Justine John A. Serdoncillo

AEM 4253/5253 Homework 2 November 1 2022

Computational Fluid Dynamics

1) a) $\frac{\partial f}{\partial x}\Big|_i$ using $\{i-2, i-1, i, i+1\}$

$$\frac{\partial f}{\partial x}\Big|_i \approx \alpha_{i-2} f_{i-2} + \alpha_{i-1} f_{i-1} + \alpha_i f_i + \alpha_{i+1} f_{i+1}$$

Taylor series expand about i

$$f_{i-2} = f_i + (-2h)\frac{\partial f}{\partial x}\Big|_i + \frac{(-2h)^2}{2!}\frac{\partial^2 f}{\partial x^2}\Big|_i + \frac{(-2h)^3}{3!}\frac{\partial^3 f}{\partial x^3}\Big|_i + \frac{(-2h)^4}{4!}\frac{\partial^4 f}{\partial x^4}\Big|_i + \dots$$

$$f_{i-1} = f_i + (-h)\frac{\partial f}{\partial x}\Big|_i + \frac{(-h)^2}{2!}\frac{\partial^2 f}{\partial x^2}\Big|_i + \frac{(-h)^3}{3!}\frac{\partial^3 f}{\partial x^3}\Big|_i + \frac{(-h)^4}{4!}\frac{\partial^4 f}{\partial x^4}\Big|_i + \dots$$

$$f_i = f_i$$

$$f_{i+1} = f_i + (h)\frac{\partial f}{\partial x}\Big|_i + \frac{(h)^2}{2!}\frac{\partial^2 f}{\partial x^2}\Big|_i + \frac{(h)^3}{3!}\frac{\partial^3 f}{\partial x^3}\Big|_i + \frac{(h)^4}{4!}\frac{\partial^4 f}{\partial x^4}\Big|_i + \dots$$

$$\alpha_{i-2} f_{i-2} + \alpha_{i-1} f_{i-1} + \alpha_i f_i + \alpha_{i+1} f_{i+1}$$

$$= f_i\left(\alpha_{i-2} + \alpha_{i-1} + \alpha_i + \alpha_{i+1}\right) \} = 0$$

$$+ \frac{\partial f}{\partial x}\Big|_i \left(-2h\alpha_{i-2} - h\alpha_{i-1} + h\alpha_{i+1}\right) \} = 1$$

$$+ \frac{\partial^2 f}{\partial x^2}\Big|_i \left(\frac{4h^2}{2}\alpha_{i-2} + \frac{h^2}{2}\alpha_{i-1} + \frac{h^2}{2}\alpha_{i+1}\right) \} = 0$$

$$+ \frac{\partial^3 f}{\partial x^3}\Big|_i \left(\frac{-8h^3}{6}\alpha_{i-2} - \frac{h^3}{6}\alpha_{i-1} + \frac{h^3}{6}\alpha_{i+1}\right) \} = 0$$

$$+ \frac{\partial^4 f}{\partial x^4}\Big|_i \left(\frac{16h^4}{24}\alpha_{i-2} + \frac{h^4}{24}\alpha_{i-1} + \frac{h^4}{24}\alpha_{i+1}\right) + \dots$$

$$\alpha_{i-2} + \alpha_{i-1} + \alpha_i + \alpha_{i+1} = 0 \qquad (1)$$

$$h(-2\alpha_{i-2} - \alpha_{i-1} + \alpha_{i+1}) = 1 \qquad (2)$$

$$\frac{h^2}{2}(4\alpha_{i-2} + \alpha_{i-1} + \alpha_{i+1}) = 0 \qquad (3)$$

$$\frac{h^3}{6}(-8\alpha_{i-2} - \alpha_{i-1} + \alpha_{i+1}) = 0 \qquad (4)$$

$\frac{h^2}{6}(1) - (4)$:

$$\frac{h^3}{6}(-2\alpha_{i-2}) - \frac{h^3}{6}(-8\alpha_{i-2}) = \frac{h^2}{6}$$

$$h\alpha_{i-2}(-2+8) = 1$$

$$\alpha_{i-2} = \frac{1}{6h}$$

$\frac{h}{3}(3) + (4)$:

$$\frac{h^3}{6}\left(\frac{4}{6h} + \alpha_{i+1} - \frac{8}{6h} + \alpha_{i+1}\right) = 0$$

