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ANALYSIS OF TIME BETWEEN FAILURES FOR REPAIRABLE COMPONENTS.(U)
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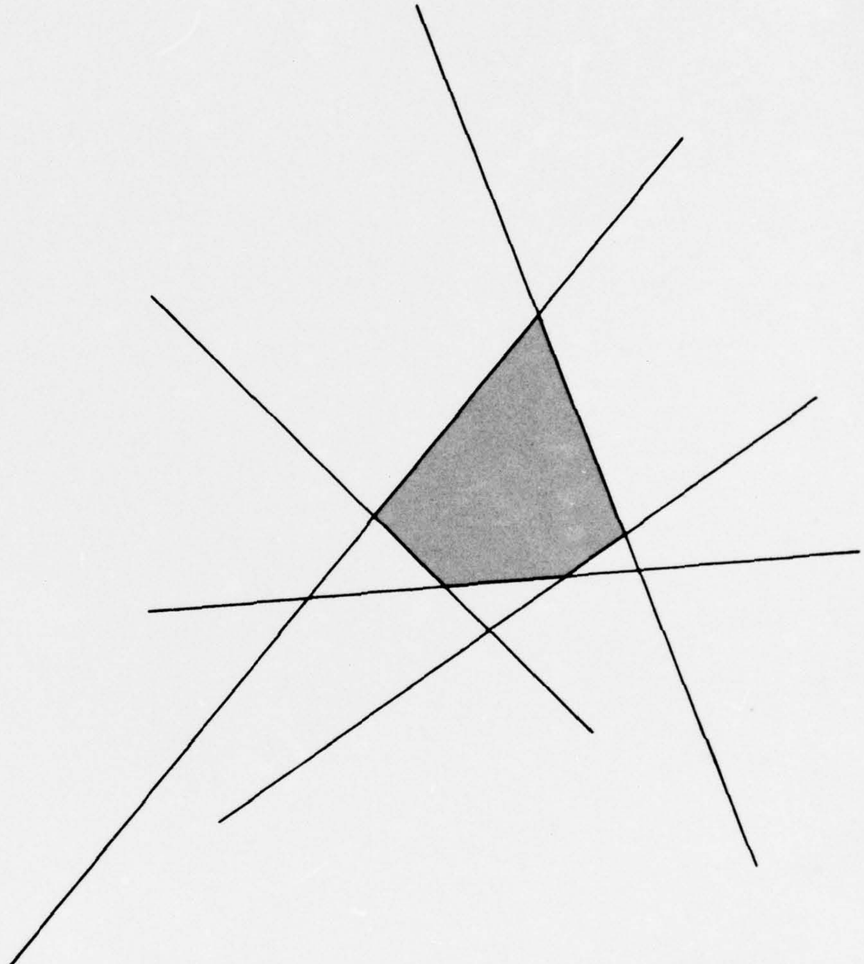
RICHARD E. BARLOW
and
BERNARD DAVIS



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ANALYSIS OF TIME BETWEEN FAILURES FOR REPAIRABLE COMPONENTS

by

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Analysis of Time Between Failures for Repairable Components

Richard E. Barlow and Bernard Davis*

Abstract. A method for analyzing time between failure data is developed. The method uses total time on test plots for a non-homogeneous Poisson process failure model. Engine failure data is used to illustrate the method. A graphical method for determining optimum replacement intervals is presented.

1. Introduction.

Most of the statistical literature concerned with analyzing failure data assumes that observations are independent and identically distributed. Although engineering reports often purport to give Mean Time Between Failure (MTBF) estimates, the estimates are, in fact only valid in general if times between failures are exponentially distributed random variables. Since this assumption is often not valid, especially for mechanical components, more sophisticated techniques are required to analyze this type of data. To focus on the kind of problem we have in mind, we will first consider some failure data on caterpillar tractor engines. The actual data is given in the appendix. The data consists of the age of the tractor at engine failure, the age of the engine at failure, and the calendar date of the failure event.

