Geometric Objects

5^{TH} WEEK, 2022

Basic Elements

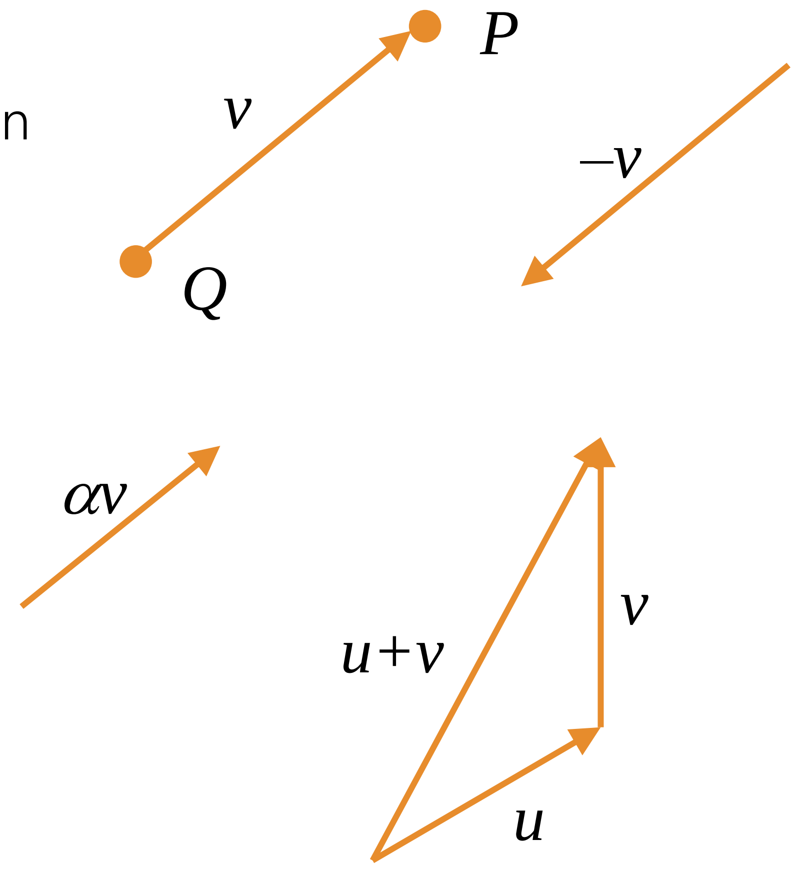
- Geometry – the study of the relationships among objects in an n-dimensional space
  - In computer graphics, objects exist in three dimensions
- Minimum set of primitives from which we can build more sophisticated objects
- Three basic elements
  - _______s
  - _______s
  - _______s

Scalars, Points, and Vectors

- ________s
  - Position in space
- ________s
  - To specify quantities such as the distance between two points
  - Ex) real numbers, complex numbers
- _______s
  - Any quantity with direction and magnitude
  - Ex) velocity, force

Vector Operations

- Every vector has an ________
  - Same magnitude but points in opposite direction
- Every vector can be multiplied by a _______
- There is a _____ vector
  - Zero magnitude, undefined orientation
- The sum of any two vectors is a _______
  - Using head-to-tail axiom



Mathematical View (1)

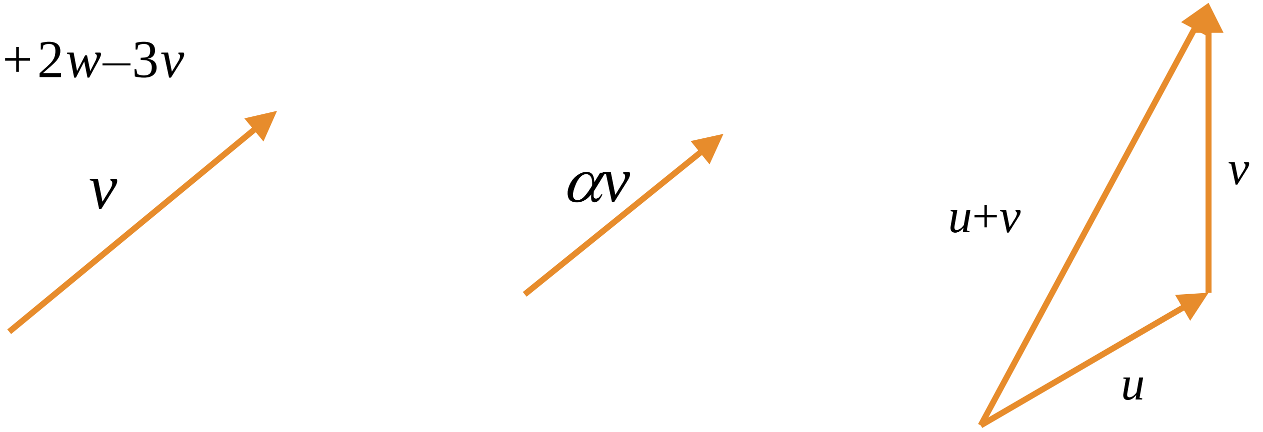
- Scalar field
  - Entities: ______s
  - Operations: _________ and __________
- (Linear) Vector space
  - Entities: ______s and scalars
  - Operations: _____-______ addition, ______-______ multiplication
- Euclidean space
  - Entities: vectors and scalars
  - Operations: vector-vector addition, vector-scalar multiplication, _______ of size or _________

