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Computational Fluid Dynamics

1) a) $\left.\frac{\partial f}{\partial x}\right|_i$ using $\{i-2, i-1, i, i+1\}$

$$\left.\frac{\partial f}{\partial x}\right|_i \approx \alpha_{i-2} f_{i-2} + \alpha_{i-1} f_{i-1} + \alpha_i f_i + \alpha_{i+1} f_{i+1}$$

Taylor series expand about i

$$f_{i-2} = f_i + (-2h)\left.\frac{\partial f}{\partial x}\right|_i + \frac{(-2h)^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_i + \frac{(-2h)^3}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_i + \frac{(-2h)^4}{4!}\left.\frac{\partial^4 f}{\partial x^4}\right|_i + \dots$$

$$f_{i-1} = f_i + (-h)\left.\frac{\partial f}{\partial x}\right|_i + \frac{(-h)^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_i + \frac{(-h)^3}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_i + \frac{(-h)^4}{4!}\left.\frac{\partial^4 f}{\partial x^4}\right|_i + \dots$$

$$f_i = f_i$$

$$f_{i+1} = f_i + (h)\left.\frac{\partial f}{\partial x}\right|_i + \frac{(h)^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_i + \frac{(h)^3}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_i + \frac{(h)^4}{4!}\left.\frac{\partial^4 f}{\partial x^4}\right|_i + \dots$$

$$\alpha_{i-2} f_{i-2} + \alpha_{i-1} f_{i-1} + \alpha_i f_i + \alpha_{i+1} f_{i+1}$$

$$= f_i\left(\alpha_{i-2} + \alpha_{i-1} + \alpha_i + \alpha_{i+1}\right) \Big\} = 0$$

$$+ \left.\frac{\partial f}{\partial x}\right|_i\left(-2h\alpha_{i-2} - h\alpha_{i-1} + h\alpha_{i+1}\right) \Big\} = 1$$

$$+ \left.\frac{\partial^2 f}{\partial x^2}\right|_i\left(\frac{4h^2}{2}\alpha_{i-2} + \frac{h^2}{2}\alpha_{i-1} + \frac{h^2}{2}\alpha_{i+1}\right) \Big\} = 0$$

$$+ \left.\frac{\partial^3 f}{\partial x^3}\right|_i\left(\frac{-8h^3}{6}\alpha_{i-2} - \frac{h^3}{6}\alpha_{i-1} + \frac{h^3}{6}\alpha_{i+1}\right) \Big\} = 0$$

$$+ \left.\frac{\partial^4 f}{\partial x^4}\right|_i\left(\frac{16h^4}{24}\alpha_{i-2} + \frac{h^4}{24}\alpha_{i-1} + \frac{h^4}{24}\alpha_{i+1}\right) + \dots$$

$$\alpha_{i-2} + \alpha_{i-1} + \alpha_i + \alpha_{i+1} = 0 \quad (1)$$

$$h(-2\alpha_{i-2} - \alpha_{i-1} + \alpha_{i+1}) = 1 \quad (2)$$

$$\frac{h^2}{2}(4\alpha_{i-2} + \alpha_{i-1} + \alpha_{i+1}) = 0 \quad (3)$$

$$\frac{h^3}{6}(-8\alpha_{i-2} - \alpha_{i-1} + \alpha_{i+1}) = 0 \quad (4)$$

$\frac{h^2}{6}(1) - (4)$:

$$\frac{h^{\cancel{3}}}{\cancel{6}}(-2\alpha_{i-2}) - \frac{h^3}{\cancel{6}}(-8\alpha_{i-2}) = \frac{h^2}{\cancel{6}}$$

$$h\alpha_{i-2}(-2+8) = 1$$

$$\alpha_{i-2} = \frac{1}{6h}$$

$\frac{h}{3}(3) + (4)$:

$$\frac{h^3}{6}\left(\frac{4}{6h} + \alpha_{i+1} - \frac{8}{6h} + \alpha_{i+1}\right) = 0$$

$$2\alpha_{i+1} = \frac{\cancel{4}^{2}}{\cancel{6}_{3}h} = \frac{1}{3h}$$

(3):

$$\frac{4}{6h} + \alpha_{i-1} + \frac{1}{3h} = 0$$

$$\alpha_{i-1} = \frac{-1}{3h} - \frac{2}{3h} = \frac{-1}{h}$$

(1):

$$\frac{1}{6h} + \left(-\frac{1}{h}\right) + \alpha_i + \left(\frac{1}{3h}\right) = 0$$

$$\alpha_i = \frac{6-1-2}{6h} = \frac{3}{6h} = \frac{1}{2h}$$

Leading term:

