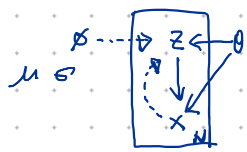
[VAE: Variational Auto-Encoder]



$$P(z|x) = \frac{P(x|z)\cdot P(z)}{P(x)}$$

① variation inference idea → optimization

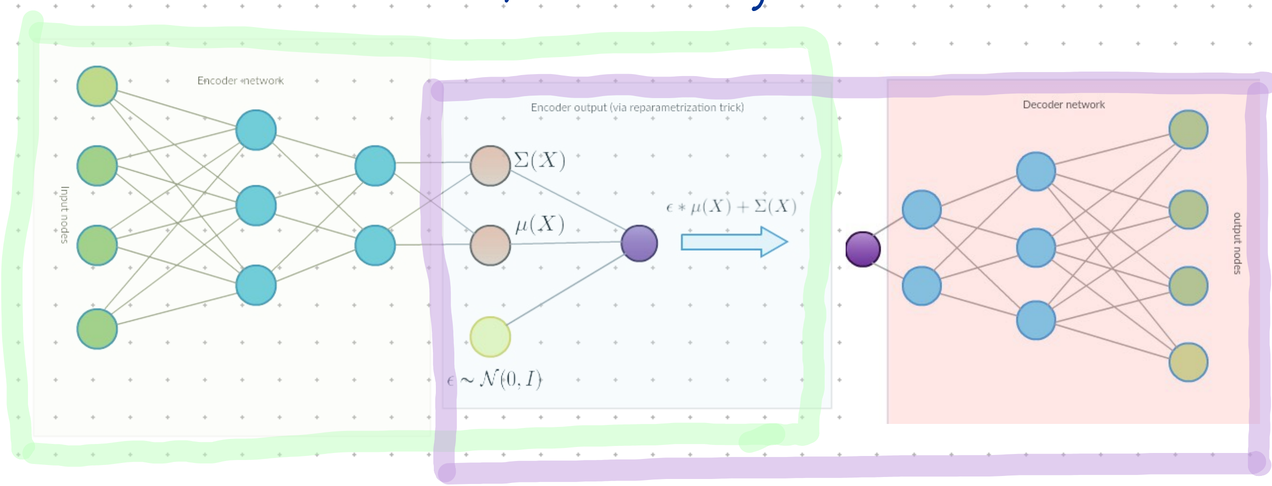
② posterior $q(z|x)$ ⟺ likelihood $P(x|z)$ → auto encoder

D_{KL}

$P_\theta(z|x)$ posterior

$q_\phi(z|x)$ variation posterior (variation approximation of posterior)

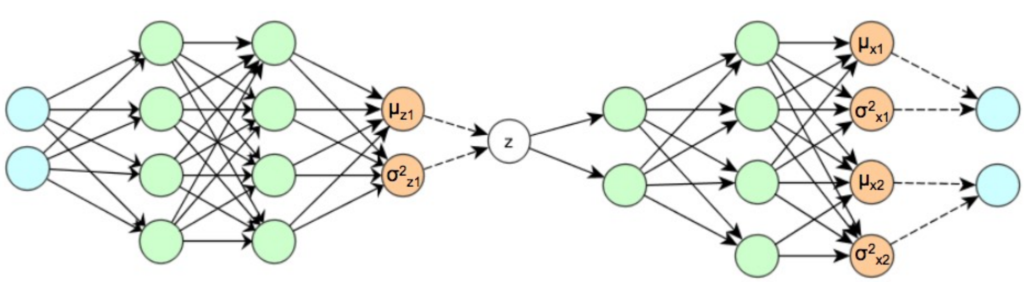
$= N(z; \mu_z(x), \sigma_z(x))$



$D_{KL}\left(N(\text{Encoder}_\phi(X))\,||\,N(0,1)\right)$ $\quad ||X - \text{Decoder}_\theta(f(\text{Encoder}_\phi(X)))||^2$

where $\text{Encoder}_\phi(X) = (\mu(X), \Sigma(X))$

and $f(\mu(X), \Sigma(X)) = \mu(X) + \Sigma(X) * \varepsilon$



Cost: Regularisation

$$-D_{KL}\left(q(z|x^{(i)})||p(z)\right) = \frac{1}{2}\sum_{j=1}^{J}\left(1+\log(\sigma_{z_j}^{(i)^2}) - \mu_{z_j}^{(i)^2} - \sigma_{z_j}^{(i)^2}\right)$$

Cost: Reproduction

$$-\log\left(p(x^{(i)}|z^{(i)})\right) = \sum_{j=1}^{D}\frac{1}{2}\log(\sigma_{x_j}^2) + \frac{(x_j^{(i)}-\mu_{x_j})^2}{2\sigma_{x_j}^2}$$

