Janez Justin
63180016

1) A_i := Number at index i is $\geq$ than nubers before it

B_i := Number at index i is $<$ than nubers before it

$i = 3, 4, \dots, n$

Define the following:

n = length of array, $n \geq 2$

$X_{A_i} = I(A_i) = \begin{cases} 1 & \text{if } A_i \\ 0 & \text{else} \end{cases}$ $\qquad X_A = \sum_{i=3}^{n} X_{A_i} + 2$

$X_{B_i} = I(B_i) = \begin{cases} 1 & \text{if } B_i \\ 0 & \text{else} \end{cases}$ $\qquad X_B = \sum_{i=3}^{n} X_{B_i} + 2$

X = expected number of nonzero elements in out

$E[X_{A_i}] = \frac{1}{i}$ → i-th number must be the biggest out of i numers.

$E[X_{B_i}] = \frac{1}{i}$ → same but looking for the smallest.

50% chance of first number being smaller

$$E[X] = \frac{1}{2} \cdot E[X_A] + \frac{1}{2} E[X_B]$$

$$= \frac{1}{2}\left(\sum_{i=3}^{n} \frac{1}{i} + 2\right) + \frac{1}{2}\left(\sum_{i=3}^{n} \frac{1}{i} + 2\right) =$$

$$= \underbrace{\sum_{i=1}^{n} \frac{1}{i}}_{\approx \ln(n) + 0.6} + 2 - \frac{1}{1} - \frac{1}{2} = \boxed{\ln(n) + 1{,}1}$$

2)

a) X - number of bits set to 1

A_i = i-th bit is set to 1

$$X_i = I(A_i) = \begin{cases} 1 & \text{if } A_i \\ 0 & \text{else} \end{cases}$$

5n calls of hash function

$$E[X_i] = P(A_i) = 1 - \underbrace{\left(\frac{n-1}{n}\right)^{5n}}_{\text{Probability of bit not being 0 after 5n updates of array}}$$

$n-1$ # of other bits

n # of all bits

$$E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \left(1 - \left(\frac{n-1}{n}\right)^{5n}\right) = \boxed{n\left(1 - \left(\frac{n-1}{n}\right)^{5n}\right)}$$

for big n

$$\approx n \cdot (1 - e^{5}) = \boxed{n \cdot 0{,}99326}$$

$$\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^{5n} = \lim_{n\to\infty} \left(\left(1 - \frac{1}{n}\right)^{n}\right)^{5} = (e^{-1})^{5}$$

b) Assumption: Number being checked is not in the structure. What is the probability of returning true?

B = for some number all 5 hash functions map to 1

5 hashes

$$P(B_i) = \boxed{\Bigg(\underbrace{1 - \left(\frac{n-1}{n}\right)^{5n}}_{\text{prob. of single bit being 1}}\Bigg)^{5}}$$

for big enough n

$$\approx (1 - e^{-5})^{5} = \boxed{0{,}96676\ldots}$$

$$\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^{5n} = \lim_{n\to\infty} \left(\left(1 - \frac{1}{n}\right)^{n}\right)^{5} = (e^{-1})^{5}$$

3) X - Number of stickers needed

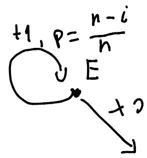
A_i = get duplicate sticker if we already have $n-i$ stickers

$A_{ij} = A_i$ if we already tried j times

$$X_{A_{ij}} = I(A_{ij}) = \begin{cases} 1 & \text{if } A_{ij} \\ 0 & \text{otherwise} \end{cases}$$

$$E[A_{ij}] = P(A_{ij}) = \left(\frac{n-i}{n}\right)^{j+1}$$

$$E[X_{A_i}] = \boxed{\sum_{j=0}^{\infty} P(A_{ij})}$$



$$E[X_{A_i}] = \frac{n-i}{n}\left(1 + E[X_{A_i}]\right) \quad / \cdot n$$

$$n \cdot E[X_{A_i}] = n-i + (n-i)E[X_{A_i}]$$

$$(\cancel{n}-\cancel{n}+i)\, E[X_{A_i}] = n-i$$

$$E[X_{A_i}] = \frac{n-i}{i} = \frac{n}{i} - 1$$

all n stickers

$$E[X] = n + \sum_{i=1}^{n}\left(\frac{n}{i} - 1\right) = \cancel{n} + n \underbrace{\sum_{i=1}^{n} \frac{1}{i}}_{\approx \ln(n) + 0.6} - \cancel{\sum_{i=1}^{n} 1} = n \cdot (\ln(n) + 0.6)$$

$$= \boxed{n \ln(n) + 0.6n}$$