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EE5761 - Image Processing & Applications — November 7, 2023

1) $\underline{\underline{A}} \in \mathbb{C}$, $M \times N$ matrix of rank $r \leq \min(M,N)$

a) Show that $\underline{\underline{A}}^* \underline{\underline{A}}$ is a Hermitian matrix & is positive semi-definite

$\underline{\underline{A}}^* = \overline{A_{ji}}$, $\underline{\underline{A}} = A_{ij}$

$$\underset{[N \times M]}{\underline{\underline{A}}^*} \; \underset{[M \times N]}{\underline{\underline{A}}} = \underset{[N \times N]}{\underline{\underline{B}}}$$

$\underline{\underline{B}}$ is hermitian if $\underline{\underline{B}} = \underline{\underline{B}}^*$ then prove $\underline{\underline{A}}^* \underline{\underline{A}} = (\underline{\underline{A}}^* \underline{\underline{A}})^*$

LHS: $\overline{\underline{\underline{A}}}^T \underline{\underline{A}}$

RHS: $\overline{(\overline{\underline{\underline{A}}}^T \underline{\underline{A}})}^T = \overline{\left(\underline{\underline{A}}^T (\overline{\underline{\underline{A}}}^T)^T\right)} = \overline{\left(\underline{\underline{A}}^T \overline{\underline{\underline{A}}}\right)} = \overline{\underline{\underline{A}}}^T \underline{\underline{A}}$

LHS = RHS

$\boxed{\underline{\underline{B}} \text{ or } \underline{\underline{A}}^* \underline{\underline{A}} \text{ is a hermitian matrix}}$

b) $\underline{\underline{A}}^* \underline{\underline{A}}$ can be diagonalized by $\underline{\underline{U}}$ $\Rightarrow \underline{\underline{B}} = \underline{\underline{U}}\, \underline{\underline{D}}\, \underline{\underline{U}}^H$. How many positive eigenvalues does $\underline{\underline{A}}^* \underline{\underline{A}}$ have?

$\underline{\underline{A}}^* \underline{\underline{A}} = \underline{\underline{B}}$ $[N \times N]$ $\boxed{\text{can have } r \text{ positive eigenvalues}}$

bp2) Q = # eigenvalues $\sigma_1^2 \geq \sigma_2^2 \geq \dots \geq \sigma_1^2 > 0$ k^{th} column of $\underline{\underline{V}}$ unitary matrix is $\underline{V_k}$

find $\|\underline{V_k}\|_2^2$ and why?

$\|V_k\|_2^2 = \underline{V}_k^H \underline{V}_k = 1$ because $\underline{V_k}$ is a column of a unitary matrix w/ property $\underline{\underline{U}}\,\underline{\underline{U}}^H = \underline{\underline{I}}$ which mean that $\underline{V_k}^H \underline{V_k}$ will correspond to a diagonal element of $\underline{\underline{I}}$

$\boxed{\|\underline{V_k}\|_2^2 = 1}$

c) Consider $\underline{u}_k = \frac{1}{\sigma_k} \underline{\underline{A}} \underline{v}_k$ for $k \in 1, 2, \dots, l$, Show that $\|\underline{u}_k\|_2^2 = 1$

$$\|\underline{u}_k\|_2^2 = \underline{u}_k^H \underline{u}_k = \left[\frac{1}{\sigma_k} \underline{v}_k^T \underline{\underline{A}}^T\right]\left[\frac{1}{\sigma_k} \underline{\underline{A}} \underline{v}_k\right] = \frac{1}{\sigma_k^2} \underline{v}_k^H \underbrace{\underline{\underline{A}}^H \underline{\underline{A}}}_{\underline{\underline{B}}} \underline{v}_k = \frac{1}{\sigma_k^2} \underline{v}_k^H \underline{\underline{B}} \underline{v}_k$$

due to diagonalizable by unitary matrices of $\underline{\underline{B}}$

due to diagonalizable by unitary matrices of $\underline{\underline{B}}$ $\underline{v}_k$ is column vector of $\underline{\underline{V}}$ & $\underline{\underline{D}}$ is diagonal matrix containing $\sigma_1^2 \dots \sigma_k^2$

then $\underline{v}_k^H \underline{\underline{B}} \underline{v}_k = D_{kk}$ & $D_{kk} = \sigma_k^2$

$[1 \times N][N \times N][N \times 1] = [1 \times 1]$

then $\|\underline{u}_k\|_2^2 = \frac{1}{\sigma_k^2}(\sigma_k^2) = 1$ $\boxed{\|\underline{u}_k\|_2^2 = 1}$:)

d) Show that $\underline{u}_k$ is an eigenvector of $\underline{\underline{A}}\underline{\underline{A}}^*$ with eigenvalue σ_k^2 ?

