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School of Mathematical Sciences

Mathematics News from the University of Nottingham

Volume 1, Number 4 (December 2022)

In this issue of the University of Nottingham Mathematics Newsletter we have:

- **Studying Mathematics at the University of Nottingham**: This issue we focus on our 4-year MMath degree and talk to a recent graduate on her experiences with it
- **Puzzle 1.4**: A graph-theoretic puzzle to help Santa efficiently deliver presents!
- **Solution to Puzzle 1.3**: The solution to last issue's puzzle
- **Interesting Mathematical Facts**: An article about Leonhard Euler
- **Hints for Puzzle 1.4**: Full details next issue
- **Useful Links**: Links to useful resources
- **Back issues**: Links to all issues of this newsletter so far

We welcome feedback, comments and suggestions. Please let us know what you found most interesting, what else you would like to see, and any other comments you have by filling in the short feedback form at https://tinyurl.com/uonmathsnewsfeedback
Alternatively, you can contact us by email at james.walton@nottingham.ac.uk

Editor: Jamie Walton

Studying mathematics at the University of Nottingham

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The four-year Mathematics MMath degree at the University of Nottingham

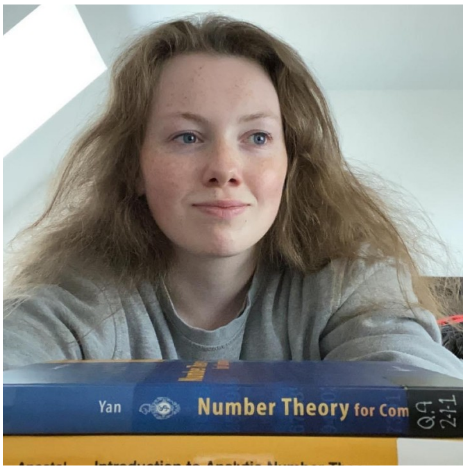
The modern world is powered by Mathematics, which continues to provide key insights into into Science, Technology and Business, and is applicable to a wide variety of careers. If you are fascinated by how abstract and beautiful ideas can help us to structure and understand the world, then a Mathematics degree might be just for you!

In Issue 2 we gave an overview of our three-year single-honours Mathematics BSc (https://tinyurl.com/uonG100), which gives our students the opportunity to learn in depth about a wide range of mathematical topics, in both Pure and Applied Mathematics, Probability and Statistics. Our four-year MMath (https://tinyurl.com/uonG103) extends the length of your course by an extra year, providing all of the same benefits but also allowing for further specialisation in your favourite subjects.

Mathematicians at the University of Nottingham work in a diverse variety of areas, such as the modelling of epidemics, Statistical Machine Learning, Algebra and Number Theory, Geometry and Symmetry, the application of Fluid Mechanics to problems in industry and Biology, Quantum Mathematics and much more. By your final year you will be learning about topics on the cutting-edge of mathematical research and have the opportunity to put what you have learned in practice, by writing a dissertation on a project that you will be researching under the guidance of an expert in your chosen field. In the process you will learn transferable skills that are highly valued by employers, such as problem solving, creativity, learning how to conduct innovative research and giving presentations on your work.

To find out more about the student experience of taking the MMath at Nottingham, I spoke to Holly Justice, who graduated this year. Holly recently started a PhD project, also at the University of Nottingham, on Mathematics Education.

Why did you choose to do a degree in Mathematics?
Maths always ‘clicked’ more than any other subject I did at school. I was happy with my grades in other subjects but with Maths I always felt like I really *understood* things, more than just being able to answer exam questions. My Maths teachers were inspirational, and at A-Level they gave me a lot of independence and let me direct my own learning a lot, which really drove my curiosity for Maths. I think they also helped me to realise that I didn’t need to have a career in mind to decide what to study at university, which I’m grateful for because I changed my mind several times about what I wanted to do in the end!



