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School of Mathematical Sciences

Mathematics News from the University of Nottingham

Volume 2, Number 1 (June 2023)

In this issue of the University of Nottingham Mathematics Newsletter we have:

We welcome feedback, comments and suggestions. Please let us know what you found most interesting, what else you would like to see, and any other comments you have by filling in the short feedback form at https://tinyurl.com/uonmathsnewsfeedback Alternatively, you can contact us by email at james.walton@nottingham.ac.uk

Editor: Jamie Walton

Maths Outreach at the University of Nottingham

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As well as teaching our own undergraduates, staff at the University of Nottingham are also active in sharing their love of their subjects within the wider community. This includes presentations at science fairs, running our Year 12 (or S5 in Scotland) summer schools, aimed at widening participation, as well as visiting schools or hosting schools groups on our campus.

The School of Mathematical Sciences at the University of Nottingham works with the AMSP (Advanced Mathematics Support Programme) for many of these events. The AMSP supports teachers and students learning Core Maths, AS/A-Level Mathematics and Further Mathematics. Two of our staff members are AMSP local area coordinators, Jerome Foley and Dr Chris Luke, who often work AMSP nationally but also with our academics locally in organising and running events.

SUMS Event

Recently we hosted a SUMS (Steps to University for Mathematical Students) event, aimed at Year 12 female Maths students who were considering studying Maths, or a Maths-related subject, at university. Although Mathematics offers excellent career prospects over all demographics, UK Government statistics show that it can be especially beneficial to the careers of women, coming third of all degree subjects in the estimated average return in salary for women by age of 29, at +41.55%. Historically the subject has suffered from gender imbalance and, thankfully, this seems to have shifted in the right direction over recent years. For example, in the School of Maths at Nottingham, 44% of the 2021–22 undergraduate intake were female, continuing an upwards trend over the last 5 years. Of course, outdated preconceptions about the subject still exist in society at large, which is why events like SUMS are important.

There were many exciting and enriching activities on the day, such as ‘slow Maths’, allowing students the joy of discovery of an area of Maths that usually only begins to be taught at university. A very popular part of the day was the ‘speed networking’ session, where participants met in groups of two or three with one of our mentors (current Maths undergraduates at Nottingham), asking them whatever they wanted about university life and taking a Maths degree. These quick-fire discussions only lasted for a few minutes per mentor, with students then moving between mentors (9 in total), letting them see the wide variety of individuals choosing to study Maths at university.

Gabriella Teriaca, our first early careers speaker. Image from a building tour of the School of Mathematical Sciences building

There were also two engaging and informative talks from recent undergraduates now working in industry, who both talked about their backgrounds, experiences of doing Maths degrees and careers since then. First was Gabriella Teriaca (pictured to the left), who graduated just a couple of years ago with a BSc in Mathematics with Statistics. Gabriella is currently working in Nottingham as a Graduate Data Analyst at Experian. I asked Gabriella why she wanted to be involved:

“I was very excited to get involved with the event to show young people the reality of studying maths and careers you can have on the back of this. It was important to me to break stereotypes of ‘maths’ students and show young people the realities of pursuing a career in maths—it is not as scary as everyone makes it out to be! I wish this opportunity was available to me when I was a student, breaking myths about university life and realising I am not alone in any reservations about further study.”

Asking what she had enjoyed about the day, Gabriella said:

“I particularly enjoyed the chance to talk to current maths students after the event, comparing lecturers and module choices—seeing how much had changed since I had been there (not a crazy long time ago…) and how much had remained exactly the same. I enjoyed the chance to go back to the university and the welcoming and warm atmosphere I was greeted with and the opportunity to speak to so many young people in helping decide on their futures.”

I also asked Gabriella if doing a Maths degree had been useful to her career:

“In my degree, I completed a group work module where our project was in credit scoring—I now work for Experian, a credit bureau, so this was definitely very convenient. More generally, the way of thinking about problems was definitely the most useful aspect that I have carried into my career. Looking at a problem in R or Python, the methods taught at university were so beneficial in approaching problems at work.”

