superposition integral becomes

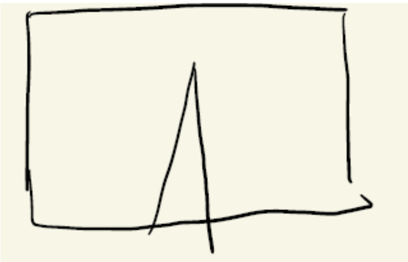
$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')h(x-x',y-y')dx'dy'$$

$$= f * h \quad (\text{or } f * *h)$$

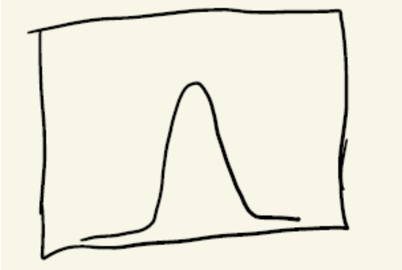
  - This is called a <u>convolution</u> integral

Resolution

object
“impulse”

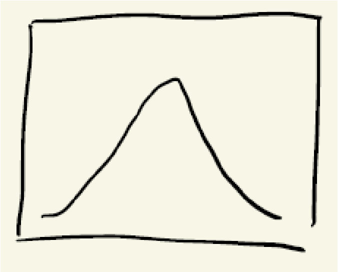


PSF

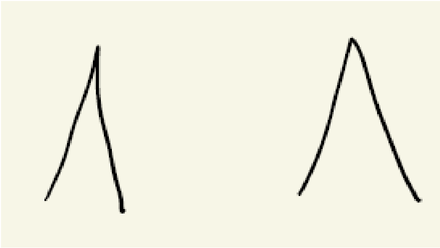


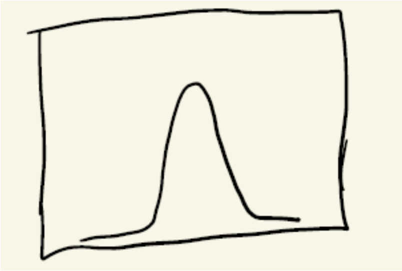
convolution $\longrightarrow$

output

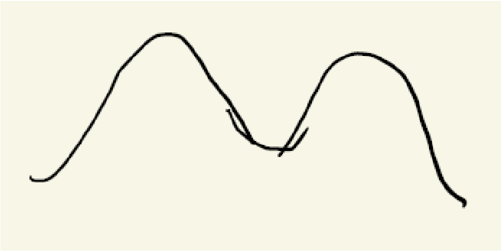


- Now suppose we have two closely spaced impulse-like objects/signals

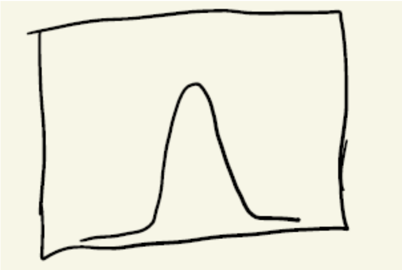




convolution $\longrightarrow$



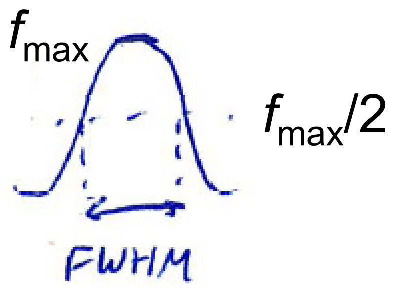
even
closer



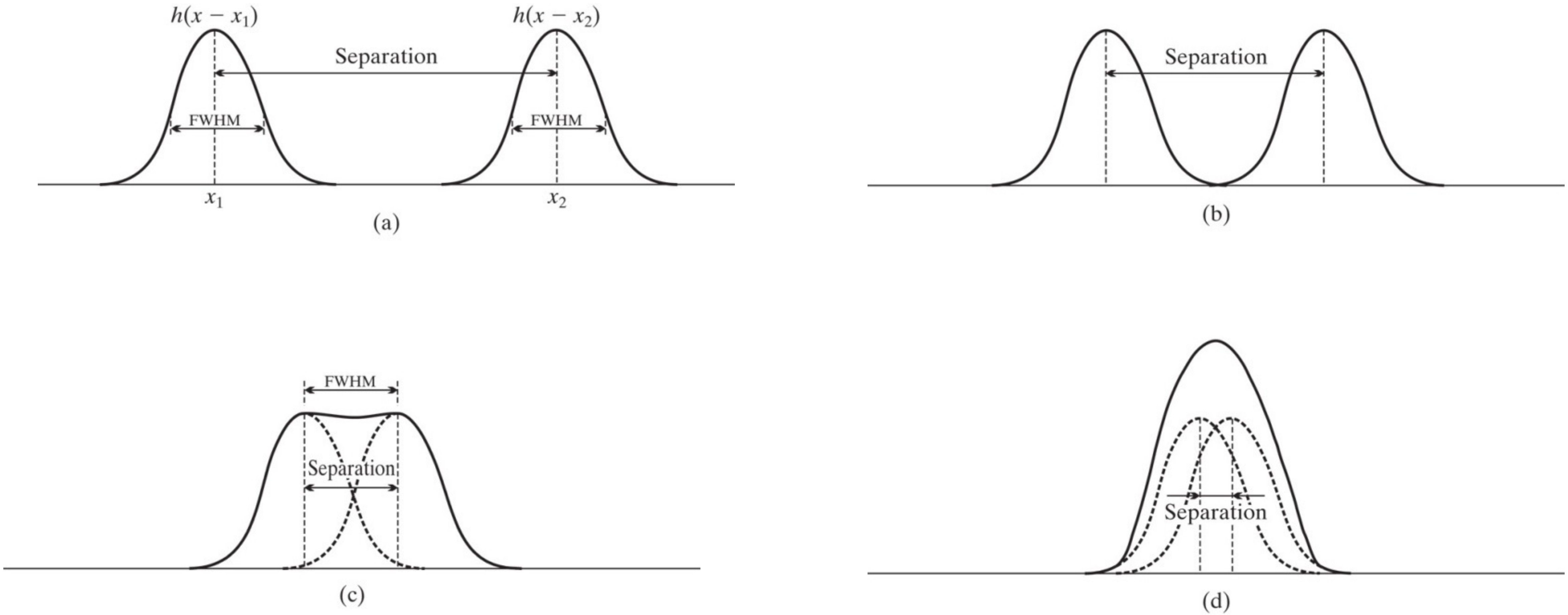
convolution $\longrightarrow$

Resolution

- This gives us a definition of resolution
  - Full width half maximum
  - Full width of the PSF at half its maximum value
  - If two impulses are separated >FWHM, then this creates two peaks at output → One can tell them apart
  - Smaller FWHM → improved resolution



Resolution



$h(x - x_1)$
$h(x - x_2)$
Separation
FWHM
FWHM
x_1
x_2
(a)
Separation
(b)
FWHM
Separation
(c)
Separation
(d)