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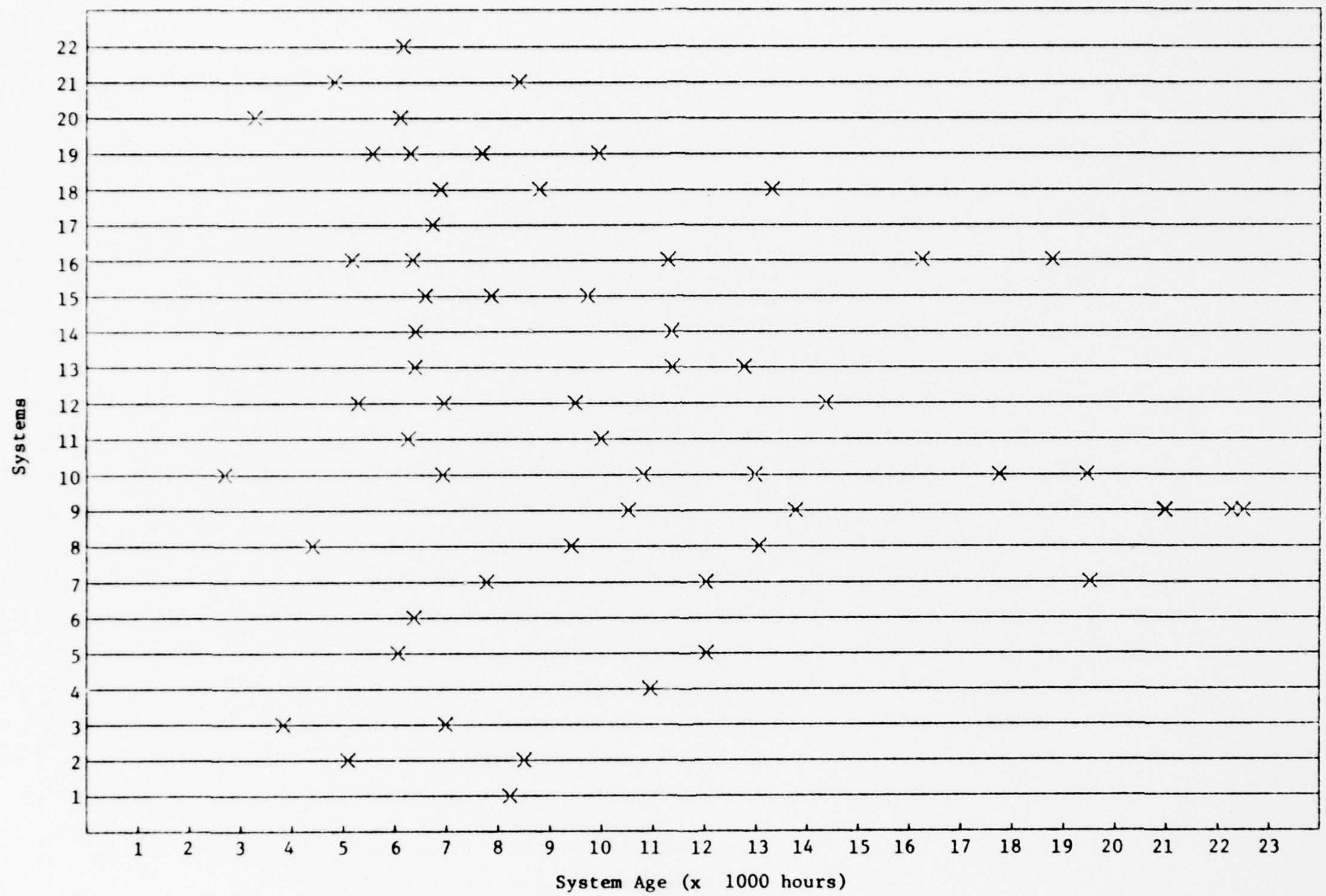
Figure 1 shows engine failure removal times on each of 22 tractors as a function of tractor age. The large number of failures at 6000 hours was due to a piston failure problem and not a planned maintenance action. In order to plan maintenance actions, we need a mathematical model for