Holly Justice, who graduated with an MMath from Nottingham in 2022

And why Nottingham?
It was a long time ago now, but when I applied for university I wanted to choose somewhere with a really good reputation for Maths, and somewhere a bit different to my home town. Other than the academic side, Nottingham is a great city with fantastic nightlife and transport links, University Park campus is beautiful and it’s always nice to see the squirrels and other wildlife when you’re walking around! Now I’ve been here 4 years and stayed on to do my PhD at the university, I can say that the Maths department is

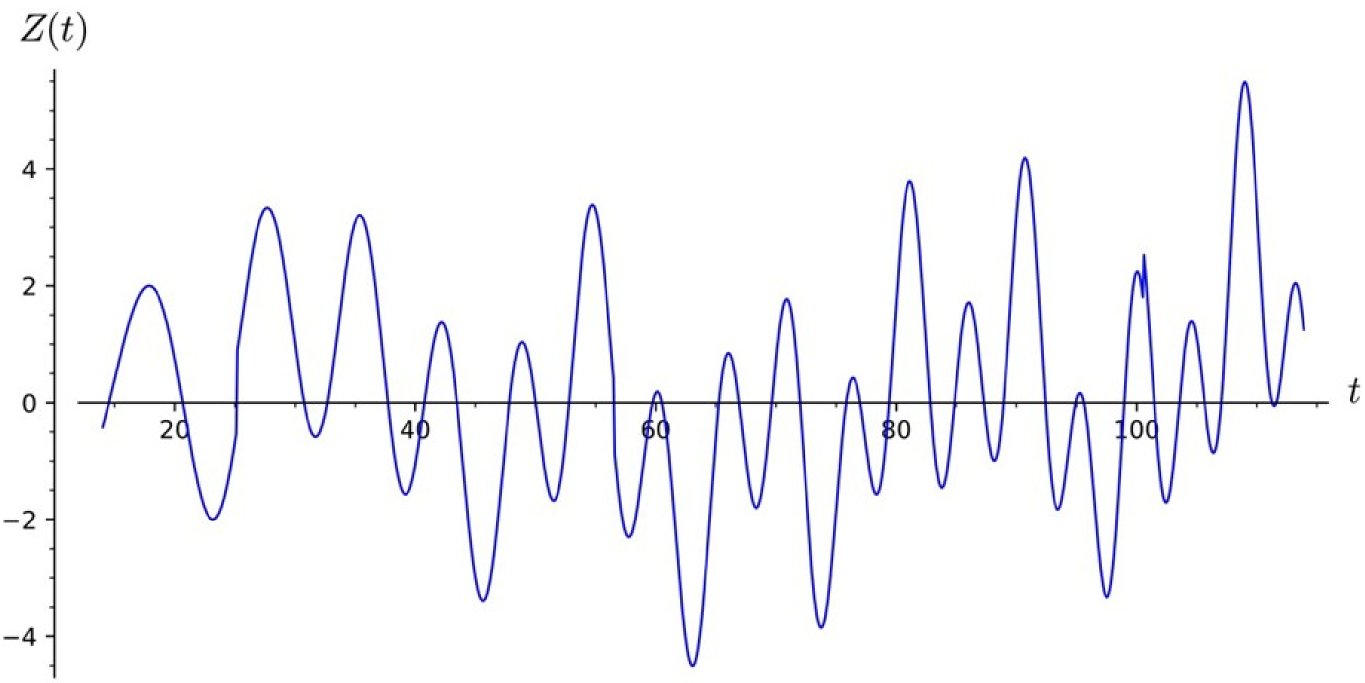
great, and there are plenty of ways to get involved other than just through lectures. I did a couple of undergraduate research internships in the department, and they were great – interesting, good to put on your CV, and a way to make some money over the summer!

What advantages do you feel you gained by taking the 4-year MMath over the 3-year BSc?
Honestly, I chose the MMath when I applied because I wanted to be at university for as long as possible and figured I would probably end up doing a master's anyway, so why not?... When I actually got to university, I realised the 4th year wasn't a commitment: I could easily transfer to the BSc if I wanted to.
I felt one of the biggest advantages of the MMath (as opposed to continuing my studies by doing a stand-alone master's degree) was the fact that I was very well equipped to access the modules in my 4th year. I was used to the teaching and assessment at UoN already, and the 4th year was the same structure as the previous years, which is not quite true for the MSc. The 4-year MMath is also set up so that you complete a 3rd year project which prepares you for the dissertation in 4th year, and I personally liked my 3rd year project so much that I chose the same supervisor for the 4th year dissertation.

