Problem Set 2 Solutions

1) a) We can define the 1D kernels for the vertical and horizontal operations separately and then, multiply them to get the final 2D kernel. The following Gaussian smoothing kernel can be used to suppress the vertical edges:

$$h_1 = \begin{bmatrix} 1/4 & 1/2 & 1/4 \end{bmatrix}$$

A simple derivative kernel can be used to detect the horizontal edges:

$$h_2 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$$

Finally, we can calculate the 2D kernel by multiplying h_1 and h_2:

$$h[m,n] = h_2^T \cdot h_1 = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 0 & 0 & 0 \\ -1/4 & -1/2 & -1/4 \end{bmatrix}$$

b) Here is the $H[w_x, w_y]$ calculation:

$$\begin{aligned} H[w_x, w_y] &= H_1[w_x] \cdot H_2[w_y] \\ &= \left(\frac{1}{4}e^{-i(-w_x)} + \frac{1}{2}e^{-i0} + \frac{1}{4}e^{-iw_x}\right)\left(1e^{-i(-w_y)} + (-1)e^{-iw_y}\right) \\ &= \frac{1}{2}\left(1 + \cos w_x\right) \cdot 2i \sin w_y \\ &= i\left(1 + \cos w_x\right) \cdot \sin w_y \end{aligned}$$

c) We can analyse the behaviour of the filter in x and y directions separately:

$$\begin{aligned} H_1[w_x] &= \frac{1}{2}\left(1 + \cos w_x\right) \\ |H_1[0]| &= 1 \\ |H_1[\pi]| &= 0 \end{aligned}$$

In the x direction, the filter allows the low-frequency components to pass while attenuating the high-frequency components of the signal. Therefore, it is **lowpass in x direction**.

$$\begin{aligned} H_2[w_y] &= 2i \sin w_y \\ |H_2[0]| &= 0 \\ |H_2[\pi/2]| &= 2 \\ |H_2[\pi]| &= 0 \end{aligned}$$

In the y direction, the filter allows the mid-frequency components to pass while attenuating the low and high-frequency components of the signal. Therefore, it is **bandpass in y direction**.

2) a) Let $F(I_i^{new}) = \sum_{j \in N(i)} |I_i^{new} - I_j|^2$. Taking the derivative with respect to I_{new_i}, we have $F'(I_i^{new}) = \sum_{j \in N(i)} 2(I_i^{new} - I_j)$. To minimize the function $F(I_i^{new})$, we should find I_i^{new} which makes the derivative 0. Setting the derivative to 0, we have:

$$\sum_{j \in N(i)} I_i^{new} = \sum_{j \in N(i)} I_j$$

Thus,

$$I_i^{new} = \frac{1}{|N(i)|} \sum_{j \in N(i)} I_j$$

b) Let $F(I_i^{new}) = \sum_{j \in N(i)} |I_i^{new} - I_j|$. Taking the derivative with respect to I_i^{new}, we have $F'(I_i^{new}) = \sum_{j \in N(i)} \text{sign}(I_i^{new} - I_j)$. Noting $|N(i)|$ is odd, the derivative is 0 when I_i^{new} is the median of $\{I_j\}_{j \in N(i)}$ since $(|N(i)| - 1)/2$ values will be less than the median (sign function evaluates to -1) and $(|N(i)| - 1)/2$ values will be greater than the median (sign function evaluates to 1). At the median, the sign function will evaluate to 0, so that the sum will be 0. In other words, the median of $\{I_j\}_{j \in N(i)}$ minimizes the function $F(I_i^{new})$.