predicting engine failures as a function of tractor age as well as engine age. Sections 2 and 3 present a technique for solving this problem. In Section 4 a graphical method for determining optimum component replacement based on component life cycle costing is presented.

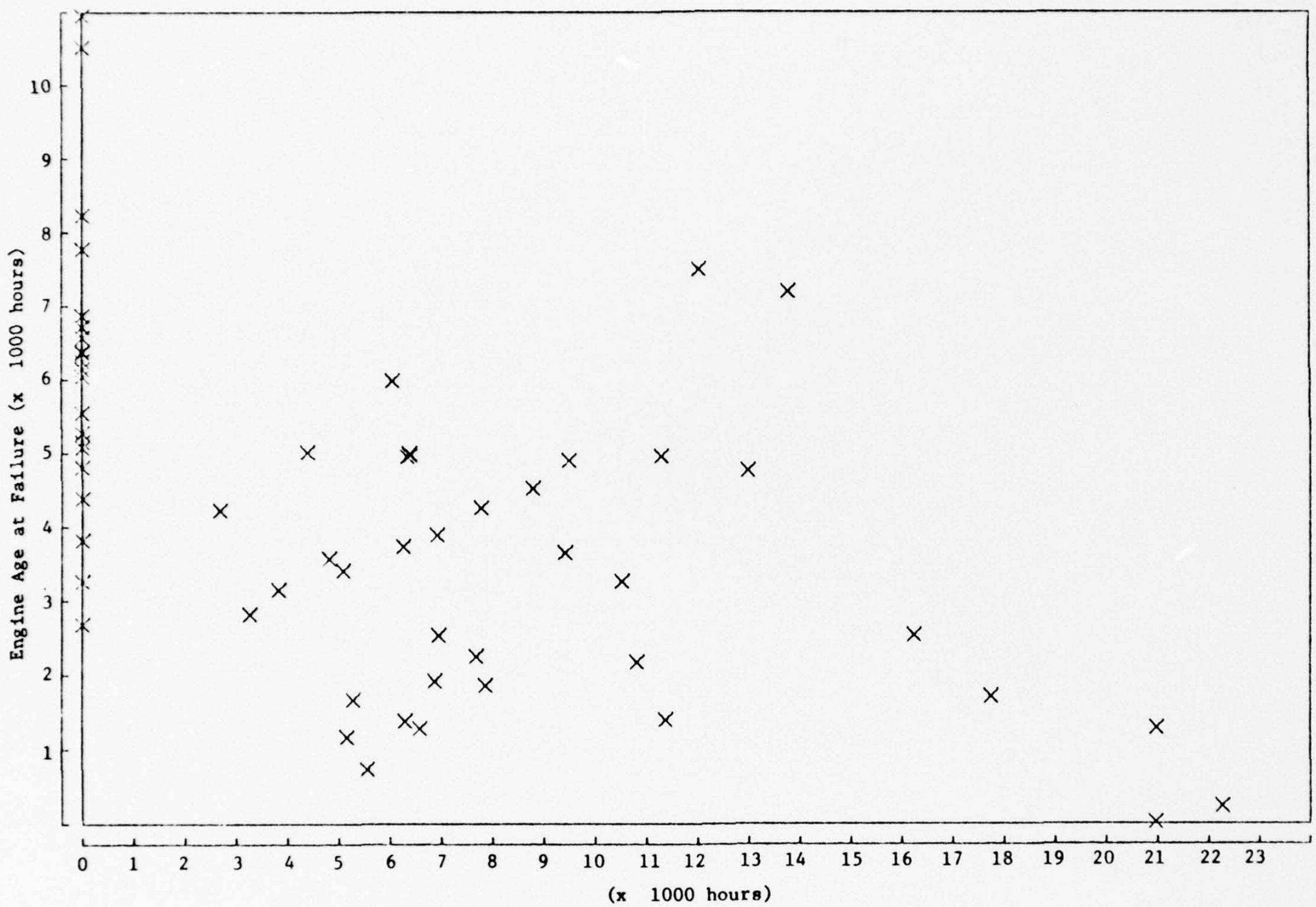
2. A Non-homogeneous Poisson Model for Times Between Failures.

