Homework #2

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1)
$$\begin{bmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{bmatrix}$$

Suppose
$$\begin{bmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{bmatrix} = \begin{bmatrix} a & & \\ b & c & \\ d & e & f \end{bmatrix}\begin{bmatrix} a & b & d \\ & c & e \\ & & f \end{bmatrix}$$
$$= \begin{bmatrix} a^2 & ab & ad \\ ab & b^2+c^2 & bd+ce \\ ad & bd+ce & d^2+e^2+f^2 \end{bmatrix}$$

$\therefore$ $a^2 = b^2+c^2 = d^2+e^2+f^2 = 1$

$ab = bd+ce = \rho$

$ad = \rho^2$

$\therefore$ $a = 1$

$b = \rho$

$c = \sqrt{1-\rho^2}$

$d = \rho^2$

$e = \rho\sqrt{1-\rho^2}$

$f = \sqrt{1-\rho^2}$

$\therefore$
$$\begin{bmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ \rho & \sqrt{1-\rho^2} & \\ \rho^2 & \rho\sqrt{1-\rho^2} & \sqrt{1-\rho^2} \end{bmatrix}\begin{bmatrix} 1 & \rho & \rho^2 \\ & \sqrt{1-\rho^2} & \rho\sqrt{1-\rho^2} \\ & & \sqrt{1-\rho^2} \end{bmatrix}$$ —— Cholesky decomposition

To simulate a sequence of trivariate normal r.v with the above Cov matrix:

$X_1 = Z_1$

$X_2 = \rho Z_1 + \sqrt{1-\rho^2}\, Z_2$

$X_3 = \rho^2 Z_1 + \rho\sqrt{1-\rho^2}\, Z_2 + \sqrt{1-\rho^2}\, Z_3$

where Z_1, Z_2, Z_3 i.i.d. $\sim N(0,1)$.

4)

(a) $$g(x) = \frac{f_n(x)}{C_n \cdot f_1(x)}$$

C_n must be chosen so that $0 \le g(x) \le 1$, $\forall x$

Consider

$$\frac{f_n(x)}{f_1(x)} = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\,\Gamma(\frac{n}{2})}\left(1+\frac{x^2}{n}\right)^{-\frac{n+1}{2}} \cdot \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2})}{\cancel{\Gamma(1)}}(1+x^2)$$

$$= \sqrt{\frac{\pi}{n}}\,\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}\,\frac{1+x^2}{(1+\frac{x^2}{n})^{\frac{n+1}{2}}}$$

Let $\sup_{\mathbb{R}} g(x) = 1$

$$\therefore \frac{1}{C_n}\sqrt{\frac{\pi}{n}}\,\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}\sup_{\mathbb{R}}\frac{1+x^2}{(1+\frac{x^2}{n})^{\frac{n+1}{2}}} = 1$$

$$\therefore C_n = \sqrt{\frac{\pi}{n}}\,\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}\sup_{\mathbb{R}}\frac{1+x^2}{(1+\frac{x^2}{n})^{\frac{n+1}{2}}}$$

Consider $\frac{1+x^2}{(1+\frac{x^2}{n})^{\frac{n+1}{2}}}$ maximizes at $x = \pm 1$,

so $$\sup_{\mathbb{R}}\frac{1+x^2}{(1+\frac{x^2}{n})^{\frac{n+1}{2}}} = \frac{2}{(1+\frac{1}{n})^{\frac{n+1}{2}}} = 2\left(1+\frac{1}{n}\right)^{-\frac{n+1}{2}}$$

$$\therefore C_n = 2\sqrt{\frac{\pi}{n}}\,\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}\left(1+\frac{1}{n}\right)^{-\frac{n+1}{2}}$$ ___ expression for smallest value of C_n

(b)

$$C_2 = \cancel{2}\sqrt{\frac{\pi}{2}}\,\frac{\cancel{\tfrac{1}{2}}\sqrt{\pi}}{1}\cdot\left(1+\frac{1}{2}\right)^{-\frac{3}{2}} = \frac{2\pi}{3\sqrt{3}}$$

$$C_3 = 2\sqrt{\frac{\pi}{3}}\,\frac{1}{\frac{1}{2}\sqrt{\pi}}\left(1+\frac{1}{3}\right)^{-\frac{4}{2}} = \frac{3\sqrt{3}}{4}$$

$$C_5 = 2\sqrt{\frac{\pi}{5}}\,\frac{2}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}\left(1+\frac{1}{5}\right)^{-\frac{6}{2}} = \frac{50\sqrt{5}}{81}$$

For C_∞, define $g(x) = \frac{\phi(x)}{C_\infty \cdot f_1(x)}$ where $\phi(x)$ is standard normal pdf

