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Exercise 1

a) Find tight asymptotic bound of the given function Fun:

```
function Fun(int n, int m)
    while n > 0 do
        for (int i = m; i > 1; i = floor(√i)) do
            RefreshState(n,i) // Θ(n)
        end for
        for (int i = n; i <= 2*n; i++) do
            UpdateParams(2*n,m,i) // Θ(1)
        end for
        n = 2*n/3
    end while
    return 0
end function
```

T_1 T_2 T_3

T_1:

\`floor(sqrt(i))\` lahko tretiramo kot \`sqrt(i)\`, ko delamo z inti

step	i
0	m
⋮	$\sqrt{m} = n^{1/2}$
⋮	$\sqrt[4]{m} = m^{1/4}$
⋮	
k	2 ali 3 $= m$
$k+1$	1

$i = 2$

$2 = m^{1/2^k}$

$\log 2 = \frac{1}{2^k} \log(m) \quad / \cdot 2^k \cdot \frac{1}{\log 2}$

$2^k = \frac{\log(m)}{\log(2)} = \log_2(m)$

$k = \log_2(\log_2(m))$

$i = 3$

$k = \log_2(\log_3(m))$

$\Rightarrow T_1(m,n) = \Theta(n) \cdot \Theta(\log(\log(m)))$
$= \Theta(n \cdot \log\log(m))$

T_2:

i gre od n do 2n, kjer ga v vsaki iteraciji povečamo za 1. Torej imamo n iteracij

$\Rightarrow T_2(m,n) = \Theta(1) \cdot \Theta(n) = \Theta(n)$

$T_1 + T_2$:

$\Theta(n \log(\log(m))) + \Theta(m) = \Theta(n \log(\log(m)))$

T_3:

$\Theta(\log_{3/2}(n)) \cdot (T_2 + T_3) = \boxed{\Theta(n \cdot \log(\log(m)) \cdot \log(n))} = T(n,m)$

b) Find tight bound of function Fun, in relation to a,b,c

```
function FUN(int a, int b, int c)
    if b+c == 0 then
        return a
    end if
    int s1 = 0;
    int d = b+c;
    for (int i=0; i<d; i++) do
        s1 += a*i + d/i
        if b <= c then
            int s2 = FUN(a/2, floor(b/2), floor(c/2))
            int s3 = FUN(2*a, floor(b/2), floor(c/2))
        else
            int s2 = FUN(a/2, floor(c/2), floor(b/2))
            int s3 = FUN(2*a, floor(c/2), floor(b/2))
        end if
    end for
    return s1+s2+s3
end function
```

$$T(a,b,c) = (b+c)\cdot\left(T\left(\tfrac{a}{2}, \left\lfloor\tfrac{b}{2}\right\rfloor, \left\lfloor\tfrac{c}{2}\right\rfloor\right) + T\left(2a, \left\lfloor\tfrac{b}{2}\right\rfloor, \left\lfloor\tfrac{c}{2}\right\rfloor\right)\right) + \Theta(1)$$

$$T(b,c) = (b+c)\cdot 2T\left(\left\lfloor\tfrac{b}{2}\right\rfloor, \left\lfloor\tfrac{c}{2}\right\rfloor\right) + \Theta(1)$$

$$= (b+c)\underbrace{\cdot 2\left(\left\lfloor\tfrac{b}{2}\right\rfloor+\left\lfloor\tfrac{c}{2}\right\rfloor\right)\cdot 2\left(\left\lfloor\tfrac{b}{4}\right\rfloor+\left\lfloor\tfrac{c}{4}\right\rfloor\right)\cdots\cdots}_{\log_2(m)}\Theta(1)$$

$m := \max(b,c)$

$$= \underbrace{2^{\log_2 m}}_{=m}\cdot \underbrace{(b+c)\left(\left\lfloor\tfrac{b}{2}\right\rfloor+\left\lfloor\tfrac{c}{2}\right\rfloor\right)\left(\left\lfloor\tfrac{b}{4}\right\rfloor+\left\lfloor\tfrac{c}{4}\right\rfloor\right)\cdots\left(\left\lfloor\tfrac{b}{m}\right\rfloor+\left\lfloor\tfrac{c}{m}\right\rfloor\right)}_{\log_2 m+1 \times \Theta(b+c)}\cdot\Theta(1)$$

$$= \Theta\left(m\cdot(b+c)^{\log_2 m+1}\right) = \boxed{\Theta\left(m\cdot(b+c)^{\log(\max(b,c))+1}\right)}$$

$$\begin{matrix} \| \\ T(a,b,c) \end{matrix}$$

c) Find upper and lower asymptotic bounds of the following algorithm, where the time complexity of function TokBatch$(x, y) = \Theta(y \log y)$

```
function PREPROCESS(string[] a, int n, int m)
    int c;
    for (int i = 0; i < n; i++) do
        c = a[i].length
        if c <= m then
            TOKBATCH(a[i], c)
        else
            for (int j = 0; j < min(c/m, m); j++) do
                TOKBATCH(a[i][j*m:(j+1)*m], m)
            end for
        end if
    end for
    return 0
end function
```

