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AEM 8202 - Fluid Mechanics 2

24 February 2023
Problem set 2

1a) find a complex potential function for a flow having exactly 3 stagnation points at the locations $(1,0), \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(-\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$

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Stagnation point means $u, v = 0$. using $w = u - iv = 0$ at stagnation point

then the complex potential function should satisfy

$$w = (z - z_0)(z - z_1)(z - z_2)$$ where $z_0, z_1$ & $z_2$ are the stagnation points

$z_0 = 1$, $z_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $z_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i = z_1^*$

$$w = (z-1)(z-z_1)(z-z_1^*) = (z-1)(z^2 \underbrace{- zz_1 - zz_1^*}_{-z(z_1+z_1^*) = -z\left(2\left(-\frac{1}{2}\right)\right) = z} + z_1 z_1^*)$$

$$z_1 z_1^* = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$= (z-1)(z^2 + z + 1) = z^3 \cancel{+ z^2} \cancel{+ z} \cancel{- z^2} \cancel{- z} - 1$$

$$w = z^3 - 1$$

$$w = \frac{dF}{dz} \Rightarrow \int w\, dz = \int dF \Rightarrow F = \int z^3 - 1\, dz = \frac{z^4}{4} - z$$

$$\boxed{F(z) = \frac{z^4}{4} - z}$$

b) express the real velocity potential $\phi(x,y)$ & $\psi(x,y)$ in terms of x & y

$$F(z) = \frac{z^4}{4} - z = \phi(x,y) + i\psi(x,y), \quad z = x+iy$$

$$\frac{(x+iy)^4}{4} - (x+iy) = \frac{x^4 + 4ix^3y + 6(i)^2x^2y^2 + 4(i)^3xy^3 + (i)^4y^4}{4} - x - iy$$

$$= \left(\frac{x^4 - 6x^2y^2 + y^4}{4} - x\right) + i\left(\frac{4x^3y - 4xy^3}{4} - y\right)$$

$$\boxed{\phi(x,y) = \frac{x^4 - 6x^2y^2 + y^4 - 4x}{4} \qquad \psi(x,y) = x^3y - xy^3 - y}$$

c) plotted in the attached code & plots

d) Write a question appropriate for an undergraduate fluid mechanics exam.

Given $\psi(x,y) = x^3y - xy^3 - y$ &

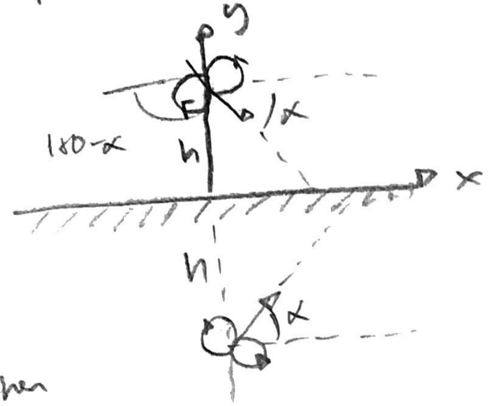
$$u = \frac{\partial\psi}{\partial y} \quad \& \quad v = \frac{-\partial\psi}{\partial x}$$

solve for $\phi(x,y)$ where

$$\frac{\partial\phi}{\partial x} = u \quad \& \quad \frac{\partial\phi}{\partial y} = v$$

2. A dipole of strength $\mu = 1$ is placed a distance $h = 1$ away from an infinite flat plate such that its axis is oriented at an angle $\alpha = \frac{\pi}{3}$ w/ respect to the horizontal. Assume that the fluid above the wall is inviscid and has density $\rho = 1$

a. Find complex potential function for this flow.




in order to generate an infinite flat plate, the method of images must be used. An additional dipole is generated below the plate & completely mirroring the dipole.