Could you describe your dissertation's research project?
My project looks at historical methods used to compute the Riemann zeta function. Computing the Riemann zeta function is really important to mathematicians because it means they can numerically verify the Riemann Hypothesis (one of the currently unsolved Millennium Problems) for a given region of the critical line in the complex plane, $s = 1/2 + it$. Even though mathematicians will never be able to completely prove the Riemann Hypothesis just by continuously finding its zeroes numerically, they could *disprove* it if it is false. All it would take is finding one zero off the critical line.

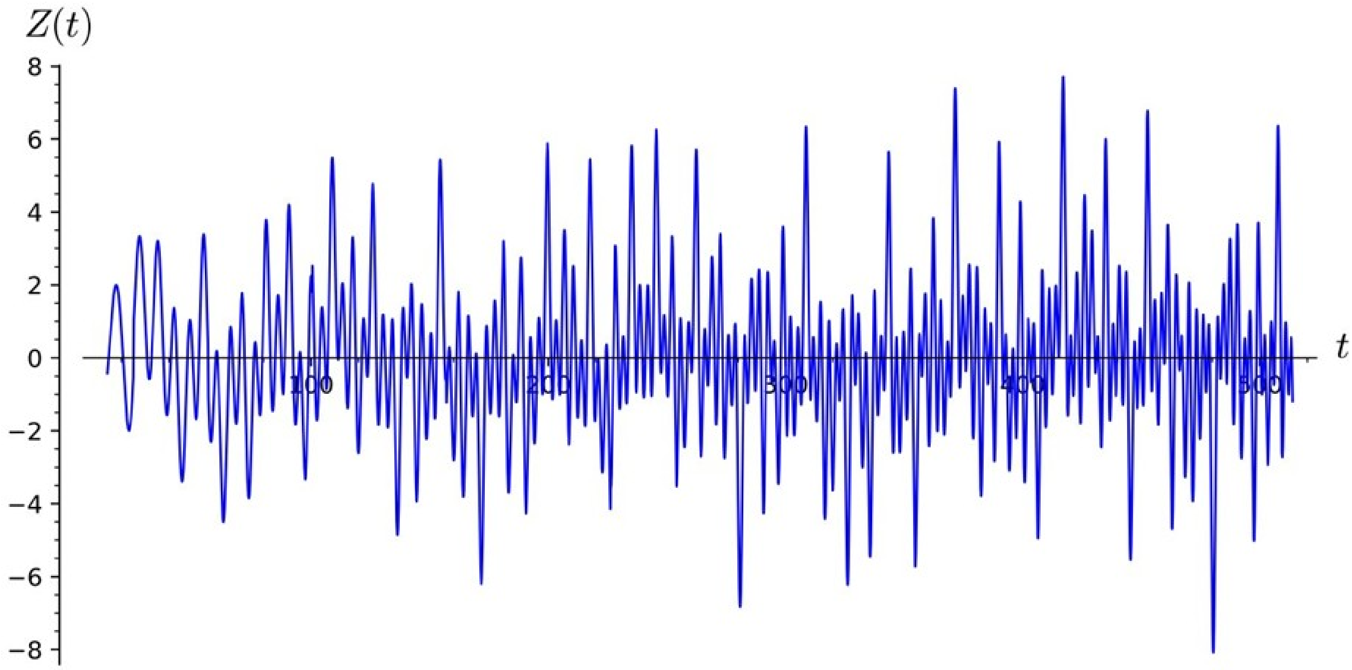
In my project, I looked at how some of these methods were derived, and I investigated the parameters which yielded the smallest error in the approximation for each of the methods.

