TBD

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Abstract. This is the abstract.

1. Introduction

We are concerned with the PDE in two dimensions

$$\partial_t \omega + u \cdot \nabla \omega = 0 \tag{1.1}$$

with

$$u \coloneqq -\nabla^\perp (-\Delta)^{-1+\alpha} \omega \tag{1.2}$$

and $\alpha \in \left(0, \frac{1}{2}\right)$, where $(x_1, x_2)^\perp \coloneqq (-x_2, x_1)$ and $\nabla^\perp \coloneqq (-\partial_{x_2}, \partial_{x_1})$.

Given $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, the velocity field $u(\omega)$ generated by ω as in (1.2) is given as

$$u(\omega; x) \coloneqq \int_{\mathbb{R}^2} \nabla^\perp K(x-y) \omega(y)\, dy,$$

where kernel $K\colon \mathbb{R}^2 \to (0, \infty]$ is defined as

$$K(x) \coloneqq \frac{c_\alpha}{2\alpha \left|x\right|^{2\alpha}}$$

for some $c_\alpha > 0$. Since ω is in $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, it can be easily seen that $u(\omega)$ is a well-defined $(1-2\alpha)$-Hölder continuous function $\mathbb{R}^2 \to \mathbb{R}^2$.

More generally, when ω is a finite signed Borel measure on $\mathbb{R}^2$, (1.2) yields

$$u(\omega; x) \coloneqq \int_{\mathbb{R}^2} \nabla^\perp K(x-y)\, d\omega(y)$$

whenever the integral converges absolutely. (The reason for considering this general case is merely because of convenience in certain aspects of developing the theory, and we will not be concerned with the well-posedness of (1.1)–(1.2) in this general setting.) When ω is an L^1 function, we identify it with the finite signed Borel measure it defines through integration with respect to the Lebesgue measure, so that the interpretations of $u(\omega)$ in both ways are consistent. Note that for any measure-preserving homeomorphism $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$, this correspondence between L^1 functions and finite signed Borel measures identifies the function $\omega \circ \Phi^{-1}$ with the pushforward measure $\Phi_* \omega$.

Our first result is local well-posedness of (1.1)–(1.2) within a class of $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ admitting a decomposition of the form

$$\omega(x) = \int_{\mathcal{L}} \mathbb{1}_{\Omega^\lambda}(x)\, d\theta(\lambda), \tag{1.3}$$

where $\mathcal{L}$ is a measurable space (whose σ-algebra is not explicitly written), θ is a σ-finite signed measure on $\mathcal{L}$, Ω is a set in the product σ-algebra of $\mathbb{R}^2 \times \mathcal{L}$, and $\Omega^\lambda \subseteq \mathbb{R}^2$ is the λ-section of Ω for each $\lambda \in \mathcal{L}$.

A natural choice of (Ω, θ) is that each Ω^λ is a super-level set of ω^+ and ω^-, and θ^+, θ^- are respectively the uniform measures on $(0, \sup \omega]$ and $[\inf \omega, 0)$, so that (1.3) is the standard layer cake representation. In this reason, we call the pair (Ω, θ), or simply Ω, a *generalized layer cake representation* of ω.

Remark. The measurable space $\mathcal{L}$ is implicit from Ω and θ, so we will suppress it in the notation. From now on, $\mathcal{L}$ always denotes the measurable space on which Ω and θ are defined.

It turns out that each Ω^λ being a super-level set of ω is not necessary for the well-posedness theory we develop. Furthermore, this abstract setting allows a simple setup for studying H^2 regularity of level sets, which our second result is mainly about, that encompasses situations where some level sets of ω may consist of multiple (or even infinitely many) disjoint curves. In these reasons, we state and prove our well-posedness result in terms of an arbitrary generalized layer cake representation, rather than the standard layer cake representation.