$$2\alpha_{i+1} = \frac{4}{6h} = \frac{1}{3h}$$

(3):

$$\frac{4}{6h} + \alpha_{i-1} + \frac{1}{3h} = 0$$

$$\alpha_{i-1} = \frac{-1}{3h} - \frac{2}{3h} = \frac{-1}{h}$$

(1):

$$\frac{1}{6h} + \left(-\frac{1}{h}\right) + \alpha_i + \left(\frac{1}{3h}\right) = 0$$

$$\alpha_i = \frac{6-1-2}{6h} = \frac{3}{6h} = \frac{1}{2h}$$

Leading term:

$$\frac{h^4}{24}\left(16\frac{1}{6h} + \left(-\frac{1}{h}\right) + \left(\frac{1}{3h}\right)\right)\frac{\partial^4 f}{\partial x^4}\Big|_i$$

$$\frac{h^3}{24}\left(\frac{12}{6}\right)\frac{\partial^4 f}{\partial x^4}\Big|_i = \frac{h^3}{12}\frac{\partial^4 f}{\partial x^4}\Big|_i$$

$$\boxed{\frac{\partial f}{\partial x}\Big|_i = \underbrace{\frac{f_{i-2} - 6f_{i-1} + 3f_i + 2f_{i+1}}{6h}}_{\text{first derivative}} + \underbrace{\frac{h^3}{12}\frac{\partial^4 f}{\partial x^4}\Big|_i}_{\text{leading truncation error}}}$$

b)

S_1 $\quad f_i' \approx \dfrac{f_i - f_{i-1}}{\Delta x} \qquad f_{i-1} = e^{Ik(x_i - \Delta x)}$

$$\frac{f_i - f_{i-1}}{\Delta x} = \frac{1}{\Delta x}\left\{ e^{Ikx_i} - e^{Ik(x_i - \Delta x)} \right\}$$

$$= \frac{1}{\Delta x} e^{Ikx_i} \left\{ 1 - e^{-Ik\Delta x} \right\}$$

$$= \frac{1}{\Delta x} e^{Ikx_i} \left\{ 1 - \left(\cos(k\Delta x) - I\sin(k\Delta x) \right) \right\}$$

$$= \frac{1}{\Delta x} e^{Ikx_i} \left[(1 - \cos(k\Delta x)) + I\sin(k\Delta x) \right]$$

$$= I e^{Ikx_i} \left[\frac{\sin(k\Delta x) - I(1 - \cos(k\Delta x))}{\Delta x} \right]$$

$$\boxed{k^* = \frac{\sin(k\Delta x) - I(1 - \cos(k\Delta x))}{\Delta x}}$$

S_2 $\quad f_i' \approx \dfrac{f_{i+1} - f_{i-1}}{2\Delta x} \qquad f_{i+1} = e^{Ik(x_i + \Delta x)}$

$f_{i-1} = e^{Ik(x_i - \Delta x)}$

$$\frac{f_{i+1} - f_{i-1}}{2\Delta x} = \frac{1}{2\Delta x}\left\{ e^{Ik(x_i + \Delta x)} - e^{Ik(x_i - \Delta x)} \right\}$$

$$= \frac{1}{2\Delta x} e^{Ikx_i} \left\{ e^{Ik\Delta x} - e^{-Ik\Delta x} \right\}$$

$$= \frac{1}{2\Delta x} e^{Ikx_i} \left\{ \left[\cancel{\cos(k\Delta x)} + I\sin(k\Delta x) \right] - \left[\cancel{\cos(k\Delta x)} - I\sin(k\Delta x) \right] \right\}$$

$$= \frac{1}{\cancel{2}\Delta x} e^{Ikx_i} \cancel{2} I \sin(k\Delta x)$$

$$= I e^{Ikx_i} \frac{\sin(k\Delta x)}{\Delta x}$$

$$\boxed{k^* = \frac{\sin(k\Delta x)}{\Delta x}}$$

S_3

$$f_i' \approx \frac{f_{i-2} - 6f_{i-1} + 3f_i + 2f_{i+1}}{6\Delta x}$$

$$= \frac{1}{6\Delta x}\left\{ e^{Ik(x_i - 2\Delta x)} - 6e^{Ik(x_i - \Delta x)} + 3e^{Ikx_i} + 2e^{Ik(x_i + \Delta x)} \right\}$$