These are some graphs of the 'Z' function, which has the really cool property that $Z(t) = 0$ whenever $\zeta(1/2 + it) = 0$ (which means we can use the Z function to locate the zeroes of the Riemann Zeta function!). The Z function is also a real function (unlike the complex-valued Riemann-Zeta function), so it is a lot easier to work with.



Holly's graph of the Z function, which can be used to find zeros of the famous Riemann zeta function

So much work has been done in this area, and there are other sophisticated methods of locating zeroes. Nevertheless, the few methods I investigated in my project helped me to gain some insight into how efficient computational methods are developed, and why they are sought after. Understanding how the methods were originally derived also helped me to really understand what the analytical continuation of a function means.



Another graph of the Z function from Holly's dissertation, shown over a wider domain, demonstrating its complicated behaviour

Overall, my project helped me to make connections between a lot of the areas of Maths I had studied over the course of my degree, and I thoroughly enjoyed being able to devote time to reading the things that I wanted to read. Admittedly, when I was completing my project, I felt like I was in the dark a lot of the time, and I struggled to move forward sometimes, but by the end everything had clicked into place, and I realised how much I had actually learned. It felt like a big accomplishment when I had finished. Not only because I understood the things I had written, but because I had learned lessons about organising my time, recording what I had read, and writing proofs without being given hints or prompts like in lectures.

Puzzle 1.4

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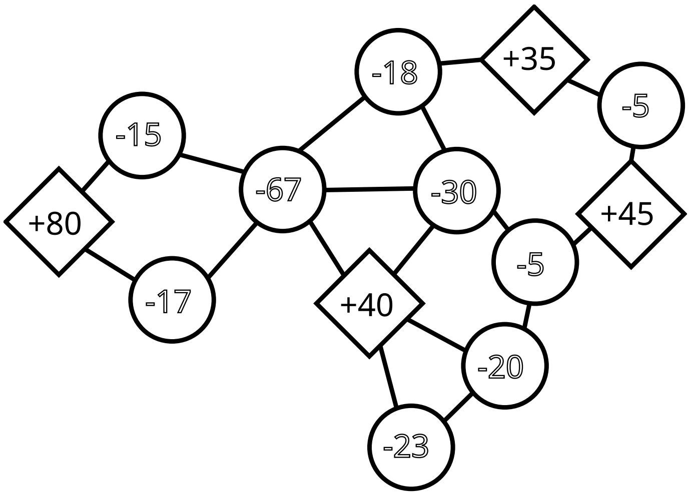
Help Santa to Deliver!

As we all know, Santa's gifts are made by elves. However, you may not know that elves do not all live in the North Pole any more: Santa recently globalised production so that toys are made across the globe. Santa will need to visit each toy factory to collect the toys on Christmas (not earlier, the elves have left it to the last minute as usual!) and then drop them off at children's homes.

Santa's helpful elven Head of Logistics, Zippy, has created a map of toy factories and regions to deliver to. The number of toys available from each factory is labeled as a positive number inside a square. The number of toys he needs to deliver to a region is labeled as a negative number in a circle. Both are measured in tens of millions (so a factory making 800 million toys is labeled "+80", and a region needing 50 million toys is labeled "-5"). The factories and regions are linked up by edges, showing a valid route Santa may take between them on Christmas, short enough for Santa's reindeer to go without breaks in-between. Being environmentally conscious, of course the elves make no more presents than necessary: the sum of all of the labels on the network is zero.

Although Santa's sleigh is magical and so can hold an unlimited number of presents, it is not able to magically produce new ones. Therefore, Santa needs to make sure he never runs out of presents on his journey.



How can Santa go around the graph without running out of presents?

Here is the puzzle!

We want to find a starting point and path around the network that ensures Santa never runs out of presents, meaning that he never needs to drop off more presents than he has already picked up and not yet delivered. Santa demands does not want to retrace his path, as he wants an efficient route, gets bored traveling the same path twice on the same night and also wants to minimise the risk of being spotted! So can you find a path that never uses any given edge more than once, delivers all of the toys and so that Santa never runs out of toys? If Santa visits a region twice (at a 'junction'), he may decide how many gifts to deliver each time.