Later in the day we had another early careers talk, from Urvi Dabhi (pictured to the right, talking at the event), who graduated five years ago with a BSc in Mathematics. She is now a Product Manager for Barclays. When I asked why she wanted to be involved, Urvi said:

Urvi Dabhi, speaking to A-Level students at the University of Nottingham at the SUMS event

“It would have been really helpful for me, when I was a student considering doing Maths at university, to hear from graduates on what it was like and how they found it. I wanted to be involved in the event in case any of the student who attended were also feeling the same way!”

On how doing Maths had helped her career, Urvi said:

“More than anything, I think my degree has been helpful in how I solve problems, whether they’re related to maths/data or not.”

KPMG National Numeracy Day Event

As part of the National Numeracy Day, we recently hosted an event joint with the multinational company KPMG (who work in professional services and accounting), for a large group of year 7/8 students from local schools.

During the day the students took part in interactive talks, starting with Rufus Roberts (a Masterclass Speaker for the Royal Institution), on “Computing and card tricks”. Later in the day they were shown around our campus and took part in two interactive sessions with university lecturers from the School of Maths: Dr Ria Symonds talked about “Investigating the mathematics of random growth” and Dr Matthew Scase explained “The Mathematics of fraud”, in particular setting fun challenges of spotting examples of fraud using the surprising Benford’s Law.

The students seemed to have a lot of fun exploring new ideas in Maths that do not necessarily come up in the curriculum but can enrich their understanding and interest in it. The schools’ students also had lots of great questions for the Q&A at the end of the day, including: “what’s your favourite number?” (personally, mine is 5, because of the surprising symmetries of some of the most important examples of some beautiful aperiodic tilings), “do you have detentions at university?” (fortunately we do not seem to need to!) and “what does Maths mean to you?”, which gathered a range of interesting responses. The panel consisted of volunteers from KPMG, current Maths undergraduates from our department (both these groups also worked with the school students throughout the day) and some of our Mathematics lecturers, including Dr Edward Hall, a main organiser who also helped to initiated the collaboration with KPMG. Our undergraduate helpers also got a lot out of the event, gaining experience in explaining Maths to a younger generation and also networking with the volunteers from KPMG over lunch.

Puzzle 2.1

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Stern's diatomic sequence

This puzzle is about a fascinating sequence, called **Stern's diatomic sequence:** $1, 1, 2, 1, 3, 2, 3, 1, \ldots$ The n^{th} term of the sequence will be denoted $a(n)$, so $a(1) = 1$, $a(2) = 1$, $a(3) = 2$, $a(4) = 1$, $a(5) = 3$ and so on. Here are the rules that generate the sequence:

1. the first term is $a(1) = 1$;
2. $a(2n) = a(n)$ (that is, the value at an even position equals that at half the position);
3. $a(2n+1) = a(n) + a(n+1)$ (that is, at odd positions the value is the sum of the two occurring at half that position rounded down and rounded up).

For example, we have that

$$a(8) = a(4) = a(2) = a(1) = 1 \text{ and}$$
$$a(7) = a(3) + a(4) = (a(1) + a(2)) + a(2) = (a(1) + a(1))) + a(1) = 3.$$

We will use this sequence to pair every natural number (the numbers $1, 2, 3, \ldots$) with **every** positive rational number! (numbers such as $1, 2, \frac{1}{2}, 3, \frac{2}{3}$ and so on). Here are the questions (hints can be found later!):

(a) To get more familiar with the sequence, **try computing some more terms:** what are $a(1), a(2), \ldots$ $a(20)$ (that is, the first 20 terms)? Do more if you're finding it fun! The first 8 terms were given earlier.

(b) We introduce an operation on pairs of numbers: given a pair (i, j), we write $F(i, j)$ for a new pair given by subtracting the smaller number from the larger one. For example, $F(9, 3) = (6, 3)$, $F(3, 7) = (3, 4)$ and $F(100, 93) = (7, 93)$. We can repeat this until both numbers are equal, for example: $(9, 7) \to (2, 7) \to (2, 5) \to (2, 3) \to (2, 1) \to (1, 1)$.
Show that if we start with any pair (i, j) then we will always eventually reach the pair (g, g) with g the greatest common divisor of i and j.

(c) Take any pair $(a(n), a(n+1))$ of consecutive terms of Stern's diatomic sequence. **Show there are exactly two pairs (i, j) with $F(i, j) = (a(n), a(n+1))$. Namely, show that these are $(i, j) = (a(2n), a(2n+1))$ and $(i, j) = (a(2n+1), a(2n+2))$.** In particular, we see that only pairs which are consecutive terms in the sequence map to other consecutive pairs in the sequence!