Ker $c \leq m$

$\Theta(m \log m)$

$\Theta(m \log m)$

Ω : $c <= m \quad \forall c$:

$$\boxed{T(n,m) = \Omega(\underbrace{n}_{\text{for Loop}} \cdot \underbrace{m \log(m)}_{\text{TokBatch}})}$$

O : $c > m$

$$T(n,m) = O(\underbrace{n}_{\text{For}} \cdot \underbrace{m}_{\text{Notranji for}} \cdot \underbrace{m \log(m)}_{\text{Tok batch}}) = \boxed{O(n m^2 \log(m)) = T(n,m)}$$

Inner for loop:

$\min(\frac{c}{m}, m)$

$\frac{c}{m} < m$

$c \in (m, m^2)$

največ m iteracij

$m < \frac{c}{m}$

m iteracij

Exercise 2

Solve the following two recurrences using Master method.

a)

$$T(n) = 6 \cdot T\left(\frac{n}{3}\right) + n^{\frac{2}{3}} \cdot \log n$$

$a = 6$

$b = 3$

$f(n) = n^{2/3} \log(n)$

$n^{\log_3 6} > n^{1,6}$

$f(n) = O(n^{1,6})$

$\Rightarrow \boxed{T(n) = \Theta\left(n^{\log_3 6}\right)}$

b)

$$T(n) = 14 \cdot T\left(\frac{n}{7}\right) + \sqrt{n^3}$$

$a = 14$

$b = 7$

$f(n) = n^{\frac{3}{2}}$

1) $n^{\log_7 14} < n^{1,4}$

$f(n) = \Omega(n^{1,4})$

2) $14 \cdot f\left(\frac{n}{7}\right) \leq c \cdot f(n)$; $c < 1$, $n > n_0$

$14 \cdot \sqrt{\frac{n^3}{7^3}} \leq c \cdot \sqrt{n^3}$

$\sqrt{\frac{4}{7} n^3} \leq c \cdot \sqrt{n^3} \quad / : \sqrt{n^3}$

$\sqrt{\frac{4}{7}} \leq c$ ✓

$\Rightarrow \boxed{T(n) = \Theta\left(n^{\frac{3}{2}}\right)}$

Exercise 3

Solve the following two recurrences using Akra-Bazzi method.

a)

$$T(n) = \frac{3}{2} \cdot T\left(\frac{n}{3}\right) + 7 \cdot T\left(\frac{n}{14}\right) + \sqrt{n}$$

$a_1 = \frac{3}{2}$ $b_1 = \frac{1}{3}$ $h_1 = 0$ $f(x) = \sqrt{n}$

$a_2 = 7$ $b_2 = \frac{1}{14}$ $h_2 = 0$

p:

$\frac{3}{2} \cdot \left(\frac{1}{3}\right)^p + 7 \cdot \left(\frac{1}{14}\right)^p = 1$

$\underline{\underline{p = 1}}$

$$T(x) = \Theta\left(n\left(1 + \underbrace{\int_1^n \frac{\sqrt{u}}{u^2}\,du}\right)\right)$$

$\hookrightarrow \int_1^n u^{-3/2}\,du = -2u^{-\frac{1}{2}}\Big|_1^n = -\frac{2}{\sqrt{n}} + 2$

$$= \Theta\left(n - \frac{2n}{\sqrt{n}} + 2n\right) = \boxed{\Theta(n) = T(n)}$$

b)

$$T(n) = T\left(\frac{n}{4}\right) + T\left(\frac{n}{5}\right) + 3 \cdot T\left(\frac{n}{11}\right) + n^2$$

p: $\left(\frac{1}{4}\right)^p + \left(\frac{1}{5}\right)^p + 3\left(\frac{1}{11}\right)^p = 1$

$p \approx 0{,}82426$

$a_1 = 1$ $b_1 = \frac{1}{4}$ $h_i = 0$ $f(n) = n^2$

$a_2 = 1$ $b_2 = 1/5$

$a_3 = 3$ $b_3 = 1/11$

$$T(n) = \Theta\left(n^p\left(1 + \underbrace{\int_1^n \frac{u^2}{u^{1+p}}\,du}\right)\right)$$