Once you have found a path for Santa for the above network, consider the general problem, for any network that satisfies the following:

- the network consists of nodes and edges between them, where each node is labeled with either a positive number (a toy factory) or negative number (a region to deliver presents to);
- the sum of nodes is zero (the number of toys made by factories equals the number to be delivered);
- there is at least one 'closed circuit', a path that visits every node, starts and ends at the same node and uses no edge more than once.

Try to find an argument as to why, in the above situation, there must *always* exist a path that never uses any edge more than once and never runs out of presents. If Santa decides he must drop off all presents the first time he enters a region, is such a path still always possible? This is not easy, so some hints will be given later in case you get stuck!

Solution to Puzzle 1.3

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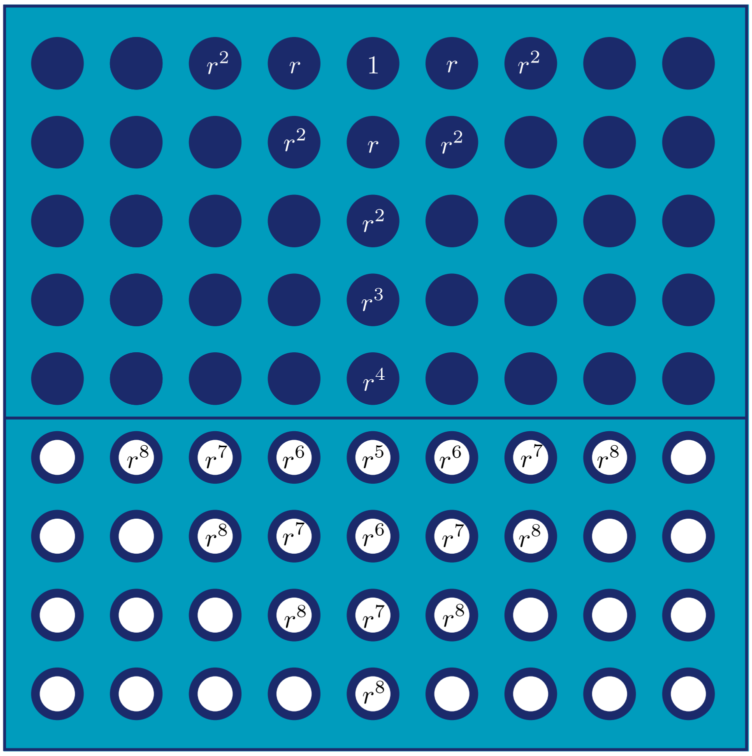
Perplexing Pegs

Solution: unattainable values!

Hopefully you managed to find a sequence of moves that gets pegs to rows 1, 2, 3 and 4. If not, have a go now! You will find that getting to row 4 takes a bit of work, in fact it takes at least 19 moves. We will explain here Conway's proof that pegs can never reach row 5. For further details, see here.

As described in the hint, we assign a 'value' to each hole, with our target getting value 1 and others getting value r^k, where k is the smallest number of (adjacent) steps to that hole (see the figure) and $r = \dfrac{\sqrt{5}-1}{2}$ is the reciprocal of the golden ratio.

Remember that a 'move' jumps one peg over another one, into an empty hole, removing the peg we jump over.



Some of the values assigned to holes (with or without pegs)

What happens to the sum of values of pegs after making a move? Consider a triple (from,over,to) of values of pegs involved in a move, where the first entry is the value of the jumping peg, the second is the value of the peg we jump over and the third is the value of the peg in its new home. Supposing the original peg has value r^k, there are three options:

1. (r^k, r^{k+1}, r^{k+2}), if the peg jumps away from the target;
2. (r^k, r^{k-1}, r^{k-2}), if the peg jumps towards the target;
3. (r^k, r^{k-1}, r^k), if the peg remains the same distance from the target (that is, jumping over the central vertical line).