(d) **Using the above parts, show that every consecutive pair $(a(n), a(n+1))$ of terms from the sequence is relatively prime, and conversely that every relatively prime pair of positive natural numbers occurs as a consecutive pair of terms from the sequence**. Remember that two numbers are relatively prime if they share no common factors other than 1.

(e) **Show that no consecutive pair from the sequence can be repeated, that is, $(a(n), a(n+1)) \neq (a(m), a(m+1))$ unless $n = m$.**

Combining parts **(d)** and **(e)** shows that **every** relatively prime pair (i, j) occurs once, and only once, in the form of a pair of consecutive terms of the sequence! Every positive rational number can be written uniquely in the form $\frac{i}{j}$, for relatively prime positive natural numbers i and j. So we have shown that the sequence

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{2}{3}, \frac{3}{2}, \frac{1}{3}, \ldots, \frac{a(n)}{a(n+1)}, \ldots$$

lists every rational number, with each appearing exactly once. This idea is related to "counting infinities", as will be explained more in the hints later and the article about Georg Cantor!

Solution to Puzzle 1.4

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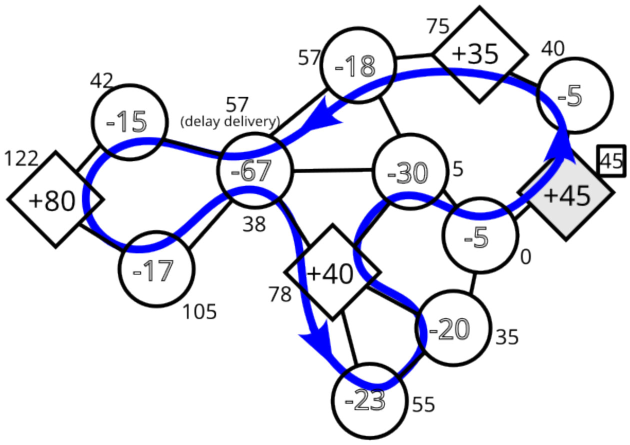
Delivering without backtracking

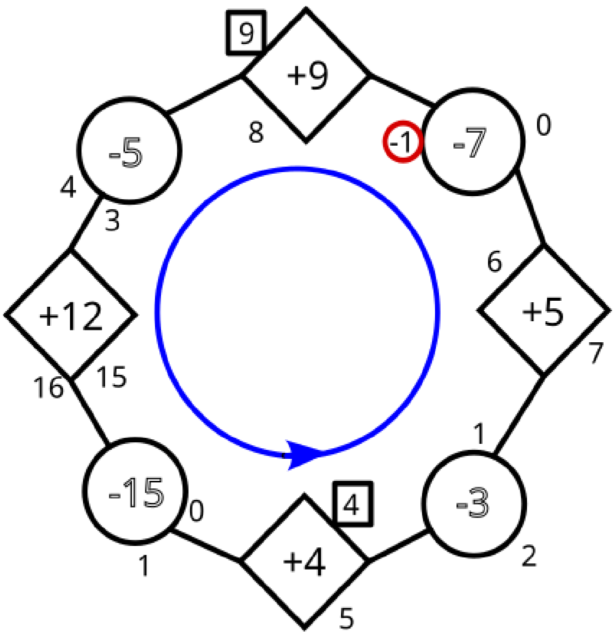
Hopefully you found some starting position and route that solved Santa's logistical problem, which needed us to pick up and deliver gifts along a network graph, never using any edge more than once. There were several possibilities, one is given on the image to the right.



A route that delivers without backtracking, starting at the grey factory.

The more interesting part of this puzzle was devising a general method for finding a valid route, for any network graph satisfying the rules. Following the hint, we first restrict to simple cyclic graphs, like the one below, and temporarily allow Santa to "go into debt" so that he can always continue his journey, even when out of gifts (in which case the number of toys he is carrying may be negative at some locations).

Start anywhere on the given cyclic network, and fix some direction (anti-clockwise, say). There will be a stop S (possibly more than one) where Santa has the fewest gifts, say n gifts. If $n = 0$ then Santa never ran out and we've already found a valid route, otherwise n is negative. The next stop, T, must be a toy factory as otherwise it would result in an even lower running. We claim that T is a valid starting position from which Santa will never run out of gifts.