$$\frac{h^4}{24}\left(16\frac{1}{6h} + \left(-\frac{1}{h}\right) + \left(\frac{1}{3h}\right)\right)\left.\frac{\partial^4 f}{\partial x^4}\right|_i$$

$$\frac{h^{\cancel{4}3}}{\cancel{24}_2}\left(\frac{\cancel{12}}{6\cancel{h}}\right)\left.\frac{\partial^4 f}{\partial x^4}\right|_i = \frac{h^3}{12}\left.\frac{\partial^4 f}{\partial x^4}\right|_i$$

$$\boxed{\left.\frac{\partial f}{\partial x}\right|_i = \underbrace{\frac{f_{i-2} - 6f_{i-1} + 3f_i + 2f_{i+1}}{6h}}_{\text{first derivative}} + \underbrace{\frac{h^3}{12}\left.\frac{\partial^4 f}{\partial x^4}\right|_i}_{\text{leading truncation error}}}$$

b)

S_1 $\quad f_i' \approx \frac{f_i - f_{i-1}}{\Delta x} \qquad f_{i-1} = e^{Ik(x_i - \Delta x)}$

$$\frac{f_i - f_{i-1}}{\Delta x} = \frac{1}{\Delta x}\left\{ e^{Ikx_i} - e^{Ik(x_i - \Delta x)} \right\}$$

$$= \frac{1}{\Delta x} e^{Ikx_i}\left\{ 1 - e^{-Ik\Delta x} \right\}$$

$$= \frac{1}{\Delta x} e^{Ikx_i}\left\{ 1 - (\cos(k\Delta x) - I\sin(k\Delta x)) \right\}$$

$$= \frac{1}{\Delta x} e^{Ikx_i}\left[(1 - \cos(k\Delta x)) + I\sin(k\Delta x) \right]$$

$$= I e^{Ikx_i}\left[\frac{\sin(k\Delta x) - I(1 - \cos(k\Delta x))}{\Delta x} \right]$$

$$\boxed{k^* = \frac{\sin(k\Delta x) - I(1 - \cos(k\Delta x))}{\Delta x}}$$

S_2 $\quad f_i' \approx \frac{f_{i+1} - f_{i-1}}{2\Delta x} \qquad f_{i+1} = e^{Ik(x_i + \Delta x)}$

$\qquad f_{i-1} = e^{Ik(x_i - \Delta x)}$

$$\frac{f_{i+1} - f_{i-1}}{2\Delta x} = \frac{1}{2\Delta x}\left\{ e^{Ik(x_i + \Delta x)} - e^{Ik(x_i - \Delta x)} \right\}$$

$$= \frac{1}{2\Delta x} e^{Ikx_i}\left\{ e^{Ik\Delta x} - e^{-Ik\Delta x} \right\}$$

$$= \frac{1}{2\Delta x} e^{Ikx_i}\left\{ \left[\cancel{\cos(k\Delta x)} + I\sin(k\Delta x) \right] - \left[\cancel{\cos(k\Delta x)} - I\sin(k\Delta x) \right] \right\}$$

$$= \frac{1}{\cancel{2}\Delta x} e^{Ikx_i}\, \cancel{2} I \sin(k\Delta x)$$

$$= I e^{Ikx_i}\frac{\sin(k\Delta x)}{\Delta x}$$

$$\boxed{k^* = \frac{\sin(k\Delta x)}{\Delta x}}$$

S_3

$$f_i' \approx \frac{f_{i-2} - 6f_{i-1} + 3f_i + 2f_{i+1}}{6\Delta x}$$

$$= \frac{1}{6\Delta x}\left\{ e^{Ik(x_i - 2\Delta x)} - 6e^{Ik(x_i - \Delta x)} + 3e^{Ikx_i} + 2e^{Ik(x_i + \Delta x)} \right\}$$

$$= \frac{e^{Ikx_i}}{6\Delta x}\left\{ e^{-2Ik\Delta x} - 6e^{-Ik\Delta x} + 3 + 2e^{Ik\Delta x} \right\}$$

$$= \frac{e^{Ikx_i}}{6\Delta x}\left\{ \begin{array}{l} \cos(2k\Delta x) - 6\cos(k\Delta x) \\ - I\sin(2k\Delta x) + 6I\sin(k\Delta x) \end{array} + 3 + \begin{array}{l} 2\cos(k\Delta x) \\ 2I\sin(k\Delta x) \end{array} \right\}$$

$$= \frac{e^{Ikx_i}}{6\Delta x}\left\{ \begin{array}{l} \cos(2k\Delta x) - 4\cos(k\Delta x) + 3 \\ + I(-\sin(2k\Delta x) + 8\sin(k\Delta x)) \end{array} \right\}$$

$$= I e^{Ikx_i}\left\{ \frac{(-\sin(2k\Delta x) + 8\sin(k\Delta x)) + I(4\cos(k\Delta x) - \cos(2k\Delta x) - 3)}{6\Delta x} \right\}$$

$$\boxed{k^* = \frac{(-\sin(2k\Delta x) + 8\sin(k\Delta x))}{6\Delta x} + I\,\frac{4\cos(k\Delta x) - \cos(2k\Delta x) - 3}{6\Delta x}}$$