In addition to merely having a decomposition (1.3), we impose a regularity condition

$$L_\theta(\Omega) \coloneqq \sup_{x \in \mathbb{R}^2} \int_{\mathcal{L}} \frac{d|\theta|(\lambda)}{d(x, \partial\Omega^\lambda)^{2\alpha}} < \infty \tag{1.4}$$

where $|\theta|$ is the total variation of θ. Note that a standard measure theory argument shows joint measurability of

$$d(x, \partial\Omega^\lambda) = \max\{d(x, \Omega^\lambda), d(x, \mathbb{R}^2 \setminus \Omega^\lambda)\}$$

in (x, λ), so $L_\theta(\Omega)$ is always well-defined (with the convention $\frac{1}{0} = \infty$ and $\infty \cdot 0 = 0$). The quantity $L_\theta(\Omega)$ arises in the context of estimating the growth of H^2 norm of a boundary curve $\partial\Omega^\lambda$ (Lemma 3.7), which was the motivation for us to study the well-posedness problem under the condition (1.4).

It turns out that any ω admitting Ω with (1.4) must be 2α-Hölder continuous. In fact, this is the best possible L^∞-type regularity condition (1.4) implies because there exists ω satisfying (1.4) whose modulus of continuity $\rho(\delta)$ is in between constant multiples of $\min\{\delta^{2\alpha}, 1\}$. In the below (Lemma 2.1) we show that (1.4) implies Lipschitz continuity of $u(\omega)$, which does not follow from mere 2α-Hölder continuity in general. This is one of the key components that lead to our well-posedness result.

On the other hand, investigating the distributional derivative of $u(\omega)$ shows that the most general condition on ω *in terms of* ρ that ensures Lipschitz continuity of $u(\omega)$ is

$$\int_0^1 \frac{\min\{\rho(\delta), 1\}}{\delta^{1+2\alpha}}\, d\delta < \infty. \tag{1.5}$$

It can be shown that (1.5) in fact implies (1.4) when (Ω, θ) is the standard layer cake representation. Since (1.5) does not hold for $\rho(\delta) = \min\{\delta^{2\alpha}, 1\}$, we see that (1.4) is strictly more general than any assumption on the modulus of continuity of ω from which Lipschitz continuity of $u(\omega)$ is guaranteed.

Since any ω in the class of functions we consider always generates Lipschitz velocity field, it is natural to consider the following notion of solutions to (1.1)–(1.2).

Definition 1.1. Let $\omega^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. A *Lagrangian solution* to (1.1)–(1.2) on a time interval $I \ni 0$ with the initial data ω^0 is a function $\omega\colon I \to L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ given as $\omega^t := \omega \circ (\Phi^t)^{-1}$, where $\Phi \in C(I; C(\mathbb{R}^2;\mathbb{R}^2))$ (with respect to the extended metric $d(F,G) := \|F-G\|_{L^\infty}$) satisfies the initial value problem

$$\begin{aligned} \partial_t \Phi^t &= u(\omega^t) \circ \Phi^t, \\ \Phi^0 &= \mathrm{Id}, \end{aligned} \tag{1.6}$$

where the time derivative is one-sided at any end-point of I, and each Φ^t is a measure-preserving homeomorphism. We call Φ the *flow map* associated to ω.

Remark. It is easy to show that any Lagrangian solution ω is a weak solution to (1.1)–(1.2) in the sense that

$$\frac{d}{dt}\int_{\mathbb{R}^2} \varphi(x)\,\omega^t(x)\,dx = \int_{\mathbb{R}^2} \big(u(\omega^t;x)\cdot\nabla\varphi(x)\big)\,\omega^t(x)\,dx$$

holds for all $\varphi \in C^1(\mathbb{R}^2)$.

Given a measure-preserving homeomorphism $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$, it can be easily seen that $\Phi_*\Omega := \{(\Phi(x),\lambda) \in \mathbb{R}^2 \times \mathcal{L} \colon (x,\lambda) \in \Omega\}$ is a generalized layer cake representation of $\omega \circ \Phi^{-1}$ and

$$\omega(\Phi^{-1}(x)) = \int_{\mathcal{L}} \mathbb{1}_{\Phi(\Omega^\lambda)}(x)\,d\theta(\lambda)$$

for $x \in \mathbb{R}^2$. Also,

$$\begin{aligned} L_\theta(\Phi_*\Omega) &= \sup_{x\in\mathbb{R}^2} \int_{\mathcal{L}} \frac{d|\theta|(\lambda)}{d(x,\partial\Phi(\Omega^\lambda))^{2\alpha}} = \sup_{x\in\mathbb{R}^2}\int_{\mathcal{L}} \frac{d|\theta|(\lambda)}{d(x,\Phi(\partial\Omega^\lambda))^{2\alpha}} \\ &= \sup_{x\in\mathbb{R}^2}\int_{\mathcal{L}} \frac{d|\theta|(\lambda)}{d(\Phi(x),\Phi(\partial\Omega^\lambda))^{2\alpha}} \le \left\|\Phi^{-1}\right\|_{\dot C^{0,1}}^{2\alpha} L_\theta(\Omega). \end{aligned} \tag{1.7}$$

Now we state our first main result.

