Problem Set 1

Due: September 22, 2023

1) [9 pts] Consider the 1D system whose input-output equation is

$$y(t) = x(t) * cos(2\pi f_c t)$$

a) Is this system linear?

b) Is this system shift-invariant?

c) Is this system BIBO stable?

2) [8 pts] a) Show that the 1D Fourier transform of the triangle function (tri) is a sinc^2 function. *Hint:* Note the tri function is the convolution of $rect$ function with itself.

$$\text{tri}(x) = \Lambda(x) \overset{\text{def}}{=} \max(1 - |x|, 0) = \begin{cases} 1 - |x|, & |x| < 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

b) Determine the energy of the sinc^2 function.

3) [12 pts] Fourier and spatial domains are closely related to each other. In this exercise, we will consider the so-called uncertainty principle (similar in spirit to Heisenberg's), which states that if a signal is contained in a small area in space, its Fourier spectrum will be wide-spread, and vice versa. We will quantify this spread. For ease of exposition, we will do this in 1D for a real signal $f(x)$.

Assume $||f(x)||^2 = 1$; and $f(x)$, $xf(x)$ and $uF(u)$ are square-integrable, where $F(u)$ is the Fourier transform of $f(x)$. Further assume

$$\int_{-\infty}^{\infty} x|f(x)|^2 dx = 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} u|F(u)|^2 du = 0$$

Define the uncertainty (or dispersion) in x and u respectively as

$$\sigma_x = \left(\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx \right)^{1/2}$$

$$\sigma_u = \left(\int_{-\infty}^{\infty} u^2 |F(u)|^2 du \right)^{1/2}$$

Show $\sigma_x \sigma_u \geq \frac{1}{4\pi}$.

Hints: Start with the definition of the norm of $f(x)$, and perform integration by parts, by carefully choosing the parts for integration. Then use Cauchy-Schwarz inequality, i.e.

$$|\langle f, g \rangle| \leq ||f||||g||,$$

and the relationship between the Fourier transform of $f'(x)$ and $F(u)$.

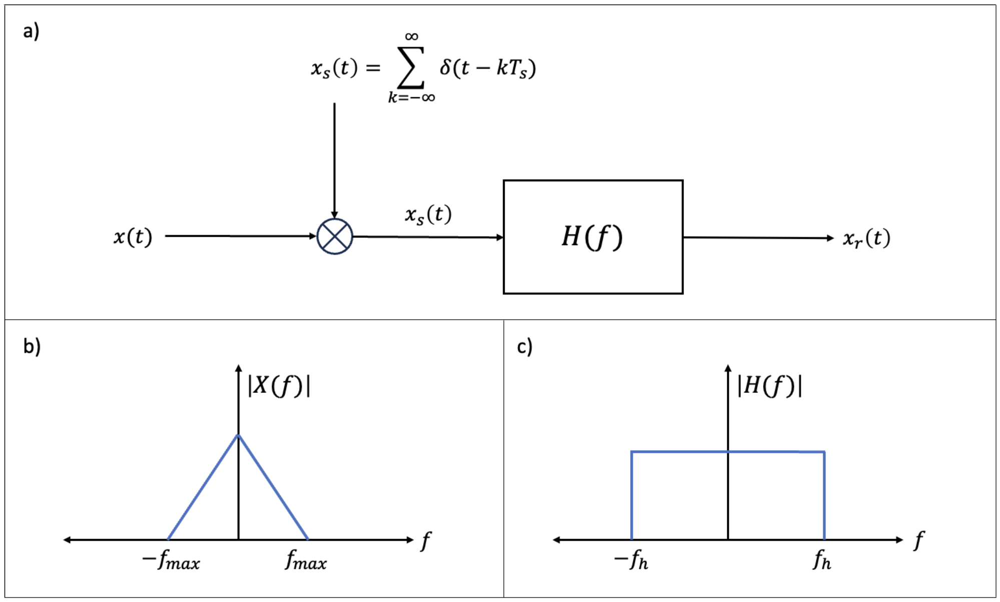

Figure 1: a) The system samples and recovers the 1D signal. b) Fourier transform of the input signal. c) Frequency response of the low pass filter.

4) [12 pts] Suppose we have a system (Figure 1) that samples the 1D signal $x(t)$ with a periodic impulse train $s(t)$, then recovers the signal from the sampled signal $x_s(t)$.

a) Plot the Fourier transform of the sampled signal $x_s(t)$.

b) What is the Nyquist criterion for the sampling period T_s so that the input signal is recoverable?

c) Determine the cutoff frequency of the low pass filter f_h that allows recovery of $x(t)$ from $x_s(t)$, and show how the filter recovers the signal.

d) (**Bonus: 6 pts**) Plot the Fourier transform of a signal that is different from $x(t)$, but is recovered as the same $x(t)$ using the same procedure.

5) [33 pts] Programming Exercise: In this exercise, you will re-generate three examples we saw in class on the Lena image (provided with the assignment). In Python, you will use the built-in NumPy package (`np.fft.fft2`, `np.fft.fftshift`, `np.fft.ifft2`, `np.fft.ifftshift`) to go between the image space and Fourier space. Use the `plt.imshow` function to display the Fourier spectra and images. While importing packages, please follow the standard style, import numpy as `np`, import matplotlib.pyplot as `plt`.

a) [3 pts] Calculate the 2D FFT of the Lena image. Plot its logarithmic magnitude spectrum, and phase spectrum.

b) [6 pts] Apply low-pass filters of size 128×128 and 32×32, and generate the corresponding images. Calculate the peak signal-to-noise ratio (PSNR) in dB for these two images compared to the original image

$$10 \log_{10} \left(\frac{\max_{m,n} |x_{\text{orig}}(m,n)|^2}{\frac{1}{mn} \sum_m \sum_n |x(m,n) - x_{\text{orig}}(m,n)|^2} \right)$$

b) [6 pts] Apply high-pass filters that are complements of the low-pass filters in (b), i.e. a high-pass filter that passes all frequencies except the central 128×128 and 32×32, and generate the corresponding images. Calculate the peak signal-to-noise ratio (PSNR) in dB for these two images compared to the original image

$$10 \log_{10} \left(\frac{\max_{m,n} |x_{\text{orig}}(m,n)|^2}{\frac{1}{mn} \sum_m \sum_n |x(m,n) - x_{\text{orig}}(m,n)|^2} \right)$$

c) [6 pts] We will assume 1 pixel in the Fourier spectrum corresponds to 1 cycle/mm. Thus the image is band-limited to 256 cycles/mm. Suppose we sample this image using two different sample spacings that violate the Nyquist criterion: i) with 1/384 mm sample spacing, ii) with 1/256 mm sample spacing in both dimensions. Assume the aliased sampled spectra are low-pass filtered with a bandwidth of 256 cycles/mm. Generate and display the corresponding images from these filtered aliased spectra (use the "subplot" feature to display the images together). Report PSNR.

d) [12 pts] We will implement the moving average filter:

- Use `scipy.signal.convolve2d` in Python or the `imfilter` function in MATLAB to apply 3×3 and 7×7 moving average filters to the Lena image. Display all three images. Report PSNR.
- Add random Gaussian noise (with $\sigma = 10$) to the original Lena image to get a degraded version. Apply the moving average filter with 3×3 and 7×7 kernels. Report the PSNR of the "denoised" versions.