For a dipole $F(z) = \frac{\mu}{z}$

to rotate

$z \rightarrow Uz$ where $U = re^{-i\theta}$, $\theta$ is rotate

to translate

$z \rightarrow z - z_0$ where $z_0$ is translation

then

$r_{1,2} = 1 \quad \theta_1 = \frac{2\pi}{3} \quad \theta_2 = \frac{-2\pi}{3}$

$z_{01} = 1i \quad z_{02} = -1i$

$$F(z) = \frac{e^{i\frac{2\pi}{3}}\mu}{z - 1i} + \frac{e^{-i\frac{2\pi}{3}}\mu}{z + 1i} = \mu\left(\frac{\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)}{z - 1i} + \frac{\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)}{z + 1i}\right)$$

$$= \mu\left(\frac{-\frac{1}{2} + \frac{\sqrt{3}}{2}i}{z - 1i} + \frac{-\frac{1}{2} - \frac{\sqrt{3}}{2}i}{z + 1i}\right) = \mu\left(\frac{\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)(z + 1i) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)(z - 1i)}{(z - 1i)(z + 1i)}\right) = \mu\left[\frac{\left(-\frac{z}{2} - \cancel{\frac{i}{2}} + \cancel{\frac{\sqrt{3}iz}{2}} - \frac{\sqrt{3}}{2}\right) + \left(-\frac{z}{2} + \cancel{\frac{i}{2}} - \cancel{\frac{\sqrt{3}iz}{2}}\right)}{z^2 + 1}\right.$$

$$= \frac{2\mu\left(-\frac{z}{2} - \frac{\sqrt{3}}{2}\right)}{z^2 + 1} = \boxed{\frac{-\mu(\sqrt{3} + z)}{z^2 + 1} = F(z)}$$

b) find location of stagnation points

Stagnation pts occur at $u, v = 0$ $\quad w = u - iv = 0$

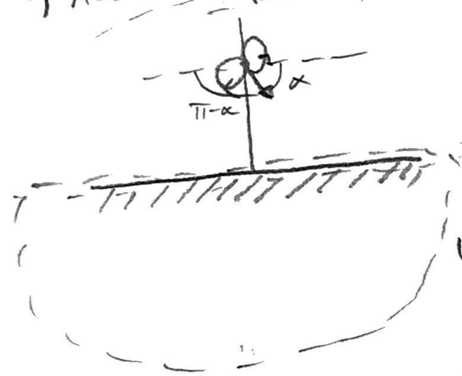
$$w = \frac{dF}{dz} = \frac{d}{dz}\left[\frac{-\mu(\sqrt{3}+z)}{z^2+1}\right] = \frac{(z^2+1)(-\mu) + (\mu\sqrt{3}+\mu z)(2z)}{(z^2+1)^2} = 0$$

$$-\cancel{\mu} z^2 - \cancel{\mu} + \cancel{\mu} 2\sqrt{3} z + 2\cancel{\mu} z^2 = 0 \Rightarrow z^2 + 2\sqrt{3} z - 1 = 0$$

$$z = \frac{-2\sqrt{3} \pm \sqrt{(2\sqrt{3})^2 - 4(1)(-1)}}{2} = \frac{-2\sqrt{3} \pm \sqrt{12+4}}{2} = \sqrt{3} \pm \frac{4}{2} = -\sqrt{3} \pm 2$$

$$\boxed{\text{Stagnation points at } z = -\sqrt{3}+2,\ z = -\sqrt{3}-2}$$

c) Assume $P_{\text{below plate}} = P_\infty$, find $F_{net}$ acting on the plate




$$F_x - iF_y = -\rho\pi\,[\Sigma]$$

$$F(z)\ \text{dipole} = O\left(\tfrac{1}{z}\right)$$

$$w = \frac{dF}{dz}\text{dipole} = \frac{d}{dz}\left(O\left(\tfrac{1}{z}\right)\right) = O\left(\tfrac{1}{z^2}\right)$$

use $\operatorname{Res}_f(z_1) = \dfrac{1}{(m-1)!}\dfrac{d^{m-1}}{dz^{m-1}}\left[(z-z_1)^m f(z)\right]\Bigg|_{z=z_1}$ #didn't really use this explicitly

