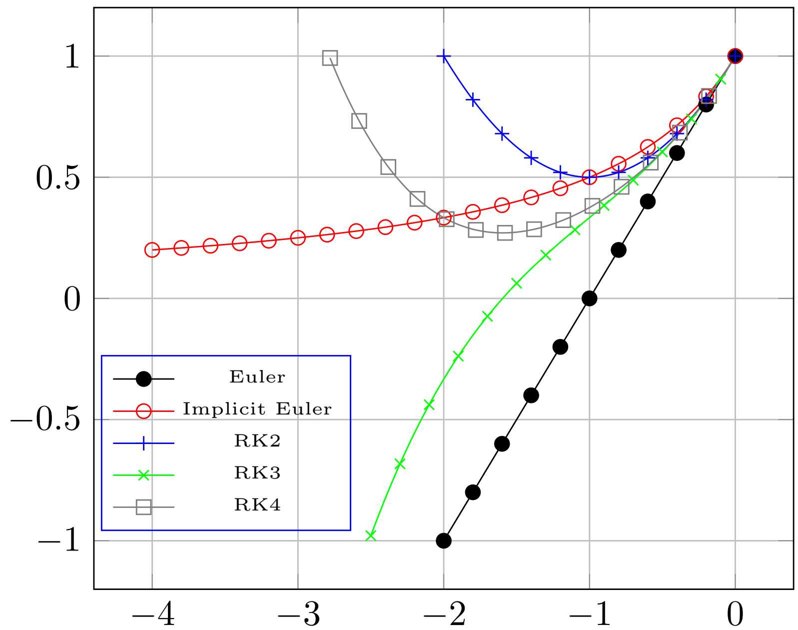
HW1 Solutions

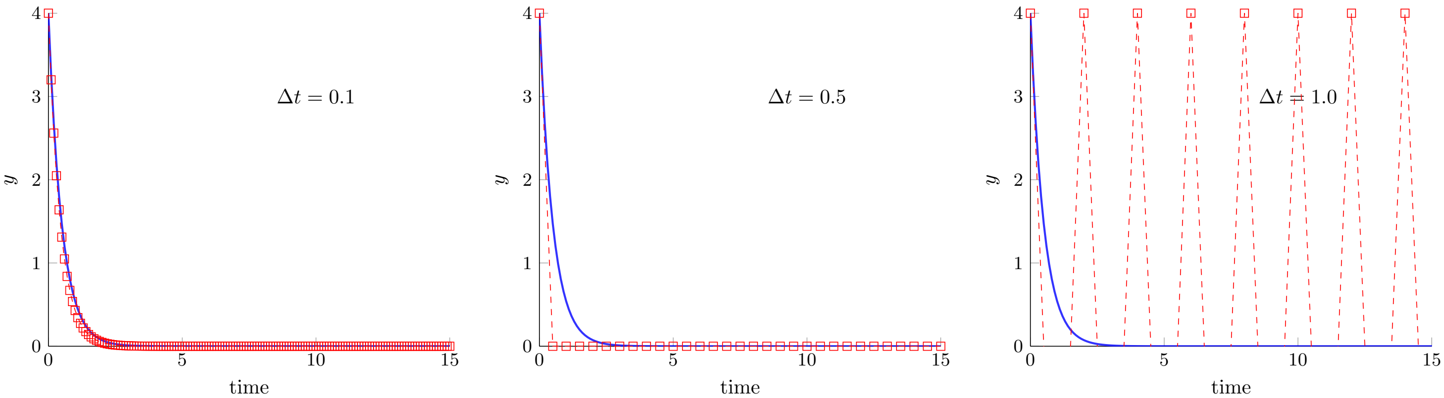
1.

a.) Plots: see next page.

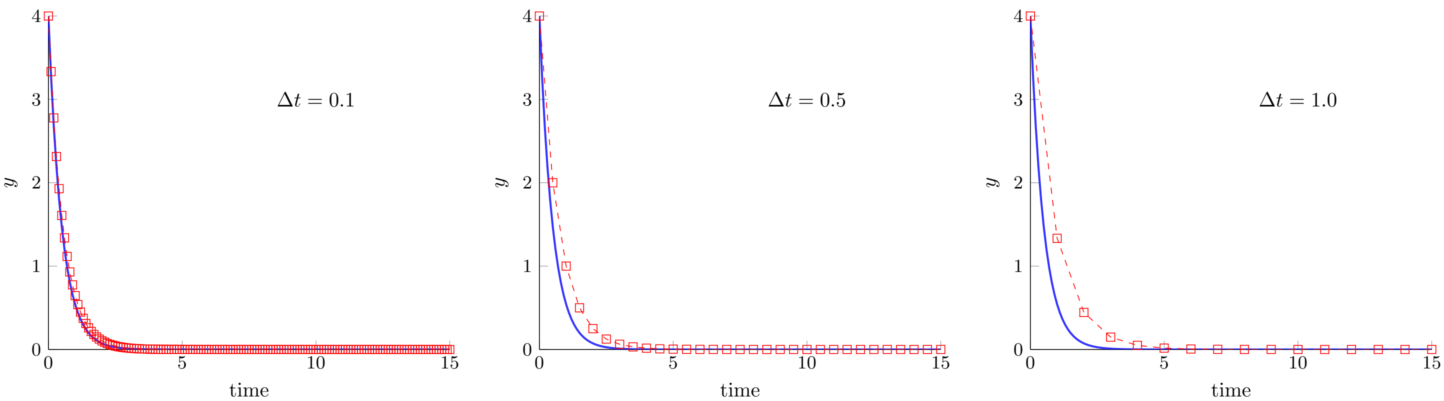
b.) Say each time step is represented by $y_{k+1} = \sigma y_k$. Clearly, if σ is negative, the solution will oscillate in sign ($y_1 = \sigma y_0,\ y_2 = \sigma^2 y_0, y_3 = \sigma^3 y_0 ...$). This could happen even though the solution is stable by our criterion ($|\sigma| < 1$). For this problem, since σ is real, we can easily find if there is a region where sign(σ) is negative and $|\sigma| < 1$ by simply plotting σ: this is shown in the figure below for several schemes.