$$= \frac{e^{Ikx_i}}{6\Delta x}\left\{ e^{-2Ik\Delta x} - 6e^{-Ik\Delta x} + 3 + 2e^{Ik\Delta x} \right\}$$

$$= \frac{e^{Ikx_i}}{6\Delta x}\left\{ \begin{matrix} \cos(2k\Delta x) - 6\cos(k\Delta x) \\ -I\sin(2k\Delta x) + 6I\sin(k\Delta x) \end{matrix} + 3 + \begin{matrix} 2\cos(k\Delta x) \\ 2I\sin(k\Delta x) \end{matrix} \right\}$$

$$= \frac{e^{Ikx_i}}{6\Delta x}\left\{ \begin{matrix} \cos(2k\Delta x) - 4\cos(k\Delta x) + 3 \\ + I(-\sin(2k\Delta x) + 8\sin(k\Delta x)) \end{matrix} \right\}$$

$$= I e^{Ikx_i} \left\{ \frac{\begin{matrix} (-\sin(2k\Delta x) + 8\sin(k\Delta x)) \\ + I(4\cos(k\Delta x) - \cos(2k\Delta x) - 3) \end{matrix}}{6\Delta x} \right\}$$

$$\boxed{k^* = \frac{(-\sin(2k\Delta x) + 8\sin(k\Delta x))}{6\Delta x} + I\,\frac{4\cos(k\Delta x) - \cos(2k\Delta x) - 3}{6\Delta x}}$$

S_4 $f_i' \approx \dfrac{-f_{i+2}+8f_{i+1}-8f_{i-1}+f_{i-2}}{12\Delta x}$

$f_{i+2}=e^{Ik(x_i+2\Delta x)}$ $\quad f_{i-2}=e^{Ik(x_i-2\Delta x)}$

$f_{i+1}=e^{Ik(x_i+\Delta x)}$ $\quad f_{i-1}=e^{Ik(x_i-\Delta x)}$

$$f_i' \approx \frac{1}{12\Delta x}\left\{-1\left(e^{Ik(x_i+2\Delta x)}\right)+8\left(e^{Ik(x_i+\Delta x)}\right)\right.$$

$$\left.-8\left(e^{Ik(x_i-\Delta x)}\right)+1\left(e^{Ik(x_i+2\Delta x)}\right)\right\}$$

$$\approx \frac{1}{12\Delta x}e^{Ikx_i}\left\{-e^{2Ik\Delta x}+8e^{Ik\Delta x}-8^{-Ik\Delta x}+e^{-2Ik\Delta x}\right\}$$

$$=\frac{e^{Ikx_i}}{12\Delta x}\left\{\begin{array}{l}-\cancel{\cos(2k\Delta x)}+\cancel{8\cos(k\Delta x)}-\cancel{8\cos(k\Delta x)}+\cancel{\cos(k\Delta x)}\\ -I\sin(2k\Delta x)+8I\sin(k\Delta x)+8I\sin(k\Delta x)-I\sin(2k\Delta x)\end{array}\right\}$$

$$=\frac{e^{Ikx_i}}{12\Delta x}\left\{-2I\sin(2k\Delta x)+16I\sin(k\Delta x)\right\}$$

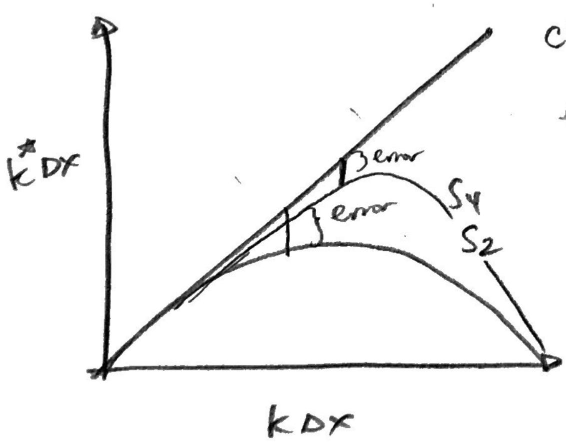
$$=Ie^{Ikx_i}\left\{\frac{-\cancel{2}\sin(2k\Delta x)+\cancel{16}^{\,8}\sin(k\Delta x)}{\cancel{12}_{\,6}\Delta x}\right\}$$