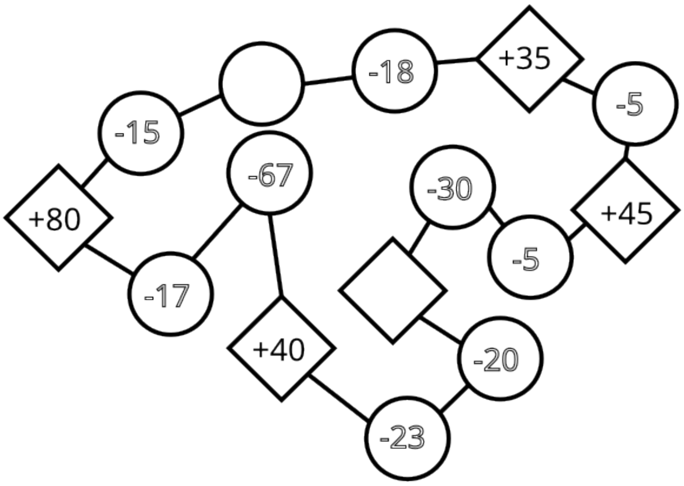
An example cyclic graph. Going anti-clockwise from the +4 and logging the number of gifts carried, we have the lowest running total at the -7 stop (where we also run into debt, so this is not a valid route). This stop is denoted S in the general argument to the right. Starting at T instead, the following +9 factory, means we will be able to get the whole way around without running out.

To see why, log the number of gifts Santa has at each stop using the original starting point, and compare it to the one starting at T. The number of gifts goes up and down between stops identically, regardless of the starting position (for a fixed direction around the cyclic graph), as the change between stops only depends on the number of toys produced or demanded at each location. This even applies around the start/end points, since the running total returns to 0 after a full cycle (as the total number of toys demanded equals that supplied, by assumption). So any two such running logs of toys will agree at every stop, after adding/subtracting the same fixed amount from each stop. In the image to the left, you can see that starting at the bottom $+4$ stop keeps a running total (written on the inside) one less than starting at the top $+9$ stop (written on the outside).

It follows that Santa will have the fewest toys at stop S for *any* given starting position! But if he starts from T, the next stop after S, then he will have 0 presents when reaching S, since this is the final stop and, by assumption, the number of toys demanded equals that supplied. So, starting from T, the fewest gifts Santa every carries is 0, that is, he never ran into debt so was always able to deliver the demanded number of gifts without running out. This route successfully delivers!

Now consider the more general problem, where the graph may not be a basic cycle. By assumption, there's a route around it that visits each stop and uses each edge at most once (recall that we're not allowed to use an edge more than once, that would be inefficient!). Then we can "unfold" such a route to a cyclic graph and treat it as above. For example, following the blue route, we can unfold the original network graph from the previous page to the one below.

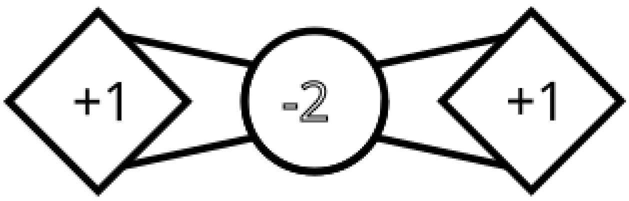
At toy factories that Santa visits twice, split up the pickup of toys however you want on the cyclic graph, and similarly at regions to be delivered to. For example, we could choose to deliver/receive all gifts at the first (or alternatively last) visit to a stop, and just pass through at other visits. A valid route around this cycle now induces a valid one on the original graph.



One way of unfolding the network graph from the previous page into a cycle. The -67 and $+40$ stops are visited twice, and in two places we decide to pass through without action (we can treat these are -0 and $+0$ stops).

Note that how you split the toys at multiply visited stops onto the cyclic graph affects the plan on how they will be delivered, and it may not work if you are required to deliver them earlier. For example, if in the graph to the left we had decided to deliver the 67 gifts immediately after the -18 stop, then we would have ran out if starting at the $+45$ stop. However, this would just change the correct starting location elsewhere, to the $+80$ stop.

