AEM 4253/5253, Fall 2022, Homework 1

Due Thursday, September 22

Notes:

(1) Clearly label the axes on all your plots (and indicate the problem # in the caption).
(2) Use colors and symbols to differentiate different solutions on the plots.

1. Consider the ODE

$$\frac{dy}{dt} = -2y, \quad y(0) = 4, \quad 0 \leq t \leq 15$$

a) Solve the equation using the following schemes: (i) Explicit Euler (ii) Implicit Euler (iii) Second order Runge-Kutta (RK2) and (iv) RK4. Use $\Delta t = 0.1, 0.5, 1.0$ and compare with the exact solution.

b) For each scheme, what is the critical time step (if any) beyond which oscillatory (but stable) solutions appear? Justify.

2. An ODE with a slightly different right hand side from the previous one is

$$\frac{dy}{dt} = -(2 + 0.01t^2)y, \quad y(0) = 4, \quad 0 \leq t \leq 15.$$

a) Estimate the maximum stable time step over the entire time domain of interest for each of the schemes in problem (**1**a).

b) Solve the equation using the same schemes as in the previous problem (with the same time steps indicated) and compare with the exact solution.

3. The equation of motion of a simple pendulum consisting of mass m attached to a string of length l is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta,$$

where positive θ is counterclockwise. For small θ, $\sin\theta \approx \theta$, and the linearized equation is then

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta.$$

Assume $g = 9.81$ m/s^2, $l = 0.6$ m, and the starting angle to be $\theta(t = 0) = 10°$.

a) Write the linearized equation as a first order system and solve for $0 \leq t \leq 6$ using (i) Euler (ii) Implicit Euler (iii) RK4. Try time steps $\Delta t = 0.15, 0.5, 1$. Discuss your results in the context of what we have learned about the accuracy and stability of these schemes. For each case, on separate plots, compare your results with the exact solution (check):

$$\theta_{exact}(t) = \theta_0 \cos\left(t\sqrt{\frac{g}{l}}\right)$$

b) Suppose there is friction. The linearized equation is now

$$\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + \frac{g}{l}\theta = 0.$$

With $c = 4$ s^{-1}, repeat part (a) with the Euler and RK4 schemes. Discuss quantitatively (as far as possible) the stability as compared with part (a). The exact solution is now:

$$\theta_{exact}(t) = \theta_0 e^{-ct/2}\left(\cos(\alpha t) + \beta\sin(\alpha t)\right), \quad \alpha = \sqrt{4g/l - c^2}/2, \quad \beta = c/\sqrt{4g/l - c^2}$$

4. Non-linear differential equations with several degrees of freedom often exhibit chaotic solutions. Chaos is associated with sensitive dependence to initial conditions; however, numerical solutions are often confined to a so-called strange attractor, which attracts solutions resulting from different initial conditions to its vicinity in the phase space. It is the sensitive dependence on initial conditions that makes many physical systems (such as weather patterns) unpredictable, and it is the attractor that does not allow physical parameters to get out of hand (e.g., very high or low temperatures, etc.) An example of a strange attractor is the Lorenz attractor, which results from the solution of the following equations:

$$
\begin{aligned}
\frac{dx}{dt} &= \sigma(y - x) \\
\frac{dy}{dt} &= rx - y - xz \\
\frac{dz}{dt} &= xy - bz.
\end{aligned}
$$

The values of σ and b are usually fixed (take $\sigma = 10$ and $b = 8/3$ in this problem) leaving r as the control parameter. For low values of r, the stable solutions are stationary. When r exceeds 24.74, the trajectories in $x - y - z$ space become irregular orbits about two particular points.

a) Solve these equations using $r = 20$. Start from point $(x, y, z) = (1, 1, 1)$, and plot the solution trajectory for $0 \leq t \leq 25$ in the xy, xz, and yz planes. Plot also x, y, and z versus t. Comment on your plots in terms of the previous discussion. Use the RK4 method with a small time step (try $\Delta t = 0.005$, maybe)

b) Observe the change in the solution by repeating (a) for $r = 28$.

c) Observe the unpredictability at $r = 28$ by overplotting two solutions versus time starting from two initially nearby points: $(6, 6, 6)$ and $(6, 6.01, 6)$.

5. The 1D wave equation,

$$\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial x} = 0,$$

on a periodic domain of length L with N uniformly distributed points, discretized (spatially) with a second-order centered finite-difference scheme

$$\left.\frac{\partial f}{\partial x}\right|_i \approx \frac{f_{i+1} - f_{i-1}}{2\Delta x}, \qquad \Delta x = \frac{L}{N-1},$$

leads to a coupled system of the form

$$\frac{d}{dt}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \cdot \\ \cdot \\ f_N \end{pmatrix} = -\frac{a}{2\Delta x}\begin{pmatrix} 0 & 1 & \dots & & & -1 \\ -1 & 0 & 1 & \dots & & \\ 0 & -1 & 0 & 1 & \dots & \\ \cdot & & & & & \\ \cdot & & & & & \\ 1 & \dots & & & -1 & 0 \end{pmatrix} = \frac{a}{2\Delta x}\boldsymbol{A}.$$

Note that $\boldsymbol{A}$ is a skew-symmetric matrix (be careful about the negative sign that's absorbed into $\boldsymbol{A}$). Similarly, the 1D heat equation,

$$\frac{\partial f}{\partial t} = \nu\frac{\partial^2 f}{\partial x^2},$$

on a periodic domain of length L with N points, discretized with a second-order centered finite-difference scheme,

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_i \approx \frac{f_{i-1} - 2f_i + f_{i+1}}{(\Delta x)^2}$$

leads to a coupled system of the form

$$\frac{d}{dt}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \cdot \\ \cdot \\ f_N \end{pmatrix} = \frac{\nu}{(\Delta x)^2}\begin{pmatrix} -2 & 1 & \dots & & & 1 \\ 1 & -2 & 1 & \dots & & \\ 0 & -1 & -2 & 1 & \dots & \\ \cdot & & & & & \\ \cdot & & & & & \\ 1 & \dots & & & 1 & -2 \end{pmatrix} = \frac{\nu}{(\Delta x)^2}\boldsymbol{B}.$$

Note that $\boldsymbol{B}$ is a symmetric matrix.

a) Set up the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ in a program for a generic value N. For various values of N (start from, say, 11, and go upto 51 in steps of 10), compute the eigenvalues of these matrices numerically. For Matlab, `eig(A)`, for e.g., will give you the eigenvalues. For Fortran/C programs, use the LAPACK package. (To be fair, this is far easier in Matlab). What are the (estimated) limiting values of the eigenvalues as $N \to \infty$?

b) Using this information, which of the time-stepping schemes that we have talked about could you use for these problems?

c) How would you go about picking a time step Δt for a stable explicit scheme for these problems (for large N) - define what this should be in terms of a and Δx for the wave equation, and in terms of ν and Δx for the heat equation.