$$\boxed{k^*=\frac{-\sin(2k\Delta x)+8\sin(k\Delta x)}{6\Delta x}}$$

---

c) $\dfrac{\partial f}{\partial t}+a\dfrac{\partial f}{\partial x}=0$

looking at $S2$ & $S4$



Choose Δx such that error = 10%

$S2$: $k^*\Delta x=\left(\dfrac{\sin(k\Delta x)}{\Delta x}\right)(\Delta x)$

$=\sin(k\Delta x)$

error% $=\dfrac{k\Delta x-k^*\Delta x}{k\Delta x}$ $\quad$ let $\star=k\Delta x$

$=\dfrac{\star-\sin(\star)}{\star}=10\%$

do graphically for LHS and solve for $\star$

$\star\approx 0.787=k\Delta x$

$$\boxed{\Delta x=\frac{0.787}{k}}$$

$S4$: $k^*\Delta x=\dfrac{-\sin(2k\Delta x)+8\sin(k\Delta x)}{6}$

error% $=\dfrac{\star+(\sin(2\star)-8\sin(\star))/6}{\star}=10\%$

$\star\approx 1.395=k\Delta x$

$$\boxed{\Delta x=\frac{1.395}{k}}$$

2) one dimensional Poisson equation

$$\frac{d^2\phi}{dx^2} = f(x)$$

2nd order discretization

$$\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{(\Delta x)^2} = f_i$$

$$\left[\phi_{i+1} = \phi_i + \Delta x\,\phi'\big|_i + \frac{(\Delta x)^2}{2}\phi''\big|_i + \frac{(\Delta x)^3}{6}\phi'''\big|_i + \frac{(\Delta x)^4}{24}\phi''''\big|_i + \frac{(\Delta x)^5}{120}\phi'''''\big|_i + \frac{(\Delta x)^6}{720}\phi''''''\big|_i + \dots\right]1$$

$$[\phi_i = \phi_i](-2)$$

$$\left[\phi_{i-1} = \phi_i - \Delta x\,\phi'\big|_i + \frac{(\Delta x)^2}{2}\phi''\big|_i - \frac{(\Delta x)^3}{6}\phi'''\big|_i + \frac{(\Delta x)^4}{24}\phi''''\big|_i + \frac{(\Delta x)^5}{120}\phi'''''\big|_i + \frac{(\Delta x)^6}{720}\phi''''''\big|_i + \dots\right]1$$

Add together

$$\frac{\cancel{2}\frac{(\Delta x)^2}{\cancel{2}}\phi''\big|_i + \cancel{2}\frac{(\Delta x)^4}{\cancel{24}\,12}\phi''''\big|_i + 2\frac{(\Delta x)^6}{720}\phi''''''\big|_i + \dots}{(\Delta x)^2} = f_i$$

$$\boxed{\phi''\big|_i + \frac{(\Delta x)^2}{12}\phi''''\big|_i + \frac{(\Delta x)^4}{360}\phi''''''\big|_i + \dots = f_i}$$

$$\phi''\big|_i \approx \alpha_{i-1}\phi_{i-1} + \alpha_i\phi_i + \alpha_{i+1}\phi_{i+1}$$

$$\left(\phi_{i-1} = \phi_i - h\frac{d\phi}{dx}\Big|_i + \frac{h^2}{2}\frac{d^2\phi}{dx^2}\Big|_i - \frac{h^3}{6}\frac{d^3\phi}{dx^3}\Big|_i + \dots\right)\alpha_{i-1}$$

$$(\phi_i = \phi_i)\alpha_i$$

$$\left(\phi_{i+1} = \phi_i + h\frac{d\phi}{dx}\Big|_i + \frac{h^2}{2}\frac{d^2\phi}{dx^2} + \frac{h^3}{6}\frac{d^3\phi}{dx^3}\Big|_i + \dots\right)\alpha_{i+1}$$