Mathematical View (2)

- _______ space
  - Entities: vectors, scalars, and ______s
  - Operations: vector-vector addition, vector-scalar multiplication, ______-______ ________, _____-_____ _____________

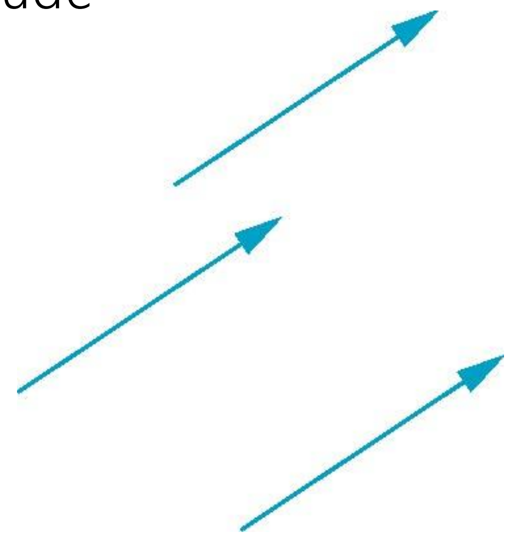
Linear Vector Spaces

- Mathematical system for manipulating vectors
- Operations
  - ______-_______ __________: $u=\alpha v$
  - ______-_______ __________: $w=u+v$
- Expression
  - Ex) $v=u+2w-3v$



Vector Lack Position

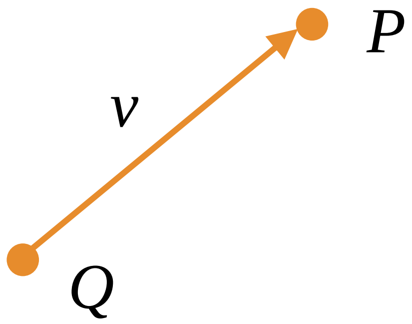
- These vectors are identical
  - Same direction and magnitude



- Vector spaces are insufficient for geometry
  - Need ______s

Points

- ________ in space
- Operations allowed between points and vectors
  - ______-_______ _________ yields a _______
  - Equivalent to ______-_______ _______



$$v = P - Q$$

$$P = v + Q$$

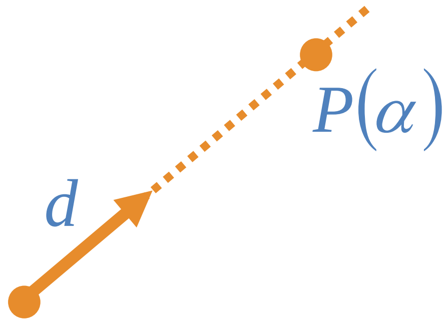
Affine Spaces

- Vector space + _______s
- Operations
  - Scalar-scalar operations
  - Scalar-vector multiplication
  - Vector-vector addition
  - Vector-_____ addition
  - ______-______ subtraction

Lines

- Considering all points of the form
  - $P(\alpha)=P_0+\alpha d$
  - Set of all points that pass through P_0 in the direction of the vector d

$$P(\alpha)=P_0+\alpha d$$



Affine Sums

- In affine space
  - O: vector-vector addition, vector-scalar multiplication, vector-point addition
  - X: _______ of two points, point-_____ multiplication
- _______ _________

$$P(\alpha) = Q + \alpha v$$

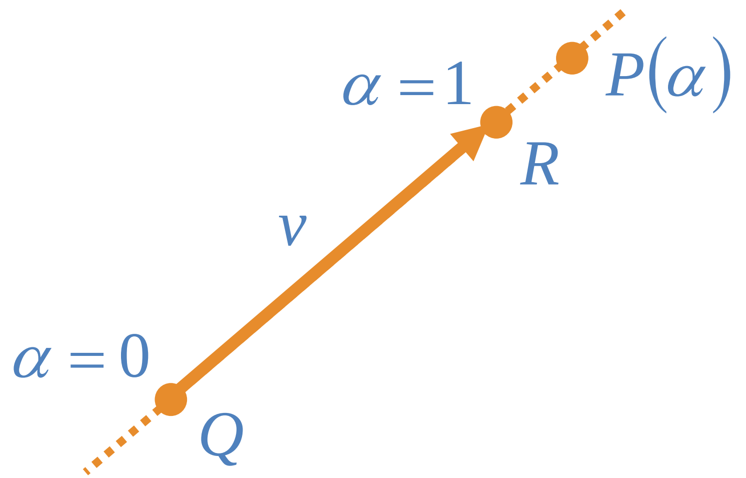
$$v = R - Q$$

$$P(\alpha) = Q + \alpha(R - Q)$$

$$= \alpha R + (1 - \alpha) Q$$

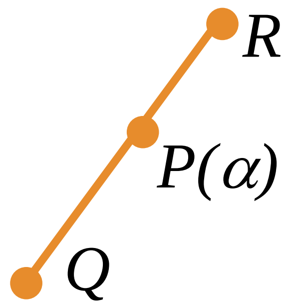
$$\boxed{P = \alpha_1 R + \alpha_2 Q}$$

$$\text{where } \alpha_1 + \alpha_2 = 1$$



Convexity

- Convex object
  - Any point lying on the line segment connecting any two points in the object is also in the object



