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Problem Set #3

April 7, 2023

Spring

FLUID MECHANICS III

1. Unsteady, 2-D flow, incompressible, viscous fluid, flat, porous plate, infinite ☆

$\frac{\partial}{\partial t} \neq 0$ $\quad$ $\frac{\partial}{\partial z} = 0$ $\quad$ $\nabla \cdot \vec{u} = 0$ $\quad$ $\nu \neq 0$ $\quad$? $\quad$ $\frac{dP}{dx} \neq 0$ $\quad$ $\frac{\partial \bar{u}}{\partial x} = 0$

The plate at rest at $t=0$, $\rightarrow$ at U {like Stoke's 1st problem}

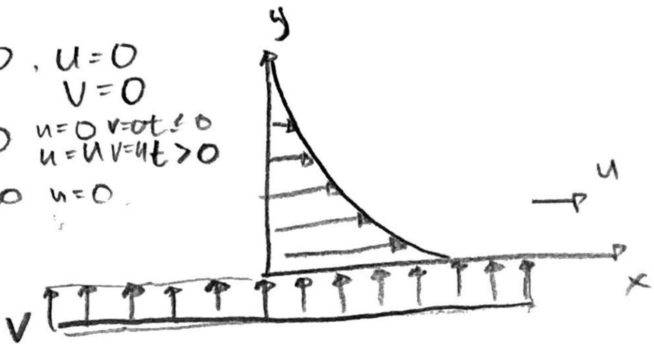
, injected uniformly through the plate with velocity V

a) find $u(y,t)$ above the plate ☆

ICs : $t \le 0$, $u = 0$, $V = 0$

BCs : $y = 0$ $\quad u=0 \; v=0 \; t \le 0$; $\quad u = U \; V = U \; t > 0$

$y = \infty \quad u = 0$



COM: inf $\quad$ 2D

$$\cancel{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}} = 0 \Rightarrow \frac{\partial v}{\partial y} = 0$$

$v = \text{const}$, using BCs $\quad v = V$

x-m

$$\frac{\partial u}{\partial t} + \cancel{u\frac{\partial u}{\partial x}}^{\text{inf}} + v\frac{\partial u}{\partial y} + \cancel{w\frac{\partial u}{\partial z}}^{\text{2D}} = \cancel{-\frac{1}{\rho}\frac{\partial p}{\partial x}}^{\text{inf}} + \nu\left[\cancel{\frac{\partial^2 u}{\partial x^2}}^{\text{inf}} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}^{\text{2D}}\right]$$

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

y-m

$$\frac{\partial v}{\partial t} + \cancel{u\frac{\partial v}{\partial x}}^{0} + \cancel{v\frac{\partial v}{\partial y}}^{0} + \cancel{w\frac{\partial v}{\partial z}}^{\text{2D}} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[\cancel{\frac{\partial^2 v}{\partial x^2}}^{\text{const}} + \cancel{\frac{\partial^2 v}{\partial y^2}}^{\text{COM}} + \cancel{\frac{\partial^2 v}{\partial z^2}}^{\text{const}}\right]$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\boxed{\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}}$$

b) if V is const in time & non-zero, then no similarity solution can exist.

$$\frac{\partial V}{\partial t}=0 \ \& \ V\neq 0$$

(9.25),(9.32)

(12.84)

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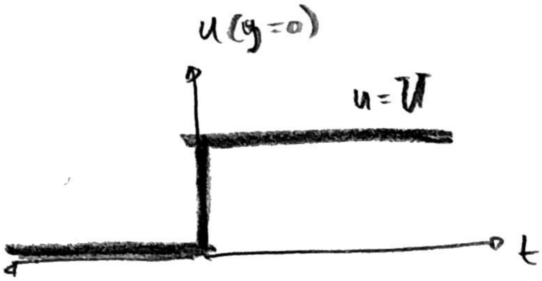
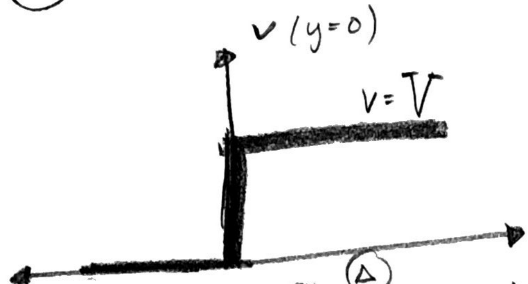
from (a) $\frac{\partial u}{\partial t}+V\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$ (★)