Let $\sup_{\mathbb{R}} g(x) = 1$

$$\frac{1}{C_\infty}\sup_{\mathbb{R}}\frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}}{\frac{1}{\pi(1+x^2)}} = 1$$

$$\therefore C_\infty = \sup_{\mathbb{R}}\frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}}{\frac{1}{\pi(1+x^2)}} = \sup_{\mathbb{R}}\frac{\pi}{\sqrt{2\pi}}(1+x^2)e^{-\frac{1}{2}x^2}$$

Consider $(1+x^2)\exp(-\frac{1}{2}x^2)$ maximizes at $x = \pm 1$, so $\sup_{\mathbb{R}}\frac{\pi}{\sqrt{2\pi}}(1+x^2)e^{-\frac{1}{2}x^2} = \sqrt{\frac{\pi}{2}}\frac{2}{\sqrt{e}} = \sqrt{\frac{2\pi}{e}}$

$$\therefore C_\infty = \sqrt{\frac{2\pi}{e}}.$$

(c) please see the printed page 10.

Homework #2

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2) Normal Generation Methods

(a) Rejection method

```
seed=1;
rand('seed',seed);
n=100;
%n=1000;
%n=10000;
X = zeros(1,n);
for i=1:n
    s=sign(rand(1)-0.5);
    u1=rand(1);
    u2=rand(1);
    x1=-log(u1);
    while (u2>exp(-0.5*(x1-1)^2))
        u1=rand(1);
        u2=rand(1);
        x1=-log(u1);
    end;
    X(i)=x1*s;
end;

qqplot(X);
```

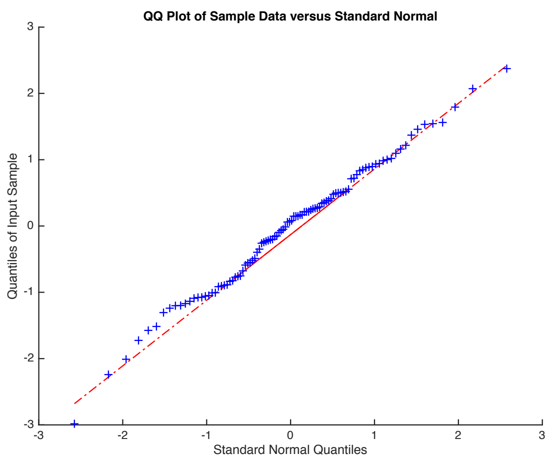

Figure 1 100 Normals from (a)

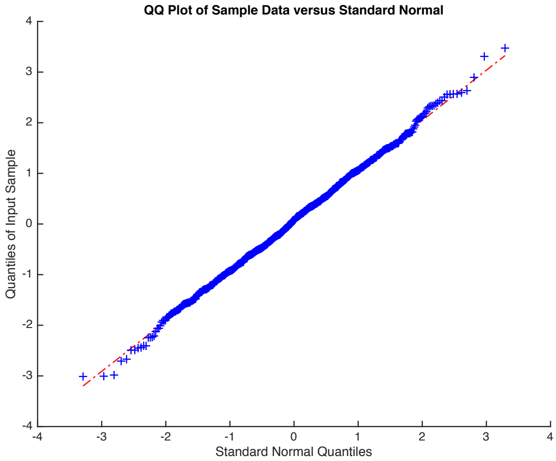

Figure 2 1000 Normals from (a)

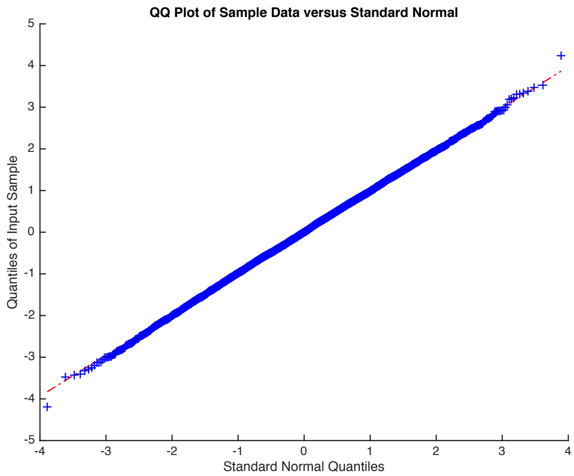

Figure 3 10000 Normals from (a)

Comment:

For n=100, some of the samples depart markedly from the linear picture. However, the results for the sample of 1000 and 10000 normals are satisfactory.