- Using affine sums
  - Convex object such as line segment

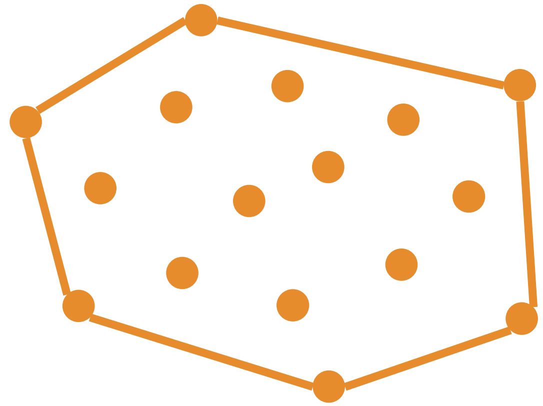
$$P(\alpha) = \alpha R + (1-\alpha)Q \quad \text{where} \quad 0 \le \alpha \le 1$$

  - Convex hull

$$\boxed{P = \alpha_1 P_1 + \alpha_2 P_2 + \ldots + \alpha_n P_n}$$

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$$

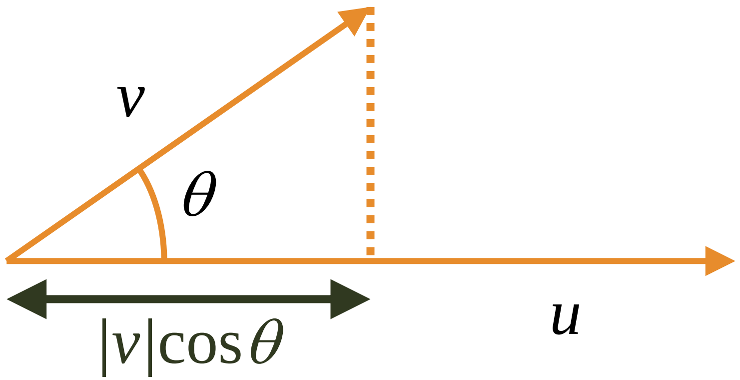
$$\alpha_i \ge 0, \quad i = 1, 2, \ldots, n$$



Dot Products

- _____ products – $u \cdot v$
  - __________: $u \cdot v = 0$
  - Magnitude of a vector: $|u|^2 = u \cdot u$
  - _____ between two vectors: $\cos\theta = \dfrac{u \cdot v}{|u||v|}$
  - Orthogonal projection: $|v|\cos\theta = u \cdot v / |u|$

$$\boxed{\begin{aligned}&\begin{pmatrix}x_1 & y_1 & z_1\end{pmatrix} \cdot \begin{pmatrix}x_2 & y_2 & z_2\end{pmatrix}\\ &= x_1x_2 + y_1y_2 + z_1z_2\end{aligned}}$$

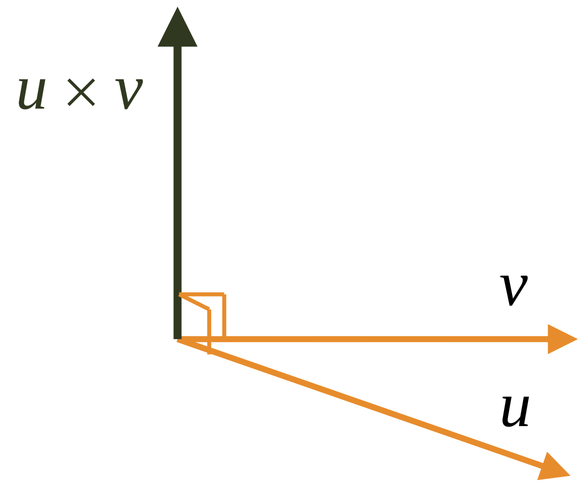


Cross Products

- _______ products – $u \times v = n$
  - Right-handed coordinates system
    - Direction of the _____ of the right hand

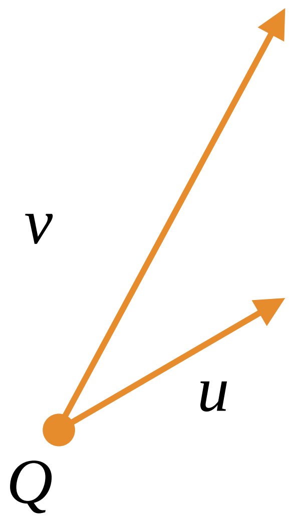
$$\left|\sin\theta\right| = \frac{\left|u \times v\right|}{\left|u\right|\left|v\right|}$$