c) This part is trickier. Again, we proceed similarly, and get the derivative $F'(I_i^{new}) = \sum_{j \in N(i)} w_i \cdot \text{sign}(I_i^{new} - I_j)$. This need not be equal to 0, so we need to estimate where the derivative changes sign (goes from negative to positive). Note the output has to be one of the samples because $F(I_i^{new})$ is piecewise linear and convex as long as $w_i \geq 0$. Let $n = |N(i)|$ for ease of notation. We first sort $\{I_j\}$ so that we have

$$I_{(1)} \leq I_{(2)} \leq \quad \dots \quad \leq I_{(n)}$$

Let the corresponding weights be $w_{(1)}$, $w_{(2)}$, ...,$w_{(n)}$. We look for k^* such that

$$\sum_{j=1}^{k^*-1} w_{(j)} + w_{(k^*)} \geq \sum_{j=k^*+1}^{n} w_{(j)}$$
$$\sum_{j=1}^{k^*-1} w_{(j)} \leq \sum_{j=k^*+1}^{n} w_{(j)} + w_{(k^*)}$$

Our output for the weighted median is $I(k^*)$, which is where the sign change of the derivative happens.

3) We will only do (b) since it is the more general case. The pdf of $\mathbf{x}_k$ is given by

$$p_k(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \mathbf{V}_k}} \exp\Big[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_k)^T\mathbf{V}_k^{-1}(\mathbf{x}-\boldsymbol{\mu}_k)\Big]$$

We can rewrite this as

$$p_k(x) = \exp\Big[\xi_k + \boldsymbol{\eta}_k^T\mathbf{x} - \frac{1}{2}\mathbf{x}^T\boldsymbol{\Lambda}_k\mathbf{x}\Big], \tag{1}$$

where $\boldsymbol{\Lambda}_k = \mathbf{V}_k^{-1}$, $\boldsymbol{\eta}_k = \mathbf{V}_k^{-1}\boldsymbol{\mu}_k$ and $\xi_k = -\frac{1}{2}\big(n\log 2\pi - \log\det\boldsymbol{\Lambda}_k + \boldsymbol{\eta}_k^T\boldsymbol{\Lambda}_k^{-1}\boldsymbol{\eta}_k\big)$. Thus,

$$\begin{aligned}\prod_k^m p_k(\mathbf{x}) &= \prod_k^m \exp\Big[\xi_k + \boldsymbol{\eta}_k^T\mathbf{x} - \frac{1}{2}\mathbf{x}^T\boldsymbol{\Lambda}_k\mathbf{x}\Big]\\ &= \exp\Bigg[\sum_k \xi_k + \big(\sum_k \boldsymbol{\eta}_k\big)^T\mathbf{x} - \frac{1}{2}\mathbf{x}^T\big(\sum_k \boldsymbol{\Lambda}_k\big)\mathbf{x}\Bigg]\end{aligned}$$

Now we define $\boldsymbol{\Omega}_m = \sum_{k=1}^m \boldsymbol{\Lambda}_k$, $\boldsymbol{\Lambda}_m = \sum_{k=1}^m \boldsymbol{\eta}_k$, and

$$\zeta_m = -\frac{1}{2}\big(n\log 2\pi - \log\det\boldsymbol{\Omega}_m + \boldsymbol{\Lambda}_m^T\boldsymbol{\Omega}_m^{-1}\boldsymbol{\Lambda}_m\big)$$

Then we re-write the product as

$$\prod_k^m p_k(\mathbf{x}) = \exp\Bigg[\big(\sum_k \zeta_k\big) - \zeta_m\Bigg]\exp\Bigg[\zeta_m + \boldsymbol{\Lambda}_m^T\mathbf{x} - \frac{1}{2}\mathbf{x}^T\boldsymbol{\Omega}_m\mathbf{x}\Bigg]$$