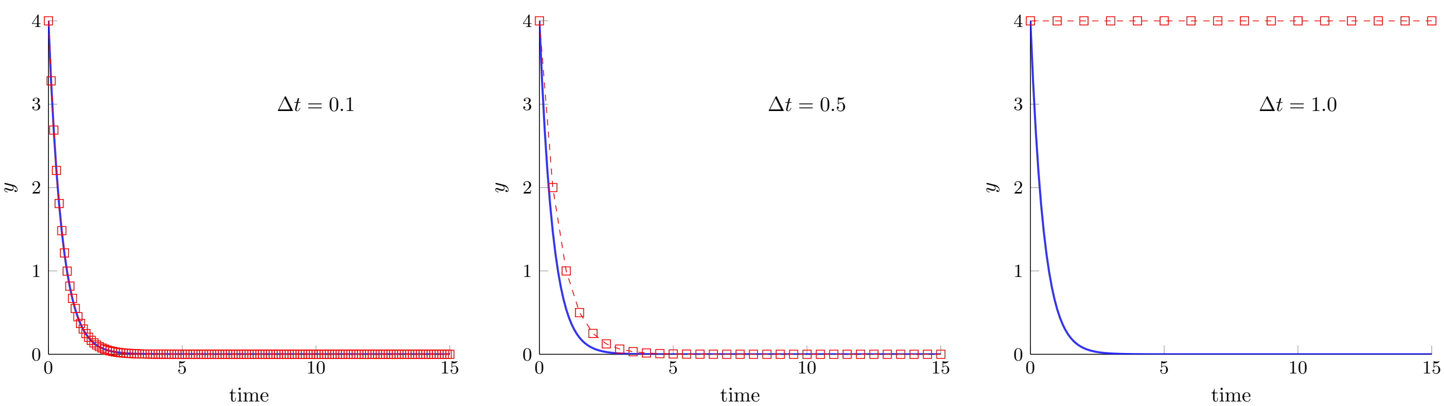
The Euler and RK3 schemes have a stable region where σ is negative. For the Euler method $|c_r|\Delta t \geq 1$ and for the RK3 method $|c_r|\Delta t \geq 1.59607$ determines this region.



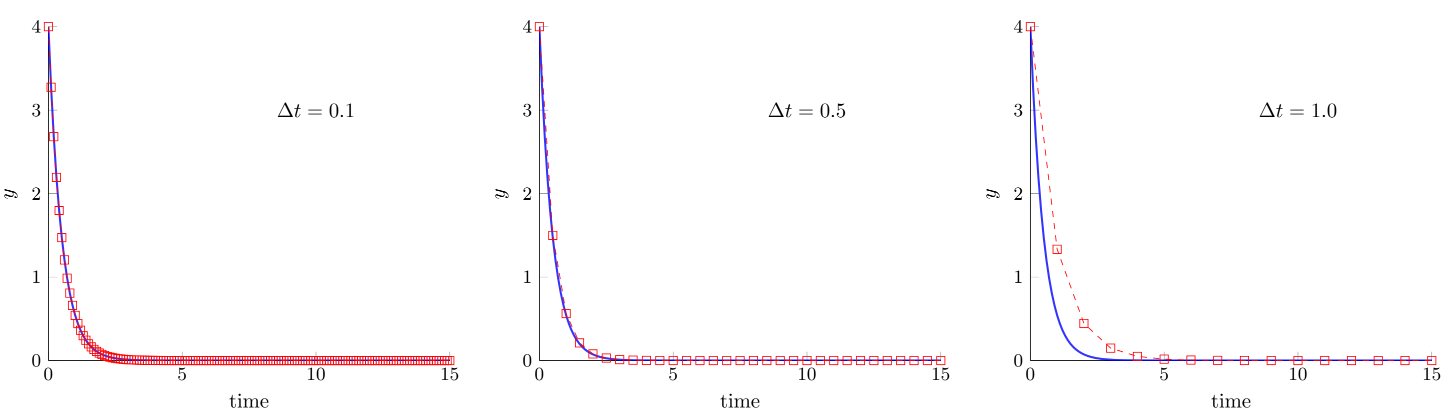
Problem 1a: Euler



Problem 1a: Implicit Euler



Problem 1a: RK2



Problem 1a: RK4

2.

a.) We can form a crude estimate by linearizing the right hand side (call it $g(y,t)$) around some point (y_p, t_p):

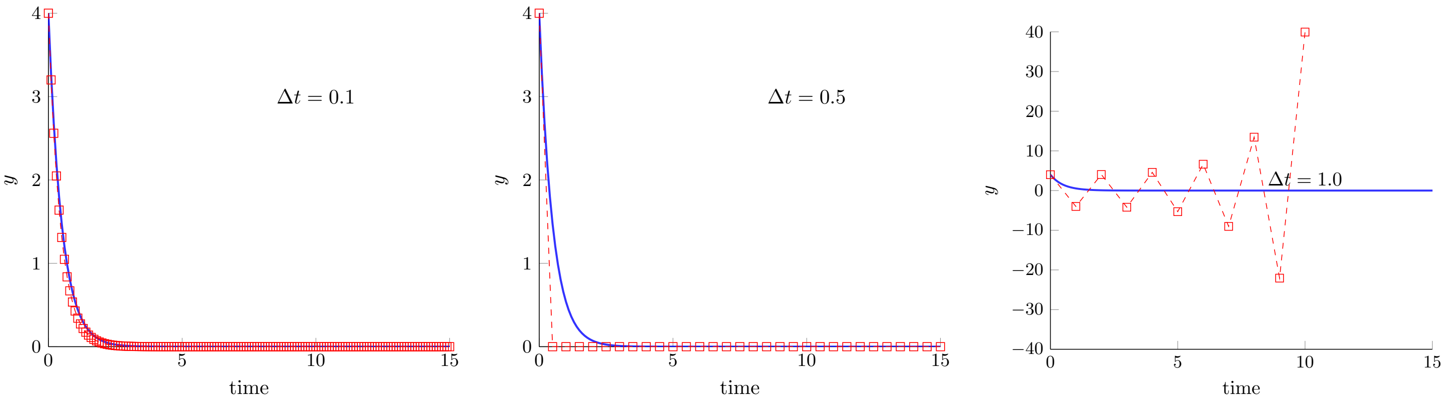
$$g(y,t) = g(y_p, t_p) + \left.\frac{\partial g}{\partial y}\right|_p (y - y_p) + \left.\frac{\partial g}{\partial t}\right|_p (t - t_p) + \cdots$$

Isolate the exponentially growing part:

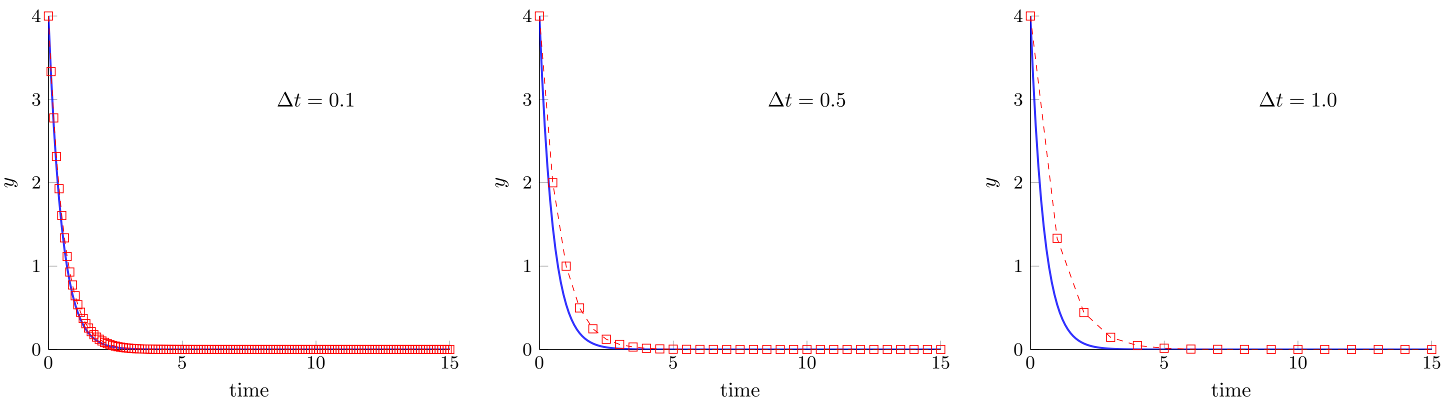
$$\left.\frac{\partial g}{\partial y}\right|_p y = -(2 + 0.01 t_p^2) y.$$

This looks like the model equation with $c = -(2 + 0.01 t_p^2)$. The maximum absolute value of c is at $t = 15$ for this problem, which gives $c_{max} = -4.25$. Use this to estimate the maximum time step. Euler and RK2: $\frac{2}{4.25} \approx 0.47$. RK4: $\frac{2.79}{4.25} \approx 0.656$. Implicit Euler should stay stable.

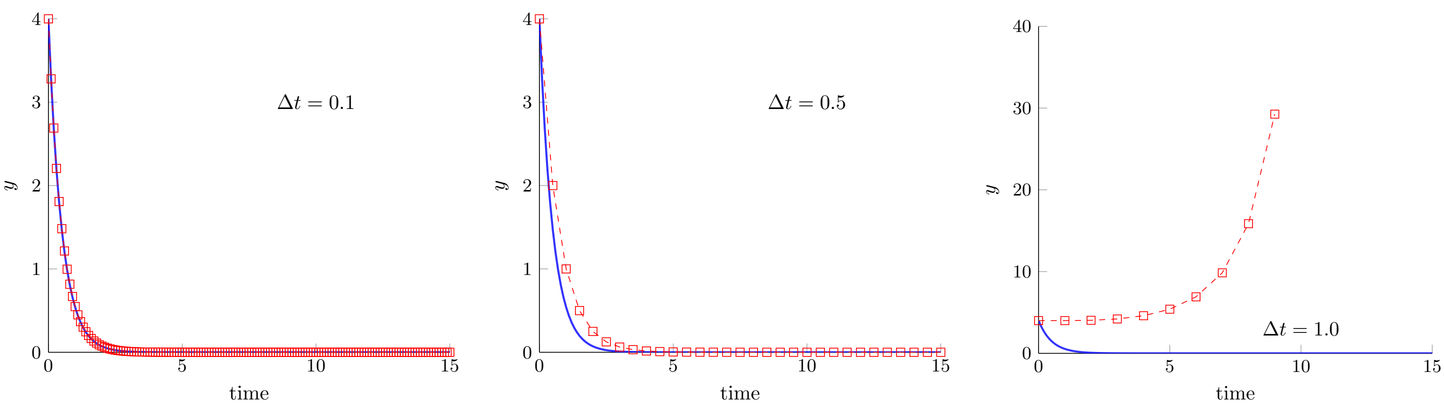
b.) Plots: see next page



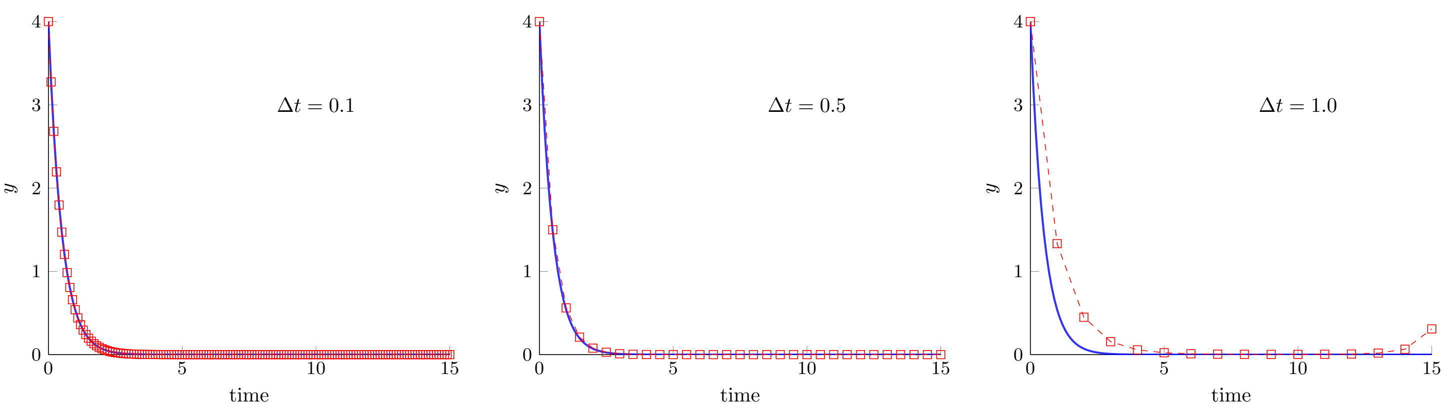
Problem 2b: Euler



Problem 2b: Implicit Euler



Problem 2b: RK2



Problem 2b: RK4

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3.

