Cahn-Hilliard-Navier-Stokes model

Governing equations

$$
\begin{aligned}
\hat{\varrho}\left(\mathbf{u}_t + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}\right) - \hat{\nu}\Delta\mathbf{u} + \boldsymbol{\nabla} p &= \hat{\varrho}\mathbf{g} - \varphi\boldsymbol{\nabla}\mu_{\mathrm{ch}},\\
\boldsymbol{\nabla}\cdot\mathbf{u} &= 0,\\
\varphi_t + \mathbf{u}\cdot\boldsymbol{\nabla}\varphi &= \boldsymbol{\nabla}\cdot\left(M\boldsymbol{\nabla}\mu_{\mathrm{ch}}\right),\\
\mu_{\mathrm{ch}} &= -\varepsilon\sigma\Delta\varphi + \frac{\sigma}{\varepsilon}\Psi'(\varphi).
\end{aligned}
$$

with zero boundary conditions for velocity $\mathbf{u}\big|_{\partial\Omega} = 0$, zero boundary flux of phase $\boldsymbol{\nabla}\varphi\cdot\mathbf{n}\big|_{\partial\Omega} = 0$ and also zero boundary flux of the chemical potential $M(\varphi)\boldsymbol{\nabla}\mu_{\mathrm{ch}}\cdot\mathbf{n}\big|_{\partial\Omega} = 0$. Initial conditions were predescribes as no velocity $\mathbf{u}(t=0) = \mathbf{0}$ and circular bubble in the bottom half of the rectangle containing a different phase $\varphi(t=0) = \varphi_0$.

The degenerate mobility is considered to take the simple form $M(\varphi) = M_0\left(1-\varphi\right)^2\varphi^2$, and the potential is of the same form, ie. $\Psi(\varphi) = \left(1-\varphi\right)^2\varphi^2$.

We considered the density, viscosity and other quantities depending on the phase to take the form

$$
\hat{\varrho}(\varphi) = \begin{cases} \hat{\varrho}_1 & ,\ \varphi < 0\\ \hat{\varrho}_1 + (\hat{\varrho}_2 - \hat{\varrho}_1)\varphi & ,\ 0 \le \varphi \le 1,\\ \hat{\varrho}_2 & ,\ \varphi > 1. \end{cases}
$$

Boundary conditions were chosen such that most of the terms in the weak formulation zero-out and we are left with this simplistic version

$$
\begin{aligned}
&\int_\Omega \hat{\varrho}\mathbf{u}_t\cdot\mathbf{v} + \int_\Omega \hat{\varrho}\left(\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}\right)\cdot\mathbf{v} + \hat{\nu}\boldsymbol{\nabla}\mathbf{u}\colon\boldsymbol{\nabla}\mathbf{v} - \int_\Omega \hat{\varrho}\mathbf{g}\cdot\mathbf{v} + \varphi\mu_{\mathrm{ch}}\boldsymbol{\nabla}\cdot\mathbf{v} + \int_\Omega q\boldsymbol{\nabla}\cdot\mathbf{u} - p\boldsymbol{\nabla}\cdot\mathbf{v}\\
&+ \int_\Omega \varphi_t\psi + \int_\Omega \left(\mathbf{u}\cdot\boldsymbol{\nabla}\varphi\right)\psi + \int_\Omega M\boldsymbol{\nabla}\mu_{\mathrm{ch}}\cdot\boldsymbol{\nabla}\psi + \int_\Omega \mu_{\mathrm{ch}}\nu_{\mathrm{ch}} - \varepsilon\sigma\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\nu_{\mathrm{ch}} - \frac{\sigma}{\varepsilon}\Psi'(\varphi)\nu_{\mathrm{ch}} = 0
\end{aligned}
$$

For stability reasons, this systems needs to be computed on a mesh with high detail and some backward method, like the backward Euler. The resulting functional renders

$$
\begin{aligned}
\int_\Omega \frac{\varphi_{i+1} - \varphi_i}{\mathrm{d}t}\psi &+ \int_\Omega \left(\theta\hat{\varrho}(\varphi_{i+1}) + (1-\theta)\hat{\varrho}(\varphi_i)\right)\frac{\mathbf{u}_{i+1} - \mathbf{u}_i}{\mathrm{d}t}\cdot\mathbf{v}\\
&+ \theta\left(F_{\text{phase}} + F_{\text{Navier-Stokes}}\right)\left(\mathbf{u}_{i+1},\, p_{i+1},\, \varphi_{i+1},\, {\mu_{\mathrm{ch}}}_{i+1}\right)\\
&+ (1-\theta)\left(F_{\text{phase}} + F_{\text{Navier-Stokes}}\right)\left(\mathbf{u}_i,\, p_i,\, \varphi_i,\, {\mu_{\mathrm{ch}}}_i\right)\\
&+ \int_\Omega q\boldsymbol{\nabla}\cdot\mathbf{u} + \int_\Omega \mu_{\mathrm{ch}}\nu_{\mathrm{ch}} - \varepsilon\sigma\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\nu_{\mathrm{ch}} - \frac{\sigma}{\varepsilon}\Psi'(\varphi)\nu_{\mathrm{ch}} = 0
\end{aligned}
$$