assume similarity solution exists: then $\eta=\eta(y,t)=\frac{By}{t^n}$ (△) & $u=At^m f(\eta)$

BCs: $u(y,0)=0$ $\quad u(0,t)=U$ (1) $\quad u(0,-t)=0$ $\quad u(\infty,t)=0$ $\quad v(0,0)=0$ $\quad v(0,t)=V$

using (1) $u(0,t)=U=At^m f(\eta)$ $\Rightarrow m=0$ $\Rightarrow u=Af(\eta)$

$$\frac{\partial u}{\partial t}=Af'\left(\frac{-Byn}{t^{n+1}}\right)=\frac{-ABf'yn}{t^{n+1}} \quad (2) \qquad \frac{\partial u}{\partial y}=Af'\left(\frac{B}{t^n}\right)=\frac{ABf'}{t^n} \quad (3)$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{AB}{t^n}f''\left(\frac{B}{t^n}\right)=\frac{AB^2f''}{t^{2n}} \quad (4)$$

plug in (2),(3) & (4) into (★)

$$\frac{-ABf'yn}{t^{n+1}}+V\frac{ABf'}{t^n}=\nu\frac{AB^2f''}{t^{2n}}$$

subst $y=\frac{\eta t^n}{B}$ from (△)

$$\frac{-\cancel{AB}f'\left(\frac{\eta t^n}{\cancel{B}}\right)n}{t^{n+1}}+V\frac{\cancel{A}Bf'}{t^n}=\nu\frac{\cancel{A}B^2f''}{t^{2n}} \Rightarrow \left[\frac{-f'\eta n}{t}+V\frac{Bf'}{t^n}=\nu\frac{B^2f''}{t^{2n}}\right]\frac{t^{2n}}{\nu B^2}$$

$$f''+\underbrace{\frac{f'\eta n t^{2n-1}}{\nu B^2}}_{t^{2n-1}\ (△)}-\underbrace{\frac{Vf't^n}{\nu B}}_{t^n \text{ if } V\neq f(t)\ (□)}=0$$

★ why not $n=1$

so (△) & (□) have same exponent?

nvm, cuz we want an ODE of just η & f

choose $n=1/2$ to make $t^{2n-1}=1$

$$f''+\frac{\eta}{2\nu B^2}f'-\frac{V}{\nu B}t^{1/2}f'=0$$

if $V\neq f(t)$ can be seen from (□)

that no similarity solutions can exist ■

c) Find a time-varying injection velocity $V(t)$ that will admit a similarity solution $f(\eta)$. Reduce PDE of (a) to an ODE for $f(\eta)$

from (a) $\quad \frac{\partial u}{\partial t} + V(t)\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$

from (b) we got $f'' + \frac{\eta}{2\nu B^2} f' - \frac{V(t)}{\nu B} t^{1/2} f' = 0$ $\quad$ let $V(t) = gCt^k$

(A) $\frac{\eta}{2\nu B^2}$

choose $\frac{1}{B^2} = 2\nu$, $\frac{1}{B} = \sqrt{2\nu}$ to make (A) num coeff = 1