a.) The exact solution is

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{l}}t\right)$$

Let $\omega = d\theta/dt$. The system is

$$\frac{d}{dt}\begin{pmatrix}\theta\\ \omega\end{pmatrix} = \begin{pmatrix}0 & 1\\ -g/l & 0\end{pmatrix}\begin{pmatrix}\theta\\ \omega\end{pmatrix}$$

The eigenvalues of the system are given by $\lambda^2 + g/l = 0 \Rightarrow \lambda = \pm\imath\sqrt{g/l}$. Note $|\lambda| = 4.04351$. For stability, we should choose methods whose stability diagrams include some part of the imaginary axis (Euler, RK2, AB2 are expected to be unstable). RK4 stability requires $\Delta t \leq\approx \frac{2.83}{4.04351} = 0.699$.

b.) The exact solution is

$$\theta(t) = \theta_0 e^{-ct/2}\left(\cos\alpha t + \beta\sin\alpha t\right),$$

where $\alpha = \sqrt{4g/l - c^2}/2$ and $\beta = c/\sqrt{4g/l - c^2}$.

The system is now

$$\frac{d}{dt}\begin{pmatrix}\theta\\ \omega\end{pmatrix} = \begin{pmatrix}0 & 1\\ -g/l & -c\end{pmatrix}\begin{pmatrix}\theta\\ \omega\end{pmatrix},$$

with eigenvalues given by

$$\lambda = \lambda_r + \imath\lambda_i = \frac{-c \pm \sqrt{c^2 - 4g/l}}{2} = -2 \pm 3.51426\imath$$

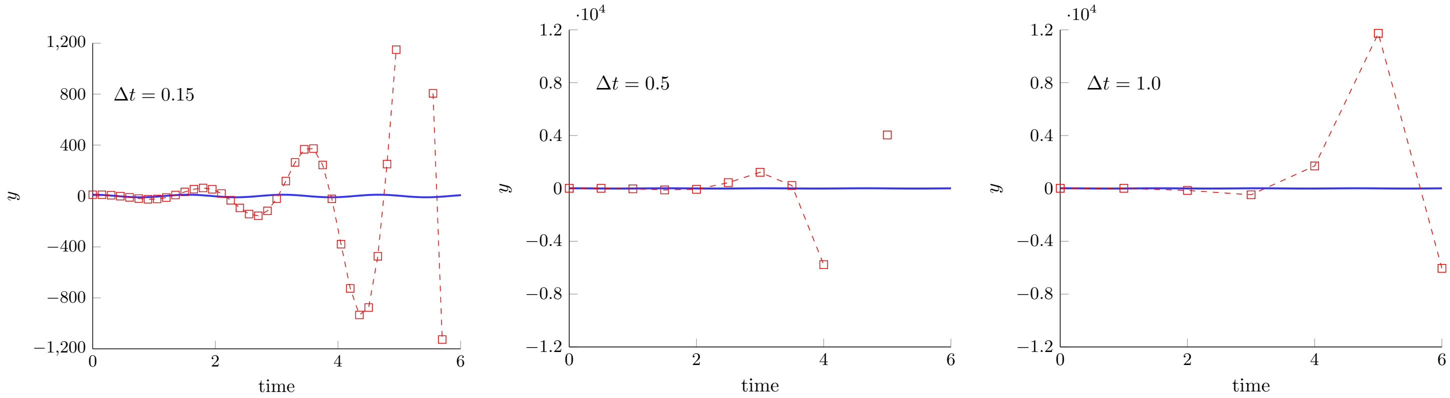
Stability for Euler requires $|1 + \lambda\Delta t| \leq 1$. Substitute for λ from above to find the condition:

$$\Delta t \leq \frac{2|\lambda_r|}{|\lambda|^2} \Rightarrow \Delta t \leq 0.2446$$

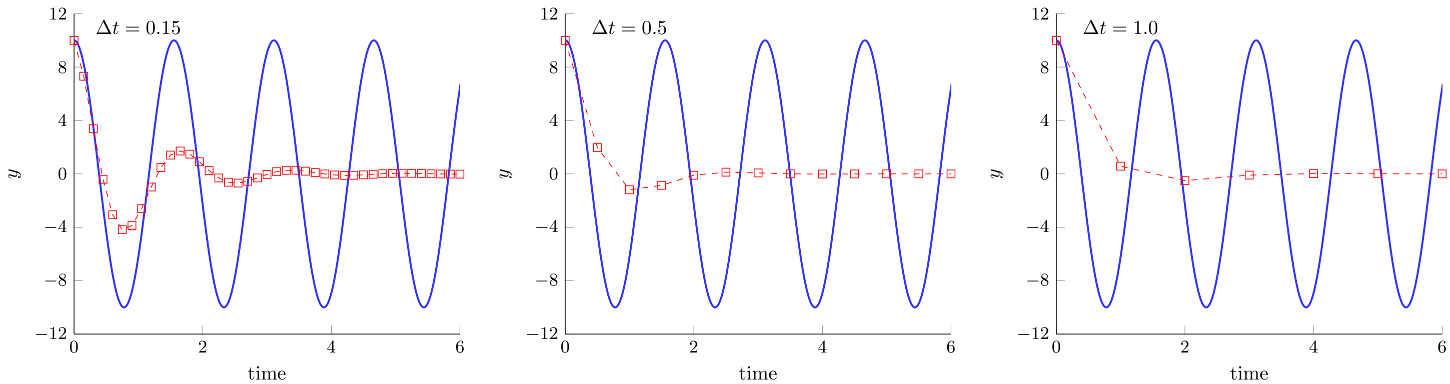
The RK4 stable time step is given by the (real, non-zero) solution of

$$|1 + \lambda\Delta t + \frac{(\lambda\Delta t)^2}{2} + \frac{(\lambda\Delta t)^3}{6} + \frac{(\lambda\Delta t)^4}{24}| \leq 1,$$