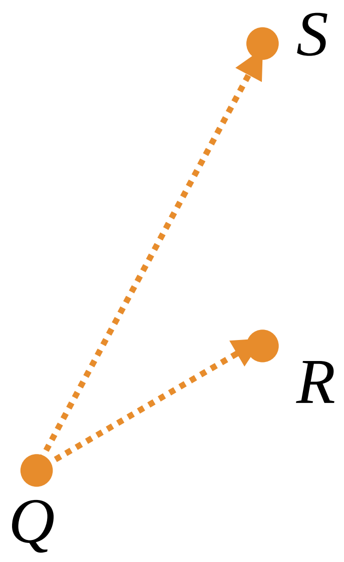
$$\begin{aligned}&\begin{pmatrix}x_1 & y_1 & z_1\end{pmatrix} \times \begin{pmatrix}x_2 & y_2 & z_2\end{pmatrix}\\ &= \begin{pmatrix}y_1 z_2 - y_2 z_1 & z_1 x_2 - z_2 x_1 & x_1 y_2 - x_2 y_1\end{pmatrix}\end{aligned}$$



Planes

- Defined by a ______ and two _______s or by three ______s



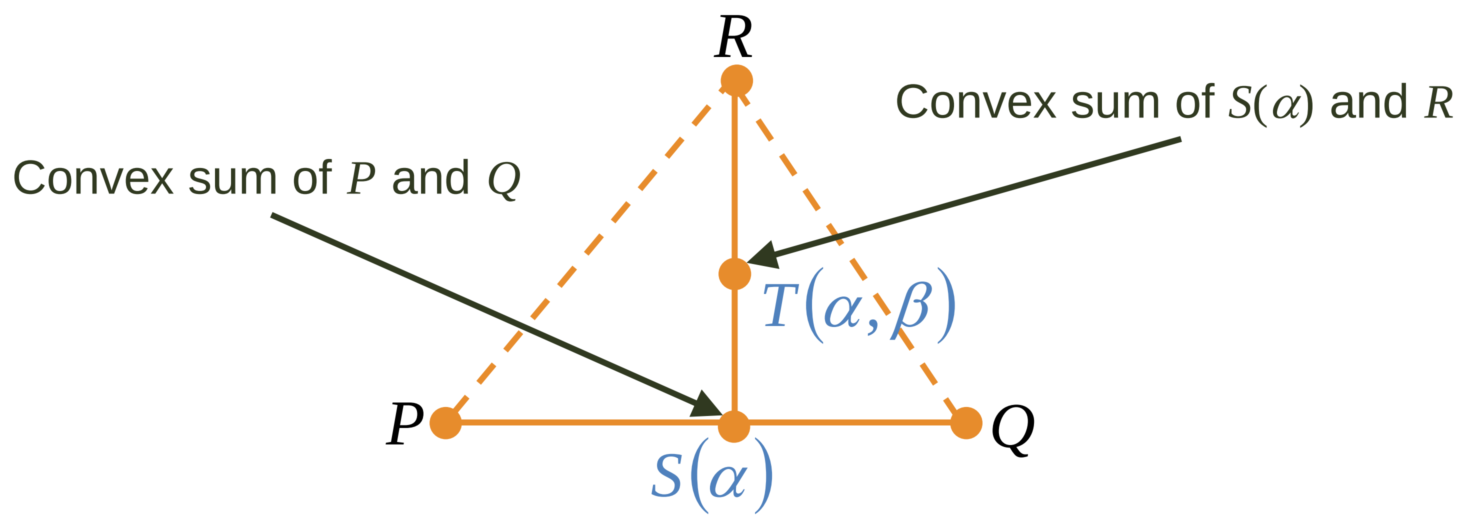


$$P(\alpha,\beta)=Q+\alpha u+\beta v$$

$$P(\alpha,\beta)=Q+\alpha(S-Q)+\beta(R-Q)$$

Triangles

$$T(\alpha,\beta)=P_0+\alpha u+\beta v,\quad 0\le\alpha,\beta\le 1$$



$$S(\alpha)=\alpha P+(1-\alpha)Q,\quad 0\le\alpha\le 1$$

$$T(\beta)=\beta S+(1-\beta)R,\quad 0\le\beta\le 1$$

$$T(\alpha,\beta)=\beta[\alpha P+(1-\alpha)Q]+(1-\beta)R$$

$$T(\alpha,\beta)=P+\beta(1-\alpha)(Q-P)+(1-\beta)(R-P)$$

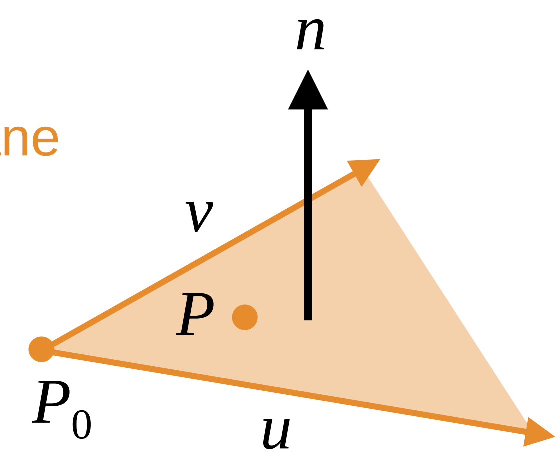
Normals

- Every plane a vector n ______ (perpendicular, orthogonal) to it
- From two vectors which form $P(\alpha,\beta)=Q+\alpha u+\beta v$, we can use the ______ product

