Justine John A Serdoncillo

AEM 8202 - Fluid Mechanics II

Feb 10, 2023

Problem Set #1

1. Show that $G = \frac{\ln r}{2\pi}$ is the free-space Green's function for the 2D Laplace equation $\nabla^2 \psi = 0$ that satisfies $\nabla^2 G = \delta(y - x)$ where $r^2 = (y_i - x_i)(y_i - x_i)$

to prove G is the Green's function then need to show that:

① $\nabla^2 G = 0$ for all r except $r \neq 0$

② $\lim\limits_{R \to 0} \iint_A \nabla^2 G \, d\underset{\sim}{x} = 1$

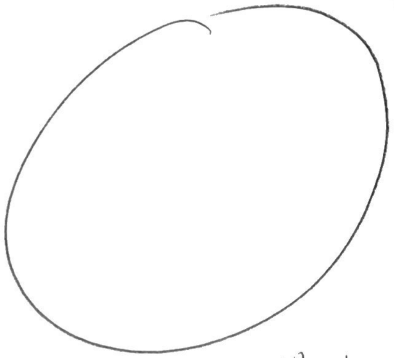
① $\nabla^2 G =$ polar laplacian $\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial G}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\left(\frac{\ln r}{2\pi}\right)\right) = \frac{1}{r}\frac{d}{dr}\left(\cancel{r}\left(\frac{1}{2\pi \cancel{r}}\right)\right) = \frac{1}{r}\frac{d}{dr}\left(\frac{1}{2\pi}\right) = \frac{1}{r}(0) = 0$ ✓✓

since $r \neq 0$

② $\lim\limits_{R \to 0} \iint_A \nabla^2 G \, d\underset{\sim}{x} = \lim\limits_{R \to 0} \oint_s \nabla G \cdot \hat{n} \, dS = \lim\limits_{R \to 0} \pi R^2 \left.\frac{dG}{dr}\right|_{r=R} = \lim\limits_{R \to 0} \cancel{2\pi R} \frac{1}{\cancel{2\pi R}} = 1$ ✓✓

2.

$\psi = 0$



ψ_2 @ surface $= -\psi_1$ to get $\psi = 0$

$$\frac{1}{2}\psi_1(y) = \oint \psi_1 \frac{\partial G}{\partial n} - G \frac{\partial \psi}{\partial n} \, ds$$

$$-\frac{1}{2}\psi_1(y) + \oint \psi_1 \frac{\partial G}{\partial n} ds = \oint G \frac{\partial \psi_1}{\partial n} ds$$

$$\begin{bmatrix} -0.5 & \frac{\partial G_{12}}{\partial n_2} dA_2 & \cdots \\ \frac{\partial G_{21}}{\partial n_1} dA_1 & -0.5 & \cdots \\ \vdots & & \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & G_{12} & G_{13} & \cdots \\ G_{21} & 0 & G_{23} & \cdots \\ \vdots & \vdots & & \end{bmatrix} \begin{bmatrix} \psi_1' \\ \psi_2' \\ \psi_3' \\ \vdots \end{bmatrix}$$

$$\underline{\underline{A}} \quad \underline{\psi} = \underline{\underline{B}} \, \underline{\psi}'$$

$$\boxed{\underline{\psi}' = \underline{\underline{B}}^{-1} \underline{\underline{A}} \, \underline{\psi}}$$

$$\frac{\partial G}{\partial n} dA = \nabla G \hat{r} \cdot \hat{n} \, dA$$

$$= \frac{\partial}{\partial r}\left(\frac{\ln r}{2\pi}\right) \hat{r} \cdot \hat{n} \, dA$$

$$= \frac{1}{2\pi r} \hat{r} \cdot \hat{n} \, dA$$