AEM 8202 Fluid Mechanics II Spring 2023

Problem Set #1
Due Fri., Feb. 10 by 11:59pm

Please hand in your completed problem sets as PDF files to the course Canvas site. Working together with others is fine (and encouraged), but you must write up each solution yourself.

1. Show that $G = \frac{\ln r}{2\pi}$ is the free-space Green's function for the 2D Laplace equation $\nabla^2\psi = 0$ that satisfies

$$\nabla^2 G = \delta(y - x),$$

where $r^2 = (y_i - x_i)(y_i - x_i)$.

2. Using this Green's function, apply the Boundary Element Method (BEM) discussed in lecture to compute the streamfunction corresponding uniform flow past a circular cylinder of radius 1. Far away from the cylinder, the flow behaves as uniform flow with unit velocity so that $\psi \longrightarrow y$ as $r \longrightarrow \infty$. The surface of the cylinder is a streamline with $\psi = 0$. To satisfy both of these boundary conditions, the total streamfunction can be thought of as a superposition $\psi = \psi_0 + \psi_1$ where $\psi_0 = y$ is uniform flow and $\psi_1$is an unknown streamfunction that decays to zero in the far field. At the surface of the cylinder, $\psi_1 = -\psi_0$ enforces the boundary condition $\psi = 0$.

Implement the BEM using the collocation (matrix) method discussed in lecture in your favorite software (e.g., matlab). This will involve discretizing the surface of the cylinder into N panels, and then forming two $N \times N$ matrices A and B that approximate

$$\frac{1}{2}\psi_1(y) = \oint \psi_1 \frac{\partial G}{\partial n} - G\frac{\partial \psi_1}{\partial n} ds,$$

or rather,

$$-\frac{1}{2}\psi_1(y) + \oint \psi_1 \frac{\partial G}{\partial n} ds = \oint G\frac{\partial \psi_1}{\partial n} ds,$$

at each panel. Use these matrices to solve for $\frac{\partial \psi_1}{\partial n}$ at each panel. Make a plot of both ψ_1 and $\frac{\partial \psi_1}{\partial n}$ as functions of the angle θ along the surface of the cylinder.

3. Now that you know both ψ_1 and $\frac{\partial \psi_1}{\partial n}$, use

$$\psi_1(y) = \oint \psi_1 \frac{\partial G}{\partial n} - G\frac{\partial \psi_1}{\partial n} ds,$$

to evaluate ψ_1 everywhere in the flow outside the cylinder. Hint: Since the number N of BEM panels is small but the number of observer locations is large, it is more efficient in matlab to treat the observer locations as "matrices" and then perform the above integral using "elementwise" operations. This allows matlab to vectorize over similar operations happening at every point in space while the outer loop is over the number of panels N. For example:

```
[X,Y] = ndgrid(-2:0.05:2, -2:0.05:2);
PSI = 0*X; % allocates memory for PSI and zeros it initially
for i1 = 1:N
      Rx = x_panel(i1) - X;
      Ry = y_panel(i1) - Y;
      Rmag = sqrt(Rx.^2 + Ry.^2);
      …
      PSI = PSI + …
end
contour(X,Y,PSI,100);
```

Note that Rx, Ry, Rmag and PSI are all matrices of the same size as X and Y. Statements such as "Rx = x_panel(i1) – X" are automatically vectorized over their elements. As a rule, avoiding multiple nested loops and using vector operations is key to efficient code in matlab (but if you need extreme speed, use Fortran or C).

Once you have computed ψ_1, add it to ψ_0, and then plot contours of ψ to visualize the flow around the cylinder. Repeat the calculations with $N = 100$ and $N = 200$ and plot contours. How does your solution change as N increases?

4. Plot streamlines of flow around a circular cylinder obtained using a complex potential function. Compare to the streamlines found in question 3. As N increases, does the BEM solution converge to that computed using potential flow theory?

5. An advantage of BEM is that it can handle arbitrary shapes. Modify your code slightly to apply BEM to uniform flow past a 4:1 ellipse (an ellipse with major axis four times longer that its minor axis). You may choose the orientation of the ellipse as you like. Create a plot of streamlines for your chosen case.

Hint: In addition to modifying the positions of each panel and normal vectors at each panel, it is likely that the lengths of panels will no longer be uniform in the case of the ellipse.