So the way we unfold our route, when a stop is visited more than once, is important to the final planned journey. If we are forced to do something different, such as always deliver to each region on the first visit, then the task may be impossible. To find such an example, we notice that we need at least three stops (otherwise there are just two stops, $+n$ and $-n$ for some number n, and we can clearly deliver by starting at $+n$). But going with 3 stops, there is the simple counter-example below. Clearly it is impossible if we are forced to always deliver all gifts when first passing through the -2 stop (as we can only initiate with a single toy, and are then forced through the -2 stop). So, for non-cyclic routes, it is sometimes necessary to delay delivery. Of course, for cyclic routes, we only pass through each region once so we can (and must) always deliver everything on a first passing.



Georg Cantor: To Infinity...

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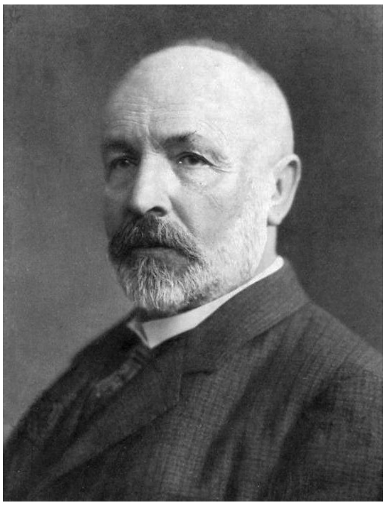
Georg Ferdinand Ludwig Philipp Cantor (1845–1918) was a Russian mathematician, most famous for his work on comparing sizes of infinities and showing that some infinities can be larger than others. More generally, he was central in the development of Set Theory.



Georg Cantor in the early 1900s, public domain

Cantor was born in 1845, in Saint Petersburg, Russia, where he lived until 1856 when he moved to Germany. He transferred from the Swiss Federal Polytechnic in Zurich to the University of Berlin in June 1863. He would submit his dissertation on Number Theory here, receiving his doctoral degree in 1867. After this he taught briefly at a girls' school in Berlin, before finally taking up a position at the University of Halle, in Germany, where he would remain for his whole career. He was made a full professor here in 1879 at the age of 34, which was remarkably young at that time for this type of position.

Set Theory can be considered as a universal language or foundation for Maths. Sets may be thought of as simply collections of objects or "elements". For example, we have the sets of real, rational or natural numbers. There are finite sets, such as $\{1, 42\}$, containing just the numbers 1 and 42, or even the (unique) empty set $\{\}$ which contains nothing! They do not have to contain numbers, they can also include (mixtures of) any formally constructed objects, including vectors, geometric objects or other sets.

Set Theory formalises ways of combining and manipulating sets. The Zermelo–Fraenkel system is the standard axiomatisation of it ("ZF", and typically even "ZFC" which is "ZF with the axiom of choice"). Some of Cantor's discoveries, which we will explain below, were initially greeted with a lot of scepticism, so one can see why it is important to establish a solid foundational footing.

Cantor's most famous discovery is that sets, even infinite ones, can be compared in size and that not all infinite sets are the same size. To understand how we might "count infinities", it is helpful to first consider the more familiar situation of counting finite sets. If I have a collection of three objects, say an apple, orange and banana, I count them by assigning each a unique number, from 1 to 3. So we map each number in the set $\{1, 2, 3\}$ to an object in our set $\{$apple, orange, banana$\}$. Of course, it should not matter which order we count them in: I could assign 1 to apple, 2 to orange and 3 to banana, but I could also have ordered my objects with banana first, then apple and orange last. Either way, I will conclude there are three objects.

So, to establish a set has three elements is to pair the elements of $\{1, 2, 3\}$ with our set. Being mathematically precise, "pairing" is choosing some function f that takes as inputs the elements of $\{1, 2, 3\}$, and for each there is an output in our set. For example, above we could let $f(1) =$ apple, $f(2) =$ orange and $f(3) =$ banana. To be a pairing, or "bijection", we must only count each element once (it would be no good if $f(1) = f(2) =$ orange, which counts the orange twice!), and each element should be mapped onto (if none of $f(1), f(2), f(3)$ equal orange then we have not counted it!). In mathematical terminology: our function should be "injective", meaning that $f(i) \neq f(j)$ for $i \neq j$, and "surjective", meaning that for all elements y in our set, there is some input x with $f(x) = y$. A function that is both injective and surjective is a "bijection", or "pairing". This easily generalises to counting finite sets of any size: for a natural number n, a finite set S has exactly n objects if there is a bijection from the set $\{1, 2, 3, \ldots, n\}$ to S. Generally, a bijection pairs the elements of its set of inputs and outputs ("domain" and "codomain"), which can thus be considered the same size (or "cardinality") as each other.