S4 $f_i' \approx \dfrac{-f_{i+2}+8f_{i+1}-8f_{i-1}+f_{i-2}}{12\Delta x}$

$f_{i+2} = e^{Ik(x_i+2\Delta x)}$ $f_{i-2} = e^{Ik(x_i-2\Delta x)}$

$f_{i+1} = e^{Ik(x_i+\Delta x)}$ $f_{i-1} = e^{Ik(x_i-\Delta x)}$

$$f_i' \approx \frac{1}{12\Delta x}\Big\{-1\left(e^{Ik(x_i+2\Delta x)}\right)+8\left(e^{Ik(x_i+\Delta x)}\right)$$

$$-8\left(e^{Ik(x_i-\Delta x)}\right)+1\left(e^{Ik(x_i+2\Delta x)}\right)\Big\}$$

$$\approx \frac{1}{12\Delta x}e^{Ikx_i}\left\{-e^{2Ik\Delta x}+8e^{Ik\Delta x}-8^{-Ik\Delta x}+e^{-2Ik\Delta x}\right\}$$

$$= \frac{e^{Ikx_i}}{12\Delta x}\left\{\begin{matrix}-\cancel{\cos(2k\Delta x)}+\cancel{8\cos(k\Delta x)}-\cancel{8\cos(k\Delta x)}+\cancel{\cos(k\Delta x)} \\ -I\sin(2k\Delta x)+8I\sin(k\Delta x)+8I\sin(k\Delta x)-I\sin(2k\Delta x)\end{matrix}\right\}$$

$$= \frac{e^{Ikx_i}}{12\Delta x}\left\{-2I\sin(2k\Delta x)+16I\sin(k\Delta x)\right\}$$

$$= Ie^{Ikx_i}\left\{\frac{-\cancel{2}\sin(2k\Delta x)+\cancel{16}^{\,8}\sin(k\Delta x)}{\cancel{12}_{\,6}\Delta x}\right\}$$

$$\boxed{k^* = \frac{-\sin(2k\Delta x)+8\sin(k\Delta x)}{6\Delta x}}$$

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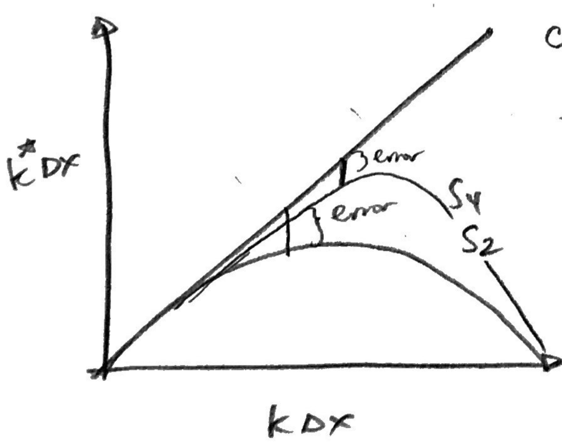
c) $\dfrac{\partial f}{\partial t}+a\dfrac{\partial f}{\partial x}=0$

looking at S2 & S4



Choose DX such that error = 10%

S2 : $k^*\Delta x = \left(\dfrac{\sin(k\Delta x)}{\Delta x}\right)(\Delta x)$

$= \sin(k\Delta x)$

error% = $\dfrac{k\Delta x - k^*\Delta x}{k\Delta x}$ let $\star = k\Delta x$

$= \dfrac{\star-\sin(\star)}{\star} = 10\%$

do graphically for LHS and solve for $\star$

$\star \approx 0.787 = k\Delta x$

$$\boxed{\Delta x = \frac{0.787}{k}}$$

S4: $k^*\Delta x = \dfrac{-\sin(2k\Delta x)+8\sin(k\Delta x)}{6}$

error% = $\dfrac{\star+(\sin(2\star)-8\sin(\star))/6}{\star} = 10\%$

$\star \approx 1.395 = k\Delta x$

$$\boxed{\Delta x = \frac{1.395}{k}}$$

2) one dimensional Poisson equation

$$\frac{d^2\phi}{dx^2} = f(x)$$

2nd order discretization

$$\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{(\Delta x)^2} = f_i$$

$$\left[\phi_{i+1} = \phi_i + \Delta x\,\phi'\Big|_i + \frac{(\Delta x)^2}{2}\phi''\Big|_i + \frac{(\Delta x)^3}{6}\phi'''\Big|_i + \frac{(\Delta x)^4}{24}\phi''''\Big|_i + \frac{(\Delta x)^5}{120}\phi'''''\Big|_i + \frac{(\Delta x)^6}{720}\phi''''''\Big|_i + \dots\right]1$$

