University of Minnesota
Driven to Discover℠

EE 5561: Image Processing and Applications

Lecture 7

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Recap of Last Lecture

- Histogram processing
- Interpolation methods
- Finished image enhancement (quickly!)
- Today: Start on statistical image processing
  - Linear algebra review

Vector spaces

- Vector space:

  A set V (a collection of vectors) defined over a field F ($\mathbb{R}$ or $\mathbb{C}$ in this course) with two operations:

  - vector addition, $\quad + : V \times V \to V$, i.e. $\mathbf{v}, \mathbf{w} \in V \to \mathbf{v} + \mathbf{w} \in V$
  - scalar multiplication $\quad \cdot : V \times F \to V$, i.e. $\mathbf{v} \in V, \alpha \in F \to \alpha\mathbf{v} \in V$

  where these operations satisfy certain properties.

- Properties of the operations
  - Vector addition is associative $\quad \mathbf{v} + (\mathbf{w} + \mathbf{y}) = (\mathbf{v} + \mathbf{w}) + \mathbf{y}$

    commutative $\quad \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$
  - Identity element exists for vector addition (i.e. zero vector)

$$\mathbf{v} + \mathbf{0} = \mathbf{v}$$

Vector spaces

- Scalar multiplication is distributive for vector addition & field addition

$$\alpha \cdot (\mathbf{v} + \mathbf{w}) = \alpha \cdot \mathbf{v} + \alpha \cdot \mathbf{w}$$
$$(\alpha + \beta) \cdot \mathbf{v} = \alpha \cdot \mathbf{v} + \beta \cdot \mathbf{v}$$

- Additive inverse exists $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$

- Scalar multiplication is associative $(\alpha\beta) \cdot \mathbf{v} = \alpha \cdot (\beta \cdot \mathbf{v})$

- Identity element exists for scalar multiplication (i.e. 1)

$$1 \cdot \mathbf{v} = \mathbf{v}$$

Linear Independence

- Definition:

A subset of vectors $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ are linearly dependent if

$\exists a_1, \ldots, a_n \in F$ (not all zero) such that

$$\sum_{k=1}^{n} a_k \mathbf{v}_k = \mathbf{0}$$

Vectors in $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ are linearly independent if

$$\sum_{k=1}^{n} a_k \mathbf{v}_k = \mathbf{0}$$

is only satisfied if $a_k = 0 \quad \forall k \in \{1, \ldots, n\}$

Spanning Set

- Span:

  The span of $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset V$ is the set of all finite linear combination of vectors in S, i.e.

$$\operatorname{span}(S) = \left\{ \sum_{k=1}^{n} \lambda_k \mathbf{v}_k \middle| n \in \mathbb{Z}^+, \mathbf{v}_k \in S, \lambda_k \in F \right\}$$

- Spanning set: If $V = \operatorname{span}(S)$ then S is called the spanning set of V.

Basis

- Basis:

  $B \subset V$ is called a basis if every element of V can be uniquely represented as a linear combination of vectors in B

  $\Updownarrow$ equivalent to

  B is a basis if its elements are linearly independent and B is a spanning set of V

- Dimension of *V* = # elements in *B*
- Note *V* can have different bases
  - e.g. For $\mathbb{R}^2$

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}, \quad B_3 = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

Normed space

- Vector space on which a norm is defined
- Norm is a real-valued function

$$||\cdot|| : V \to \mathbb{R} \text{ such that}$$

1. $||\mathbf{v}|| \geq 0$ and $||\mathbf{v}|| = 0 \Leftrightarrow \mathbf{v} = \mathbf{0}$
2. $||\alpha \cdot \mathbf{v}|| = |\alpha| \cdot ||\mathbf{v}||$
3. $|\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$ triangle inequality

Inner Product Space

- Already alluded to it while studying Fourier analysis
- Inner product is a map

$$\langle \cdot, \cdot \rangle : V \times V \to F \text{ such that}$$

1. $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$ — conjugate symmetry

2. $\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \bar{\alpha} \langle \mathbf{x}, \mathbf{y} \rangle$