(b) Generalized lambda distribution

```
Y = zeros(1,n);
for i=1:n
    x=rand(1);
    Y(i)=0+(x^0.1349-(1-x)^0.1349)/0.1975;
end

qqplot(Y);
```

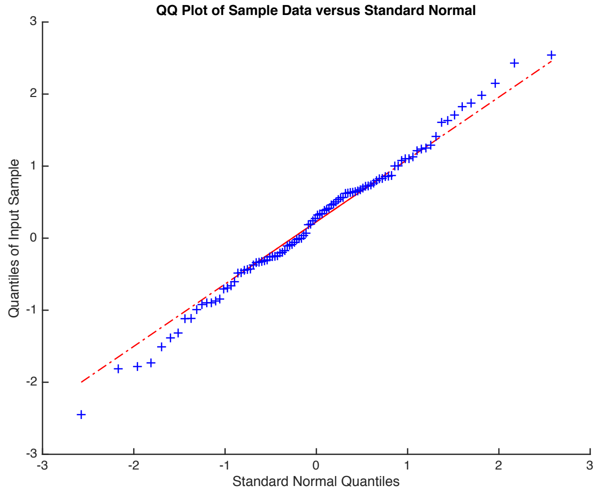

Figure 4 100 Normals from (b)

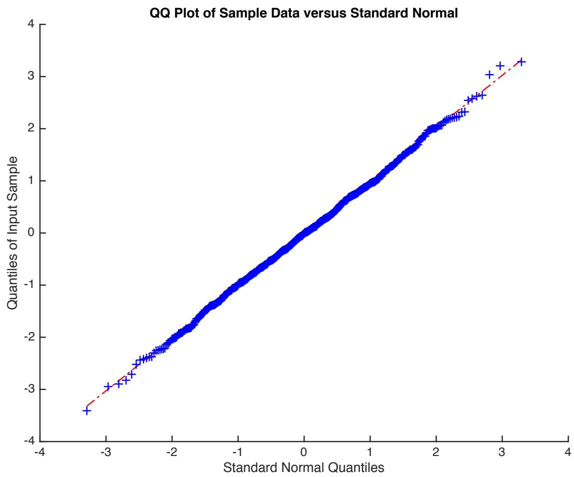

Figure 5 1000 Normals from (b)

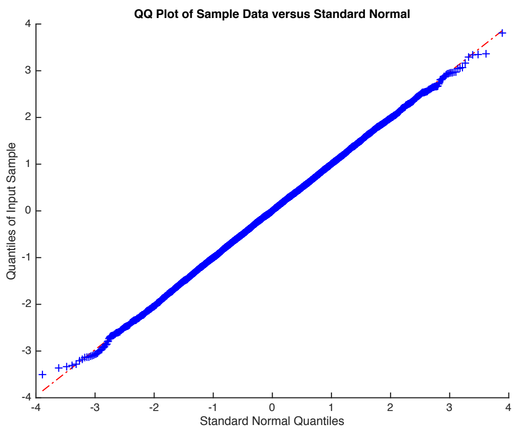

Figure 6 10000 Normals from (b)

Comment:

For n=100, some of the samples depart markedly from the linear picture. (Seems to have heavier tails) However, the results for the sample of 1000 and 10000 normals are satisfactory.

(c) Litterman-Winkelmann weighted normal distribution

```
Z= zeros(1,n);
for i=1:n
    u=rand(1);
    z=randn(1);
    if (u<=0.82)
        Z(i)=0.6*z;
    else
        Z(i)=1.98*z;
    end
end

qqplot(Z);
```

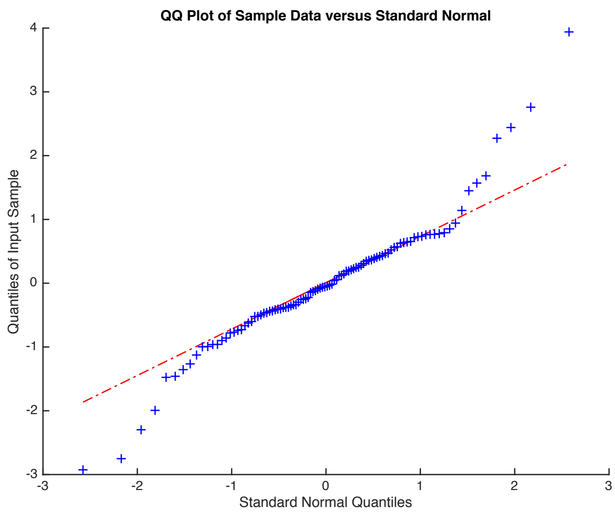

Figure 7 100 weighted Normals from (c)

