# Cahn-Hilliard model

The Cahn-Hilliard model can be formulated as a parabolic equation which is typically used to model phase separation in binary mixtures.

$$\partial_t c = \Delta \left(c^3 - c - \gamma \Delta c\right)$$

The order of this equation can be, however, lowered, yielding two coupled second-order partial differential evolution equations in terms of the so-called phase fraction $\phi$, which represents the concentration of one of the components in the binary mixture and the chemical potential $\mu$

$$\frac{\partial \phi}{\partial t} = \nabla \cdot (b(\phi) \nabla \mu) \quad \text{and} \quad \mu = -\gamma \Delta \phi + f'(\phi). \tag{1}$$

The Time evolution $\partial_t \phi$ was approximated using Crank-Nicholson scheme. We studied the above system in a unit square $\Omega = (0,\, 1)^2$ with periodic boundary conditions, in other words, we were solving the system on a torus. The initial data for the phase fraction $\phi$ were given as[1]

$$\phi_0(x,\, y) = \cancelto{0.1}{0.3} \sin(4\pi x) \sin(2\pi y) + \cancelto{0.6}{0.1}. \tag{2}$$

The interface parameter $\gamma$ was set to $\gamma = 0.003$, the double well potential $f(\phi)$ and the mobility functions were defined as

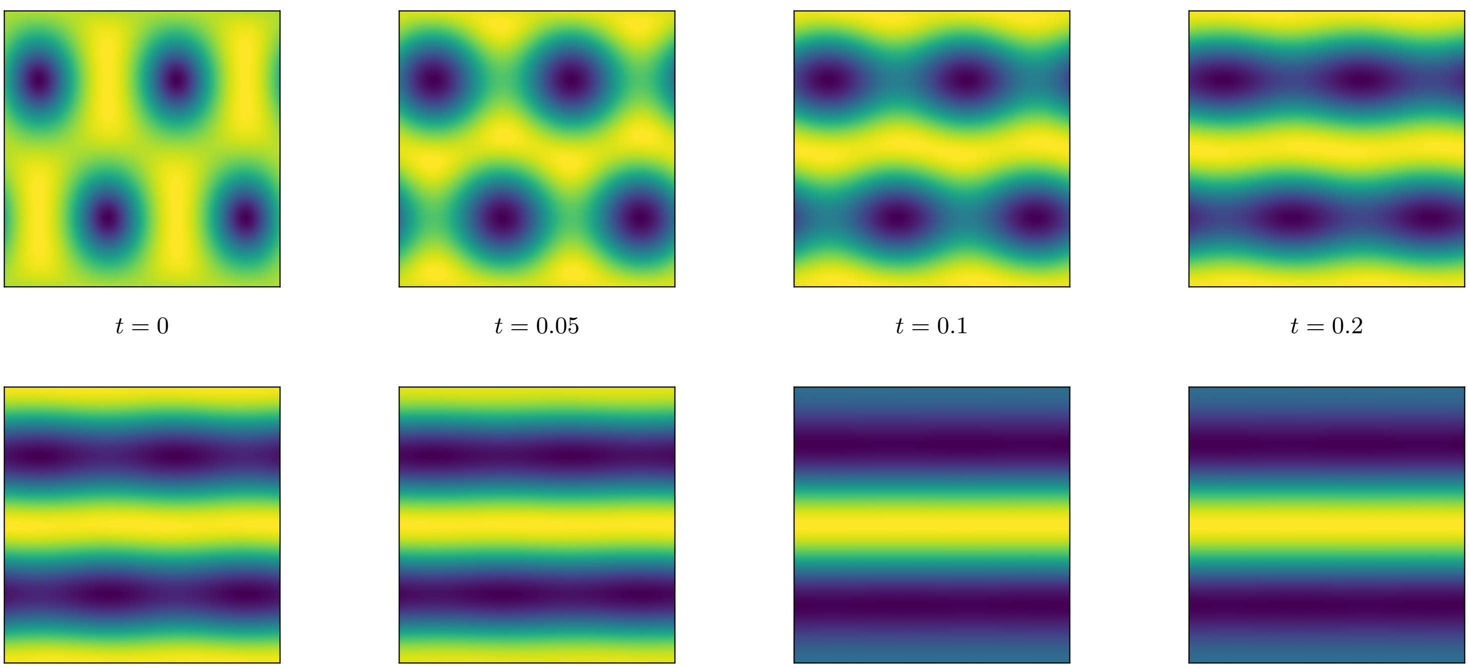
$$f(\phi) = 0.3 \left(\phi - 0.99\right)^2 \left(\phi + 0.99\right)^2 \quad \text{and} \quad b(\phi) = \left(1 - \phi\right)^2 \left(1 + \phi\right)^2 + 10^{-3}. \tag{3}$$

In order to solve the posed problem, we need to reformulate it first into the weak form

$$\langle \partial_t \phi,\, v\rangle + \langle b(\phi) \nabla \mu,\, \nabla v\rangle = 0 \quad \text{and} \quad \langle \mu - f'(\phi(t)),\, w\rangle - \langle \gamma \nabla \phi,\, \nabla w\rangle. \tag{4}$$

As per the original article [1], the model was computed with mixed elements and the Newton method was used to solve the corresponding `NonlinearVariationalProblem`.

Below we present the snapshots from the time evolution on the square as well as one instance of the model on a torus.



$t = 0$ $t = 0.05$ $t = 0.1$ $t = 0.2$

$t = 0.3$ $t = 0.4$ $t = 0.5$ $t = 1$

Figure 1: Snapshots from time evolution of the Cahn-Hilliard model on a unit square with periodic boundary conditions.

# References


[1] Aaron Brunk et al. "Stability and discretization error analysis for the Cahn-Hilliard system via relative energy estimates". In: *ESAIM: Mathematical Modelling and Numerical Analysis* 57.3 (May 2023), pp. 1297–1322. ISSN: 2804-7214. DOI: `10.1051/m2an/2023017`. URL: `http://dx.doi.org/10.1051/m2an/2023017`.


[1] The values stated in the paper did not seem to correspond to the presented figures and so were adjusted in our simulations.