for a general case $\quad \theta_1 = \pi - \alpha \quad \theta_2 = \pi + \alpha$

$$F(z) = \frac{e^{i(\pi-\alpha)}}{z-1i} + \frac{e^{i(\pi+\alpha)}}{z+1i} = \frac{\cos(\pi-\alpha) + i\sin(\pi-\alpha)}{z-1i} + \frac{\cos(\pi+\alpha) + i\sin(\pi+\alpha)}{z+1i}$$

let $c = \cos(\alpha)$, $s = \sin(\alpha)$

$\cos(\pi-\alpha) = -\cos(\alpha) \qquad \sin(\pi-\alpha) = \sin(\alpha)$

$\cos(\pi+\alpha) = -\cos(-\alpha) = -\cos(\alpha) \qquad \sin(\pi+\alpha) = -\sin(\alpha)$

$$F(z) = \frac{-c+is}{z-1i} + \frac{-c-is}{z+1i}$$

$$w = \frac{dF(z)}{dz} = -\left(\frac{-c+is}{(z-1i)^2}\right) + -\left(\frac{(-c-is)}{(z+1i)^2}\right) = \frac{c-is}{(z-1i)^2} + \frac{c+is}{(z+1i)^2}$$

<u>Blasius Thm</u>

$$F_x - iF_y = \frac{i\rho}{2} \oint w^2 dz$$

<u>Laurent Series</u>

$$w^2 = f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_j)^n$$

<u>Residue theorem</u>

$$\oint f(z)dz = 2\pi i \sum_{j=1}^{m} a_{-1}$$

$$w^2 = \underbrace{\frac{(c-is)^2}{(z-1i)^4}}_{\text{no } \frac{a_{-1}}{z+1i} \text{ term}} + \underbrace{\frac{(c+is)^2}{(z+1i)^4}}_{A} + \underbrace{\frac{2(c-is)(c+is)}{(z-1i)^2(z+1i)^2}}_{B}$$

using the following theorems just need to find the $a_{-1}$ for the singularity at $z=-1i$ which is in the control volume

$\frac{a_{-1}}{z+1i}$ want

$$A = \frac{c^2+2cis+s^2}{(z+1i)^4} = \frac{1+2cis}{(z+1i)^4}$$ can't decompose further, no $\frac{a_{-1}}{z+1i}$ term

$$B = \frac{2(c+is)(c-is)}{(z-1i)^2(z+1i)^2} = \frac{2(c^2+s^2)}{(z-1i)^2(z+1i)^2} = \frac{2}{(z-1i)^2(z+1i)^2}$$ use partial fraction decomposition

$$\frac{2}{(z-1i)^2(z+1i)^2} = \frac{A}{(z-1i)} + \frac{B}{(z-1i)^2} + \frac{C}{(z+1i)} + \frac{D}{(z+1i)^2} \Rightarrow 2 = A(z+1i)^2(z-1i) + B(z+i)^2 + C(z+1i)(z-1i)^2 + D(z-i)^2$$

if $z=i$ : $2 = B(2i)^2 \Rightarrow 2=-4B \rightarrow B = \frac{-1}{2}$

$z=-i$ : $2 = D(-2i)^2 \Rightarrow 2=-4D \Rightarrow D=-1/2$

$z=0$ : $2 = A(i)^2(-i) + B(i)^2 + c(i)(-i)^2 + D(-i)^2 = Ai - B - Ci - D = Ai - Ci + 1$

(★) $1=(A-C)i \Rightarrow A-C = \frac{1}{i} = -i$

$z=2i$ : $2 = A(3i)^2(i) + B(3i)^2 + C(3i)(i)^2 + D(i)^2 = -9Ai - 9B - 3Ci - D = 2$

$-9Ai - 3Ci + \cancel{5} = \cancel{2}$ (3) $\Rightarrow 3 = 9Ai + 3Ci \Rightarrow 1 = 3Ai + Ci$ (D)

add (★) & (D) $2 = 4Ai$ $\quad A = \frac{1}{2i} = -i/2$ $\quad C = i/2$ want this cuz $\frac{a_{-1}}{z+i}$ term

$$F_x - iF_y = -\rho\pi[\Sigma] = -\rho\pi\frac{i}{2}$$

$F_x = 0 \quad F_y = \frac{\rho\pi}{2}$

$$\boxed{F_{net} = \frac{\rho\pi}{2} \text{ in the } +y \text{ direction}}$$

d) How does the net force on the plate depend on the angle $\alpha$? Find the net force on the plate $F_{net}(\alpha)$ as a function of $\alpha$, where $0 \leq \alpha \leq 2\pi$