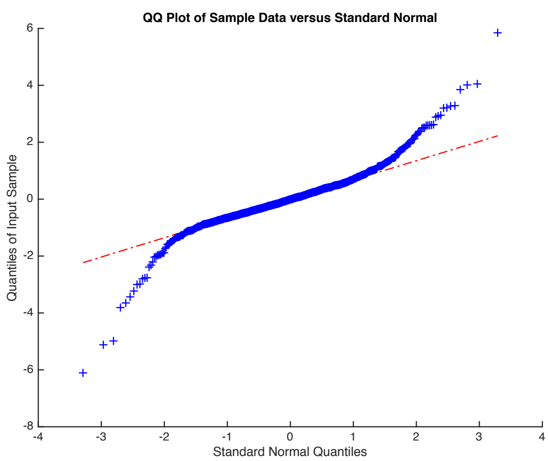

Figure 8 1000 weighted Normals from (c)

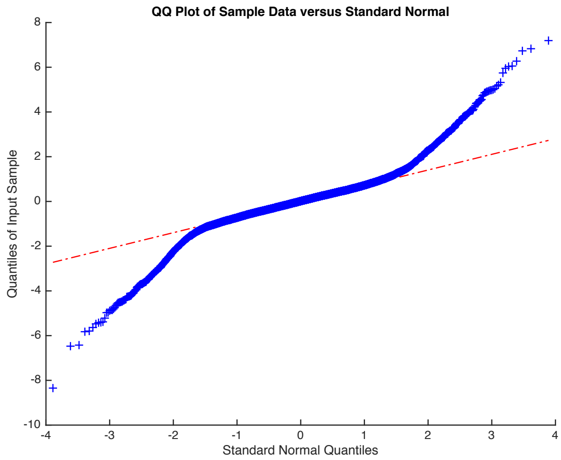

Figure 9 10000 weighted Normals from (c)

Comment:
The Litterman-Winkelmann weighted normal distribution has much heavier tails compared to normal distribution. This is obvious in the plots for n=100,1000 and 10000.

3) Bivariate data and copulas

(a) Bivariate normal

```
x=zeros(2,1000);
rho=0;% or 0.4, 0.8
x(1,:)=randn(1,1000);
x(2,:)=rho.*x(1,:)+sqrt(1-rho^2).*randn(1,1000);
scatter(x(1,:),x(2,:));
```

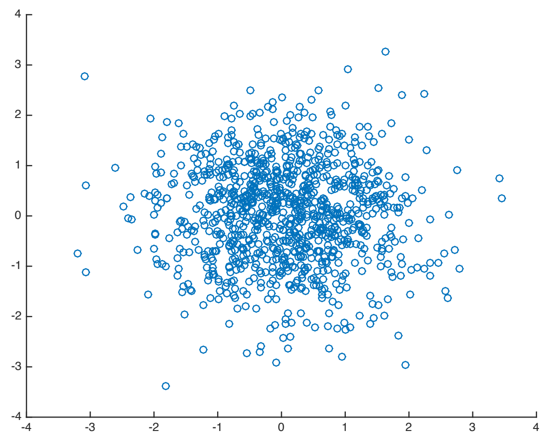

Figure 10 Scatter plot of 1000 bivariate normal with correlation=0

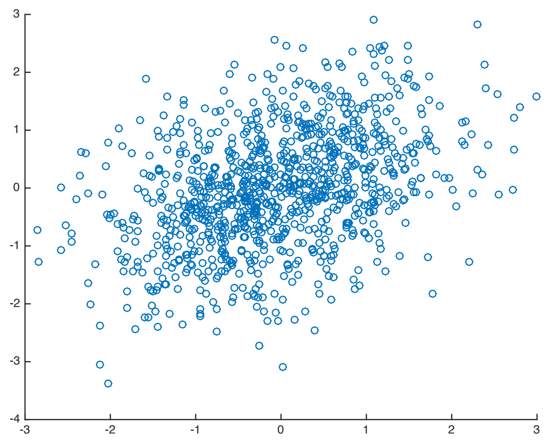

Figure 11 Scatter plot of 1000 bivariate normal with correlation=0.4

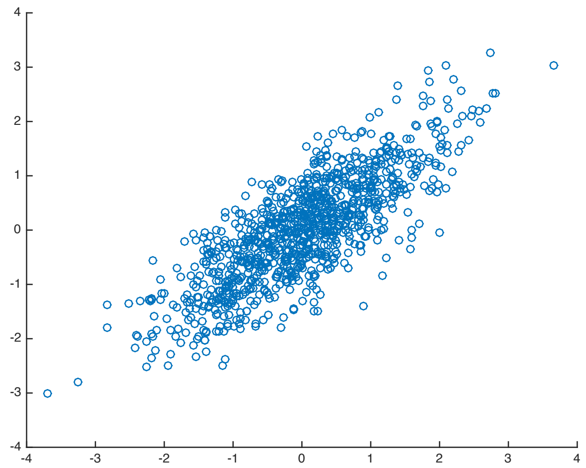

Figure 12 Scatter plot of 1000 bivariate normal with correlation=0.8

(b) Bivariate t distribution

```
y=zeros(2,1000);
z=randn(2,1000); % Z with N(0,I) distribution
rho=0; % or 0.4, 0.8
y(1,:)=z(1,:);
y(2,:)=rho*z(1,:)+sqrt(1-rho^2)*z(2,:);
s=gamrnd(5/2,1/2,2,1000);
T=sqrt(5./s).*y;
scatter(T(1,:),T(2,:));
```

