NOTES FOR "HIGH-DIMENSIONAL STATISTICS"

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Notes for the book: Martin J. Wainwright, High-Dimensional Statistics–A Non-Asymptotic Viewpoint, 2019.

1. Chapter 3

Exercise 3.10.
(a). ($\Rightarrow$)

$$\begin{aligned}
[\mathrm{vol}(A+B)]^{1/n} &= \left[\mathrm{vol}\left(\mathrm{vol}(A)\frac{A}{\mathrm{vol}(A)} + \mathrm{vol}(B)\frac{B}{\mathrm{vol}(B)}\right)\right]^{1/n} \\
&= (\mathrm{vol}(A)+\mathrm{vol}(B))\left[\mathrm{vol}\left(\frac{\mathrm{vol}(A)}{\mathrm{vol}(A)+\mathrm{vol}(B)}\frac{A}{\mathrm{vol}(A)} + \frac{\mathrm{vol}(B)}{\mathrm{vol}(A)+\mathrm{vol}(B)}\frac{B}{\mathrm{vol}(B)}\right)\right]^{1/n} \\
&\geq (\mathrm{vol}(A)+\mathrm{vol}(B))\left(\frac{\mathrm{vol}(A)}{\mathrm{vol}(A)+\mathrm{vol}(B)}\left(\frac{A}{\mathrm{vol}(A)^n}\right)^{1/n} + \frac{\mathrm{vol}(B)}{\mathrm{vol}(A)+\mathrm{vol}(B)}\left(\frac{B}{\mathrm{vol}(B)^n}\right)^{1/n}\right) \\
&= \mathrm{vol}(A)^{1/n} + \mathrm{vol}(B)^{1/n}.
\end{aligned}$$

($\Leftarrow$)

$$\begin{aligned}
(vol)(\lambda C + (1-\lambda)D)^{1/n} &\geq \mathrm{vol}(\lambda C)^{1/n} + \mathrm{vol}((1-\lambda)D)^{1/n} \\
&= \lambda(vol)(C)^{1/n} + (1-\lambda)\mathrm{vol}(D)^{1/n}
\end{aligned}$$

(b). For this problem, our condition is

$$\mathrm{vol}(\lambda C + (1-\lambda)D)^{1/n} \geq \lambda\mathrm{vol}(C)^{1/n} + (1-\lambda)\mathrm{vol}(D)^{1/n}.$$

Under this condition, we have

$$\begin{aligned}
\mathrm{vol}(\lambda C + (1-\lambda)D) &\geq \left(\lambda\mathrm{vol}(C)^{1/n} + (1-\lambda)\mathrm{vol}(D)^{1/n}\right)^n \\
&\geq 2^{n-1}\left(\lambda^n\mathrm{vol}(C) + (1-\lambda)^n\mathrm{vol}(D)\right) \\
&= \frac{2^n\lambda^n}{2}\mathrm{vol}(C) + \frac{2^n(1-\lambda)^n}{2}\mathrm{vol}(D).
\end{aligned} \tag{1.1}$$

Recalling the Young's inequality with ϵ, we know that

$$\mathrm{vol}(C)^{\lambda}\mathrm{vol}(D)^{1-\lambda} \leq \epsilon\mathrm{vol}(C) + C(\epsilon)\mathrm{vol}(D), \tag{1.2}$$

where

$$C(\epsilon) = \left(\frac{\lambda}{\epsilon}\right)^{\frac{\lambda}{1-\lambda}}(1-\lambda).$$

Comparing the estimates (1.1) and (1.2), we need the following two inequalities hold true

$$\frac{2^n\lambda^n}{2} \geq \epsilon, \quad \frac{2^n(1-\lambda^n)}{2} \geq (1-\lambda)\left(\frac{\lambda}{\epsilon}\right)^{\frac{\lambda}{1-\lambda}}. \tag{1.3}$$

The second inequality can be simplified to

$$\epsilon \geq \frac{\lambda}{2^{\frac{(n-1)(1-\lambda)}{\lambda}}(1-\lambda)^{\frac{(n-1)(1-\lambda)}{\lambda}}}.$$

That is to say, we need

$$\frac{2^n\lambda^n}{2} \geq \frac{\lambda}{2^{\frac{(n-1)(1-\lambda)}{\lambda}}(1-\lambda)^{\frac{(n-1)(1-\lambda)}{\lambda}}},$$

which can be reduced to

$$\log 2 \geq \lambda \log\frac{1}{\lambda} + (1-\lambda)\log\frac{1}{1-\lambda}.$$

The above inequality obviously holds true for $\lambda \in [0,1]$.

(c). For this problem, our condition is

$$\mathrm{vol}(\lambda C + (1-\lambda)D) \geq [\mathrm{vol}(C)]^{\lambda}[\mathrm{vol}(D)]^{1-\lambda}.$$

Using the above condition, we directly arrive at

$$\mathrm{vol}(\lambda\frac{C}{\mathrm{vol}(C)^{1/n}} + (1-\lambda)\frac{D}{\mathrm{vol}(D)^{1/n}}) \geq 1. \tag{1.4}$$

Using (1.4), denoting

$$M_1 = \frac{\mathrm{vol}(\lambda C)^{1/n}}{\mathrm{vol}(\lambda C)^{1/n} + \mathrm{vol}((1-\lambda)D)^{1/n}},$$

we have

$$\begin{aligned}
&\mathrm{vol}(\lambda C + (1-\lambda)D)^{1/n}\\
=&\mathrm{vol}\left(M_1\frac{\lambda C}{\mathrm{vol}(\lambda C)^{1/n}} + (1-M_1)\frac{(1-\lambda)D}{\mathrm{vol}((1-\lambda)D)^{1/n}}\right)^{1/n}\left[\mathrm{vol}(\lambda C)^{1/n} + \mathrm{vol}((1-\lambda)D)^{1/n}\right]\\
\geq&\left[\mathrm{vol}(\lambda C)^{1/n} + \mathrm{vol}((1-\lambda)D)^{1/n}\right]\\
=&\lambda\mathrm{vol}(C)^{1/n} + (1-\lambda)\mathrm{vol}(D)^{1/n}.
\end{aligned}$$

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