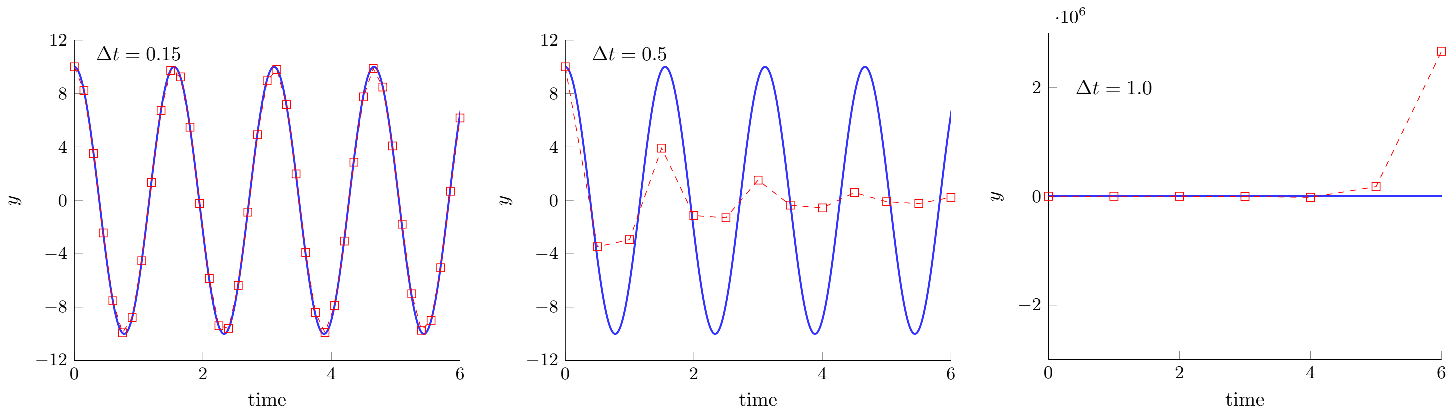
(solving this is challenging: software like Mathematica will help). The solution is $\Delta t \leq 0.64901$.