Eigenvalue & Eigenvector has a property $\underline{\underline{B}} \underline{c} = \lambda \underline{c}$

plugging in $\underline{\underline{B}} = \underline{\underline{A}}\underline{\underline{A}}^*$, $\underline{c} = \underline{u}_k$ & $\lambda = \sigma_k^2$

$$\underline{\underline{A}}\underline{\underline{A}}^* \underline{u}_k = \sigma_k^2 \underline{u}_k$$

RHS: $\sigma_k^2 \left[\frac{1}{\sigma_k} \underline{\underline{A}} \underline{v}_k\right] = \sigma_k \underline{\underline{A}} \underline{v}_k$ LHC: $\underline{\underline{A}}\underline{\underline{A}}^* \frac{1}{\sigma_k} \underline{\underline{A}} \underline{v}_k$

$\frac{1}{\sigma_k} \underline{\underline{A}} \underline{\underline{A}}^* \underline{\underline{A}} \underline{v}_k = \sigma_k \underline{\underline{A}} \underline{v}_k$ True if $\underline{v}_k$ is eigenvector of A^*A w/ eigenvalue σ_k^2 which is definition of $\underline{\underline{V}}$ & $\underline{v}_k$

Same

$$\frac{1}{\sigma_k} \underline{\underline{A}} (\sigma_k^2 \underline{v}_k) = \sigma_k \underline{\underline{A}} \underline{v}_k$$

$\boxed{\underline{u}_k \text{ is eigenvector of } \underline{\underline{A}}\underline{\underline{A}}^* \text{ w/ eigenvalue} = \sigma_k^2}$

e) let $\underline{\underline{V}}_\ell$ $[N\times\ell]$ matrix whose k^{th} column is $\underline{v}_k$ above, & $\underline{\underline{U}}_\ell$ $[M\times\ell]$ whose k^{th} column is u_k above. Let Σ_ℓ be the diagonal matrix w/ diagonal entry σ_k

Show that $\underline{\underline{A}}\,\underline{\underline{V}}_\ell = \underline{\underline{U}}_\ell \underline{\underline{\Sigma}}_\ell$

$\underline{\underline{A}}\,\underline{\underline{V}}_\ell = \underline{\underline{U}}_\ell \underline{\underline{\Sigma}}_\ell$

$\times N][N\times L] = [M\times L][L\times L]$

$[M\times L] = [M\times L]$ ✓✓

Take k^{th} column of $\underline{\underline{V}}_\ell$ we have $\underline{v}_k$ & from previous sections we know that

$\underline{u}_k = \frac{1}{\sigma_k}\underline{\underline{A}}\,\underline{v}_k$

$\underline{\underline{A}}\,\underline{v}_k = \underline{u}_k \sigma_k$

Concatenating all columns we get that

$$\boxed{\underline{\underline{A}}\,\underline{\underline{V}}_\ell = \underline{\underline{U}}_\ell \underline{\underline{\Sigma}}_\ell}$$ ✓✓

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f) Now Show that $\underline{\underline{\Sigma}}_\ell = \underline{\underline{U}}^* \underline{\underline{A}}\,\underline{\underline{V}}_\ell$

from above we know that $\underline{\underline{A}}\,\underline{\underline{V}}_\ell = \underline{\underline{U}}_\ell \underline{\underline{\Sigma}}_\ell$ then plug in becomes

$\underline{\Sigma}_\ell = \underline{\underline{U}}_\ell^* \underline{\underline{U}}\,\underline{\underline{\Sigma}}_\ell$ from part c the proof implied that $\underline{\underline{U}}^*_\ell \underline{\underline{U}} = \underline{\underline{I}}$

then

$$\underline{\underline{\Sigma}}_\ell = \underline{\underline{I}}\,\underline{\underline{\Sigma}}_\ell$$

$$\boxed{\underline{\Sigma}_\ell = \underline{\Sigma}_\ell}$$