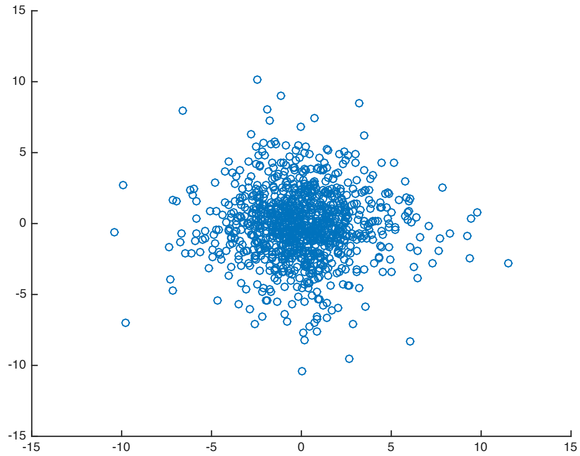

Figure 13 Scatter plot of 1000 bivariate t_5 with correlation=0

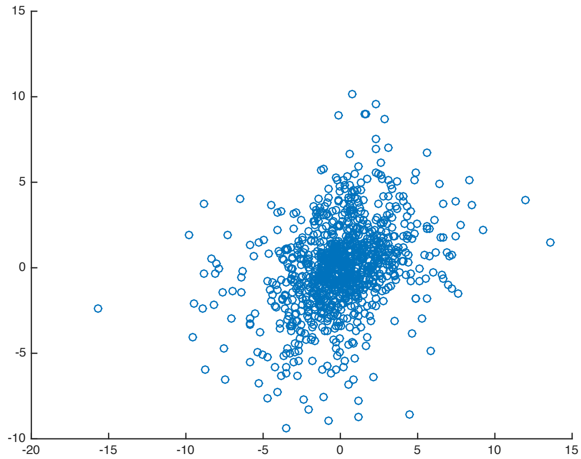

Figure 14 Scatter plot of 1000 bivariate t_5 with correlation=0.4

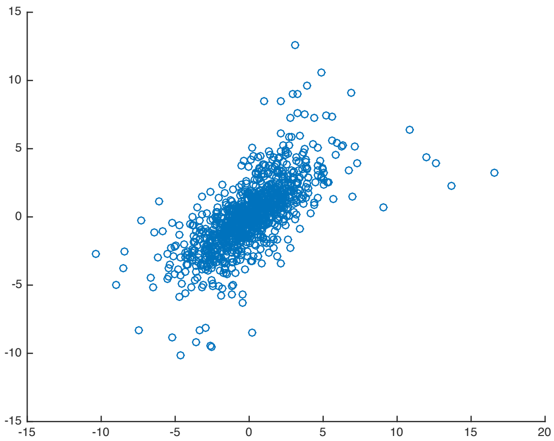

Figure 15 Scatter plot of 1000 bivariate t_5 with correlation=0.8

Comment:
The plots of bivariate t distribution changes in a similar pattern as bivariate normal distribution as the correlation increase. However, it seems that there are more outlier points in the scatter plot for bivariate t distribution.

(c) Gaussian copula

```
rho=0; % or 0.4, 0.8
z=randn(2,1000); % Z with N(0,1) distribution
y(1,:)=z(1,:);
y(2,:)=rho*z(1,:)+sqrt(1-rho^2)*z(2,:);
U=normcdf(y);
% exp(1) cdf: 1-exp(-x)
X=-log(1-U);
scatter(X(1,:),X(2,:));
```