$$P - P_0 = \alpha u + \beta v$$

$$n = u \times v$$ normal to the plane

$$\boxed{n \cdot (P - P_0) = 0}$$



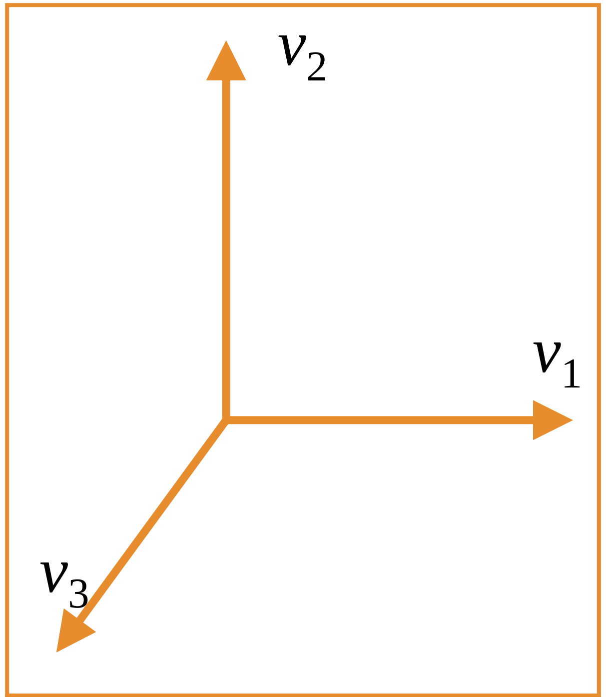
Coordinate Systems (1)

- 3D vector space

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

  - Scalar __________ : $\alpha_1, \alpha_2, \alpha_3$
  - ______ vector : v_1, v_2, v_3
    - Defining a coordinate system
    - The origin: fixed reference point
  - Representation: column matrix

$$\mathbf{a} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}^T$$



Coordinate System

Coordinate Systems (2)

- Example: $v = 2v_1 + 3v_2 + 4v_3$

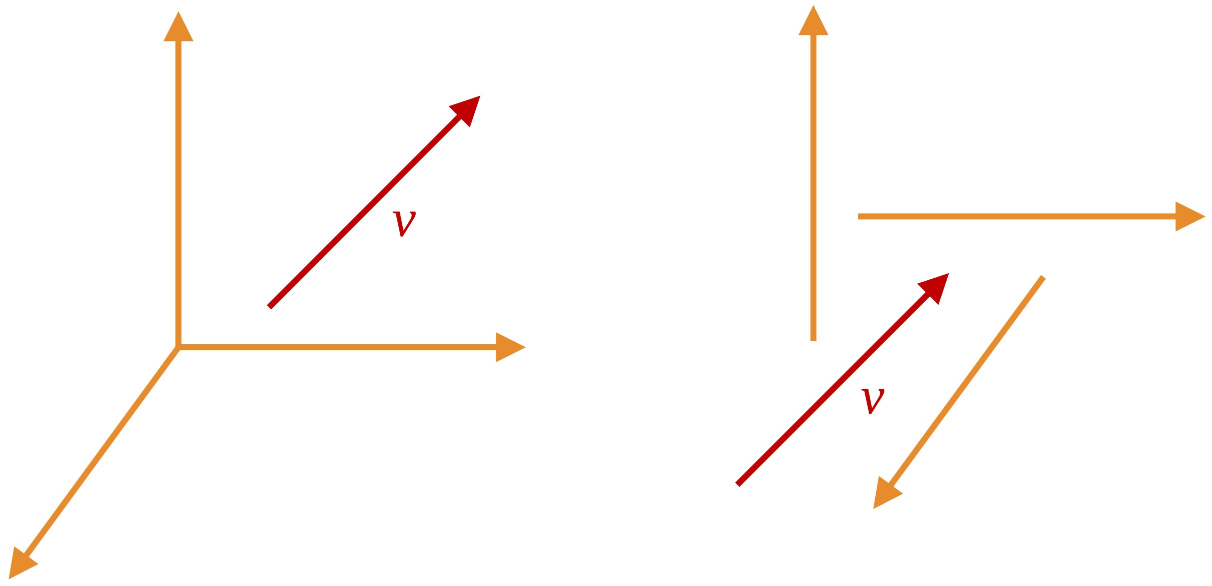
  $\mathbf{a} = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}^T$

  - Note that this representation is with respect to a particular basis
  - In OpenGL, we start by representing vectors using object basis but later the system needs a representation in terms of the camera or eye basis

    ➔ "Change of Basis"

Coordinate Systems (3)

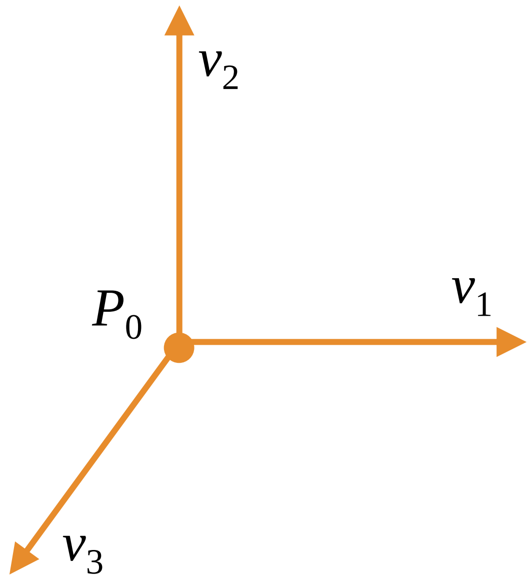
- Which is correct?