Based on Equation 1, this is a *scaled Gaussian pdf* with covariance matrix

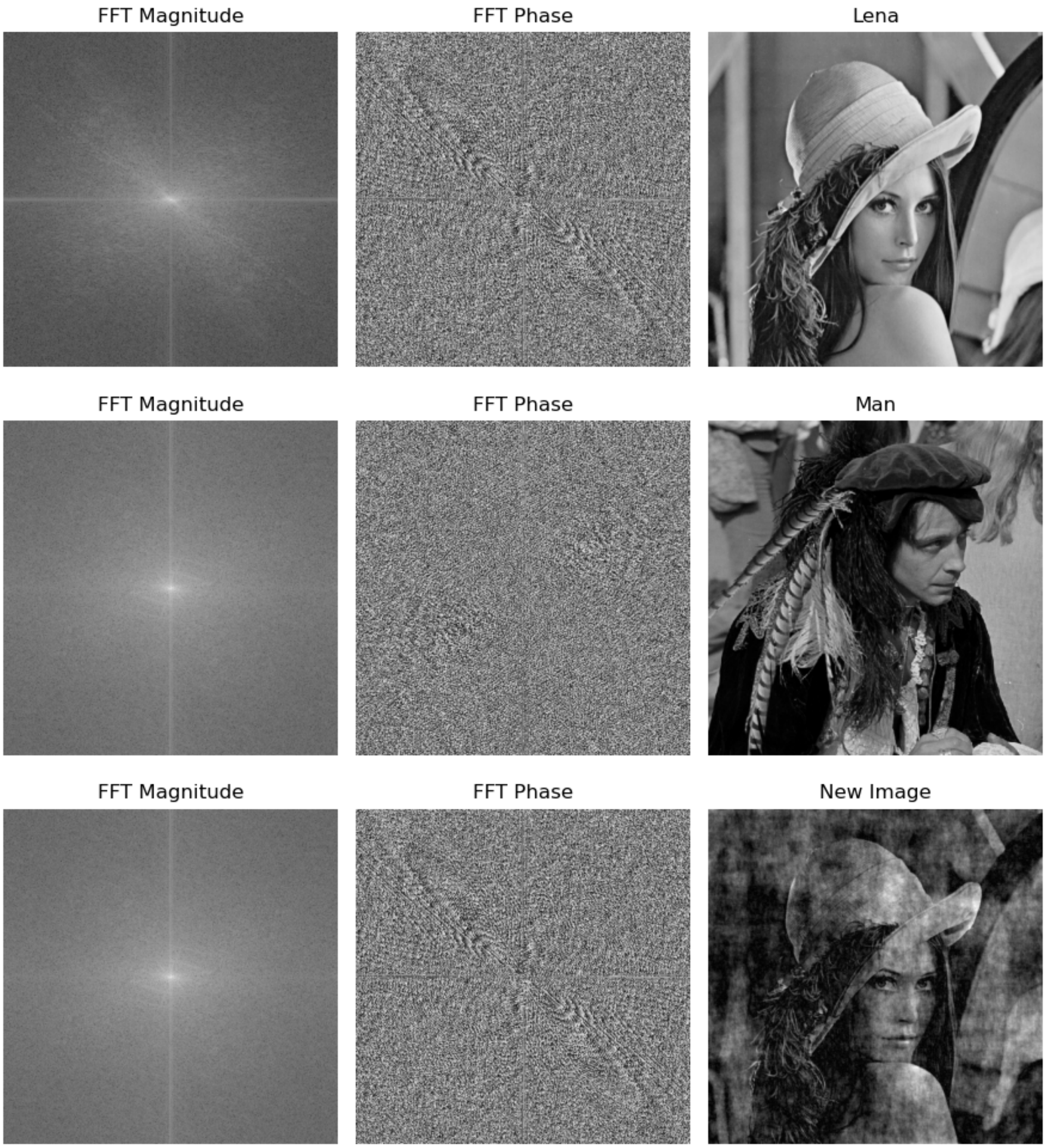
$$\mathbf{K}_m = \boldsymbol{\Omega}_m^{-1} = \left(\sum_{k=1}^m \boldsymbol{\Lambda}_k\right)^{-1} = \left(\sum_{k=1}^m \mathbf{V}_k^{-1}\right)^{-1}$$

and mean vector

$$\boldsymbol{\nu}_m = \boldsymbol{\Omega}_m^{-1}\boldsymbol{\lambda}_m = \mathbf{K}_m\left(\sum_{k=1}^m \mathbf{V}_k^{-1}\boldsymbol{\mu}_k\right)$$