$$\phi_i(\alpha_{i-1} + \alpha_i + \alpha_{i+1})\ \} = 0 \quad (1)$$

$$+ \frac{d\phi}{dx}\Big|_i h(-\alpha_{i-1} + \alpha_{i+1})\ \} = 0 \quad (2)$$

$$+ \frac{d^2\phi}{dx^2}\Big|_i \frac{h^2}{2}(\alpha_{i-1} + \alpha_{i+1})\ \} = 1 \quad (3)$$

$$+ \dots$$

$$\frac{h}{2}(2)+(3) \Rightarrow 2\alpha_{i+1} = \frac{2}{h^2} \Rightarrow \alpha_{i+1} = \frac{1}{h^2}$$

$$(2) \Rightarrow \alpha_{i-1} = \frac{1}{h^2} \qquad (1) \Rightarrow \alpha_i = \frac{-2}{h^2}$$

$$\phi''\big|_i \approx \frac{\phi_{i-1} - 2\phi_i + \phi_{i+1}}{h^2}$$

looking at the original ode

$$\phi''\big|_i = f_i$$

then $\phi''''\big|_i = f_i''$

$$f_i'' \approx \alpha_{i-1}f_{i-1} + \alpha_i f_i + \alpha_{i+1}f_{i+1}$$

solve for α_{i-1}, α_i & α_{i+1}

similar process to earlier

$$f_i'' \approx \frac{f_{i-1} - 2f_i + f_{i+1}}{h^2}$$

plug all into the modified equation

$$\frac{\phi_{i-1} - 2\phi_i + \phi_{i+1}}{(\Delta x)^2} + \frac{\cancel{(\Delta x)^2}}{12}\left(\frac{f_{i-1} - 2f_i + f_{i+1}}{\cancel{(\Delta x)^2}}\right)$$

$$+ \frac{(\Delta x)^6}{360}\phi''''''\big|_i + \dots = f_i$$

Can't deal with ϕ'''''' even at f'''' cuz of the stencil

$$\boxed{\frac{\phi_{i-1} - 2\phi_i + \phi_{i+1}}{(\Delta x)^2} + \frac{(\Delta x)^6}{360}\phi''''''\big|_i + \dots = f_i + \frac{f_{i-1} - 2f_i + f_{i+1}}{12}}$$

Homework 2

Number 1 b)

The figures below are the plots of the real and imaginary parts of the modified wave number.

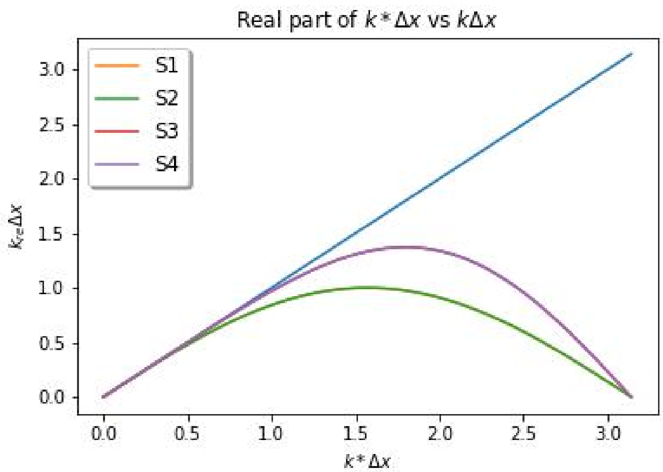


Figure 1: Real part of the modified wave number

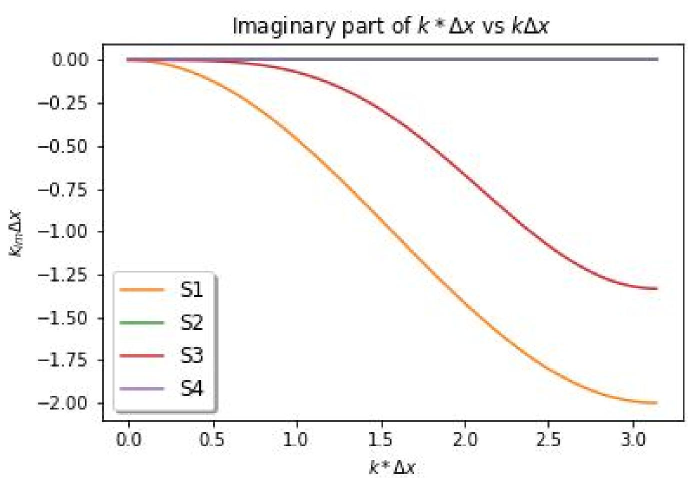


Figure 2: Imaginary part of the modified wave number

Number 1 c)

The figure below plots the percent error as the wave number is changed. It can be seen that the intersection values correspond to the limit wave number to get the desired accuracy. These numbers correspond to around 0.787 and 1.395 for the S2 and S4 schemes respectively.