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in Part c) I used a general case for $\alpha$ not just $\alpha = \pi/3$
it was seen than $F_{net} \neq f(\alpha)$ but was constant. This is as expected because the singularity did not change at all

$$\boxed{F_{net}(\alpha) = \frac{\rho\pi}{2} \text{ in } +y \text{ direction for } 0 \leq \alpha \leq 2\pi}$$

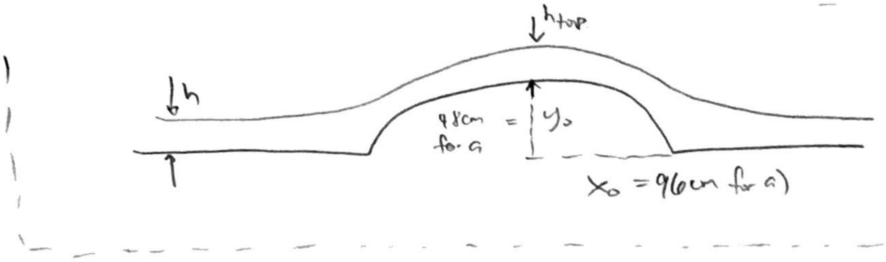
3) Consider a greenhouse in the shape of a semi-elliptical cylinder of width 192 cm and height 48 cm.

a) find an expression for $h_{top}$ in terms of $x_0, y_0, h$

b) given $h = 17$ cm, find $h_{top}$

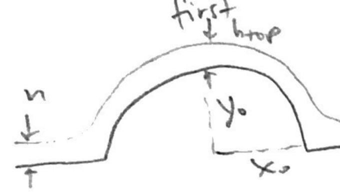
c) now given $x_0 = 48$ cm and $y_0 = 96$ cm find $h_{top}$ given $h = 17$ cm, compare w/ b)

c) plot streamlines of b) & c) & Attempt to use BEM solver to compute streamlines & compare




a) solving for a semi-regular cylider first

$x_0 = y_0 = a$




this is a flow over a circular cylinder with and use a joukowski transform from $\varsigma$ to $z$

$$z = \varsigma + \frac{b^2}{\varsigma}$$

for the circular cylider

$$F(z) = z + \frac{a^2}{z} \qquad z = re^{i\theta} = r\cos\theta + i\sin\theta$$

$$= re^{i\theta} + \frac{a^2}{r}e^{-i\theta}$$

$$= (r\cos\theta + ir\sin\theta) + \frac{a^2}{r}(\cos\theta - i\sin\theta)$$

$$= \underbrace{(r\cos\theta + a^2 r\cos\theta)}_{\phi} + i\underbrace{\left(r\sin\theta - \frac{a^2}{r}\sin\theta\right)}_{\psi}$$

$$\psi = r\sin\theta - \frac{a^2\sin\theta}{r} = y - \frac{a^2\sin\theta}{r}$$

$$\psi(r \to \infty, y = h, \theta \to \pi) = h$$

$h_{top}$ point occurs on the same streamline as above

$$\psi\left(r = h_{top}, \theta = \frac{\pi}{2}\right) = h$$

$$h_{top}\sin\left(\frac{\pi}{2}\right) - \frac{a^2}{h_{top_\varsigma}}\sin\left(\frac{\pi}{2}\right) = h$$

$$h_{top_\varsigma}^2 - a^2 - h_{top_\varsigma} h = 0$$

use quadratic eqn, use only positive value

$$h_{top_\varsigma} = \frac{h + \sqrt{h^2 + 4a^2}}{2}$$

$$F(z) = z + \frac{a^2}{z}$$

now need to get a in terms of $x_0$ & $y_0$

for ellipse case $x_0 = a + \frac{b^2}{a}$ $\quad y_0 = a - \frac{b^2}{a}$

$$x_0 + y_0 = 2a \quad \Rightarrow \quad a = \frac{x_0 + y_0}{2}$$

$$h_{top_\varsigma} = \frac{h}{2} + \sqrt{\frac{h^2}{4} + \frac{\cancel{4}}{\cancel{4}}\left(\frac{x_0 + y_0}{2}\right)^2}$$