- Both are because vectors have no fixed location

Frames (1)

- A coordinate system is insufficient to represent points
- In affine space, we can add a single point, the _______, to the basis vectors to form a _______



Frames (2)

- ______ set of vectors and a particular point P_0
  - More general representation
  - Fixing the origin at P_0

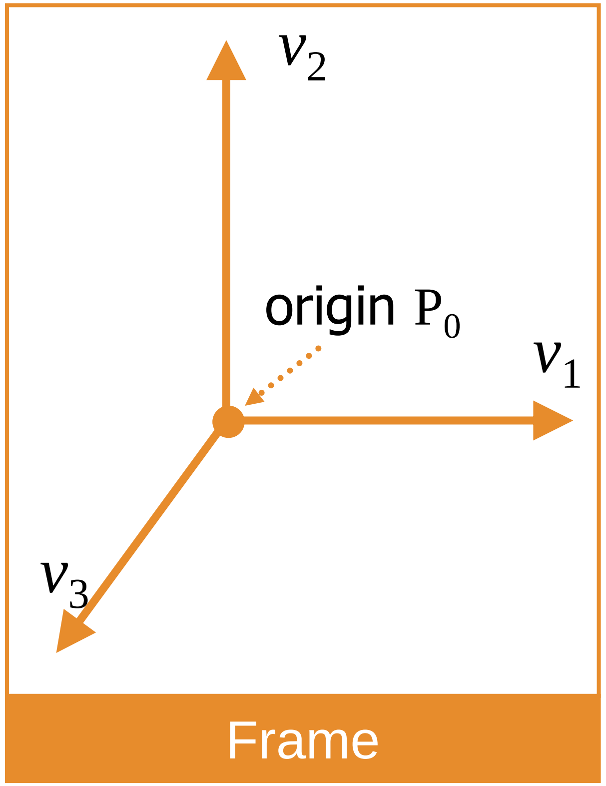
    ______ : $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$

    ______ : $P = P_0 + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3$

  - Homogeneous coordinates

    ______ : $v = [\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad 0]^T$

    ______ : $P = [\beta_1 \quad \beta_2 \quad \beta_3 \quad 1]^T$



Homogeneous Coordinates (1)

$$P = \begin{bmatrix} x_w & y_w & z_w & w \end{bmatrix}^T$$

- If $w=0$, the representation is that of a _______
- Otherwise ($w \neq 0$), we return a three dimensional _____ by

$$P = \begin{bmatrix} x_w \\ y_w \\ z_w \\ w \end{bmatrix} = \begin{bmatrix} x_w/w \\ y_w/w \\ z_w/w \\ w/w \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- To represent points and vectors with ________s but maintain a distinction between points and vectors

Homogeneous Coordinates (2)

- Homogeneous coordinates are key to all computer graphics systems
  - All standard transformation (rotation, translation, scaling) can be implemented with multiplications of _×_ matrices
  - Hardware pipeline works with 4 dimensional representations
  - For orthographic viewing, we can maintain $w=0$ for ________s and $w=1$ for _______s
  - For perspective viewing, we need a _____________ _________

Changes of Coordinate Systems (1)

- Two basis: { v_1, v_2, v_3 }, { u_1, u_2, u_3 }

$$u_1 = \gamma_{11}v_1 + \gamma_{12}v_2 + \gamma_{13}v_3$$
$$u_2 = \gamma_{21}v_1 + \gamma_{22}v_2 + \gamma_{23}v_3$$
$$u_3 = \gamma_{31}v_1 + \gamma_{32}v_2 + \gamma_{33}v_3$$

matrix

$$M = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

$$\mathbf{u} = M\ \mathbf{v}$$

Changes of Coordinate Systems (2)

- Vector: w

$$w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

$$w = \mathbf{a}^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \text{ where } \mathbf{a} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \Rightarrow w = \mathbf{b}^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \text{ where } \mathbf{b} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

$$w = \mathbf{b}^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{b}^T M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{a}^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