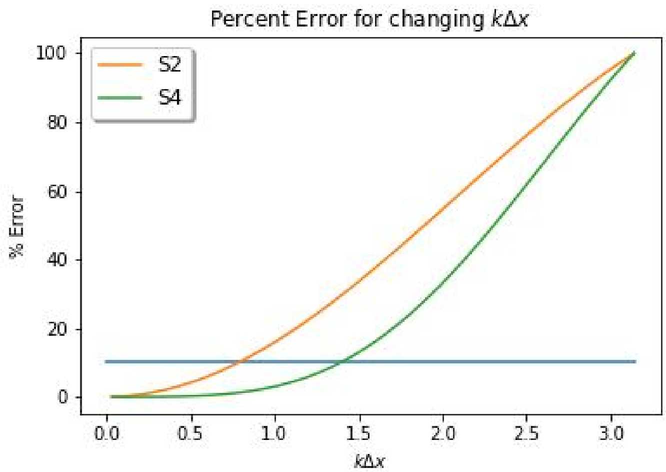


Figure 3: Percent Error of the modified wave number

Number 3 a)

The figures below are the plots of the advection equation with the Gaussian function and square pulse equation as the two different initial conditions. The spatial schemes used were the ones in number 1 while the time stepping scheme used is Runge-Kutta 3rd order.