[1] approach option(1/2)

$$D_{KL}(q_\phi(z|x) \| p_\theta(z|x))$$

$$= \int q_\phi(z|x) \log \frac{q_\phi(z|x)}{p_\theta(z|x)}$$

$$p_\theta(z|x) = \frac{p_\theta(x|z)\, p_\theta(z)}{p_\theta(x)}$$

$$= \int q_\phi(z|x) \log \frac{p_\theta(x)\; q_\phi(z|x)}{p_\theta(x|z) \cdot p_\theta(z)}$$

$$= \int q_\phi(z|x) \log p_\theta(x)\, dz - \int q_\phi(z|x) \cdot p_\theta(x|z)\, dz + \int q_\phi(z|x) \log \frac{q_\phi(z|x)}{p_\theta(z)}\, dz$$

$$= \log p_\theta(x) - \mathbb{E}_{q_\phi(z|x)}[p_\theta(x|z)] + D_{KL}(q_\phi(z|x) \| p_\theta(z))$$

$$\therefore \log p_\theta(x) = \boxed{\mathbb{E}_{q_\phi(z|x)}[p_\theta(x|z)] - D_{KL}(q_\phi(z|x) \| p_\theta(z))} + \boxed{D_{KL}(q_\phi(z|x) \| p_\theta(z|x))}$$

ELBO : Evidence Lower BOund $\Uparrow$ maximization

≥ 0 $\Downarrow$ minimization

[2] approach option(2/2)

$$\log p_\theta(x)$$

$$= \int q_\phi(z|x)\, dz \cdot \log(p_\theta(x))$$

$$p_\theta(z|x) = \frac{p_\theta(x|z) \cdot p_\theta(z)}{p_\theta(x)}$$

$$= \int q_\phi(z|x)\, dz \cdot \log\left(\frac{p_\theta(x|z) \cdot p_\theta(z)}{p_\theta(z|x)}\right)$$

$$= \int q_\phi(z|x)\, dz \cdot \log\left(\frac{p_\theta(x|z) \cdot p_\theta(z)}{q_\phi(z|x)} \cdot \frac{q_\phi(z|x)}{p_\theta(z|x)}\right)$$

$$= \int q_\phi(z|x) \cdot \log\left(\frac{p_\theta(x|z) \cdot p_\theta(z)}{q_\phi(z|x)}\right) dz + \int q_\phi(z|x) \cdot \log\left(\frac{q_\phi(z|x)}{p_\theta(z|x)}\right) dz$$

$$= \int q_\phi(z|x) \cdot \log p_\theta(x|z)\, dz - \int q_\phi(z|x) \cdot \log\left(\frac{q_\phi(z|x)}{p_\theta(z)}\right) dz + \int q_\phi(z|x) \cdot \log\left(\frac{q_\phi(z|x)}{p_\theta(z|x)}\right) dz$$

$$= \boxed{\mathbb{E}_{q_\phi(z|x)}[p_\theta(x|z)] - D_{KL}(q_\phi(z|x) \| p_\theta(z))} + \boxed{D_{KL}(q_\phi(z|x) \| p_\theta(z|x))}$$

ELBO : Evidence Lower BOund $\Uparrow$ maximization

≥ 0 $\Downarrow$ minimization

3 $\mathcal{L}$: Lower Variational Bound of the log likelihood

$$\mathcal{L}(\theta, \phi : x)$$

$$(\theta^*, \phi^*) = \underset{\theta\ \phi}{\operatorname{argmax}} (\theta, \phi : x)$$

$$\therefore \mathcal{L}(\theta, \phi : x) = \mathbb{E}_{q_\phi(z|x)}\left[P_\theta(x|z)\right] - D_{KL}\left(q_\phi(z|x) \| P_\theta(z)\right)$$

$$= \boxed{\mathbb{E}_{q_\phi(z|x^{(i)})}\left[P_\theta(x^{(i)}|z)\right]} \boxed{- D_{KL}\left(q_\phi(z|x^{(i)}) \| P_\theta(z)\right)}$$

Reconstruction

Regularization

$\log(1)$, if $x^{(i)}$ gets always reconstructed perfectly (z produces $x^{(i)}$)

$$\approx \frac{1}{L}\sum_{i=1}^{L} \log p(x^{(i)} | z^{(i,l)})$$

$$\approx -\log p(x^{(i)} | z^{(i,l)})$$

$$\simeq \sum_{j=1}^{D} \frac{1}{2}\log(\sigma_{x_j}^2) + \frac{(x_j^{(i)} - \mu_{x_j})^2}{2\sigma_{x_j}^2}$$

$P(z)$ is always a simple prior $N(0,1)$

$q(z|x)$ is Gaussian with parameter$(\mu^{(i)}, \sigma^{(i)})$ determined by NN

$$\therefore -D_{KL}\left[N(\mu(x), \Sigma(x)) \| N(0,1)\right]$$

$$= \frac{1}{2}\sum_{j=1}^{J}\left(1 + \log\left(\sigma_{z_j}^{(i)^2}\right) - \mu_{z_j}^{(i)^2} - \sigma_{z_j}^{(i)^2}\right)$$

(cf Numerical Approximation

$$\int p(x)\cdot f(x)\, dx = \mathbb{E}_{x\sim p(x)}[f(x)] \approx \frac{1}{k}\sum_{i=0}^{k}[f(x_i)]_{x_i \sim p(x)}$$)

4 Reparameterization Trick

$$z^{(i,l)} \sim N(\mu^{(i)}, \sigma^{2(i)})$$

$$z^{(i,l)} = \mu^{(i)} + \sigma^{(i)} \odot \epsilon_{(i)} \quad \epsilon_{(i)} \sim N(0,1)$$