Notice that the above ideas still make sense for infinite sets: if we can find a bijection between two sets S and T, then we have paired the elements of each of them and thus they can be considered "the same size", whether they are finite or infinite sets. In particular, we can ask if there is a bijection from the set $\{1, 2, 3, 4, 5, ...\}$ of all natural numbers to some given set S. Doing this would be to assign the numbers 1,2, 3, … to elements of S, never counting any element more than once, and making sure every element is counted. In this case, we call S "countably infinite". Intuitively, this means we can enumerate the elements of S in an "infinite list", with a first, second, third, … element, with every element appearing exactly once.

Some peculiar things can happen for infinite sets which never happen for finite sets. If two finite sets S and T are equal in size, then of course removing one or more elements from T will make it a different, smaller size to S. This is not true for infinite sets! For example, suppose I take the set of all natural numbers but then remove 1, to get the set $S = \{2, 3, 4, ...\}$ of all natural numbers excluding 1. This does not affect its size! Indeed, consider the bijection $f(i) = i + 1$, which maps the set $\{1, 2, 3, ...\}$ of all natural numbers onto S. This pairs the elements of each set together, so they must be the same size. Related here is Hilbert's Hotel: a hotel with an infinite number of rooms can always accommodate one more guest, by shifting everyone one room to the right, leaving space for one more. See here for a video explaining this topic in more detail.

Perhaps more surprising is that we can even remove an infinite number of elements without affecting a set's size. For example, remove all of the odd natural numbers to obtain the set $S = \{2, 4, 6, ...\}$ of all even natural numbers. The bijection $f(n) = 2n$ maps $\{1, 2, 3, ...\}$ onto S, establishing that S is still the same size as the full set of natural numbers, not smaller (or "half the size"), as one might first expect.

In Puzzle 2.1 we showed how we could write down a rule that generates an infinite list of positive rational numbers, with each appearing exactly once. Hence, taking the bijection that sends a natural number n to the n^{th} element of the list establishes that the positive rational numbers are still merely countably infinite i.e., the same size of infinity as just the natural numbers. This is quite surprising, as the rational numbers densely fill the number line, so one may, incorrectly, think there are "more" of them!

The above may lead you to think that all infinite sets have the same size, something many used to believe. However, this is far from the case: there are an infinite number of different sizes of infinity! The infinite set of all real numbers (numbers whose decimal expansions do not have to repeat) is **not** countably infinite. Although not his first approach, Cantor's most famous proof for this follows his diagonal argument (which proves a more general principle): Suppose, for a contradiction, that the natural numbers and reals are sets are the same size i.e., there is a bijection f from the natural numbers to the real numbers. In particular, f is surjective: we "count" each real number at least once, none is missed (that is, for each real number x, we have $f(n) = x$ for some natural number n). We will show that cannot be true, by finding a real number x that was not counted. For each n, let $b_n = 2$ if the n^{th} digit of $f(n)$ is 1 and otherwise let $b_n = 1$ (we make decimal expansions unique representations of real numbers: whenever there is a choice, we take the expansion to end in recurring 0's rather than 9's). Then $x = 0.b_1b_2b_3\cdots$ is a real number that cannot have been counted! Indeed, for any n, we cannot have that $f(n) = x$ for some n since, by design, the n^{th} digit of $f(n)$ was taken to be different to b_n, the n^{th} digit of x. Thus, there can never have been such a surjection (and thus bijection) f and the set of natural numbers is smaller than the set of real numbers!