Theorem 1.2. *Let $\omega^0 \in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ admit a generalized layer cake representation (Ω,θ) such that $L_\theta(\Omega) < \infty$. Then there is an open interval $I \ni 0$ and a Lagrangian solution $\omega\colon I \to L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ to (1.1)–(1.2) with initial data ω^0 and the associated flow map $\Phi \in C(I; C(\mathbb{R}^2;\mathbb{R}^2))$ such that $\sup_{t\in J} L_\theta(\Phi^t_*\Omega) < \infty$ for any compact interval $J \subseteq I$. This solution is unique and independent of the choice of (Ω,θ), and for any compact interval $J \subseteq I$ we have $\sup_{t\in J}\max\{\|\Phi^t\|_{\dot C^{0,1}}, \|(\Phi^t)^{-1}\|_{\dot C^{0,1}}\} < \infty$. If we let I be the maximal interval as above and T is an endpoint of I, then either $|T| = \infty$ or $\lim_{t\to T} L_\theta(\Phi^t_*\Omega) = \infty$.*

Our second result shows that in the setting of Theorem 1.2, if each $\partial\Omega^\lambda$ is an H^2 curve and certain additional assumptions are satisfied, then these properties must be retained by the solution provided by Theorem 1.2, at least for a short time. To state it precisely, let us give the following definitions. We refer to [1] for notions related to the space of closed plane curves ($\mathrm{CC}(\mathbb{R}^2)$, $\mathrm{PSC}(\mathbb{R}^2)$, $\mathrm{im}(\,\cdot\,)$, $\ell(\,\cdot\,)$, $\|\cdot\|_{\dot H^2}$ and $\Delta(\,\cdot\,,\,\cdot\,)$).

Definition 1.3. Let $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Then a generalized layer cake representation (Ω, θ) of ω is said to be *composed of simple closed curves* if for each $\lambda \in \mathcal{L}$, Ω^λ is a bounded open set and $\partial\Omega^\lambda = \operatorname{im}(z^\lambda)$ for some $z^\lambda \in \operatorname{PSC}(\mathbb{R}^2)$. In this case, we define

$$Q(\Omega) \coloneqq \sup_{\lambda\in\mathcal{L}} \ell(z^\lambda) \left\|z^\lambda\right\|_{\dot{H}^2}^2, \quad R_\theta(\Omega) \coloneqq \sup_{\lambda\in\mathcal{L}} \ell(z^\lambda)^{1/2} \overline{\int_{\mathcal{L}}} \frac{d|\theta|(\lambda')}{\ell(z^{\lambda'})^{1/2}\Delta(z^\lambda, z^{\lambda'})^{2\alpha}}.$$

Here, $\overline{\int_{\mathcal{L}}} f(\lambda)\, d|\theta|(\lambda)$ for any function $f\colon \mathcal{L} \to [0,\infty]$ is the *upper Lebesgue integral* of f; that is,

$$\overline{\int_{\mathcal{L}}} f(\lambda)\, d|\theta|(\lambda) \coloneqq \inf_g \int_{\mathcal{L}} g(\lambda)\, d|\theta|(\lambda)$$

where $g\colon \mathcal{L} \to [0,\infty]$ ranges over all measurable functions bounded below by f.