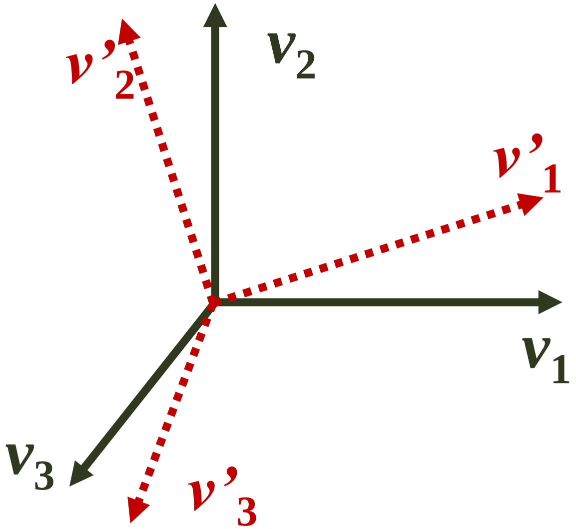
$$\boxed{\mathbf{a} = M^T \mathbf{b}}$$

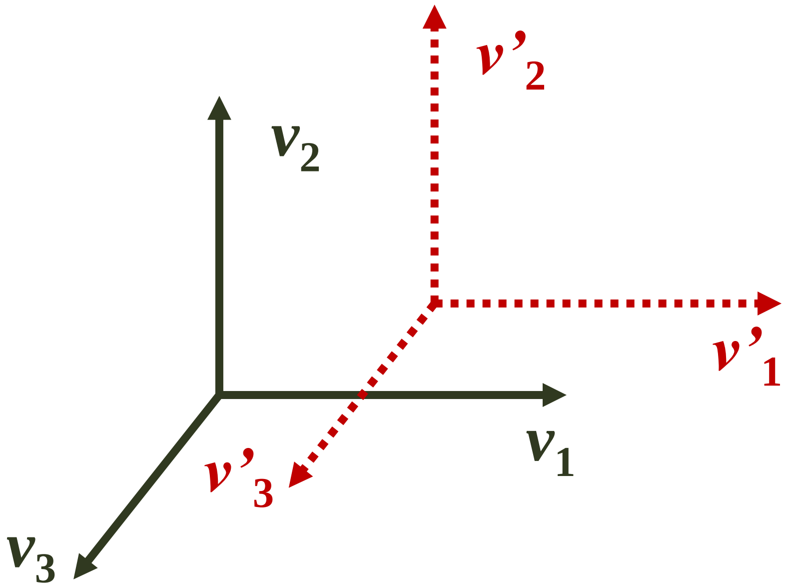
$$\boxed{\mathbf{b} = \left(M^T\right)^{-1} \mathbf{a}}$$

Changes of Basis

- Origin unchanged
  - Rotation and scaling of a set of basis vectors
- Origin changed
  - Translation of the origin, or change of frame

➔ ______________ coordinates





Example of Change Basis (1)

- Suppose a vector: w 🡐 $\mathbf{a}$ = [1 2 3]^T
  - Three basis vectors : v_1, v_2, v_3

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$w = v_1 + 2v_2 + 3v_3$$

  - New basis : u_1, u_2, u_3

$$u_1 = v_1$$
$$u_2 = v_1 + v_2$$
$$u_3 = v_1 + v_2 + v_3$$

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Example (2)

- Change of basis

$$\mathbf{b} = \left(M^T\right)^{-1}\mathbf{a}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

$$w = -u_1 - u_2 + 3u_3$$

Homogeneous Coordinates (1)

- Confusion between points and vectors !!

$$P = \begin{bmatrix} x & y & z \end{bmatrix}^T, \quad w = \begin{bmatrix} \delta_1 & \delta_2 & \delta_3 \end{bmatrix}^T$$

- Point P and vector w in frame (v_1, v_2, v_3, P_0)

$$P = P_0 + xv_1 + yv_2 + zv_3 \qquad w = \delta_1 v_1 + \delta_2 v_2 + \delta_3 v_3$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$P = \begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix} \qquad w = \begin{bmatrix} \delta_1 & \delta_2 & \delta_3 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\boxed{P = \begin{bmatrix} x & y & z & 1 \end{bmatrix}^T} \qquad \boxed{w = \begin{bmatrix} \delta_1 & \delta_2 & \delta_3 & 0 \end{bmatrix}^T}$$

Homogeneous Coordinates (2)

- Change of frames (v_1, v_2, v_3, P_0), (u_1, u_2, u_3, Q_0)

$$u_1 = \gamma_{11}v_1 + \gamma_{12}v_2 + \gamma_{13}v_3$$
$$u_2 = \gamma_{21}v_1 + \gamma_{22}v_2 + \gamma_{23}v_3$$
$$u_3 = \gamma_{31}v_1 + \gamma_{32}v_2 + \gamma_{33}v_3$$
$$Q_0 = \gamma_{41}v_1 + \gamma_{42}v_2 + \gamma_{43}v_3 + P_0$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

$$M = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0 \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & 0 \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & 1 \end{bmatrix}$$

$$\mathbf{b}^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = \mathbf{b}^T M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix} = \mathbf{a}^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

$$\boxed{\mathbf{b} = \left(\mathbf{M}^T\right)^{-1} \mathbf{a}}$$

Example of Change in Frames

- Change of frames (v_1, v_2, v_3, P_0), (u_1, u_2, u_3, Q_0)

$$u_1 = v_1$$
$$u_2 = v_1 + v_2$$
$$u_3 = v_1 + v_2 + v_3$$
$$Q_0 = P_0$$

$$\Rightarrow \quad M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Point $P = [1\ \ 2\ \ 3\ \ 1]^T \rightarrow P' = [-1\ -1\ \ 3\ \ 1]^T$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ 1 \end{bmatrix}$$