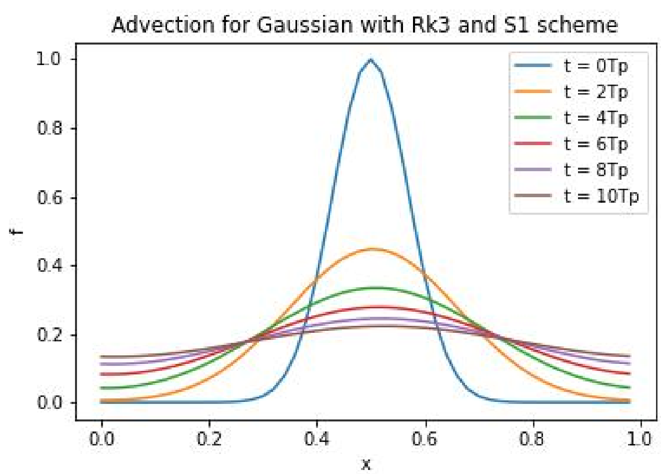


Figure 4: Gaussian advection using Rk3 and S1

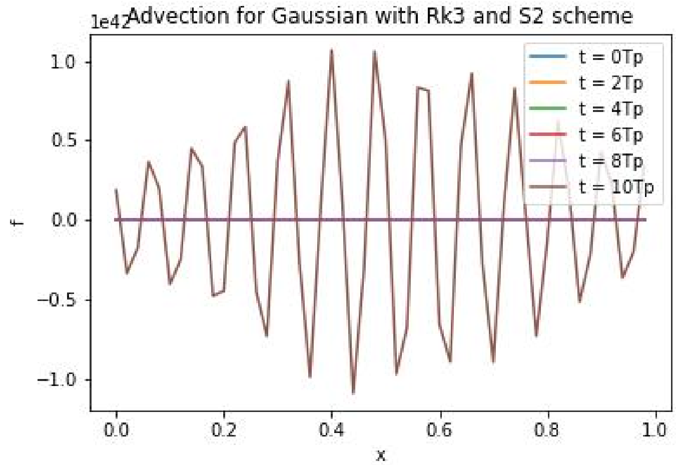


Figure 5: Gaussian advection using Rk3 and S2

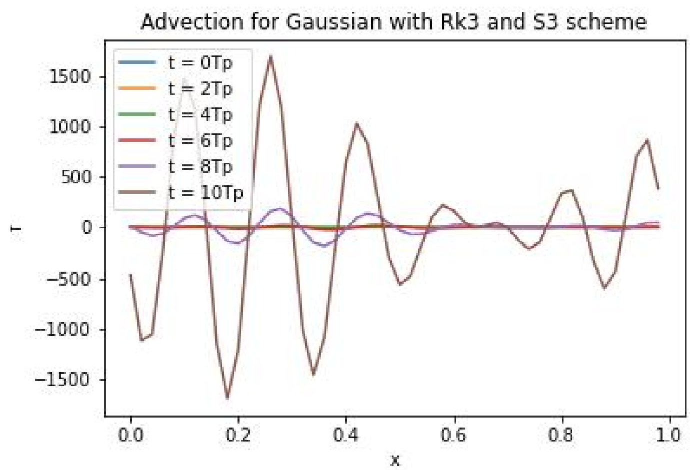


Figure 6: Gaussian advection using Rk3 and S3

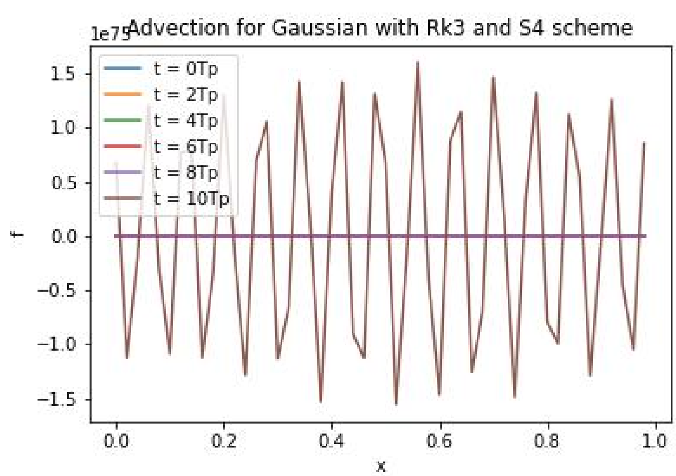


Figure 7: Gaussian advection using Rk3 and S4

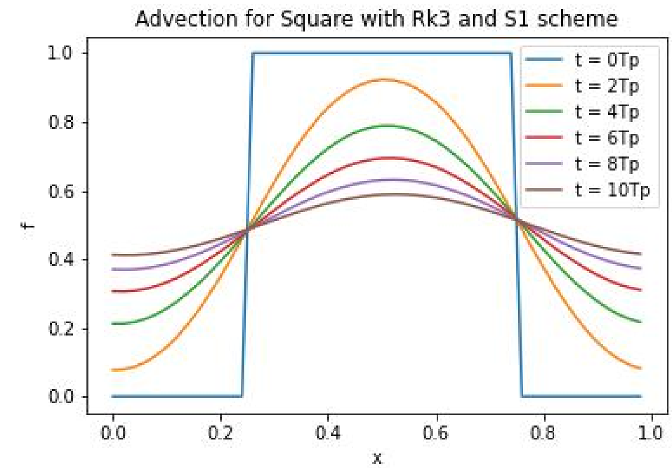


Figure 8: Square Pulse advection using Rk3 and S1

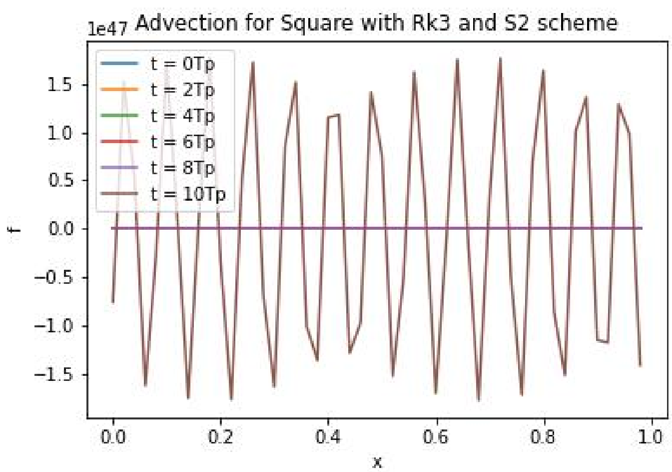


Figure 9: Square Pulse advection using Rk3 and S2

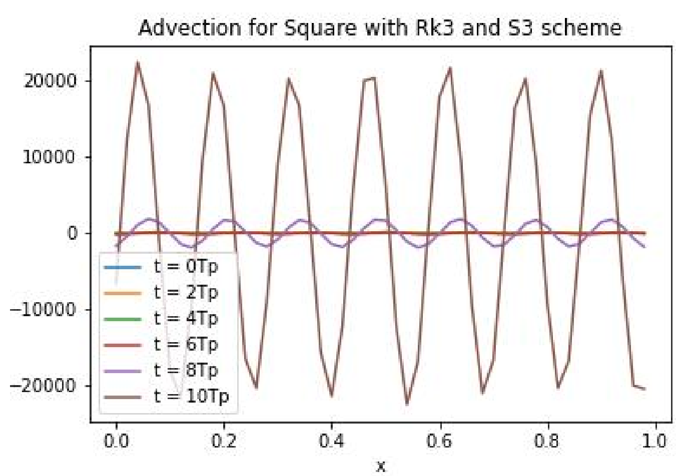


Figure 10: Square Pulse advection using Rk3 and S3

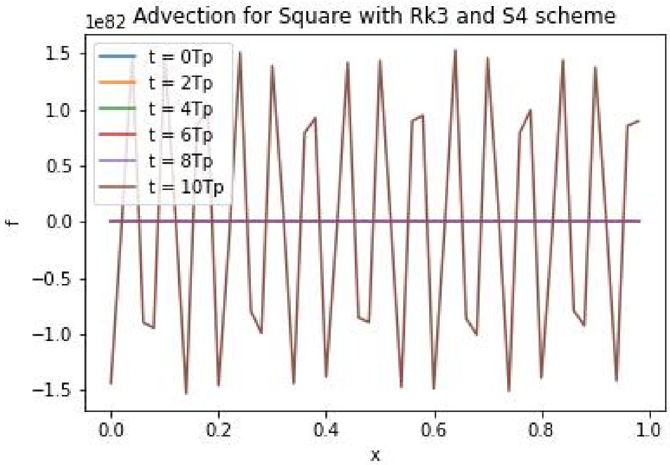


Figure 11: Square Pulse advection using Rk3 and S4

Number 3 b)

From the figures above, it can be seen that most of the plots have a pretty random oscillating solution at the higher time steps. However, the S1 schemes produced clear plots but the amplitude decreases as time increases. This is as shown in class as dissipation in the S1 schemes.