In the first case, since we remove the middle peg, the difference between the old and new values is $r^k - (r^{k+2} - r^{k+1}) = r^k(-r^2 + r + 1) = r^k(2r) > 0$, recalling that $r^2 + r = 1$ and thus $r^2 = 1 - r$. In other words, the previous value of pegs was larger, so the total value has decreased. That makes sense: we lost one peg, and the one we moved has jumped to a spot of lower value.

In the second case, jumping towards the target, we calculate that the change in value is $r^k - (r^{k-2} - r^{k-1}) = r^{k-2}(r^2 + r - 1) = 0$. That is, the value does not change.

Finally, if the peg jumps to another hole of the same value then its contribution does not change but we lose the middle peg of value r^{k-1}, so in this case the total value decreases.

We see that moves can only keep the value the same or make it lower!

How many points do we have to play with at the start? We have a useful sum:

$$r^k + r^{k+1} + r^{k+2} + r^{k+3} + \cdots = \frac{r^k}{1-r} = \frac{r^k}{r^2} = r^{k-2}.$$

The above is the sum of an infinite geometric progression. This has been calculated using the formula in the hints, using $a = r^k$ and common factors between terms being r, and using $1 - r = r^2$.

This lets us work out the sum of values along any row. For example, the sum of values along the top row of starting pegs (which is 5 steps below the target) is given by r^5 (the peg in the centre) plus two lots of $r^6 + r^7 + r^8 + \cdots = r^4$. So along this row the total value is $r^5 + 2r^4 = r^4(r+2)$. By a similar calculation, the value of the row n steps below the target is $r^{n-1}(r+2)$.

So, summing over all rows, the starting pegs have total value:

$$r^4(r+2) + r^5(r+2) + r^6(r+2) + r^7(r+2) + \cdots .$$

This is again an infinite sum over a geometric progression! The ratio of successive terms is r and the starting term is $a = r^4(2+r)$, so that the starting value of all pegs is:

$$\frac{r^4(r+2)}{1-r} = \frac{r^4(r+2)}{r^2} = r^2(r+2).$$

Using $r^2 + r = 1$ twice, we find that this works out as:

$$r^2(r+2) = r(r^2 + r + r) = r(r+1) = r^2 + r = 1$$

Now, suppose we managed to reach our target on row 5. Then the value of all pegs after making these moves must be strictly greater than 1. Indeed, the target peg itself already has value 1. and there will be other pegs left over that also contribute (infinitely many of which we have not even touched!). But we have seen that moves never increase the score and we only start with pegs of total value 1, so it is impossible to raise our value above 1. We conclude that we can never reach row 5.

Leonhard Euler: Bridges and Graphs

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Leonhard Euler (1707–1783) was a Swiss mathematician, considered to be one of the greatest of all time. As well as making contributions to many different areas of Mathematics, he also worked on its application to areas such as Astronomy, Engineering, Optics, Fluid Dynamics and the Theory of Music.

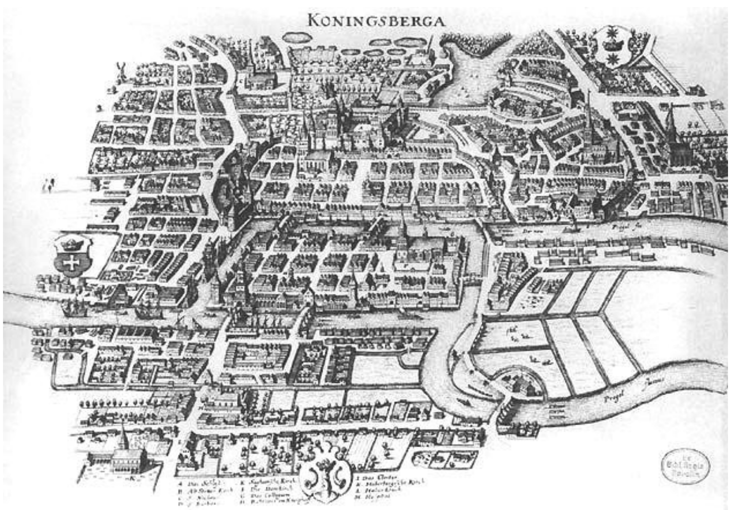
Euler (pronounced "oy-ler") was born in Basel, Switzerland. Later he remained there as a university student… not much later though, Euler began studying at the University of Basel at just 13 years old! Although remarkably young, this was less unusual than it is today. Much of his adulthood was spent in Saint Petersburg, in the Russian Empire. He became almost totally blind in his right eye in 1738, and then almost totally blind in his left eye some 30 years later, but took it remarkably well: "Now I will have fewer distractions". And, true to his word, he remained prolific in writing mathematical papers for his entire life.