   $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$

3. $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ — positive semi-definite

   $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$

Inner Product Space

- Example: In Euclidean spaces (e.g. $\mathbb{C}^n$), dot product is the typical inner product we will use

$$\left\langle \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\rangle = \sum_{k=1}^{n} \overline{x_k} y_k$$

– Norms on inner product spaces

- Based on the inner product

$$||\mathbf{x}|| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$$

- This defines the l_2 norm on Euclidean spaces

$$||\mathbf{x}||_2 = \sqrt{\sum_{k=1}^{n} |x_k|^2} = \left(\sum_{k=1}^{n} |x_k|^2 \right)^{\frac{1}{2}}$$

saw this when talking about the mean filter

Orthogonality

- Two vectors $\mathbf{u}, \mathbf{v}$ in an inner product space are orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$
- A basis $\{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is orthonormal if

$$\langle \mathbf{b}_i, \mathbf{b}_j \rangle = 0 \qquad i \neq j$$ orthogonality

and

$$\langle \mathbf{b}_i, \mathbf{b}_i \rangle = 1$$ orthonormality

Linear Transforms

- A mapping $T : V \to W$, between two vector spaces such that

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \qquad \forall \mathbf{u}, \mathbf{v} \in V$$

$$T(\alpha \mathbf{v}) = \alpha T(\mathbf{v}) \qquad \forall \alpha \in F, \mathbf{v} \in V$$

- If *V*, *W* are finite-dimensional (they are in this course), and a basis is defined for both, then $T : V \to W$ can be represented as a matrix
  - Typically we'll use the canonical basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for this representation
  - Here the j^{th} coordinate of $\mathbf{e}_i$ is 1 if $j = i$ (and 0 otherwise)

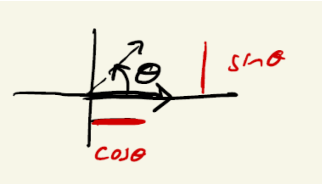
Linear Transforms

– Example:

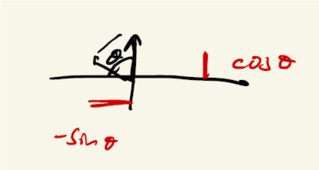
$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ is a rotation by } \theta \text{ counter clockwise}$$

- Let T be this operation

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$



$$T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$



Linear Transforms

– Why important?
  - Linear operation on 2D images can be written in matrix-vector notation
  - Mentioned this while talking about DFT/FFT
  - Image as 2D array

$$\begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & & \\ a_{M1} & \cdots & a_{MN} \end{bmatrix} \in V \qquad \longleftarrow \qquad \mathbf{v} \in V$$

images as vectors

$$\Big\downarrow \text{ linear operation}$$

new image

$$\begin{bmatrix} b_{11} & \cdots & b_{1N} \\ \vdots & & \\ b_{M1} & \cdots & b_{MN} \end{bmatrix} \qquad \longleftarrow \qquad \mathbf{Av} \in W$$

A is a matrix acting on vectors

Matrices

- An $m \times n$ matrix has m rows, n columns
- A_{ij} is the $(i,j)^{th}$ entry of $\mathbf{A}$
- $\mathbf{I}_n$ is the $n \times n$ identity matrix (we will drop the subscript when the dimension is implicit) with

$$I_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\mathbf{I}\mathbf{v} = \mathbf{v}$$

- Inverse of $n \times n$ matrix $\mathbf{A}$ is $\mathbf{A}^{-1}$ such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Matrices

- Matrix multiplication

$$\underset{m \times p}{\mathbf{C}} = \underset{m \times n}{\mathbf{A}}\,\underset{n \times p}{\mathbf{B}}$$