$f'' + \eta f' - gCt^k t^{1/2}\sqrt{\frac{2}{\nu}} f' = 0$

(B)

choose $k = -1/2$ to make $t^k t^{1/2} = 1$

$C = \sqrt{\frac{\nu}{2}}$ to make (B) num coeff = 1·g

$f'' + \eta f' - f' = 0 \quad \rightarrow \quad f'' + g(\eta - 1) f' = 0$

if the velocity is $V(t) = g\sqrt{\frac{\nu}{2}} t^{1/2}$ (float)

then a similarity solution for the ODE

$$f'' + g(\eta-1) f' = 0$$

can be found

$\frac{\alpha+2}{2}ff''+(\ )$ (F-S)

2. Falkner-Skan solutions to Laminar BL equ's, self similar profiles

$u = Cx^{\alpha}$, figure 10.8 BV & E ODEs 5.1.1 AHB / BVPs 4.11.1 FENA [example 4.9] } RK do 1up & iterate

use shooting method to numerically solve

$\alpha = -0.0904,\ 0,\ \frac{1}{3},\ \frac{1}{9},\ 1$

-0.0904 ↓ incipient separation; 0 ↓ Blasius BL

[example 10.2]

displacement thickness (10.16) $\int_0^\infty \left(1-\frac{u}{U_e}\right)dy$

- get $f''(0)$ (wall shear stress) and shape factor $H = \frac{\delta^*}{\theta}$ (10.40) for each α

momentum thickness (10.17) $\int_{y=0}^\infty \frac{u}{U_e}\left(1-\frac{u}{U_e}\right)dy$

The Falkner-Skan equation is $f''' + \frac{n+1}{2}ff'' - nf'^2 + n = 0$ & $f = f(\eta)$

BCs $f(0)=0$
$f'(0)=0$
$f''(0)=?$ ↓ shoot to get
$f'(\infty)=1$

$$\frac{d}{d\eta}\begin{bmatrix} f \\ f' \\ f'' \end{bmatrix} = \begin{bmatrix} f' \\ f'' \\ -\frac{(n+1)}{2}ff'' + nf'^2 - n \end{bmatrix}$$

using $\alpha = -0.0904$ gets weird results

get $f''(0)$ through shooting method

The following $f''(0)$ or wall shear stress was found

-0.05

$\alpha =$	~~-0.0904~~	0	1/3	1/9	1
$f''(0)$	0	0.3321	0.7574	0.5118	1.2326

to calculate H, need to get δ^* & θ for each α

this F-S is similar to Blasius then $\eta = \frac{\sqrt{\frac{c}{\nu}}\,y}{x^{(1+\alpha)/2}}$ with $n = \frac{1-\alpha}{2}$

$$\int_0^\infty \left(1-\frac{u}{U_e}\right)dy = \delta^*$$

$$u = \frac{\partial\psi}{\partial y} = \sqrt{c\nu}\, f x^{\frac{1-\alpha}{2}}$$

or maybe just do numerically

trapezoid

$$H = \frac{\delta^*}{\theta} \qquad \delta^* = \int_0^\infty \left(1 - \frac{u}{u_e}\right) dy = \int_0^{\eta} 1 - f' \, d\eta$$

$$\theta = \int_0^\infty \frac{u}{u_e}\left(1 - \frac{u}{u_e}\right) dy = \int_0^\infty f'(1 - f')\, d\eta$$

solve numerically

The following Shape Factors were found

α	H
-0.0904	4.0282
0	2.5931
$1/3$	2.2893
$1/9$	2.409
1	2.2334

3. Boundary Layer

2D, steady, semi-infinite, flat plate, incompressible, non-Newtonian $\tau \neq C = f\left(\frac{du}{dy}\right)$

$\frac{\partial}{\partial z}=0$ $\frac{\partial}{\partial t}=0$ $\frac{d\bar{u}}{dx}=0$? $\nabla\cdot\vec{u}=0$ $\mu = K\left(\frac{du}{dy}\right)^{1/2}$

flow consistency index

shear thickening

hypersonic boundary layers

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a) How does BL scale with L? grow faster / slower than Blasius BL?