Although things like Hilbert's hotel with infinitely many rooms cannot really exist, infinite sets are ubiquitous in Mathematics, so being able to rigorously deal with them is very important. Cantor's ideas faced opposition at the time from a few prominent and influential mathematicians, most notably Leopold Kronecker. Sadly, this stifled Cantor's career and caused him a great deal of depression. It seems there has been a great philosophical shift in how modern Mathematics progresses, where interesting but surprising results are celebrated and accepted, so long as they are rigorously proved from the underlying axioms. And, indeed, Cantor's theories are now established and accepted Mathematics, with ideas of (un)countable infinity typically appearing in the first year of an undergraduate Mathematics degree.

More on Stern’s Diatomic Sequence and Hints for Puzzle 2.1

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Back to Puzzle 1.4

Stern’s remarkable sequence

Hopefully you managed to compute many more terms of the sequence in part (a) of this week’s puzzle. Listing out ratios of consecutive terms gives the following:

$$\frac{1}{1},\frac{2}{1},\frac{1}{2},\frac{3}{1},\frac{2}{3},\frac{3}{2},\frac{1}{3},\frac{4}{1},\frac{3}{4},\frac{5}{3},\frac{2}{5},\frac{5}{2},\frac{3}{5},\frac{4}{3},\frac{1}{4},\frac{5}{1},\frac{4}{5},\frac{7}{4},\frac{3}{7},\frac{8}{3},\frac{5}{8},\frac{7}{5},\frac{2}{7},\frac{7}{2},\frac{5}{7},\frac{8}{5},\frac{3}{8},\frac{7}{3},\frac{4}{7},\frac{5}{4},\frac{1}{5},\dots$$

Amazingly, the result of the puzzle above is that every positive rational number appears exactly once in this sequence! This means that you can pair the “natural” or “counting” numbers $1, 2, 3, \dots$ with the positive rationals, so they represent the same size of infinity (see the earlier article on Cantor), that is, there are just as many natural numbers as there are rational numbers! In fact, there are much easier ways to establish this, but this sequence gives a remarkably beautiful and explicit way, generated by a few neat rules.

The sequence has many other fascinating properties, too many to list here, but I would like to list some favourites. The sequence is often presented via “Stern’s diatomic array”, which is given by starting with two $1s$ on the top row and then, to produce the next row, copying the previous row and placing between each pair of numbers their sum:

$$\begin{array}{ccccccccccccccccc}
1 & & & & & & & & & & & & & & & & 1 \\
1 & & & & & & & & 2 & & & & & & & & 1 \\
1 & & & & 3 & & & & 2 & & & & 3 & & & & 1 \\
1 & & 4 & & 3 & & 5 & & 2 & & 5 & & 3 & & 4 & & 1 \\
1 & 5 & 4 & 7 & 3 & 8 & 5 & 7 & 2 & 7 & 5 & 8 & 3 & 7 & 4 & 5 & 1
\end{array}$$

If you ignore the right-hand ones and read out the sequence, we get back Stern’s sequence: $1, 1, 2, 1, 3, 2$, 3, 1, 4, … The n^{th} row contains $2^{n-1}+1$ terms and sums to $3^{n-1}+1$. It follows from the symmetry of the initial row (and the symmetry of our rules) that each row is palindromic (it reads the same forwards and backwards). If we forget the 1s on the right-hand side and write the terms out row-by-row, forgetting the spaces, we get the “crushed array”:

$$\begin{array}{llllllllllllllll}
1 \\
1 & 2 \\
1 & 3 & 2 & 3 \\
1 & 4 & 3 & 5 & 2 & 5 & 3 & 4 \\
1 & 5 & 4 & 7 & 3 & 8 & 5 & 7 & 2 & 7 & 5 & 8 & 3 & 7 & 4 & 5 \\
1 & 6 & 5 & 9 & 4 & 11 & 7 & 10 & 3 & 11 & 8 & 13 & 5 & 12 & 7 & 9\cdots
\end{array}$$

The maximum entries in each row give the sequence $1, 2, 3, 5, 8, 13, 21, 34, \dots$, which turns out to be the Fibonacci sequence! (except missing the first 1). This is the sequence where we add the previous two numbers to get the next one. It also turns out that the numbers in each column increase by the same successive difference all the way down (they are “arithmetic progressions”): for example, the first column stays the same (successive differences of 0), the second and third increase by 1 each step down, the fourth increases by 2, the eighth increases by 3 etc. If you write out these successive differences, you recover Stern’s diatomic sequence!