For our second result, we will impose in addition to $L_\theta(\Omega) < \infty$ that Ω is composed of simple closed curves and $Q(\Omega), R_\theta(\Omega) < \infty$. The condition $Q(\Omega) < \infty$ ensures a form of *scaling-invariant* uniform H^2 regularity of z^λ's. (Recall that for $a > 0$ and $\gamma \in \operatorname{CC}(\mathbb{R}^2)$, $\|a\gamma\|_{\dot{H}^2}^2 = \frac{1}{a}\|\gamma\|_{\dot{H}^2}^2$ while $\ell(a\gamma) = a\ell(\gamma)$.) The condition $R_\theta(\Omega) < \infty$ on the other hand controls how densely z^λ's of *different scales* can be packed together. More specifically, it prevents too many small z^λ's to be placed near a large z^λ. Interpreting z^λ's as level sets of ω, this in effect rules out too sharp "pinched tops/bottoms". (More elaboration of why this is needed?)

Next we state our second main result.

Theorem 1.4. *Let $\omega^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ admit a generalized layer cake representation (Ω, θ) composed of simple closed curves such that $L_\theta(\Omega), Q(\Omega), R_\theta(\Omega) < \infty$. Let $\omega\colon I \to L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ be the Lagrangian solution to (1.1)–(1.2) from Theorem 1.2. Then $\Phi_*^t\Omega$ is composed of simple closed curves for each $t \in I$, $\sup_{t\in J} R_\theta(\Phi_*^t\Omega) < \infty$ for each compact interval $J \subseteq I$, and the set*

$$\left\{t \in I\colon Q(\Phi_*^t\Omega) < \infty\right\}$$

is open.

Consider some smooth even $\chi\colon \mathbb{R} \to \mathbb{R}$ such that $0 \le \chi \le 1$, $\chi \equiv 1$ on $\mathbb{R} \setminus (-1,1)$, and $0 \notin \operatorname{supp}\chi$. For each $\varepsilon > 0$, let $K_\varepsilon(x) \coloneqq \chi\left(\frac{|x|}{\varepsilon}\right) K(x)$. Note that for any $n \in \mathbb{Z}_{\ge 0}$, there is $C_{\alpha,n}$ that only depends on α, n such that the norm of the n-linear form $D^n K_\varepsilon(x)$ is bounded by $\frac{C_{\alpha,n}}{\max\{|x|,\varepsilon\}^{n+2\alpha}}$. For any finite signed Borel measure ω on $\mathbb{R}^2$ we now define the mollified velocity field

$$u_\varepsilon(\omega; x) \coloneqq \int_{\mathbb{R}^2} \nabla^\perp K_\varepsilon(x-y)\, d\omega(y)$$

for $x \in \mathbb{R}^2$. Since $\nabla^\perp K_\varepsilon$ is a smooth function whose all derivatives vanish at infinity, this integral is always well-defined and $u_\varepsilon(\omega)$ is a smooth function such that

$$\left\|D^k(u_\varepsilon(\omega))\right\|_{L^\infty} \le \left\|D^k(\nabla^\perp K_\varepsilon)\right\|_{L^\infty} \|\omega\|_{\mathrm{TV}} \tag{1.8}$$

for all $k \in \mathbb{Z}_{\ge 0}$.

2. Proof of Theorem 1.2

We start with some estimates on the velocity field $u(\omega)$ in terms of $L_\theta(\Omega)$, where (Ω,θ) is a generalized layer cake representation of ω. All constants C_α below can change from one inequality to another, but they always only depend on α.

Lemma 2.1. *There is C_α such that for any $\omega \in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ with a generalized layer cake representation (Ω,θ), and for any $\varepsilon>0$ and $x\in\mathbb{R}^2$, we have*

$$|D(u_\varepsilon(\omega))(x)| \le C_\alpha \int_{\mathcal{L}} \frac{d|\theta|(\lambda)}{d(x,\partial\Omega^\lambda)^{2\alpha}}.$$

Therefore,

$$\|u_\varepsilon(\omega)\|_{\dot C^{0,1}} \le C_\alpha L_\theta(\Omega) \qquad \textit{and} \qquad \|u(\omega)\|_{\dot C^{0,1}} \le C_\alpha L_\theta(\Omega).$$

Proof. For each $\varepsilon>0$ and $x,h\in\mathbb{R}^2$, oddness of $\nabla^\perp K_\varepsilon$ shows that

$$u_\varepsilon(\omega;x+h)-u_\varepsilon(\omega;x) = \int_{\mathbb{R}^2}\left(\nabla^\perp K_\varepsilon(x+h-y)-\nabla^\perp K_\varepsilon(x-y)\right)(\omega(y)-\omega(x))\,dy.$$