Another notable thing can be seen is through the S3 scheme because the oscillation is non-uniform and there is a change in size of the amplitude as well. By only plotting the first few Tp, we find the figure below. It can be seen that there are now waves beside the main wave. This is explained in class as dispersion and the increase in amplitude can be attributed to diffusion. The stencil is biased towards the left which explains why the dispersion is asymmetric.

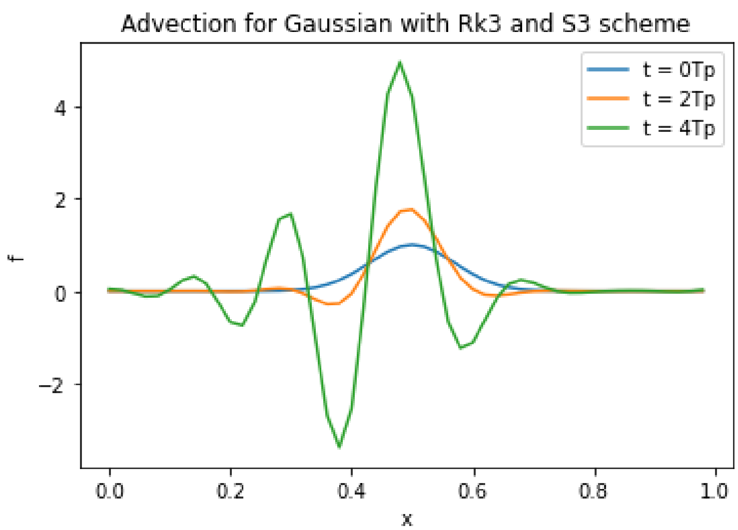


Figure 12: Gaussian showing clear Dispersion

Lastly, S2 and S4 are both central order schemes which in class discussion is not ideal because the eigenvalues are all imaginary and adds some positive diffusion. This is clearly shown as the plots look pretty random with the large oscillations.