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

  - Note this is not commutative, i.e. in general $\mathbf{AB} \neq \mathbf{BA}$

Matrices

– Matrix transpose

$$\mathbf{B} = \mathbf{A}^T$$

A is *m*×*n*, then **B** is *n*×*m*

$$B_{ij} = A_{ji}$$

- Note $\left(\mathbf{A}^T\right)^T = \mathbf{A}$

– For complex matrices, we use conjugate/Hermitian transpose

$$\mathbf{B} = \mathbf{A}^* \quad \text{or} \quad \mathbf{A}^H$$

$$B_{ij} = \overline{A_{ji}}$$

Null Space

– Null space of matrix **A** is

$$N(\mathbf{A}) = \{\mathbf{x} \in V | \mathbf{A}\mathbf{x} = \mathbf{0}\}$$

is a linear subspace of *V*

Subspace is a non-empty subset of *V* that

i) Contains **0**

ii) Closed under vector addition and scalar multiplication

Range/Column Space

- Range/column space of matrix **A** is

$$R(\mathbf{A}) = \{\mathbf{A}\mathbf{x} | \mathbf{x} \in V\}$$

Span of the columns of **A**

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix}$$

$\mathbf{a}_i$: i^{th} column of **A**

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \sum_{k=1}^{n} x_k \mathbf{a}_k$$

scalars

vectors, i.e. columns of **A**

- rank(**A**) = dimension of the column space of **A**

Eigenvalues, eigenvectors, special matrices

- Eigenvalue and eigenvector of a *square* matrix **A** are a scalar λ and a vector **v** respectively such that

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

- Diagonal matrix: Square matrix **D** where $D_{ij} = 0$ if i≠j

$$\mathbf{D}\mathbf{x} = \begin{bmatrix} D_{11}x_1 \\ D_{22}x_2 \\ \vdots \\ D_{nn}x_n \end{bmatrix} \longleftarrow$$

or elementwise multiplication between vectors

$$\begin{bmatrix} D_{11} \\ D_{22} \\ \vdots \\ D_{nn} \end{bmatrix} \text{ and } \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Special matrices

- Unitary matrix: Square **U** such that $\mathbf{U}\mathbf{U}^H = \mathbf{U}^H\mathbf{U} = \mathbf{I} \rightarrow \mathbf{U}^H = \mathbf{U}^{-1}$
  - What does $\mathbf{U}\mathbf{U}^H = \mathbf{I}$ mean?
    - Rows of **U** are orthonormal
  - What does $\mathbf{U}^H\mathbf{U} = \mathbf{I}$ mean?
    - Columns of **U** are orthonormal
  - Example: A properly normalized DFT matrix is unitary. Let $W_N = e^{-i\frac{2\pi}{N}}$

$$\mathbf{U} = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N & \cdots & W_N^{N-1} \\ \vdots & \vdots & & \vdots \\ 1 & W_N^{N-1} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \triangleq \mathbf{F}$$

Special matrices

- Symmetric matrix: Square $\mathbf{A}$ such that $\mathbf{A} = \mathbf{A}^T$

  Hermitian matrix: Square $\mathbf{A}$ such that $\mathbf{A} = \mathbf{A}^H$

  - These types of matrices are diagonalized by unitary matrices
  - i.e. There exists a unitary $\mathbf{U}$ and diagonal $\mathbf{D}$, s.t. $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^H$

Special matrices

− Circulant matrix: Each row is rotated relative to the previous one

$$\mathbf{C} = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & \cdots & c_2 \\ \vdots & & & & \vdots \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}$$

- Note this implements circular convolution
- Circulant matrices are diagonalizable by the DFT matrix. In particular

$$\mathbf{C} = \mathbf{F}^H \mathbf{D}_c \mathbf{F}$$

diagonal elements are given by $\mathbf{d}_c = \mathbf{F} \begin{bmatrix} c_0 \\ c_{n-1} \\ \vdots \\ c_1 \end{bmatrix}$

Special matrices

- This means the following

$$\mathbf{C}\mathbf{x} = \mathbf{F}^H \mathbf{D}_c \underbrace{\mathbf{F}\mathbf{x}}_{\text{FT of } \mathbf{x}} \quad = \mathbf{F}^H \left(\mathbf{D}_c \hat{\mathbf{x}}\right)$$

circular convolution of $\begin{bmatrix} c_0 \\ c_{n-1} \\ \vdots \\ c_1 \end{bmatrix}$ and $\mathbf{x}$

convolution kernel

inverse FT

Elementwise multiplication between FT of the convolution kernel ($\mathbf{d}_c$) and FT of x

i.e. Implements convolutions using FT!