Replacing h by sh with $s\in\mathbb{R}$, and then taking $s\to 0$ yields

$$\begin{aligned} D(u_\varepsilon(\omega))(x)h &= \int_{\mathbb{R}^2} D(\nabla^\perp K_\varepsilon)(x-y)h\,(\omega(y)-\omega(x))\,dy \\ &= \int_{\mathbb{R}^2}\int_{\mathcal{L}} D(\nabla^\perp K_\varepsilon)(x-y)h\,(\mathbb{1}_{\Omega^\lambda}(y)-\mathbb{1}_{\Omega^\lambda}(x))\,d\theta(\lambda)\,dy. \end{aligned}$$

Note that $\mathbb{1}_{\Omega^\lambda}(y)-\mathbb{1}_{\Omega^\lambda}(x)\neq 0$ implies $|x-y|\ge d(x,\partial\Omega^\lambda)$, so

$$|D(u_\varepsilon(\omega))(x)| \le \int_{\mathcal{L}}\int_{|x-y|\ge d(x,\partial\Omega^\lambda)} \frac{C_\alpha}{|x-y|^{2+2\alpha}}\,dy\,d|\theta|(\lambda) \le \int_{\mathcal{L}} \frac{C_\alpha}{d(x,\partial\Omega^\lambda)^{2\alpha}} d|\theta|(\lambda).$$

This proves the first and second claims, and the third follows by taking $\varepsilon\to 0^+$. ☐

Lemma 2.2. *There is C_α such that for any $\omega\in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ with generalized layer cake representations (Ω_i,θ_i) and any measure-preserving homeomorphisms $\Phi_i\colon \mathbb{R}^2\to\mathbb{R}^2$ $(i=1,2)$,*

$$\|u(\Phi_{1*}\omega)-u(\Phi_{2*}\omega)\|_{L^\infty} \le C_\alpha\left(L_{\theta_1}(\Phi_{1*}\Omega_1)+L_{\theta_2}(\Phi_{2*}\Omega_2)\right)\|\Phi_1-\Phi_2\|_{L^\infty}.$$

Proof. Let $\mathcal{L}_i$ the measurable space associated to (Ω_i,θ_i) and $\omega_i := \omega\circ\Phi_i^{-1}$ for $i=1,2$. Let $d:=\|\Phi_1-\Phi_2\|_{L^\infty}$ and fix any $x\in\mathbb{R}^2$. Then $u(\Phi_{1*}\omega;x)-u(\Phi_{2*}\omega;x)$ is the sum of

$$\begin{aligned} I_1 &:= \int_{|x-y|\le 2d} \nabla^\perp K(x-y)(\omega_1(y)-\omega_2(y))\,dy, \\ I_2 &:= \int_{|x-y|>2d} \nabla^\perp K(x-y)(\omega_1(y)-\omega_2(y))\,dy. \end{aligned}$$

Estimate for I_1. By oddness of $\nabla^\perp K$, we have

$$I_1 = \int_{|x-y|\le 2d} \nabla^\perp K(x-y)(\omega_1(y)-\omega_1(x))\,dy - \int_{|x-y|\le 2d} \nabla^\perp K(x-y)(\omega_2(y)-\omega_2(x))\,dy.$$

Since

$$I_3 \coloneqq \int_{|x-y|\le 2d} \frac{|\omega_1(y)-\omega_1(x)|}{|x-y|^{1+2\alpha}}\,dy \le \int_{\mathcal{L}_1}\int_{|x-y|\le 2d} \frac{\left|\mathbb{1}_{\Phi_1(\Omega_1^\lambda)}(y)-\mathbb{1}_{\Phi_1(\Omega_1^\lambda)}(x)\right|}{|x-y|^{1+2\alpha}}\,dy\,d|\theta_1|(\lambda)$$

and (as in the proof of Lemma 2.1) we have $|x-y| \ge d(x,\partial\Phi_1(\Omega_1^\lambda))$ whenever the last integrand is nonzero, we see that

$$I_3 \le \int_{\mathcal{L}_1}\int_{|x-y|\le 2d} \frac{1}{|x-y|\ d(x,\partial\Phi_1(\Omega_1^\lambda))^{2\alpha}}\,dy\,d|\theta_1|(\lambda) \le 4\pi L_{\theta_1}(\Phi_{1*}\Omega_1)d.$$