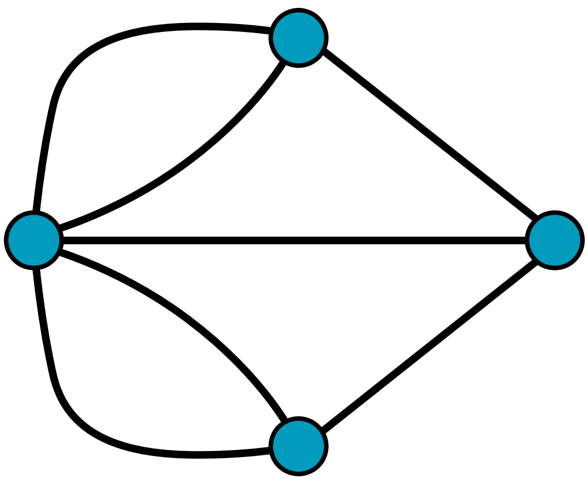
"Leonhard Euler.jpg", portrait by Jakob Emanuel Handmann (1718–1781), public domain

Euler invented a lot of the mathematical notation you may be familiar with. This includes using Σ (capital "Sigma", the Greek letter for S) for sums, i for the imaginary unit, $f(x)$ for function application and, most famously, the letter e for Euler's number, as used in the exponential function e^x.

Tying in to this issue's puzzle, Euler's solution to the Seven Bridges of Königsberg is regarded as the first ever in Graph Theory. By a 'graph', unfortunately Mathematicians can mean one of two very different things! Firstly, there is the perhaps more commonly known notion of the graph which sketches a function as a curve (like those on pages 3 and 4). The second notion, which we now restrict our attention to, is a kind of 'network'. Such a graph consists of 'vertices' (or 'nodes'), visualised as individual points, along with 'edges', visualised as arcs that have endpoints at the vertices. The particular way we might sketch a graph is not important, what is matters is the network of connections between vertices via the edges.

The problem of the Seven Bridges of Königsberg asks: is it possible to cross each of the seven bridges in Königsberg exactly once, never crossing back over a bridge you have already used? Today, Königsberg is called Kaliningrad, is in Russia and only has two of the original seven bridges. So we are instead considering the 18th century Prussian city of Königsberg, which had seven bridges, as illustrated in the figure below.





Bridges of Königsberg and its representation as a graph (both public domain)

The precise lengths and angles involved are irrelevant: all that matters is the underlying network of connections of the landmasses made by the bridges. So it is useful to strip away unnecessary detail from the problem, replacing landmasses with vertices and bridges with edges connecting them, that is, to visualise the problem on a graph. In this way, Euler's ideas also sowed the seeds for the development of Topology, the 'rubber geometry' study of properties of spaces that are preserved under stretching, changes of lengths or angles. The extension of his idea of the Euler characteristic is also considered as one of the origins of topology, as one of the first topological invariants. This is the formula $V - E + F = 2$, which holds for decompositions of the sphere into V vertices, E edges and F faces (for instance, the icosahedron has 12 vertices, 30 edges and 20 equilateral triangle faces, and 12-30+20 = 2). This formula also applies to 'planar graphs', graphs drawn in the plane without crossing edges, which defines vertices, edges and the 'faces' or regions the edges cut the plane into (including one 'infinite' face).