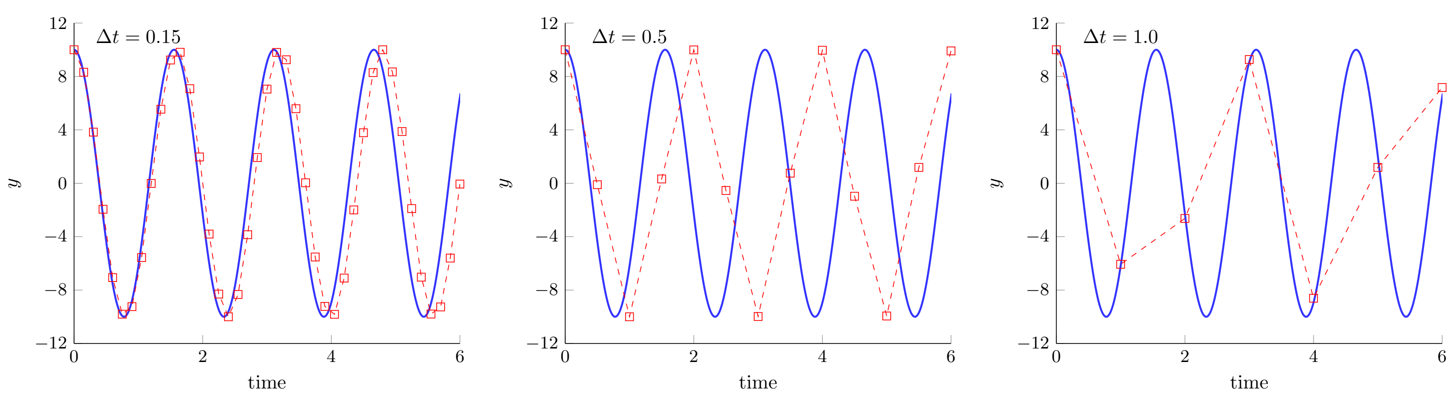
Problem 3a: Euler



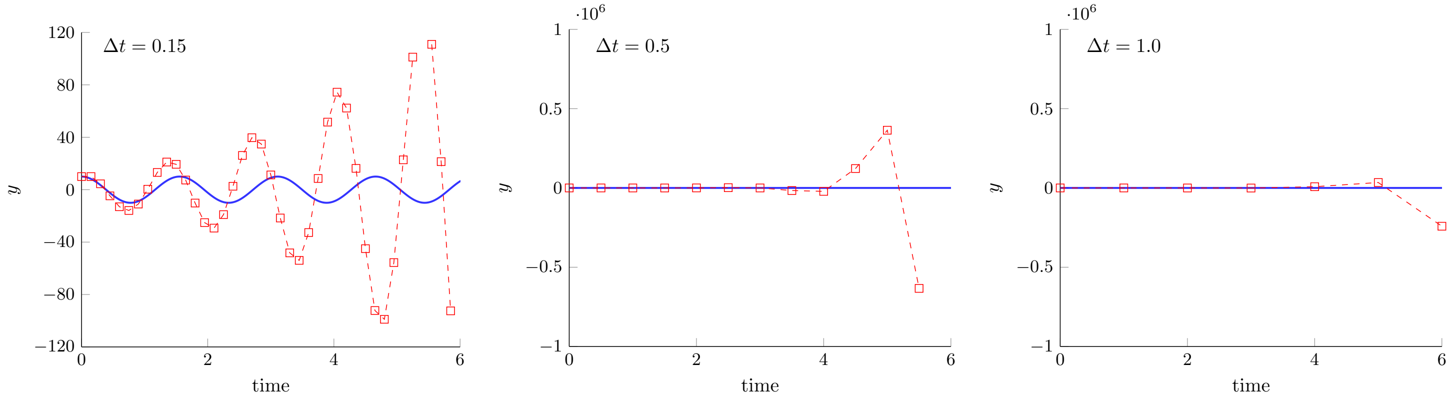
Problem 3a: Implicit Euler



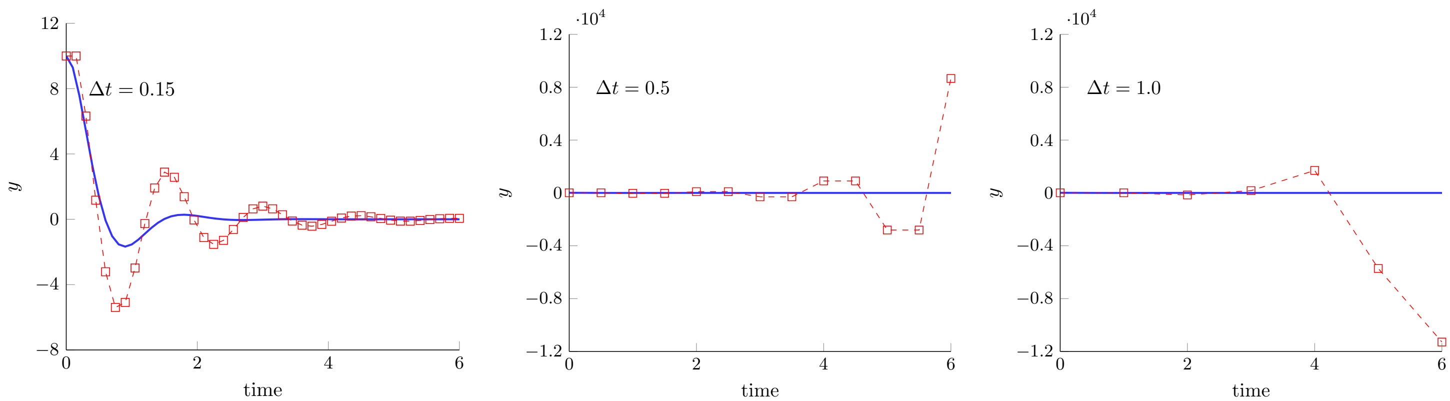
Problem 3a: RK4



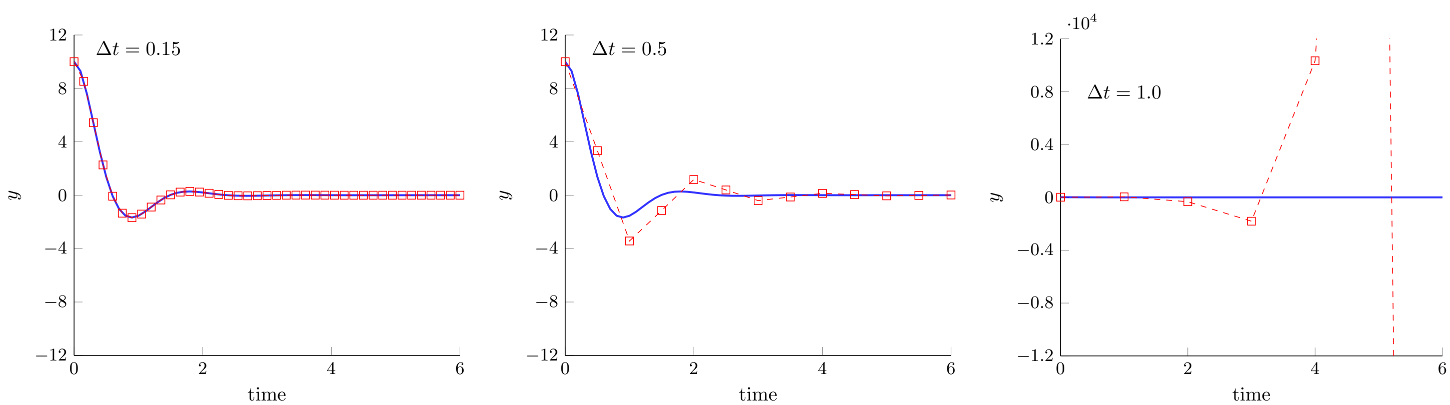
Problem 3a: Crank-Nicolson



Problem 3a: AB-2 using Explicit Euler as first step

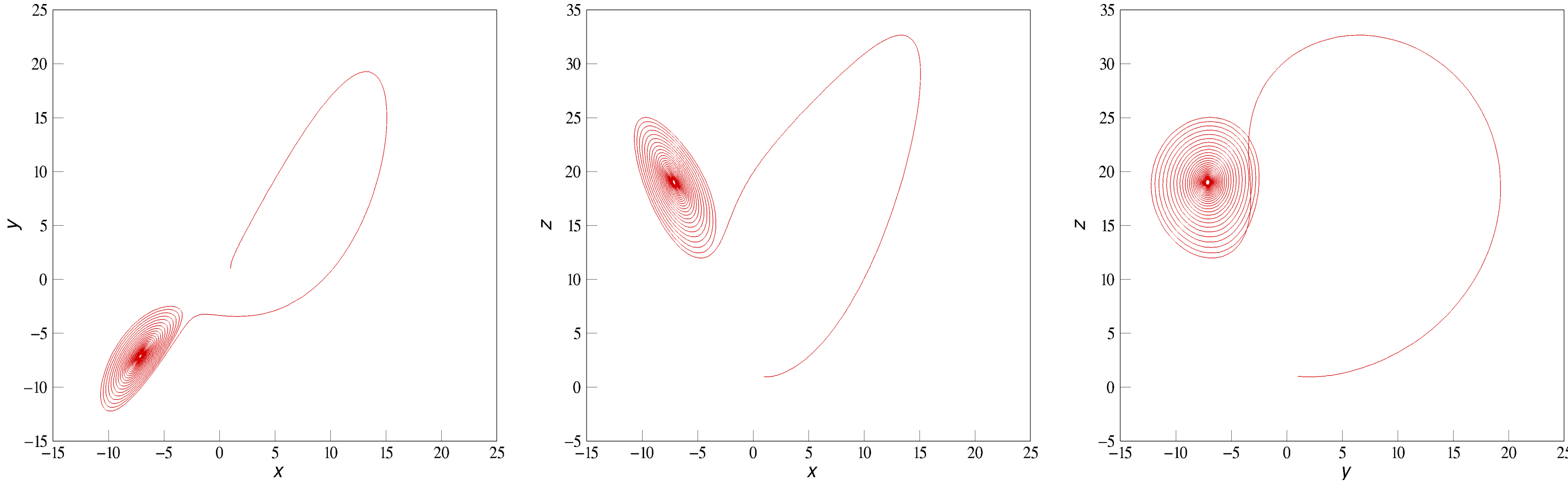


Problem 3b: Euler

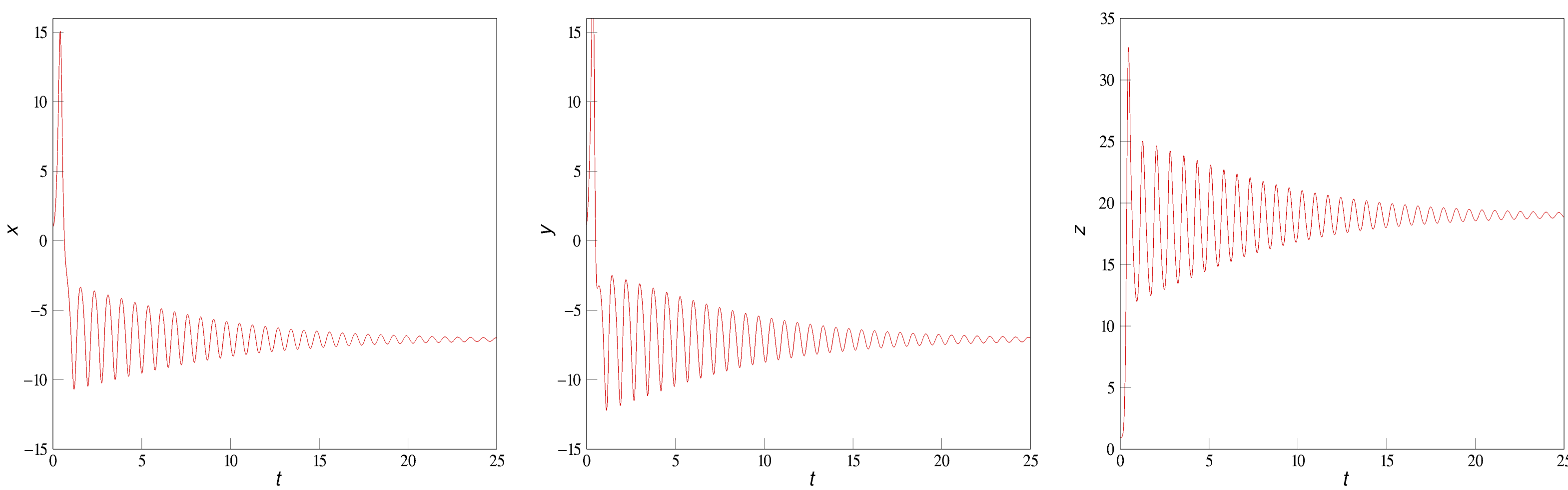


Problem 3b: RK4

PROBLEM 4a

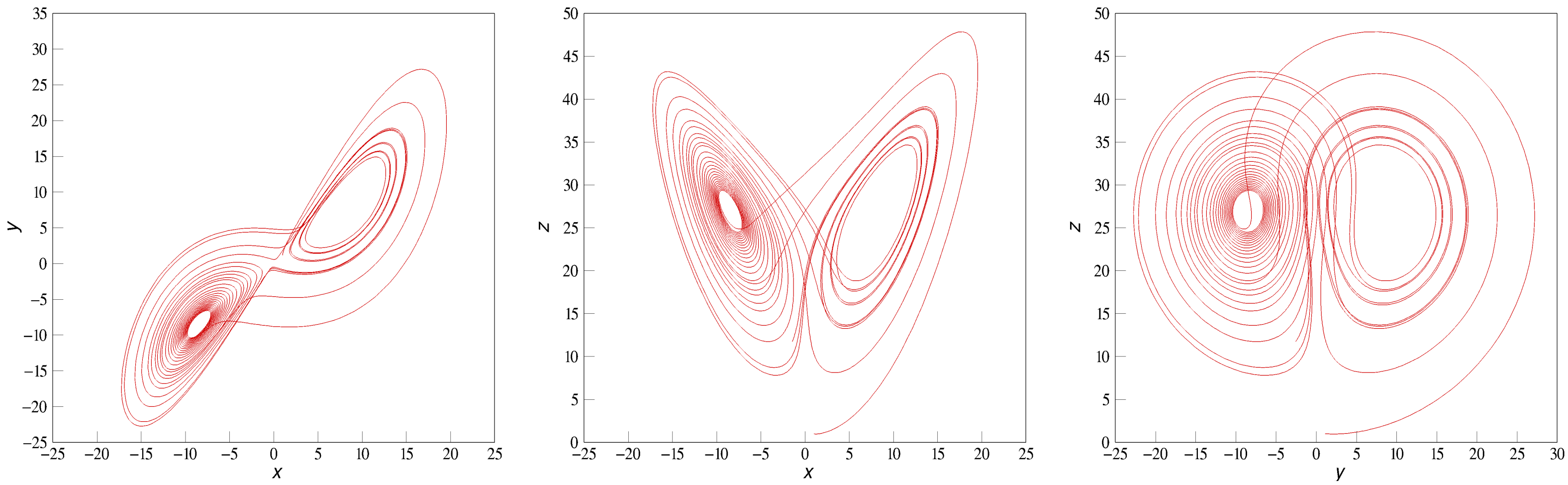


$x-y$, $x-z$ and $y-z$ plots with initial condition $(1,1,1)$ and $r=20$

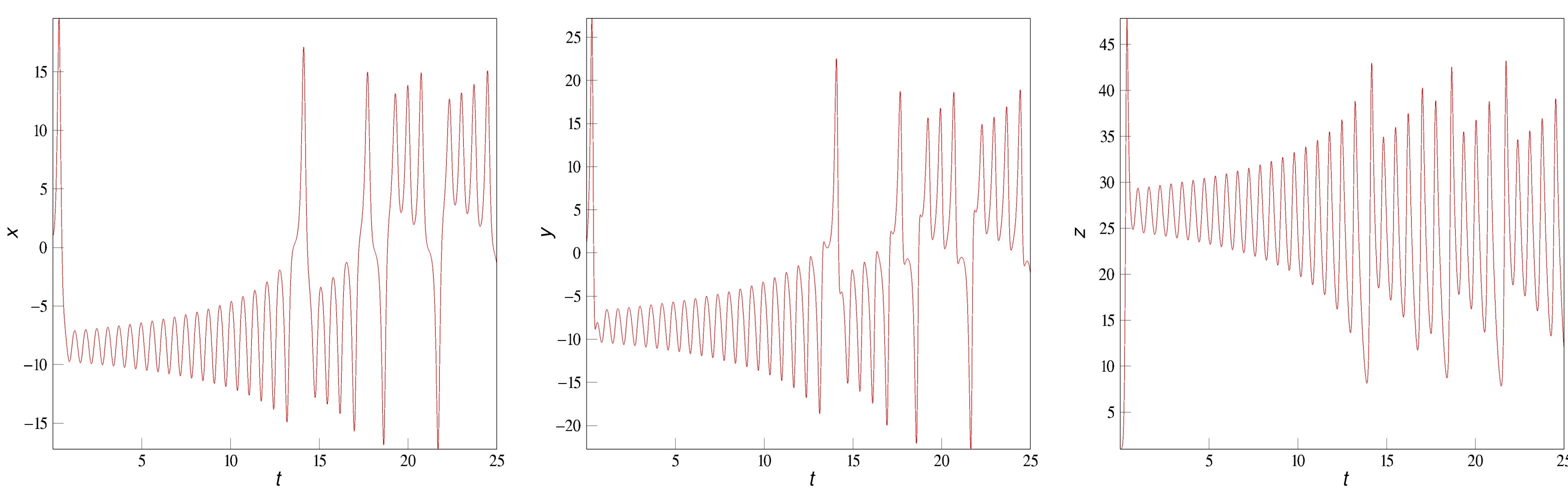


$t-x$, $t-y$ and $t-z$ plots with initial condition $(1,1,1)$ and $r=20$

PROBLEM 4b

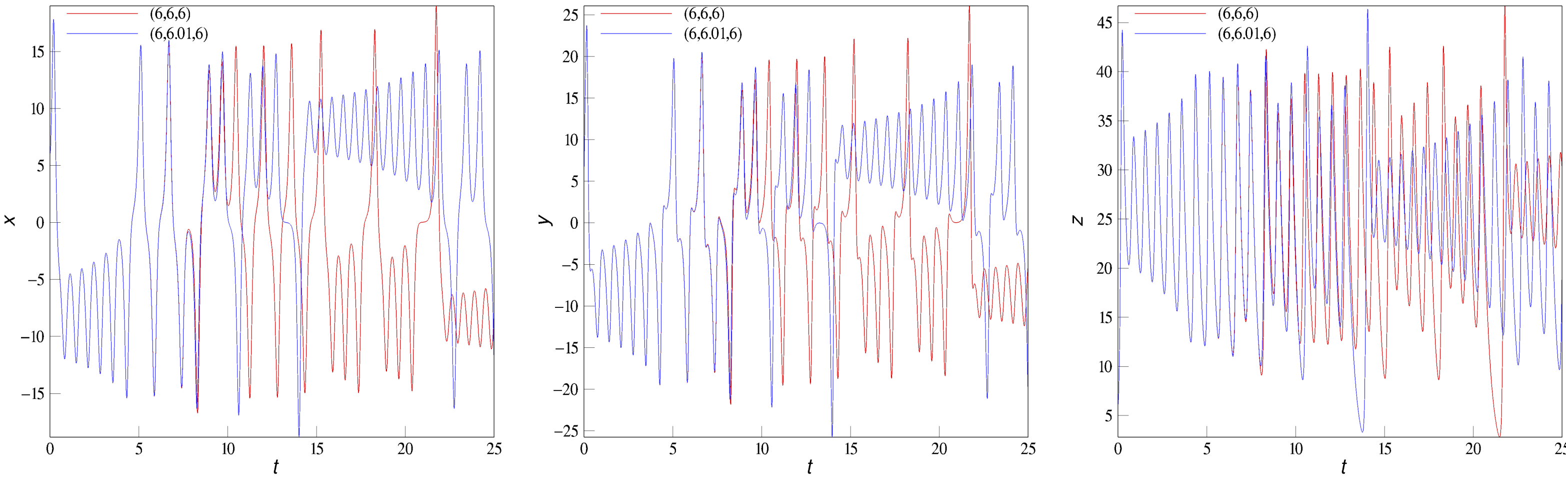


$x-y$, $x-z$ and $y-z$ plots with initial condition $(1,1,1)$ and $r=28$



$t-x$, $t-y$ and $t-z$ plots with initial condition $(1,1,1)$ and $r=28$

PROBLEM 4c



$t - x$, $t - y$ and $t - z$ plots with initial conditions $(6, 6, 6)$ (red) and $(6, 6.01, 6)$ (blue) for $r = 28$

PROBLEM 5a

The max/min values of the real/imag parts of the eigenvalues of $\boldsymbol{A}$ are listed below for $5 \leq N \leq 75$. They are purely imaginary and seem to lie in the range $\lambda \in (-2\imath, 2\imath)$

```
     N      min(imag)  max(imag) min(real) max(real)
   ==================================================