In examining 59 engine failures on 22 D9G-66A caterpillar tractors we found that engine age at failure depended on the operating age of the tractor when the engine was last placed in the tractor. Except for the original engines in new tractors, engines replacing failed engines were often repaired engines. Figure 2 is a plot of engine age at failure versus tractor age when the engine was last placed in the tractor. Thus crosses on the y-axis corresponding to $x = 0$ are ages at failure of the original engines. It is fairly clear from Figure 2 that original engines tend to have a longer mean life than repaired engines. The mean life of original engines is 6149 hours versus 3241 hours for repaired engines. The standard deviation in both cases is about 2000 hours.



OBSERVED ENGINE FAILURE TIMES

FIGURE 1



TRACTOR AGE IN OPERATING HOURS WHEN ENGINE INSTALLED

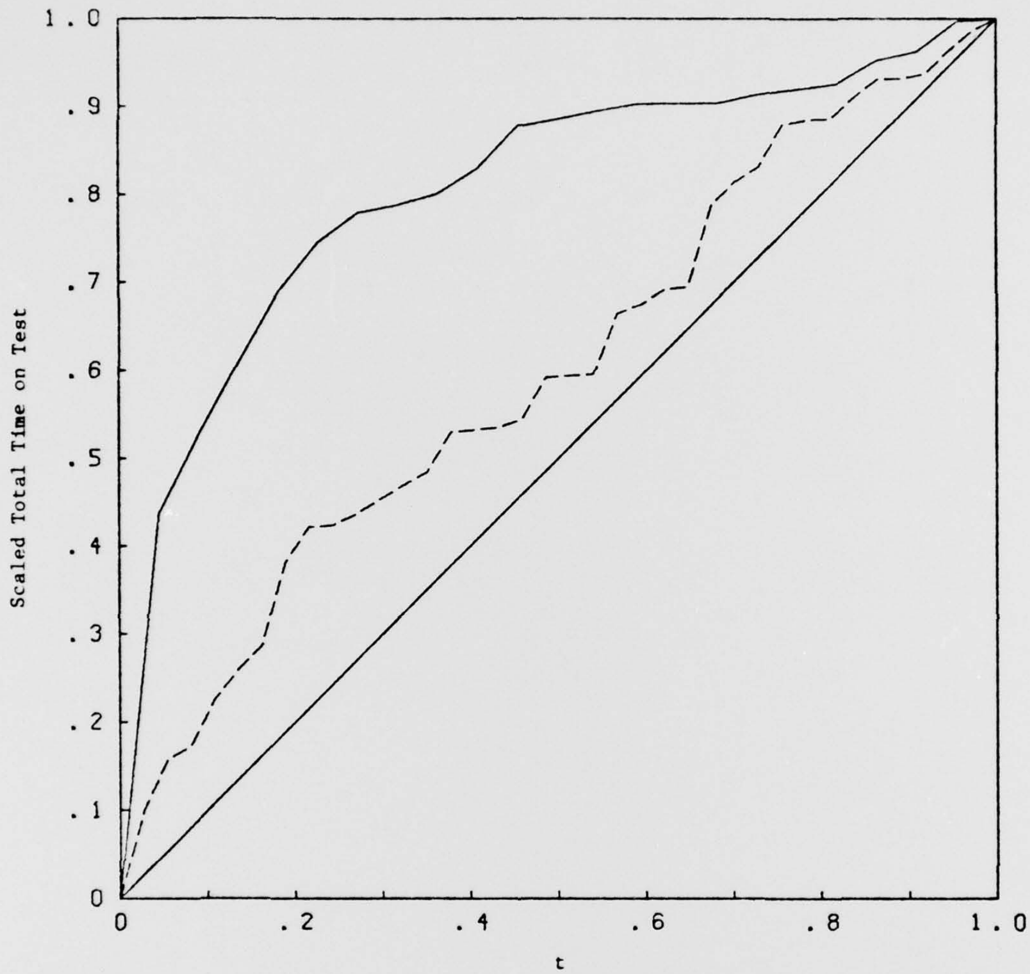
FIGURE 2

Figure 3 shows a total time on test plot for original (new) engines and also a plot for repaired engines. The fact that the new engine plot is more strongly concave indicates that the life of repaired engines is more nearly exponential. [See Barlow and Campo (1975) for a discussion of total time on test plots.]

Analysis of Failure Events from Independent Processes. Intuitively, engine failure processes will depend on both tractor age and engine age. However, Figures 2 and 3 suggest that tractor age may be the more significant variable in modelling the engine failure processes. We assume that the successive failure events of engines, say, in a given tractor can be described probabilistically by a non-homogeneous Poisson process. If $N(t)$ is the number of engine failures in $[0,t]$, for a particular tractor, then

$$P[N(t) = k] = \frac{[\Lambda(t)]^k}{k!} e^{-\Lambda(t)}$$

for $k = 0,1,2, \ldots$ where $\Lambda(t)$ is the mean number of engine failures in $[0,t]$. [See Çinlar (1975) for an introduction to non-homogeneous Poisson processes.] Since we do not know $\Lambda(t)$, it must be estimated from the data. Our approach is to use an appropriate total time on test plot to make a preliminary model identification. (The model is the analytic form of $\Lambda(t)$.) A final model identification will be made using a Bayesian approach.



TOTAL TIME ON TEST PLOTS

D9G-66A CATERPILLAR TRACTOR ENGINES

$\bar{x}_{New}$ = 6149 hours $\quad$ $\bar{x}_{Old}$ = 3241 hours

$S_{x_{New}}$ = 2000 hours $\quad$ $S_{x_{Old}}$ = 1842 hours

n = 22 $\quad$ n = 37

FIGURE 3

The superposition of n independent non-homogeneous Poisson processes each with mean function, $\Lambda(t)$, will again be a non-homogeneous Poisson process with mean function $n\Lambda(t)$. Now let each process run for the same time interval $[0,T]$. Let

$$Z_{(1)} \leq Z_{(2)} \leq \cdots \leq Z_{(N(T))}$$

be the ordered superposed event times on a common age axis, where $N(T)$ is the total number of events in $[0,T]$. In Figure 1, if all points were superposed on the x-axis, the ordered values would correspond to $Z_{(i)}$'s . (In our case, however, we do not actually observe engine failures over the same tractor age interval $[0,T]$ for each tractor. We make this assumption in order to derive our theoretical result.)

Let $n(u)$ be the number of processes under observation at tractor (or system) age u . In our example (see Figure 1), $n(0) = 22$ and $n(u) = 22$ up to about age 6000 hours at which age it drops to 21, etc. Finally, at age $u = 22507$ hours, $n(u) = 0$. The <u>scaled total time on test plot</u> for the non-homogeneous Poisson process model is a plot of

$$\frac{\int_0^{Z_{(i)}} n(u)du}{\int_0^{Z_{(N(T))}} n(u)du} \quad \text{versus} \quad \frac{i}{N(T)} ,$$

for $i = 1,2, \ldots, N(T)$. Figure 4 is a Total Time on Test Plot of the data in Appendix 1. We have used linear interpolation to produce a smooth plot. For this data, $T = 22507$ hours.

The following theorem is the basis for our preliminary model identification procedure.

Theorem 2.1. Assume that n independent non-homogeneous Poisson processes are observed on $[0,T]$ where T is fixed. Then, for $0 \leq p \leq 1$.