Solving linear equations

- Consider $\mathbf{b} = \mathbf{Ax}$ if $\mathbf{A}$ is an $n \times n$ invertible matrix → $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$
- If A is $m \times n$ with $m > n$ → system is overdetermined

  $m < n$ → system is underdetermined

- Solving overdetermined systems is about minimizing the "error"
- We usually consider the l_2 norm as the error, i.e.

$$\min_{\mathbf{x}} ||\mathbf{b} - \mathbf{Ax}||_2^2$$

Solving linear equations

– In this setting, i.e. A is *m*×*n* with *m*>*n*, we will proceed as

$$\begin{aligned} ||\mathbf{b} - \mathbf{A}\mathbf{x}||_2^2 &= \langle \mathbf{b} - \mathbf{A}\mathbf{x}, \mathbf{b} - \mathbf{A}\mathbf{x} \rangle \\ &= \langle \mathbf{b}, \mathbf{b} \rangle - 2\langle \mathbf{b}, \mathbf{A}\mathbf{x} \rangle + \langle \mathbf{A}\mathbf{x}, \mathbf{A}\mathbf{x} \rangle \end{aligned}$$

does not
depend on **x**

- Thus $\min_{\mathbf{x}} ||\mathbf{b} - \mathbf{A}\mathbf{x}||_2^2 \quad \Leftrightarrow \quad \min_{\mathbf{x}} -2\langle \mathbf{b}, \mathbf{A}\mathbf{x} \rangle + \langle \mathbf{A}\mathbf{x}, \mathbf{A}\mathbf{x} \rangle$

- Find minimum by taking the derivative with respect to **x**, and setting it to 0.

Solving linear equations

- How to take these derivatives in this case?
  - e.g. First term

$$\langle \mathbf{b}, \mathbf{A}\mathbf{x} \rangle = \mathbf{b}^T \mathbf{A}\mathbf{x}$$

let $\mathbf{c} = \mathbf{A}^T \mathbf{b}$

$$= \mathbf{c}^T \mathbf{x} = \sum_k c_k x_k$$

  - Now we look at

$$\frac{\partial}{\partial x_j} \langle \mathbf{b}, \mathbf{A}\mathbf{x} \rangle = \frac{\partial}{\partial x_j} \left(\sum_k c_k x_k \right) = c_j$$

  - Thus

$$\nabla_{\mathbf{x}} \langle \mathbf{b}, \mathbf{A}\mathbf{x} \rangle = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{c} = \mathbf{A}^T \mathbf{b}$$

Solving linear equations

- How to take these derivatives in this case?
  - Similarly $\nabla_{\mathbf{x}} \langle \mathbf{Ax}, \mathbf{Ax} \rangle = 2\mathbf{A}^T\mathbf{Ax}$

- Thus we have

$$\nabla_{\mathbf{x}} ||\mathbf{b} - \mathbf{Ax}||_2^2 = -2\mathbf{A}^T\mathbf{b} + 2\mathbf{A}^T\mathbf{Ax} = \mathbf{0}$$

$$\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$$

$$\mathbf{x} = \underbrace{\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T}_{\text{pseudoinverse of } \mathbf{A}}\mathbf{b}$$

- This is called the *least squares* solution

Recap

- Linear algebra review
  - Vector spaces, bases
  - Norms, inner products
  - Matrix representation of linear transformations
  - Matrix transpose/Hermitian, inverse, null space, range space, rank
  - Diagonal, symmetric, unitary, circulant matrices; diagonalization
  - Least squares
- Next class:
  - Use linear algebra to derive a new transform (wavelets)
  - Energy compaction
  - Random vectors review
  - Then we will proceed with image restoration