$$h_{top_\varsigma} = \frac{h}{2} + \sqrt{\left(\frac{h}{2}\right)^2 + \left(\frac{x_0 + y_0}{2}\right)^2}$$

now need to use the joukowski transform

$$h_{top\,z} = h_{top_\varsigma} i + \frac{b^2}{h_{top_\varsigma} i}$$

$$x_0 - y_0 = \frac{2b^2}{a} \qquad b = \sqrt{\frac{(x_0 - y_0)a}{2}}$$

$$b = \sqrt{\frac{(x_0 - y_0)\left(\frac{x_0 + y_0}{2}\right)}{2}} = \frac{\sqrt{x_0^2 - y_0^2}}{2}$$

$$h_{top\,z} = \frac{h}{2} + \sqrt{\left(\frac{h}{2}\right)^2 + (x_0 + y_0)^2} + \frac{\frac{x_0^2 - y_0^2}{4}}{\frac{h}{2} + \sqrt{\left(\frac{h}{2}\right)^2 + (x_0 + y_0)^2}\, i} \cdot \frac{i}{i}$$

$$h_{top} = h_{top\,z} - y_0$$

$$\boxed{h_{top} = \frac{h}{2} + \sqrt{\left(\frac{h}{2}\right)^2 + \left(\frac{x_0+y_0}{2}\right)^2} \;\; \frac{-(x_0^2-y_0^2)/4}{\frac{h}{2}+\sqrt{\left(\frac{h}{2}\right)^2+\left(\frac{x_0+y_0}{2}\right)^2}} - y_0}$$

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b) plug in $h = 17\,cm$

$x_0 = 96\,cm,\; y_0 = 48\,cm$

$$h_{top} = \frac{17}{2} + \sqrt{\left(\frac{17}{2}\right)^2 + \left(\frac{96+48}{2}\right)^2} \;\; \frac{-\left((96^2-48^2)/4\right)}{\frac{17}{2}+\sqrt{\left(\frac{17}{2}\right)^2+\left(\frac{96+48}{2}\right)^2}} - 48$$

$= 11.6667$

$$\boxed{h_{top} = 11.67\,cm}$$

c) new case plug in $h = 17\,cm$

$x_0 = 48\,cm \quad y_0 = 96\,cm$

$$h_{top} = \frac{17}{2} + \sqrt{\left(\frac{17}{2}\right)^2 + \left(\frac{96+48}{2}\right)^2} \;\; \frac{-(48^2-96^2)/4}{\frac{17}{2}+\sqrt{\left(\frac{17}{2}\right)^2+\left(\frac{96+48}{2}\right)^2}} - 96$$

$= 6.3333$

$$\boxed{h_{top} = 6.333\,cm}$$

$\cancel{h_{topB}}$ ① $= h_{topC}$ ① $= \square$

let $\Delta = 48,\; 2\Delta = 96,\; \square = h_{top}$

$(x_0)^2 - (y_0)^2$ for $h_{topB} = (2\Delta)^2 - \Delta^2 = 3\Delta^2$

$h_{topC} = \Delta^2 - (2\Delta)^2 = -3\Delta^2$

$h_{topB} + h_{topC}$ [illegible]

$=$ [illegible]

$$\boxed{h_{top\,B} \text{ is } > h_{top\,C}}$$

$h_{topB} + h_{topC}$

$$= \left[\frac{h}{2} + \square - \frac{(3\Delta^2)/4}{\frac{h}{2}+\square} - \Delta\right] + \left[\frac{h}{2} + \square - \frac{(-3\Delta^2)/4}{\frac{h}{2}+\square} - 2\Delta\right]$$

$= h + 2\square - 3\Delta$

nothing special

d) plots in python

also BEM method