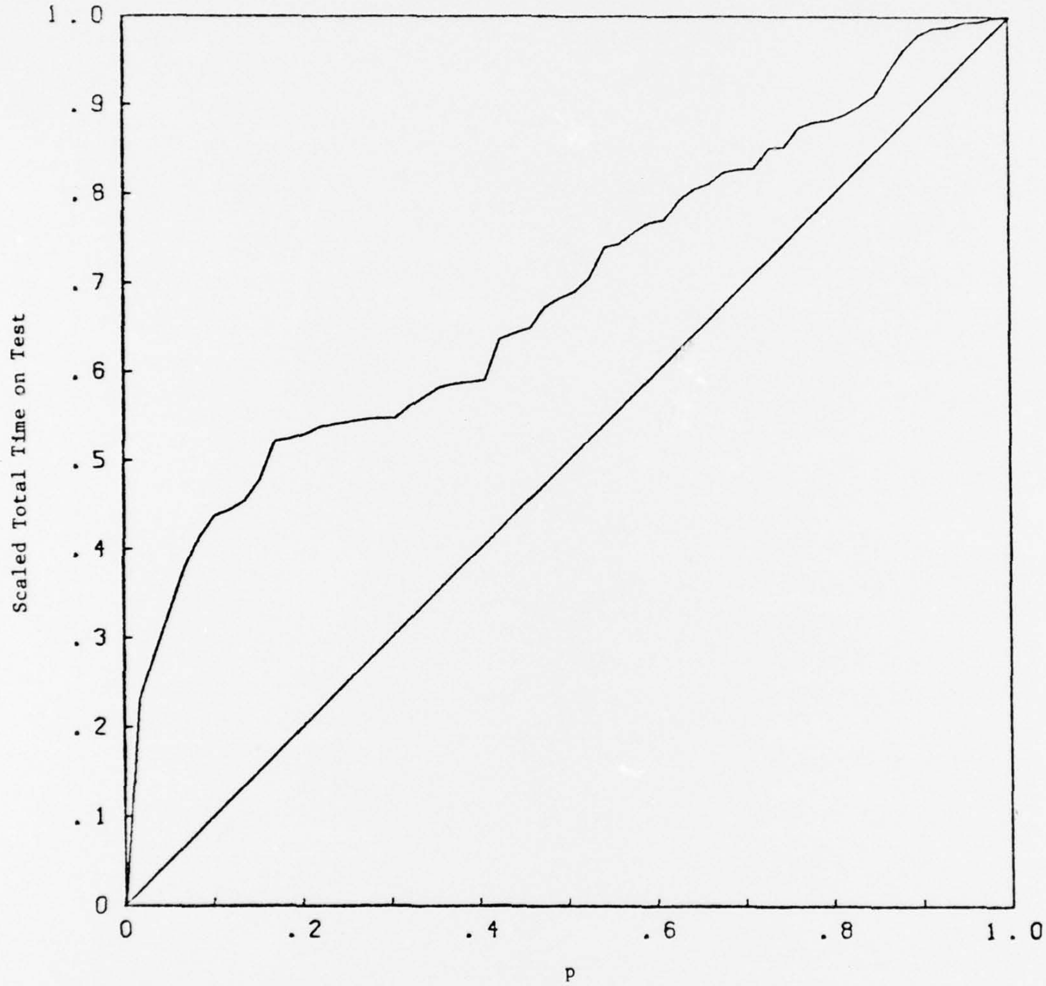
$$\lim_{n\to\infty} \frac{Z_{([pN(T)])}}{Z_{(N(T))}} = \frac{\Lambda^{-1}(p\Lambda(T))}{\Lambda^{-1}(\Lambda(T))}$$

$$= \frac{\Lambda^{-1}(p\Lambda(T))}{T}$$

almost surely. $[pN(T)]$ denotes the largest integer in $pN(T)$. [Note that $n(u) = n$ for $0 \leq u \leq T$; i.e., all processes are completely observed.]

Proof. Let $Y_1, Y_2, \ldots, Y_k, \ldots$ be independent identically distributed exponential random variables each with unit mean. Let $S_k = \sum_{i=1}^{k} Y_i$ and $\Lambda_n(x) = n\Lambda(x)$. Then

$$Z_{([pN(T)])} \overset{d}{=} \Lambda_n^{-1}\left(S_{([pN(T)])}\right)$$



TOTAL TIME ON TEST PLOT

FOR NON-HOMOGENEOUS POISSON MODEL - ENGINE FAILURE DATA

FIGURE 4

where $\stackrel{d}{=}$ means equal to in distribution. Also

$$Z_{([pN(T)])} \stackrel{d}{=} \Lambda^{-1}\left(p \frac{S_{[pN(T)]}}{[pN(T)]} \frac{[pN(T)]}{pn}\right)$$

since $\Lambda_n(x) = n\Lambda(x)$ implies

$$\Lambda_n^{-1}(y) = \Lambda^{-1}\left(\frac{y}{n}\right) .$$

Now $N(T) \to \infty$ almost surely as $n \to \infty$ so that

$$\frac{S_{[pN(T)]}}{[pN(T)]} \to 1$$

almost surely as $n \to \infty$. Since $EN(T) = n\Lambda(T)$, we see that

$$\frac{N(T)}{n} \to \Lambda(T)$$

almost surely as $n \to \infty$. It follows that

$$Z_{([pN(T)])} \to \Lambda^{-1}(p\Lambda(T))$$

almost surely as $n \to \infty$.

Q.E.D.