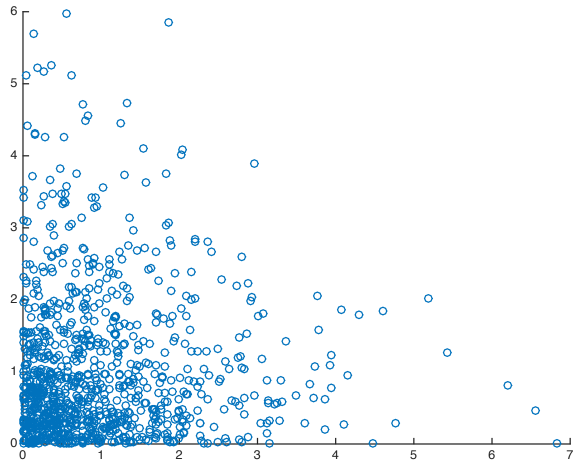

Figure 16 Scatter plot generated by 1000 Gaussian copulas with correlation=0

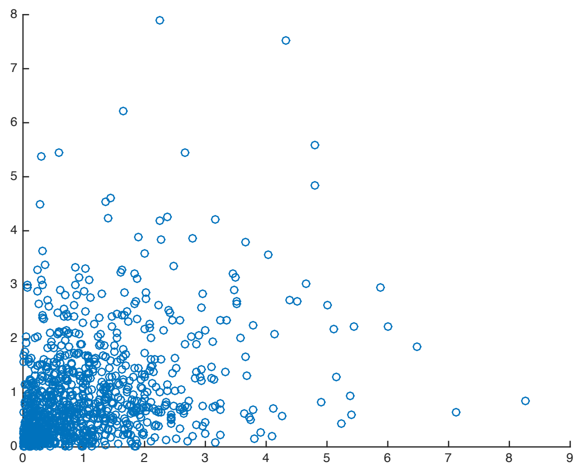

Figure 17 Scatter plot generated by 1000 Gaussian copulas with correlation=0.4

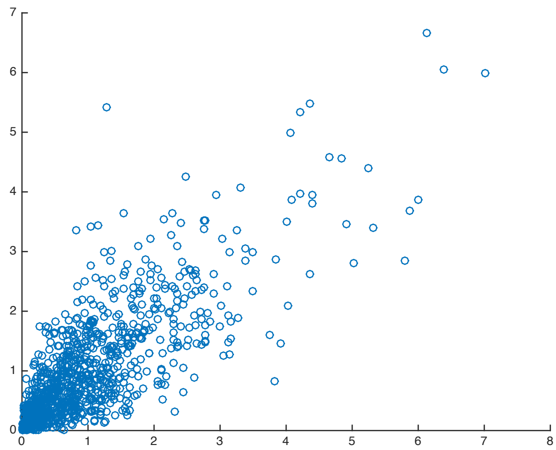

Figure 18 Scatter plot generated by 1000 Gaussian copulas with correlation=0.8

Comment: As correlation increase, points in the scatter plot seems to gather more tightly around the 45 degree line.

(d) t_5 copula

```
rho=0.8; % or 0.4, 0.8
z=randn(2,1000); % Z with N(0,1) distribution
y(1,:)=z(1,:);
y(2,:)=rho*z(1,:)+sqrt(1-rho^2)*z(2,:);
s=gamrnd(5/2,1/2,2,1000);
w=sqrt(5./s).*y;
% t copula
U=tcdf(w,5);
% exp(1) cdf: 1-exp(-x)
X=-log(1-U);
scatter(X(1,:),X(2,:));
```