The same argument for ω_2 in place of ω_1 now yields

$$|I_1| \le C_\alpha\left(L_{\theta_1}(\Phi_{1*}\Omega_1)+L_{\theta_2}(\Phi_{2*}\Omega_2)\right)d. \tag{2.1}$$

Estimate for I_2. For each $R>2d$ let

$$I_2^R \coloneqq \int_{2d<|x-y|\le R} \nabla^\perp K(x-y)(\omega_1(y)-\omega_2(y))\,dy,$$

so that $I_2=\lim_{R\to\infty} I_2^R$. Fix $R>2d$, and then Φ_i being measure-preserving yields

$$\begin{aligned}
I_2^R &= \int_{2d<|x-y|\le R} \nabla^\perp K(x-y)(\omega_1(y)-\omega_1(x))\,dy \\
&\qquad - \int_{2d<|x-y|\le R} \nabla^\perp K(x-y)(\omega_2(y)-\omega_1(x))\,dy \\
&= \int_{2d<|x-\Phi_1(y)|\le R} \nabla^\perp K(x-\Phi_1(y))(\omega(y)-\omega_1(x))\,dy \\
&\qquad - \int_{2d<|x-\Phi_2(y)|\le R} \nabla^\perp K(x-\Phi_2(y))(\omega(y)-\omega_1(x))\,dy \\
&= \int_{|x-\Phi_1(y)|,|x-\Phi_2(y)|\in(2d,R]} \left[\nabla^\perp K(x-\Phi_1(y))-\nabla^\perp K(x-\Phi_2(y))\right](\omega(y)-\omega_1(x))\,dy \\
&\qquad + \int_{|x-\Phi_2(y)|\le 2d<|x-\Phi_1(y)|\le R} \nabla^\perp K(x-\Phi_1(y))(\omega(y)-\omega_1(x))\,dy \\
&\qquad - \int_{|x-\Phi_1(y)|\le 2d<|x-\Phi_2(y)|\le R} \nabla^\perp K(x-\Phi_2(y))(\omega(y)-\omega_1(x))\,dy \\
&\qquad + \int_{2d<|x-\Phi_1(y)|\le R<|x-\Phi_2(y)|} \nabla^\perp K(x-\Phi_1(y))(\omega(y)-\omega_1(x))\,dy \\
&\qquad - \int_{2d<|x-\Phi_2(y)|\le R<|x-\Phi_1(y)|} \nabla^\perp K(x-\Phi_2(y))(\omega(y)-\omega_1(x))\,dy.
\end{aligned}$$

Let us denote the integrals on the right-hand side I_4, I_5, I_6, I_7, I_8 (in the order of appearance).

To estimate I_4, note that for any y in the domain of integration we have

$$\min_{\eta\in[0,1]} |x-(1-\eta)\Phi_1(y)-\eta\Phi_2(y)| \ge |x-\Phi_1(y)|-d \ge \frac{|x-\Phi_1(y)|}{2},$$

so the mean value theorem shows that

$$\left|\nabla^\perp K(x-\Phi_1(y)) - \nabla^\perp K(x-\Phi_2(y))\right| \le \frac{C_\alpha d}{|x-\Phi_1(y)|^{2+2\alpha}}.$$

The change of variables formula now yields

$$\begin{aligned}|I_4| &\le \int_{|x-y|>2d} \frac{C_\alpha d\,|\omega_1(y)-\omega_1(x)|}{|x-y|^{2+2\alpha}}\,dy \\ &\le \int_{\mathcal{L}_1}\int_{|x-y|>2d} \frac{C_\alpha d\left|\mathbb{1}_{\Phi_1(\Omega_1^\lambda)}(y)-\mathbb{1}_{\Phi_1(\Omega_1^\lambda)}(x)\right|}{|x-y|^{2+2\alpha}}\,dy\,d|\theta_1|(\lambda).\end{aligned}$$