If $\lambda(u)$, $u \geq 0$ is the intensity function for our failure process then $\Lambda(x) = \int_0^x \lambda(u)du$ is the expected number of failures in

$[0,x]$. If $\lambda(u) = \alpha\lambda^{\alpha}u^{\alpha-1}$, then $\frac{\Lambda^{-1}(p\Lambda(T))}{T} = p^{1/\alpha}$ for $0 \leq p \leq 1$. In this case $\frac{\Lambda^{-1}(p\Lambda(T))}{T}$ is concave in $0 \leq p \leq 1$ for $\alpha > 1$ and convex in $0 \leq p \leq 1$ for $0 < \alpha < 1$. This does not appear to be a good model for the plot in Figure 4. If

$$\lambda(u) = \frac{u^{\alpha-1}e^{-\beta u}}{\int_u^{\infty} w^{\alpha-1}e^{-\beta w}dw}$$

then $\lambda(u)$ is a gamma intensity function and $\Lambda^{-1}(x)$ will be approximately linear for large values of x . For this reason, the gamma intensity function with $\alpha > 1$ may be a better model for the plot in Figure 4.

Since the plot in Figure 4 was based on incomplete data (i.e., not all tractors were observed for 22507 hours), Theorem 2.1 does not strictly apply. However we can make a valid comparison to a homogeneous Poisson process using the following theorem.

Theorem 2.2. If $\frac{\Lambda(x)}{x}$ is nondecreasing in $x \geq 0$ and $n(x) \leq \frac{1}{x}\int_0^x n(u)du$ almost surely, then conditional on $N(T) = N$

$$\frac{\int_0^{Z_{(i)}} n(u)du}{\int_0^{Z_{(N(T))}} n(u)du} \underset{st}{\geq} U_{i:N-1}$$

where $U_{i:N-1}$ is the i-th order statistic from $N - 1$ independent uniform $[0,1]$ random variables. $\left(\underset{st}{\geq}\right.$ means stochastically greater or equal than.$\left.\right)$

Note that if $\lambda(u)$ is nondecreasing, then $\Lambda(x) = \int_0^x \lambda(u)du$ satisfies the condition of Theorem 2.2. It follows from Theorem 2.2., that if the failure rate is nondecreasing, then the scaled total time on test plot will tend to lie above the 45° line $\left(\text{since } EU_{i:N-1} = \frac{i}{N}\right)$. Figure 4 indicates an increasing failure rate process. The distribution of crossings of the scaled total time on test plot in the case of a homogeneous Poisson model has been derived by Bo Bergman (1976). The proof of Theorem 2.2 is similar to that of Theorem 2 in Barlow and Proschan (1969) and will not be given here.

3. A Bayesian Approach to Model Identification. In Section 2 we discussed a method for preliminary model identification. On the basis of Figure 4 and the discussion in Section 2, we could choose a gamma failure process model; i.e., $\Lambda(x) = \int_0^x \lambda(u)du$ where

$$\lambda(u) = \frac{\beta^{\alpha} u^{\alpha-1} e^{-\beta u}}{\int_u^{\infty} \beta^{\alpha} w^{\alpha-1} e^{-\beta w} dw} . \tag{3.1}$$

The parameters α and β are to be estimated. Given the model, the likelihood best summarizes the information in the data concerning the parameters [cf. Basu (1975)].

Let $n(u)$ be the number of systems (e.g., tractors) under observation at age u and $N(T) = N$, the total number of failures. The likelihood function, given $\underline{Z} = (Z_{(1)} \leq Z_{(2)} \leq \cdots \leq Z_{(N)})$ is easily found to be

$$L(\alpha,\beta \mid \underline{Z}) = \left[\prod_{i=1}^{N} n(Z^-_{(i)})\lambda(Z_{(i)})\right] \exp\left[-\int_0^{Z_{(N)}} n(u)\lambda(u)du\right]$$

where $n(Z^-_{(i)})$ is the number of systems under observation just prior to the i-th observed failure and $\lambda(u)$ is given by (3.1). Given a prior density $\pi_o(\alpha,\beta)$, the posterior density on α, β is

$$\pi(\alpha,\beta \mid \underline{Z}) = \frac{L(\alpha,\beta \mid \underline{Z})\pi_o(\alpha,\beta)}{\int_0^{\infty}\int_0^{\infty} L(\alpha,\beta \mid \underline{Z})\pi_o(\alpha,\beta)d\alpha d\beta}$$

by Bayes Theorem. If a diffuse prior; i.e., $\pi_o(\alpha,\beta) \equiv c$ is chosen, the $\pi(\alpha,\beta \mid \underline{Z})$ is proportional to the likelihood function. For illustrative purposes we could graph $L(\alpha,\beta \mid \underline{Z})$ for the data of Appendix 1. The maximum likelihood estimates can be obtained from contour plots of the likelihood function.

4. Graphical Determination of Optimum Replacement Policies. For our non-homogeneous Poisson process model, $\Lambda(t)$ is the expected number of component (engine) failures in $[0,t]$. Let c_1 be the average cost of repairing the component and c_2 the cost of buying a new replacement component. Then, if a new component is bought at time t, the long run average cost of replacing components is