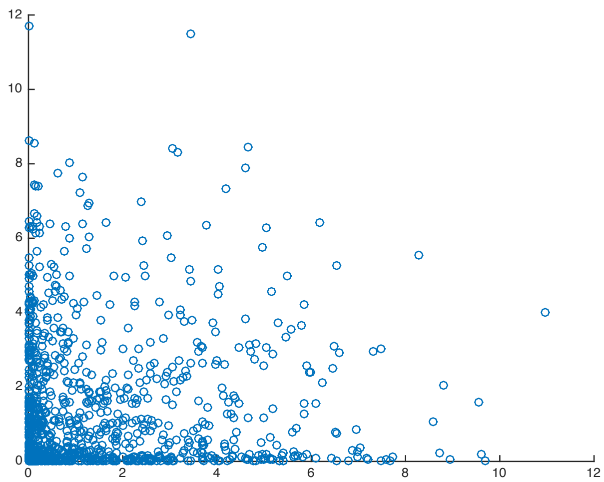

Figure 19 Scatter plot generated by 1000 t_5 copulas with correlation=0

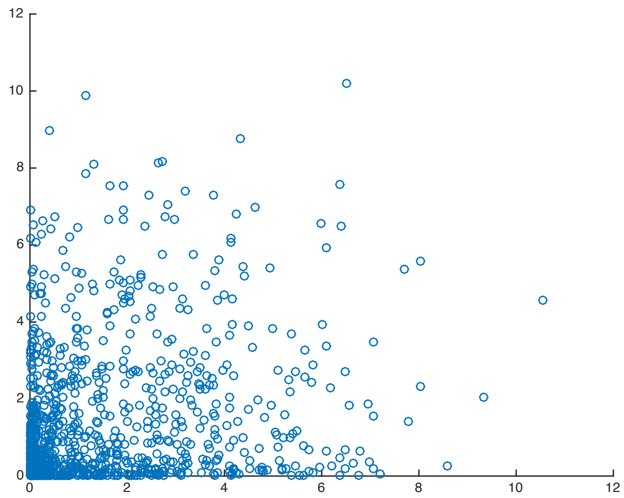

Figure 20 Scatter plot generated by 1000 t_5 copulas with correlation=0.4

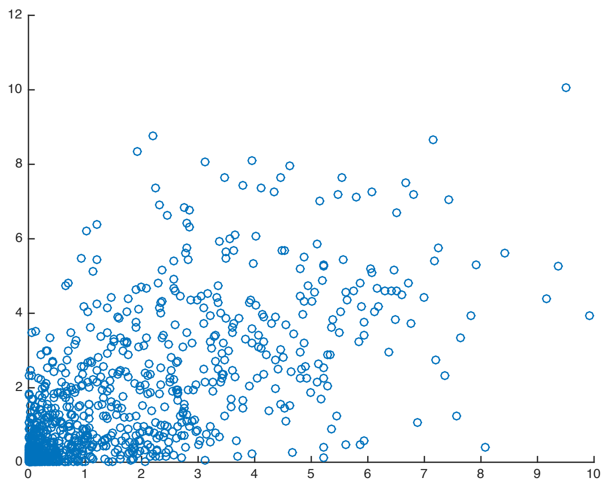

Figure 21 Scatter plot generated by 1000 t_5 copulas with correlation=0.8

Comment: The plots generated by t copula changes in a similar pattern as generated by Gaussian copula as the correlation increase. However, it seems that the points in scatter plots do not gather as "tight" as they do in scatter plots generated by Gaussian copulas.

4) Generate t-variables using rejection

(c) Rejection algorithm

```
n=1; % or 3, 5, 10, 30
m=1000;
x=zeros(1,m);
for i=1:m
    y=tan(rand(1).*pi-pi./2); % cauchy dist.
    u=rand(1);
```

```
    % C=sqrt(2*pi/e)
    while
(u>1/sqrt(2*pi/exp(1))*gamma((1+n)/2)/sqrt((n*pi)*gamma(n/2))
*(1+y.^2./n).^(n+1)/2/(1/(pi*(1+y.^2))))
        y=tan(rand(1).*pi-pi./2);
        u=rand(1);
    end
    x(i)=y;
end
qqplot(x);
%xlim([-2 2]);
```