$$[\phi_i = \phi_i](-2)$$

$$\left[\phi_{i-1} = \phi_i - \Delta x\,\phi'\Big|_i + \frac{(\Delta x)^2}{2}\phi''\Big|_i - \frac{(\Delta x)^3}{6}\phi'''\Big|_i + \frac{(\Delta x)^4}{24}\phi''''\Big|_i + \frac{(\Delta x)^5}{120}\phi'''''\Big|_i + \frac{(\Delta x)^6}{720}\phi''''''\Big|_i + \dots\right]1$$

Add together

$$\frac{\cancel{2}\frac{(\Delta x)^2}{\cancel{2}}\phi''\Big|_i + \cancel{2}\frac{(\Delta x)^4}{24_{12}}\phi''''\Big|_i + 2\frac{(\Delta x)^6}{720}\phi''''''\Big|_i + \dots}{(\Delta x)^2} = f_i$$

$$\boxed{\phi''\Big|_i + \frac{(\Delta x)^2}{12}\phi''''\Big|_i + \frac{(\Delta x)^4}{360}\phi''''''\Big|_i + \dots = f_i}$$

$$\phi''\Big|_i \approx \alpha_{i-1}\phi_{i-1} + \alpha_i\phi_i + \alpha_{i+1}\phi_{i+1}$$

$$\left(\phi_{i-1} = \phi_i - h\frac{d\phi}{dx}\Big|_i + \frac{h^2}{2}\frac{d^2\phi}{dx^2}\Big|_i - \frac{h^3}{6}\frac{d^3\phi}{dx^3}\Big|_i + \dots\right)\alpha_{i-1}$$

$$(\phi_i = \phi_i)\alpha_i$$

$$\left(\phi_{i+1} = \phi_i + h\frac{d\phi}{dx}\Big|_i + \frac{h^2}{2}\frac{d^2\phi}{dx^2} + \frac{h^3}{6}\frac{d^3\phi}{dx^3}\Big|_i + \dots\right)\alpha_{i+1}$$

$$\phi_i(\alpha_{i-1} + \alpha_i + \alpha_{i+1})\ \} = 0 \quad (1)$$

$$+ \frac{d\phi}{dx}\Big|_i h(-\alpha_{i-1} + \alpha_{i+1})\ \} = 0 \quad (2)$$

$$+ \frac{d^2\phi}{dx^2}\Big|_i \frac{h^2}{2}(\alpha_{i-1} + \alpha_{i+1})\ \} = 1 \quad (3)$$

$$+ \dots$$

$$\frac{h}{2}(2) + (3) \Rightarrow 2\alpha_{i+1} = \frac{2}{h^2} \Rightarrow \alpha_{i+1} = \frac{1}{h^2}$$

$$(2) \Rightarrow \alpha_{i-1} = \frac{1}{h^2} \quad (1) \Rightarrow \alpha_i = \frac{-2}{h^2}$$

$$\phi''\Big|_i \approx \frac{\phi_{i-1} - 2\phi_i + \phi_{i+1}}{h^2}$$

looking at the original ode

$$\phi''\Big|_i = f_i$$

then $\phi''''\Big|_i = f_i''$

$$f_i'' \approx \alpha_{i-1}f_{i-1} + \alpha_i f_i + \alpha_{i+1}f_{i+1}$$

solve for α_{i-1}, α_i & α_{i+1}

similar process to earlier

$$f_i'' \approx \frac{f_{i-1} - 2f_i + f_{i+1}}{h^2}$$

plug all into the modified equation

$$\frac{\phi_{i-1} - 2\phi_i + \phi_{i+1}}{(\Delta x)^2} + \frac{\cancel{(\Delta x)^2}}{12}\left(\frac{f_{i-1} - 2f_i + f_{i+1}}{\cancel{(\Delta x)^2}}\right) + \frac{(\Delta x)^6}{360}\phi''''''\Big|_i + \dots = f_i$$

Can't deal with ϕ'''''' even at f'''' cuz of the stencil

$$\boxed{\frac{\phi_{i-1} - 2\phi_i + \phi_{i+1}}{(\Delta x)^2} + \frac{(\Delta x)^6}{360}\phi''''''\Big|_i + \dots = f_i + \frac{f_{i-1} - 2f_i + f_{i+1}}{12}}$$