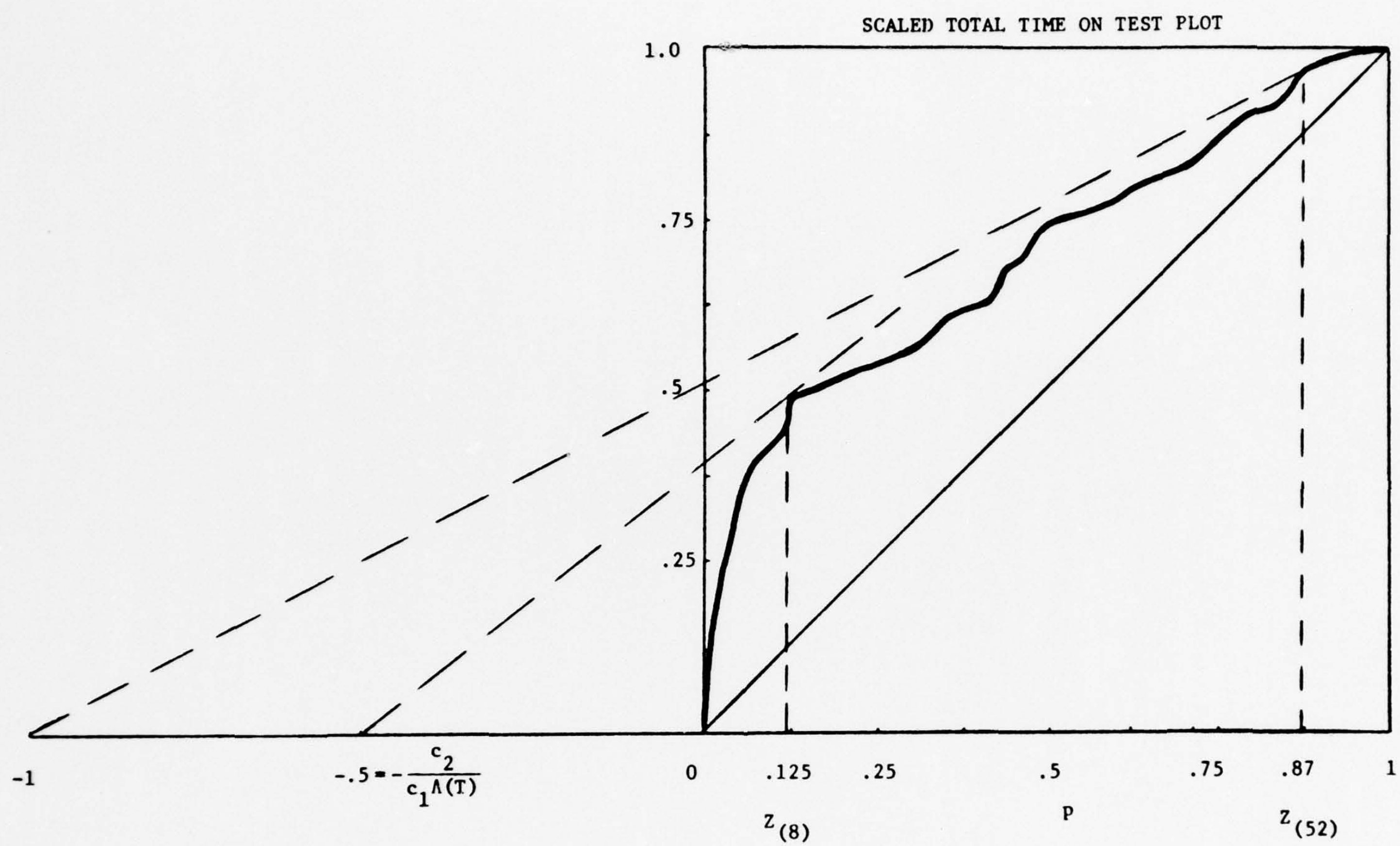
$$C(t) = \frac{c_1\Lambda(t) + c_2}{t} .$$

The numerator is just the expected cost of a life cycle of length t. Let t_o be the optimum replacement age, if it exists; i.e.,

$$C(t_o) = \underset{t>0}{\text{Minimum}} \frac{c_1\Lambda(t) + c_2}{t} .$$

If c_1 is interpreted as the average time to repair the component and c_2 is the average time to replace with a new component, then $C(t)$ is the long run average unavailability where an old component is replaced by a new component at age t.

To determine t_o graphically using the scaled total time on test plot as in Figure 4, first plot $-c_2/c_1\Lambda(T)$ on the x-axis [see Figure 5]. ($\Lambda(T)$ is estimated from the data as in the previous



MINIMUM COST REPLACEMENT SOLUTIONS

FOR $\frac{c_2}{c_1 \Lambda(T)} = 1$ AND $\frac{c_2}{c_1 \Lambda(T)} = .5$

FIGURE 5

section.) Construct a tangent to the scaled total time on test plot as in Figure 5. The projection of the tangent point on the x-axis will correspond to a ratio of the form $i_o/N(T)$. The solution is $Z_{(i_o)}$; i.e., $t_o = Z_{(i_o)}$ will minimize the average life cycle cost per unit time against $\Lambda(t)$ estimated (implicitly) by the scaled total time on test plot. This solution is completely analogous to that of Bergman (1977) for a different model.