$\hookrightarrow \int_1^n u^{1-p}\,du = \frac{1}{(2-p)}\left[u^{2-p}\right]_1^n = \frac{n^{2-p} - 1}{(2-p)}$

$$= \Theta\left(n^p + n^p\left(\frac{n^{2-p} - 1}{(2-p)}\right)\right) = \Theta\left(n^p + \frac{1}{(2-p)} \cdot n^2 - n^p\right) = \boxed{\Theta(n^2) = T(n)}$$

Exercise 4

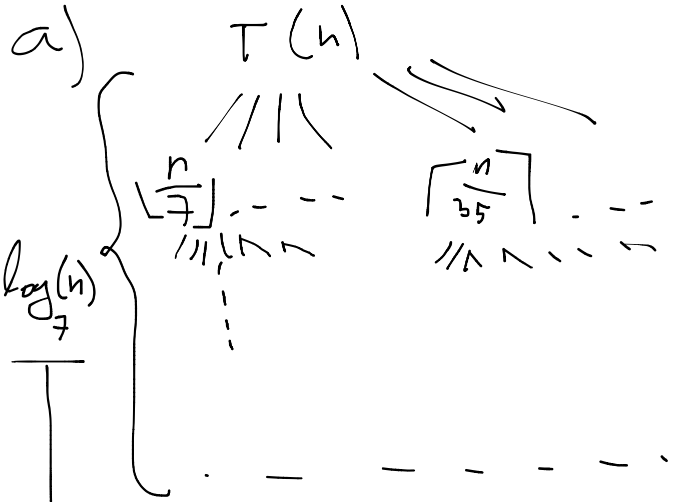
a) Estimate upper and lower asymptotic bounds of the following recurrence using the tree method:

$$T(n) = 4 \cdot T\left(\left\lfloor \frac{n}{7} \right\rfloor\right) + 3 \cdot T\left(\left\lceil \frac{n}{35} \right\rceil\right) + n, \qquad n > 1$$

$T(0) = 1$

$T(1) = 1$

a)



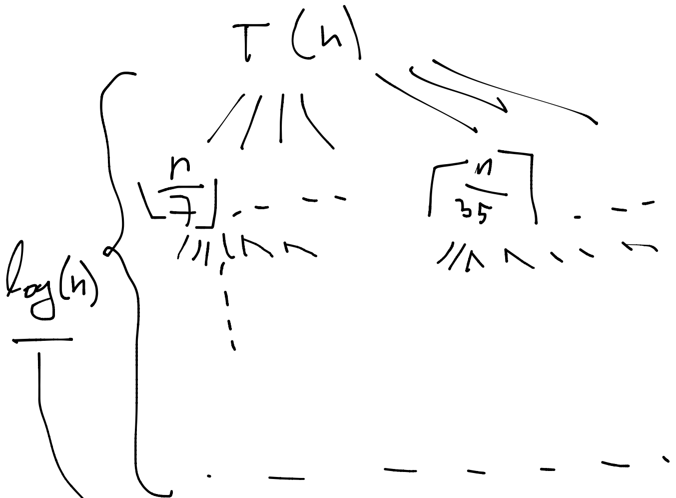
Σ

n

$3 \cdot O(n) + 4\, O(n) = O(n)$

$C \cdot O(n) = O(n)$

$T(n) = O(\log n \cdot n)$



Σ

n

$3 \cdot \Omega(n) + 4\, \Omega(n) = \Omega(n)$

$C \cdot \Omega(n) = \Omega(n)$

$T(n) = \Omega(\log n \cdot n)$

b) prove results in a) using substitution method

O: Predpostavka: $T(n) \leq c \log n \cdot n$

$n = \ ! \quad 4 \cdot 3 + \quad = 9$

$n = 18: \quad 4 \cdot T(2) + 3 \cdot T(1) + 18 = 57 \leq c \cdot \log$

$$c \geq \frac{57}{18 \log 18}$$

rekurzija:

$$T(n) \leq 4 \cdot T\left(\frac{n}{7}\right) + 3 \cdot T\left(\frac{n+35}{35}\right) + n$$

$$= \frac{4c}{7}(\log n - \log 7) n + \frac{3c}{35}\Big(\log(n+35) - \log(35)\Big)(n+35) + n$$

$$\leq \frac{4c}{7} \log n \cdot n + \frac{3c}{35}\left(\log(n+35)\right)(n+35) + n \leq c \cdot \log n \cdot n \quad / : cn$$

$$\frac{4}{7}\log(n) + \frac{3}{35}\underbrace{\log(n+35)}_{\leq \log(3n)}\underbrace{\left(1 + \frac{35}{n}\right)}_{\leq 3} + \frac{1}{c} \leq \log n$$