infinite plate

0 due to scaling $\delta \ll L$

COM $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

x-mom $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$

y-mom $u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+\frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$

scale using

$u^*=\frac{u}{U_\infty}$ $x^*=\frac{x}{L}$ $y^*=\frac{y}{\delta}$ $v^*=\frac{v}{V_{max}}$ $P^*=\frac{P}{\rho U_\infty^2}$

COM $\underset{①}{\frac{U_\infty}{L}\frac{\partial u^*}{\partial x^*}}+\underset{②}{\frac{V_{max}}{\delta}\frac{\partial v^*}{\partial y^*}}=0 \Rightarrow \frac{V_{max}}{\delta}\sim\frac{U_\infty}{L} \Rightarrow V_{max}\sim\frac{U_\infty \delta}{L}$ ③

x-com $\frac{U_\infty^2}{L}u^*\frac{\partial u^*}{\partial x^*}+\frac{U_\infty V_{max}}{\delta}v^*\frac{\partial u^*}{\partial y^*}=-\frac{1}{\cancel{\rho}}\frac{\cancel{\rho}U_\infty^2}{L}\frac{\partial P^*}{\partial x^*}$

$$+\frac{K}{\rho}\left(\frac{U_\infty}{\delta}\frac{\partial u^*}{\partial y^*}\right)^{1/2}\left[\underbrace{\frac{U_\infty}{L^2}\frac{\partial^2 u^*}{\partial x^{*2}}}_{④}+\underbrace{\frac{U_\infty}{\delta^2}\frac{\partial^2 u^*}{\partial y^{*2}}}_{⑤}\right]$$

①: $\frac{U_\infty^2}{L}$; ②: $\frac{U_\infty U_\infty \cancel{\delta}}{\cancel{\delta} L}=\frac{U_\infty^2}{L}$ ③ $\frac{-U_\infty^2}{L}$

④ $\frac{K U_\infty^{3/2}}{\rho\,\delta^{1/2}L^2}=\frac{U_\infty^2}{L}\underbrace{\left(\frac{K}{\rho U^{1/2}\delta^{1/2}L}\right)}_{\ll \text{ because } L^2 \text{ is big}}$

⑤ $\frac{K U_\infty^{3/2}}{\rho\,\delta^{5/2}}=\frac{U_\infty^2}{L}\left(\frac{kL}{\rho\,U^{1/2}\delta^{5/2}}\right)$ ⑦

BL is δ

in terms of L

Can remove U_∞

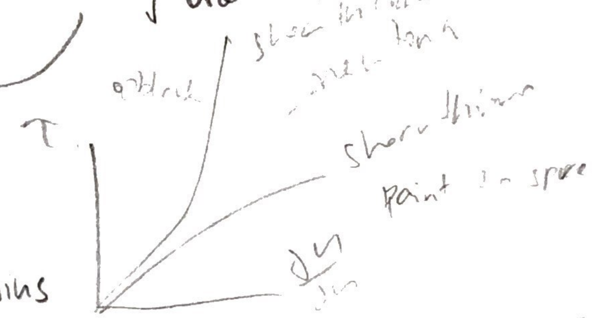
⑦ must be $O(1)$

$$\frac{\frac{kL}{\rho u^{1/2}\delta^{5/2}}}{u_\infty^2/L} \sim 1 \Rightarrow \frac{L}{u^{1/2}} \sim \delta^{5/2}$$

$$\delta^5 \sim \frac{L^2}{u} \qquad \boxed{\delta \sim \sqrt[5]{\frac{L^2}{u}}}$$ compare w/ $\delta \sim \sqrt{\frac{\nu L}{u_\infty}} \sim \sqrt{\frac{\nu x}{L}}$

$$\boxed{\delta \sim L^{2/5}} < L^{1/2}$$

scales lower than Blasius BL



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b) Show that similarity solution exists for this BL by finding ODE of $f(\eta)$

$$\psi = A f x^m, \quad \eta = \frac{By}{x^n}$$

going back to x-momentum equation we have:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} = \frac{k}{\rho}\left(\frac{\partial u}{\partial y}\right)^{1/2}\frac{\partial^2 u}{\partial y^2} \quad (*)$$