Turning back to the Bridges of Königsberg, our question may now be considered in terms of a graph: can we find a path in the graph that goes over each edge exactly once? Euler noticed that, if you can cross each edge exactly once, then every vertex must have even 'degree', which is the number of edges connected to that vertex. Or, more precisely, either every vertex must have even degree, or all but exactly two vertices must have even degree, with the two exceptions being the vertices you start and end the walk on, if they happen to be different. Indeed, for vertices along your path, you need to first 'enter' that vertex from one edge, then 'leave' from a different edge, so there are two edges involved in that visit that can never be used again. If you start and end on different vertices, then each visit similarly accounts for two edges, leaving just the first edge you use (for the starting vertex) or the last edge you use (for the ending vertex), so these must have odd degree.

Since you also need to use all edges, it follows that there must be either 0 or 2 vertices with odd degree, with all of the others being even degree. But the graph for the Bridges of Königsberg has 5 vertices which are all odd degree (3 have degree 3, the other has degree 5). So there does not exist an 'Eulerian trail', that is, a path along the graph that uses each edge exactly once.

Remarkably, the converse of the above also holds: assuming our graph is 'connected' (we can reach any from any other using a path of edges) then if every vertex of a graph has even degree there must be an Eulerian trail. And this must in fact be a 'cycle' (a path that starts and finishes on the same vertex). If there are exactly two vertices with odd degree, then there is always an Eulerian trail that starts at one of the odd degree vertices and finishes at the other. There are simple algorithms to actually find these cycles and trails when they exist.

Graph Theory remains an important area of Mathematics, with many modern applications considering huge and complicated graphs. For instance, imagine the graph of a social network site, where the vertices are individuals that are connected by edges if they are friends on that website. Understanding large graphs such as this can help us to understand how information or influence spreads. Similarly, it can help us to understand how viruses spread, by considering graphs of individuals and connections between those they may contact. As we've seen in this issue's puzzle, graphs are important in logistics: one could label edges with lengths (this makes a 'weighted graph'), and then ask for the shortest path that visits each vertex. This is called the Travelling Salesman Problem. For large graphs, it is a very difficult computational problem to find shortest routes, but one of great interest to delivery companies! Graph Theory is also important in Computer Science. For instance, the PageRank Algorithm used by Google's search engine involves webgraphs, given by links between webpages.

Hints for Puzzle 1.4

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Hopefully you helped find Santa a route for this Christmas. However, next year the graph might be more complicated, so we should try to give a general argument as to why it can always be done! Here are some hints:

- First, instead of a general graph, suppose we just have a 'cyclic graph'. In other words, the edges form a single loop, broken up by the various vertices (labeled with either positive or negative numbers, all summing to zero).
- To solve the problem on cyclic graphs, imagine that Santa was allowed to start with more gifts than he needs to deliver. Take any vertex and any direction around the cycle and think about how the number of gifts in his sleigh changes, which will go up and down as he passes through toy factories and regions. Looking at the cumulative total at each location, does this suggest a good place to start?
- If you manage to solve the problem for cyclic graphs, why does that give a solution for any graph containing a cycle using any given edge at most once?
- If Santa is forced to deliver all presents to a region at a fork when first entering it, then you can find graphs where the task is impossible. Of course, you will need at least one factory and at least two regions...

Useful Links

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Here are some links to useful resources.

Study with us!
Full details of our mathematics courses and our entry requirements
https://tinyurl.com/mathscourseuon

Careers
What do our mathematics graduates do?
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Here is a list of our issues so far, with QR codes and a brief summary of the contents of each issue.

Volume 1, Number 1	Information on the careers of University of Nottingham mathematics graduates, Puzzle 1.1 (about prime numbers), a short article about Edmund Landau and Landau's problems, links to useful resources
Volume 1, Number 2	Information about our three-year single-honours mathematics BSc degree, Puzzle 1.2 (about discs), the solution to Puzzle 1.1, an article about Alice Roth and Swiss cheeses, links to useful resources, back issues
Volume 1, Number 3	Maths graduation celebration and prizegiving, spotlight on Bindi Brook, Puzzle 1.3 (about peg solitaire), the solution to Puzzle 1.2, an article about John Horton Conway, links to useful resources, back issues