The following graphical technique is valid if all failure processes are observed throughout $[0,T]$ and the number of processes, n, is large. To verify the above solution, let $p_o = i_o/N(T)$ and note that by construction

$$\frac{\Lambda^{-1}(p_o\Lambda(T))}{p_o + \frac{c_2}{c_1\Lambda(T)}} \geq \frac{\Lambda^{-1}(p\Lambda(T))}{p + \frac{c_2}{c_1\Lambda(T)}}$$

for $0 \leq p \leq 1$. Hence

$$\frac{p + \frac{c_2}{c_1\Lambda(T)}}{\Lambda^{-1}(p\Lambda(T))} \geq \frac{p_o + \frac{c_2}{c_1\Lambda(T)}}{\Lambda^{-1}(p_o\Lambda(T))} .$$

Now let $t = \Lambda^{-1}(p\Lambda(T))$ and $t_o = \Lambda^{-1}(p_o\Lambda(T))$ so that $p = \frac{\Lambda(t)}{\Lambda(T)}$ and $p_o = \frac{\Lambda(t_o)}{\Lambda(T)}$.

Hence

$$\frac{c_1\Lambda(t) + c_2}{t} \geq \frac{c_1\Lambda(t_o) + c_2}{t_o}$$

which implies t_o is the optimum replacement interval. But $t_o = \Lambda^{-1}(p_o\Lambda(T)) \sim Z_{([p_oN(T)])}$ by Theorem 2.1 Hence $Z_{(i_o)}$ will be (approximately) the optimum replacement age.

Acknowledgement. We would like to thank William Vesely of Nuclear Regulatory Commission for suggesting this problem and Paul Teicholz of the Guy F. Atkinson Construction Company for supplying the data.

REFERENCES

[1] R. E. BARLOW and R. CAMPO, Total time on test processes and applications to failure data analysis, in Reliability and Fault Tree Analysis, edited by R. E. Barlow, J. Fussell and N. Singpurwalla, Conference Volume, SIAM, Philadelphia, 1975.

[2] R. E. BARLOW and F. PROSCHAN, A note on tests for monotone failure rate based on incomplete data, Annals of Mathematical Statistics, 40(1969), No. 2, pp. 595-600.

[3] D. BASU, Statistical information and likelihood, Sankhyā, 37, Series A, Part 1, pp. 1-71.

[4] B. BERGMAN, Crossings in the total time on test plot, Technical Report, University of Lund, Department of Mathematical Statistics, LUNFD6/(NFMS-3043)/1-21/(1976).

[5] B. BERGMAN, Some graphical methods for maintenance planning, Proceedings 1977 Annual Reliability and Maintainability Symposium, Philadelphia, 1977.

[6] E. ÇINLAR, Introduction to Stochastic Processes, Prentice-Hall, Inc., 1975.

Appendix 1. Hours on Tractor and Engine at the Time of Failure and the Date of Failure

Tractor	Engine	Hrs. on Tractor	Hrs. on Engine	Date of Failure
1	1	8230	8230	06-16-71
2	1	5085	5085	04-16-70
2	2	8501	3416	06-24-71
3	1	3826	3826	10-11-71
3	2	6983	3157	11-10-72
4	1	10950	10950	05-08-72
5	1	6052	6052	06-01-70
5	2	12040	5988	08-21-74
6	1	6367	6367	06-07-71
7	1	7774	7774	08-10-70
7	2	12035	4261	01-11-72
7	3	19520	7485	12-28-73
8	1	4394	4394	08-08-69
8	2	9415	5021	02-24-71
8	3	13069	3654	03-08-72
9	1	10517	10517	09-21-70
9	2	13783	3266	07-12-71
9	3	20970	7187	07-11-73
9	4	20988	18	08-14-73
9	5	22273	1285	03-12-74
9	6	22507	234	04-16-74
10	1	2690	2690	05-08-67
10	2	6922	4232	04-30-70
10	3	10815	3893	10-20-71
10	4	12988	2173	06-14-72
10	5	17751	4763	11-19-73
10	6	19458	1707	08-15-74
11	1	6259	6259	03-26-68
11	2	9994	3735	02-14-72
12	1	5278	5278	06-28-65
12	2	6949	1671	07-26-66
12	3	9484	2535	10-09-67
12	4	14383	4899	03-04-71
13	1	6378	6378	08-01-66
13	2	11374	4996	05-11-72
13	3	12771	1397	01-09-73
14	1	6385	6385	09-14-66
14	2	11359	4974	05-15-70

Tractor	Engine	Hrs. on Tractor	Hrs. on Engine	Date of Failure
15	1	6578	6578	08-03-66
15	2	7860	1282	03-22-67
15	3	9719	1859	05-22-68
16	1	5161	5161	04-15-65
16	2	6332	1171	11-12-65
16	3	11288	4956	11-04-69
16	4	16249	4961	07-11-72
16	5	18780	2531	06-27-73
17	1	6717	6717	10-26-66
18	1	6869	6869	11-01-67
18	2	8790	1921	12-10-68
18	3	13315	4525	05-08-71
19	1	5556	5556	04-03-67
19	2	6293	737	08-09-67
19	3	7679	1386	05-18-68
19	4	9931	2252	12-20-72
20	1	3268	3268	07-30-71
20	2	6091	2823	05-31-72
21	1	4815	4815	01-21-72
21	2	8388	3573	03-12-73
22	1	6150	6150	10-31-69