$$u = \frac{\partial \psi}{\partial y} = Ax^m f'\left(\frac{B}{x^n}\right) = ABf' \qquad \text{choose } AB = U_\infty$$

$$\frac{\partial \psi}{\partial x} = A\left[f m x^{m-1} + x^m f'\left(\frac{-nBy}{x^{n+1}}\right)\right] = Anx^{n-1}(f - \eta f')$$

$$\lim_{y\to\infty} u = U_\infty \Rightarrow \lim_{\eta\to 0} Ax^m f'\frac{B}{x^n} = U_\infty \quad \text{then } m \text{ must be } = n$$

$$\frac{\partial \eta}{\partial y} = \frac{B}{x^n} \qquad \frac{\partial \eta}{\partial x} = \frac{-nBy}{x^{n+1}} = \frac{-nB\eta x^{n}}{Bx^{n+1}} = \frac{-n\eta}{x}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \frac{k}{\rho}\left(\frac{\partial^2 \psi}{\partial y^2}\right)^{1/2}\left(\frac{\partial^3 \psi}{\partial y^3}\right)$$

$$\frac{\partial^2\psi}{\partial y^2} = Ax^m f''\left(\frac{B}{x^n}\right)^2 = \frac{AB^2 f''}{x^n} \qquad \frac{\partial^3\psi}{\partial y^3} = Ax^m f'''\left(\frac{B}{x^n}\right)^3 = \frac{AB^3 f'''}{x^{2n}}$$

$$\frac{\partial^2\psi}{\partial x \partial y} = ABf''\left(\frac{-n\eta}{x}\right) = \frac{-nAB\eta f''}{x}$$

plug in everything to ☆

$$(ABf')\left(\frac{-nAB\eta f''}{x}\right) - \left(Anx^{n-1}(f-\eta f')\right)\left(\frac{AB^2 f''}{x^n}\right) = \frac{k}{\rho}\left(\frac{AB^2 f''}{x^n}\right)^{1/2}\left(\frac{AB^3 f'''}{x^{2n}}\right)$$

$$\left[\cancel{\frac{-nA^2B^2\eta f' f''}{x}} - A^2nx^{-1}B^2 ff'' + \cancel{A^2nx^{-1}B^2 f''\eta f'} = \frac{k}{\rho}\frac{A^{3/2}B^4 f''^{1/2} f'''}{x^{5n/2}}\right]\frac{\rho}{k}\frac{x^{5n/2}}{A^{3/2}B^4}$$

$$f''' f''^{1/2} + \frac{\rho}{k}\frac{A^{1/2}}{B^2} n x^{\frac{5n}{2}-1} ff'' = 0 \qquad \text{choose } n = \frac{2}{5} \text{ to make } x^0$$

$$\text{choose } \frac{A^{1/2}}{B^2} = \frac{5k}{2\rho}$$

$$f''' f''^{1/2} + \frac{\rho}{k}\frac{2}{5}\frac{A^{1/2}}{B^2} ff'' = 0 \qquad AB = U_\infty$$

$$B = \frac{U_\infty}{A} \Rightarrow \frac{1}{B^2} = \frac{A^2}{U_\infty^2} \quad \Rightarrow \text{plug in to} \quad \frac{A^{1/2}A^2}{U_\infty^2} = \frac{5k}{2\rho}$$

$$A^{5/2} = \frac{U_\infty^2 5k}{2\rho} \qquad A = \sqrt[5]{\frac{25U_\infty^4 k^2}{4\rho^2}} \qquad B = \sqrt[5]{\frac{4\rho^2 U_\infty}{25k^2}}$$