Another Example

- Change of frames (v_1, v_2, v_3, P_0), (u_1, u_2, u_3, Q_0)

$$u_1 = v_1$$
$$u_2 = v_1 + v_2$$
$$u_3 = v_1 + v_2 + v_3$$
$$Q_0 = P_0 + v_1 + 2v_2 + 3v_3$$

$$\Rightarrow \quad M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

- Point $P = [1 \;\; 2 \;\; 3 \;\; 1]^T \rightarrow P' = [0 \;\; 0 \;\; 0 \;\; 1]^T$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

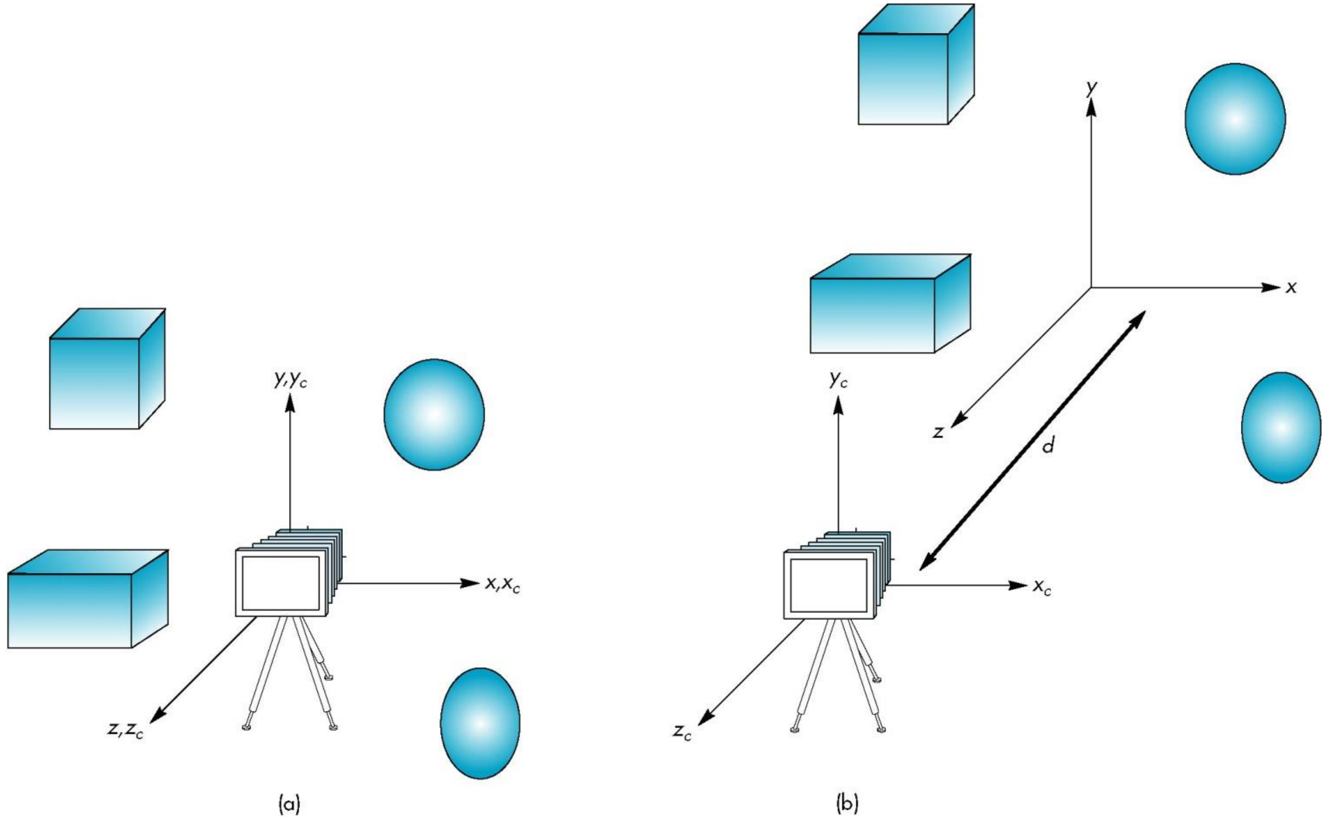
Frames in WebGL (1)

- Six representations embedded in the WebGL pipeline
  - ________ (or _______) coordinates
  - ________ coordinates
  - ___ (or ______) coordinates
  - _____ coordinates
  - ___________ ________s coordinates
  - _______ (or _______) coordinates
- Change in frames are defined by _×_ matrices
  - Sequence of transformations

Frames in WebGL (2)

- Moving the camera frame relative to the object frame

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Summary

- Basic elements – ______s, ______s, ______s
- Scalar fields, linear vector spaces, Euclidean spaces, affine spaces
  - _____s – $P(\alpha) = P_0 + \alpha d$
  - _____s – $P(\alpha, \beta) = Q + \alpha u + \beta v$
- ___ product – $\cos\theta = \dfrac{u \cdot v}{|u||v|}$ ______ product – $|\sin\theta| = \dfrac{|u \times v|}{|u||v|}$
- Frames = ______ + _______
  - Homogenous coordinates ➔ ______ : $v = [\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad 0]^T$

    ______ : $P = [\beta_1 \quad \beta_2 \quad \beta_3 \quad 1]^T$