$\boxed{n > 17,5}$

$$\frac{4}{7}\log(n) + \frac{9}{35}\log(n) + \frac{9\log 3}{35} + \frac{1}{c} \leq \log n$$

$$\frac{29}{35}\log(n) + \frac{9\log 3}{35} + \frac{1}{c} \leq \log n$$

$\square$

Ω: Predpostavka: $T(n) \geq c \log(n) \cdot n$

$n = 20$: $T(n) = 57 \geq c \log(n) \cdot n = c \log(20) \cdot 20 \Rightarrow c \leq \frac{57}{20 \log(20)}$

rekurzija:

$$T(n) \geq 4T\left(\frac{n}{7} - 1\right) + 3T\left(\frac{n}{35}\right) + n$$

$$\geq \frac{4c}{7}\left(\log(n-7) - \log(7)\right)(n-7) + \frac{3c}{35}\left(\log(n) - \log(35)\right)n + n \geq c \log n \cdot n \quad /:cn$$

$$\frac{4}{7}\left(\underbrace{\log(n-7)}_{\geq \log\left(\frac{n}{2}\right)} - \log(7)\right)\left(1 - \frac{7}{n}\right) + \frac{3}{35}\left(\log(n) - \log(35)\right) + \frac{1}{c} \geq \log n$$

$$\frac{4}{7}\left(\log(n) - \log(14)\right)\left(1 - \frac{7}{n}\right) + \frac{3}{35}\log(n) - \frac{3}{35}\log(35) + \frac{1}{c} \geq \log(n)$$

Drži za c dovolj blizu 0.

Exercise 5

Solve the following two recurrences using annihilator method. Find the exact solutions.

a)

$$T(n) = 4 \cdot T(n-1) + n - 2; \qquad T(0) = 3$$

$T(1) = 12 + 1 - 2 = 11$

$T(2) = 44$

$T(n+1) - 4T(n) = -n+2$

$(E-4)\,T(n) = -n+2$

$\underbrace{(E-1)^2(E-4)}\,T(n) = 0$

$\rightarrow \quad T(n) = (\alpha n + \beta)1^n + \gamma 4^n$

$T(0) = \beta + \gamma = 3 \quad \rightarrow \quad \beta = 3 - \gamma = \frac{2}{9}$

$T(1) = \alpha + \beta + 4\gamma = 11 \quad \rightarrow \quad \alpha = 8 - 3\gamma = -\frac{1}{3}$

$T(2) = 2\alpha + \beta + 16\gamma = 44$

$16 - 6\gamma + 3 - \gamma + 16\gamma = 44$

$9\gamma = 25$

$\gamma = \frac{25}{9}$

$$\Rightarrow \boxed{T(n) = -\frac{1}{3}n + \frac{2}{9} + \frac{25}{9}4^n}$$

b)

$$T(n) = 10 \cdot T(n-1) - 32 \cdot T(n-2) + 38 \cdot T(n-3) - 15 \cdot T(n-4)$$
$$T(0) = 0$$
$$T(1) = 2$$
$$T(2) = 10$$
$$T(3) = 23$$

$$T(n+4) - 10\,T(n+3) + 32\,T(n+2) - 38\,T(n+1) + 15\,T(n) = 0$$

$$(E^4 - 10E^3 + 32E^2 - 38E + 15)\,T(n) = 0$$

Nizle:

	1	−10	+32	−38	15
		1	−9	23	−15
1	1	−9	23	−15 ‖	0
		1	−8	15	
1	1	−8	15 ‖	0	

$E^2 - 8E + 15$
$(E-3)(E-5)$

$$\underbrace{(E-3)(E-5)(E-1)^2}\;T(n) = 0$$

$$\hookrightarrow\quad T(n) = A\,3^n + B\,5^n + C\,n + D$$

$$T(0) = 0 = A + B + D \longrightarrow D = -A - B = -\frac{87}{32}$$
$$T(1) = 2 = 3A + 5B + C + D \longrightarrow C = 2 - 2A - 4B = -\frac{21}{8}$$
$$T(2) = 10 = 9A + 25B + 2C + D \longrightarrow A = \frac{6}{4} - 4B = \frac{25}{8}$$
$$T(3) = 23 = 27A + 125B + 3C + D \longrightarrow B = -\frac{13}{32}$$

$$\boxed{T(n) = \frac{25}{8} \cdot 3^n - \frac{13}{32}\,5^n - \frac{21}{8}\,n - \frac{87}{32}}$$