There is also a fascinating connection to “hyperbinary expansions”. We usually write numbers in decimal, where an expansion like 32763 means $3\times10^4+2\times10^3+7\times10^2+6\times10^1+3\times10^0$ i.e., each digit (which is one of 0, …, 9) corresponds to a power of 10. In binary expansions, each digit is either 0 or 1 and corresponds to a power of 2. For example, the binary number 1011010 is, in decimal notation, the number $1 \times 2^6 + 0 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 90$.

Just as every natural number has a unique decimal expansion, it also has a unique binary expansion. But now suppose that we consider binary expansions that are not limited to using each power of 2 at most once, rather we may use each at most twice. Equivalently, our "digits" can now be 0, 1 or 2. Such an expansion is called a **hyperbinary expansion** and they are no longer always unique for a given number. For example, 10 has five hyperbinary expansions: $10 = 8 + 2,\ 8 + 1 + 1,\ 4 + 4 + 2,\ 4 + 4 + 1 + 1$ and $4 + 2 + 2 + 1 + 1$. Try finding the number of hyperbinary expansions for each of the numbers from 1 up to 10. Amazingly, you will see a connection with Stern's sequence! The n^{th} term of Stern's sequence, $a(n)$, is equal to the number of hyperbinary expansions of $n - 1$ (for this good reason, sometimes people index Stern's sequence differently, by shifting each term down one place). So, for example, the 11^{th} term of Stern's sequence is $a(11) = 5$, the number of hyperbinary expansions of the number 10.

Some hints for Puzzle 2.1

(b) If a has a factor g, that means we can write $a = kg$ for some integer k. If i and j share some factor g, then why does the pair $F(i, j)$ still share that factor?

(c) Take any non-equal pair i and j and consider separately the cases where $i < j$ and $i > j$. In the first case we have $F(i, j) = (i, j - i)$ and in the second $F(i, j) = (i - j, j)$. Set these equal to $(a(n), a(n + 1))$ and then use the rule generating Stern's sequence.

(d) Any consecutive pair of terms from the sequence is either of the form $(a(2n), a(2n + 1))$ or $(a(2n + 1), a(2n + 2))$. By the previous part, applying F takes this to an earlier consecutive pair. How does part (b) combine with this to show that every pair of consecutive terms from the sequence is relatively prime?

For the second part of the question, suppose that instead you start with some given relatively prime pair. Why does part (b) show that iterating F enough times takes you to a consecutive pair from the sequence, and why does (c) then show your original pair was a consecutive pair from the sequence?

(e) This is the trickiest part! Firstly, why can $(1, 1)$ only appear once, as the first pair of consecutive terms? Next, suppose that a different pair repeats, say $(a(n), a(n + 1)) = (a(m), a(m + 1))$ with $n \neq m$ and n as small as possible, that is, with $(a(n), a(n + 1))$ the earliest pair that repeats itself somewhere later. Since we also have $F(a(n), a(n + 1)) = F(a(m), a(m + 1))$, why can we conclude that $m = n + 1$? But this runs into a contradiction. Why?

The ideas for the puzzle and further interesting properties on Stern's sequence came from a variety of sources, most notably Dr Sam Northshield's article and interesting discussions with Dr Chris Luke.

Useful Links

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Here are some links to useful resources.

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Volume 1, Number 1	Information on the careers of University of Nottingham mathematics graduates, Puzzle 1.1 (about prime numbers), a short article about Edmund Landau and Landau's problems, links to useful resources
Volume 1, Number 2	Information about our three-year single-honours mathematics BSc degree, Puzzle 1.2 (about discs), the solution to Puzzle 1.1, an article about Alice Roth and Swiss cheeses, links to useful resources, back issues
Volume 1, Number 3	Maths graduation celebration and prizegiving, spotlight on Bindi Brook, Puzzle 1.3 (about peg solitaire), the solution to Puzzle 1.2, an article about John Horton Conway, links to useful resources, back issues
Volume 1, Number 4	Information about our four-year MMath degree, Puzzle 1.4 (on Santa's logistics), the solution to Puzzle 1.3, an article about Leonhard Euler, links to useful resources, back issues
Volume 2, Number 1	Recent Maths Outreach events at the University of Nottingham, Puzzle 2.1 (Stern's Diatomic Sequence), an article about Georg Cantor, links to useful resources, back issues