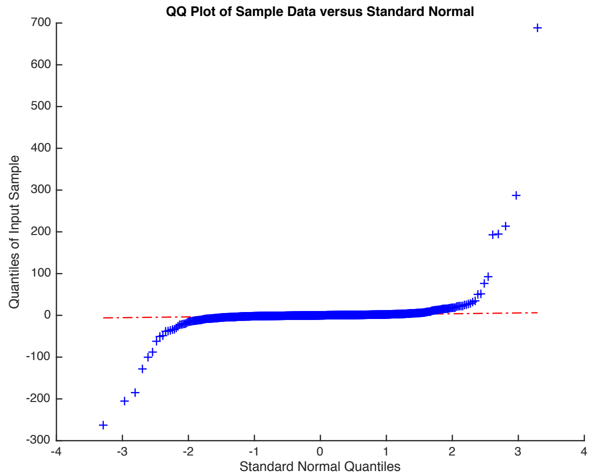

Figure 22 Scatter plot of 1000 t_1

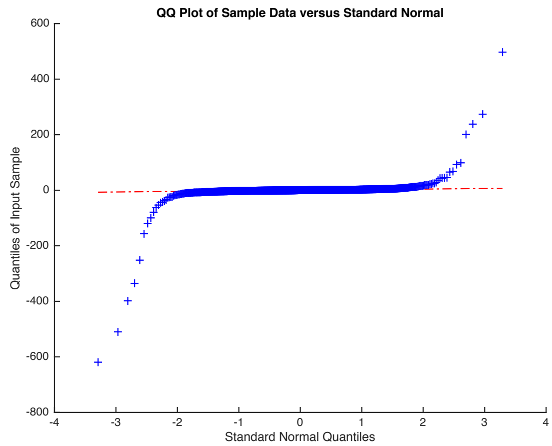

Figure 23 Scatter plot of 1000 t_3

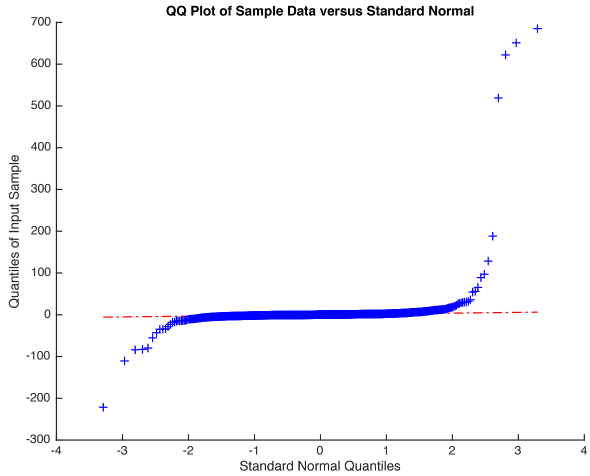

Figure 24 Scatter plot of 1000 t_5

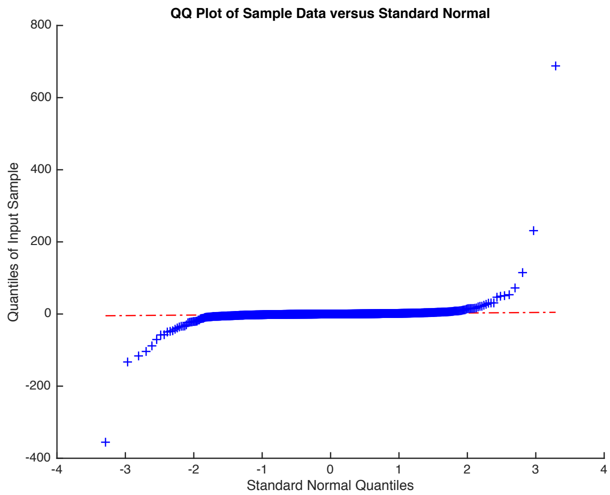

Figure 25 Scatter plot of 1000 t_{10}

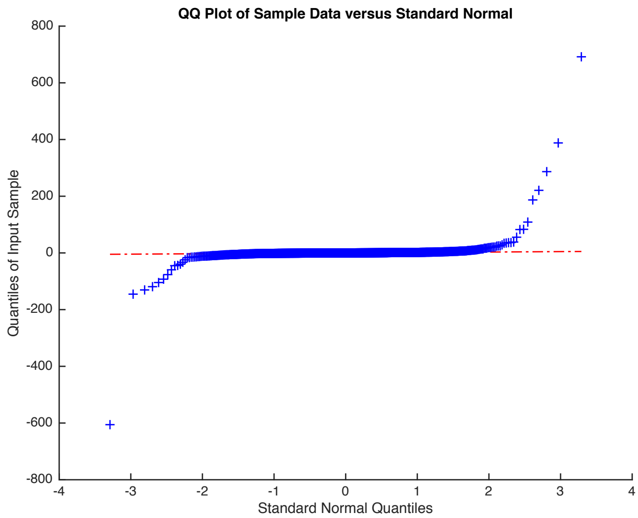

Figure 26 Scatter plot of 1000 t_{30}

Comment: It's obvious from the plots that t distribution has much heavier tails than normal distribution

5) Antithetic Variables on Black Scholes

a) Price using standard MC methods

```
n=1000;
Z=randn(1,n);
S0=100;
T=1;
r=0.05;
sig=0.1;
K=95; % or 100, 105
S=S0*exp((r-1/2*sig^2)*T+sig*sqrt(T).*Z);
C=max(S-K,zeros(1,n))*exp(-r*T);
Cbar=mean(C)
stderr=std(C)

% real value
dplus=(log(S0/K)+(r+sig^2/2)*T)/(sig*sqrt(T));
dminus=dplus-sig*sqrt(T);
realC=S0*normcdf(dplus)-normcdf(dminus)*K*exp(-r*T)
```

The results are as follows:

	K=95	K=100	K=105
Exact price	10.4053	6.8050	4.0461
Simulation price	10.1856	6.8078	4.1672
Standard error	0.2725	0.2483	0.1956

b) Price using antithetic variables

```
K=95; % or 100, 105
Z2=-Z;
S2=S0*exp((r-1/2*sig^2)*T+sig*sqrt(T).*Z2);
C2=max(S2-K,zeros(1,n))*exp(-r*T);
newC=(C+C2)./2;
newCbar=mean(newC)
newstderr=std(newC)
```

The comparison is as follos:

	K=95	K=100	K=105
Exact price	10.4053	6.8050	4.0461
Standard MC method	10.1856(0.2725)	6.8078(0.2483)	4.1672(0.1956)
Antithetic variables	10.4292(0.0726)	7.0341(0.0946)	4.0395(0.1028)

We can conclude from the table above that, for K=95 or 100 or 105, the method using antithetic variables gives a much precise result than standard Monte Carlo method. In addition, standard errors are smaller in antithetic variables method compared to standard Monte Carlo method.