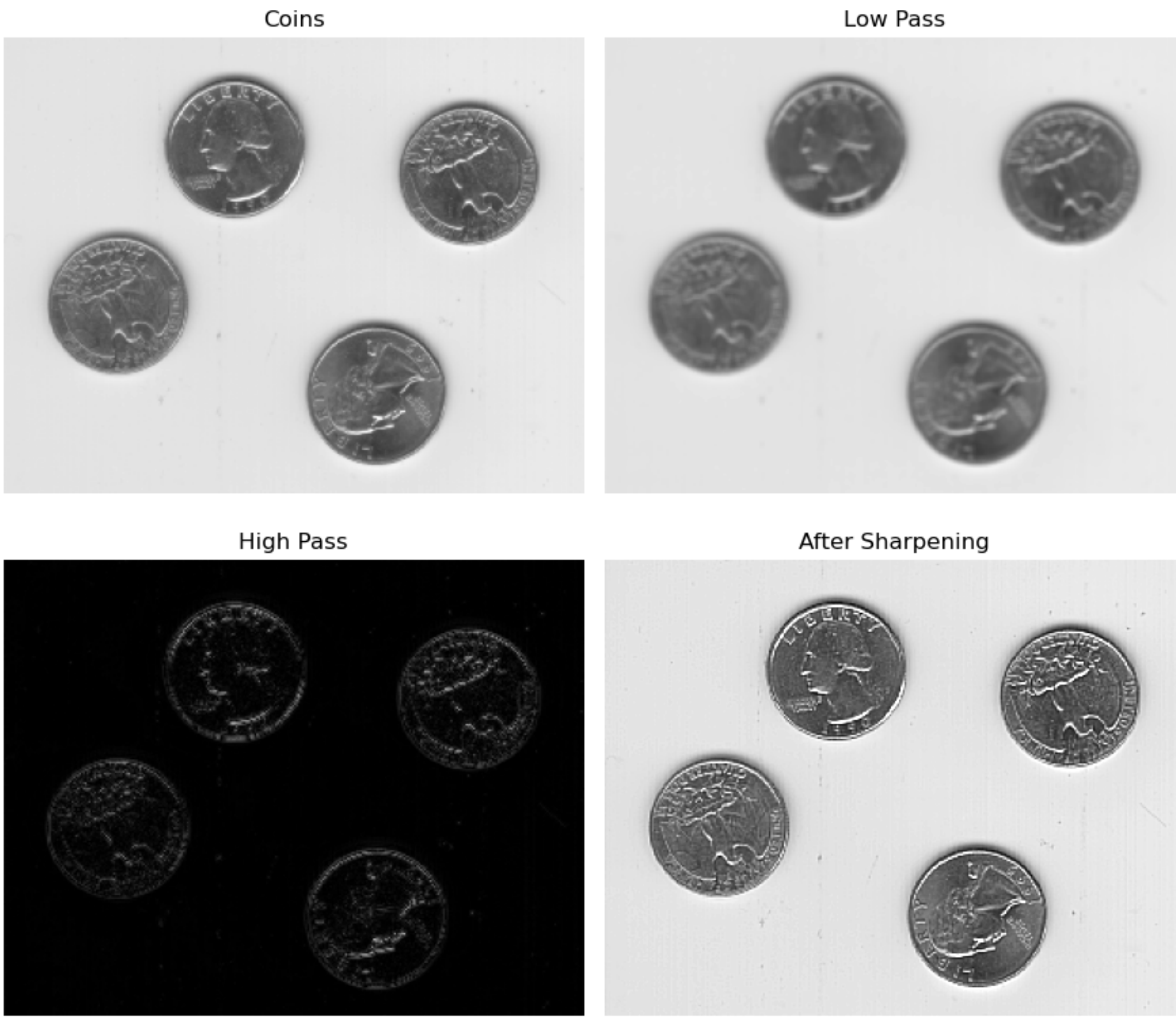
The scaling factor is also a Gaussian function.

4) Programming Exercise: Code is attached.



(a)

(b)



(c)