    5.0000    -1.9021     1.9021    -0.0000     0.0000
   15.0000    -1.9890     1.9890    -0.0000     0.0000
   25.0000    -1.9961     1.9961    -0.0000     0.0000
   35.0000    -1.9980     1.9980    -0.0000     0.0000
   45.0000    -1.9988     1.9988    -0.0000     0.0000
   55.0000    -1.9992     1.9992    -0.0000     0.0000
   65.0000    -1.9994     1.9994    -0.0000     0.0000
   75.0000    -1.9996     1.9996    -0.0000     0.0000
```

The max/min values of the real/imag parts of the eigenvalues of $\boldsymbol{B}$ are listed below for $5 \leq N \leq 75$. They are purely real and seem to lie in the range $\lambda \in (-4, 0)$

```
     N      min(imag)  max(imag) min(real) max(real)
   ==================================================
    5.0000          0          0    -3.6180    -0.0000
   15.0000          0          0    -3.9563     0.0000
   25.0000          0          0    -3.9842     0.0000
   35.0000          0          0    -3.9919    -0.0000
   45.0000          0          0    -3.9951     0.0000
   55.0000          0          0    -3.9967     0.0000
   65.0000          0          0    -3.9977    -0.0000
   75.0000          0          0    -3.9982     0.0000
```

PROBLEM 5b