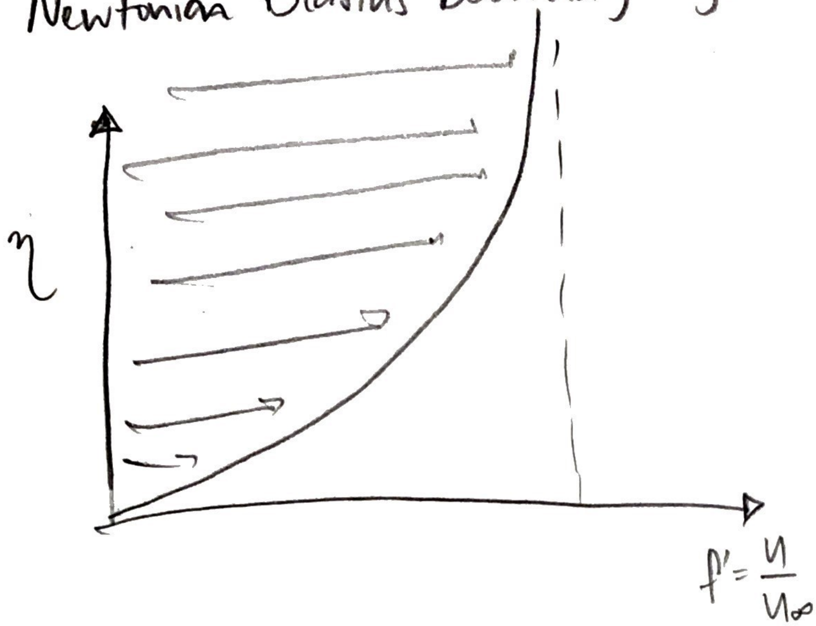
A similarity solution exists which results to the following ODE

$$\boxed{f''' f''^{1/2} + ff'' = 0}$$

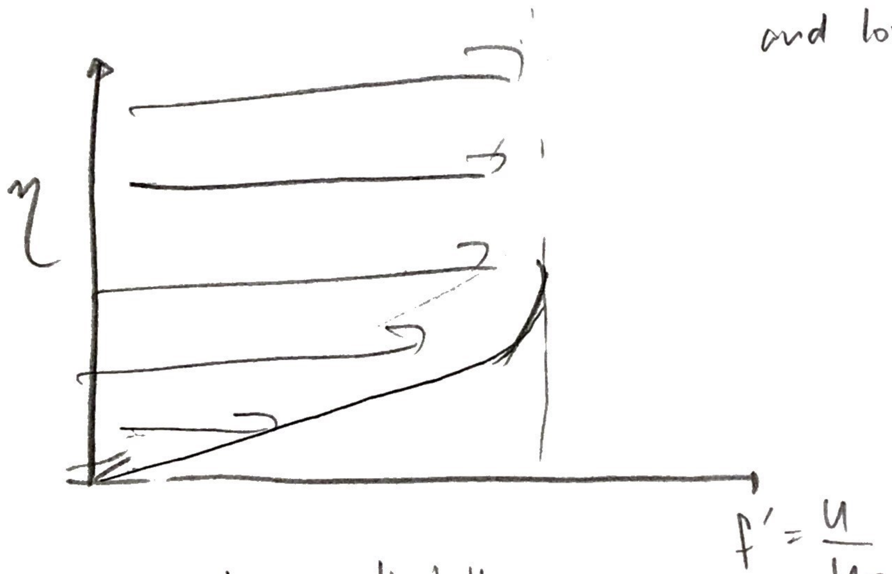
c) Use shooting method to solve the ODE in part B

got $f''(0) = 0.619$

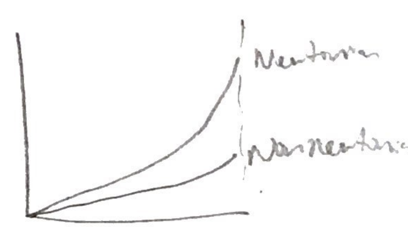
Newtonian Blasius boundary layer:



Special Non Newtonian boundary layer from python



and lower too



can be seen that this pretty linear close to the wall