For the wave equation discretization given, clearly, any scheme with a stability diagram that doesn't include a portion of the imaginary axis will be unstable. So we should not use explicit Euler, RK2 or AB2. The RK3/4 or implicit Euler schemes will be stable. For the heat equation discretization given, since the eigenvalues are purely real, and all the schemes we've talked about include a part of the negative real axis, any of the methods will be stable for a suitable choice of Δt.

PROBLEM 5c

For the wave equation discretization given, we found the eigenvalues of $\boldsymbol{A}$: call them $\lambda_j, \quad j \in (1, N)$. The factor $-a/(2\Delta x)$ in front of $\boldsymbol{A}$ scales the eigenvalues: the decoupled equations will thus have coefficients $c_j = -a/(2\Delta x)\lambda_j$. Each of these decoupled equations is like the model equation. We want to pick Δt such that $c_j \Delta t$ lies in the stability region of the time stepping scheme chosen for all j. It is enough to consider the max/min values of λ_j for this. For the wave equation, for the RK4 scheme, we will want to ensure $\Delta t < 2.79\Delta x/(|a|)$.

Similarly, for the heat equation discretization given, we need to include the factor $\nu/(\Delta x)^2$. For example, for the explicit Euler scheme, we will need a time step $\Delta t \leq (\Delta x)^2/(2\nu)$. You should compare with the von Neumann analysis results we will get later in the class.