Again, $|x-y|\ge d(x,\partial\Phi_1(\Omega_1^\lambda))$ holds whenever the last integrand is nonzero, so

$$|I_4| \le \int_{\mathcal{L}_1}\int_{|x-y|\ge d(x,\partial\Phi_1(\Omega_1^\lambda))} \frac{C_\alpha d}{|x-y|^{2+2\alpha}}\,dy\,d|\theta_1|(\lambda) \le C_\alpha L_{\theta_1}(\Phi_{1*}\Omega_1)d.$$

For I_5, note that for any y in the domain of integration we have

$$|x-\Phi_1(y)| \le |x-\Phi_2(y)|+d \le 3d.$$

By applying again change of variables we obtain

$$|I_5| \le \int_{|x-y|\le 3d} \frac{C_\alpha\,|\omega_1(y)-\omega_1(x)|}{|x-y|^{1+2\alpha}}\,dy,$$

so the same argument as in the estimate for I_3 shows $|I_5|\le C_\alpha L_{\theta_1}(\Phi_{1*}\Omega_1)d$. And clearly

$$|I_6| \le \int_{|x-\Phi_1(y)|\le 2d} \frac{C_\alpha\,|\omega(y)-\omega_1(x)|}{|x-\Phi_1(y)|^{1+2\alpha}}\,dy = \int_{|x-y|\le 2d} \frac{C_\alpha\,|\omega_1(y)-\omega_1(x)|}{|x-y|^{1+2\alpha}}\,dy,$$

so again $|I_6|\le C_\alpha L_{\theta_1}(\Phi_{1*}\Omega_1)d$.

To estimate I_7, note that for any y in the domain of integration we have

$$|x-\Phi_1(y)| \ge |x-\Phi_2(y)|-d > R-d,$$

so the change of variables formula yields

$$|I_7| \le \int_{R-d<|x-y|\le R} \frac{C_\alpha\,\|\omega\|_{L^\infty}}{|x-y|^{1+2\alpha}}\,dy \le \frac{C_\alpha\,\|\omega\|_{L^\infty}\,d}{R^{2\alpha}}$$

because $R>2d$. In the same way we also obtain $|I_8|\le \frac{C_\alpha\|\omega\|_{L^\infty}d}{R^{2\alpha}}$.

Collecting the above estimates and letting $R\to\infty$, we see that $|I_2|\le C_\alpha L_{\theta_1}(\Phi_{1*}\Omega_1)d$. This and (2.1) now hold uniformly in $x\in\mathbb{R}^2$, finishing the proof. □

Lemma 2.3. *There is C_α such that for any $\omega\in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ and $\varepsilon>0$ we have*

$$\|u(\omega)-u_\varepsilon(\omega)\|_{L^\infty} \le C_\alpha\,\|\omega\|_{L^\infty}\,\varepsilon^{1-2\alpha}.$$

Proof. Since $\nabla^\perp K_\varepsilon(x) = \nabla^\perp K(x)$ when $|x| \geq \varepsilon$, for any $x \in \mathbb{R}^2$ we have

$$\begin{aligned}|u(\omega;x) - u_\varepsilon(\omega;x)| &\leq \int_{|x-y|\leq\varepsilon} \left|\nabla^\perp K(x-y) - \nabla^\perp K_\varepsilon(x-y)\right| \|\omega\|_{L^\infty}\, dy \\ &\leq \int_{|x-y|\leq\varepsilon} \frac{C_\alpha \|\omega\|_{L^\infty}}{|x-y|^{1+2\alpha}}\, dy = C_\alpha \|\omega\|_{L^\infty} \varepsilon^{1-2\alpha}.\end{aligned}$$

□

Now, fix any initial data $\omega^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ admitting a generalized layer cake representation (Ω, θ) with $L_\theta(\Omega) < \infty$. Fix any $\varepsilon > 0$ and consider the ODE

$$\begin{aligned}\partial_t \Psi^t_\varepsilon &= u_\varepsilon((\mathrm{Id} + \Psi^t_\varepsilon)_* \omega^0) \circ (\mathrm{Id} + \Psi^t_\varepsilon), \\ \Psi^0_\varepsilon &= 0\end{aligned} \tag{2.2}$$

with $\Psi^t_\varepsilon \in BC(\mathbb{R}^2;\mathbb{R}^2)$ (the space of bounded continuous functions from $\mathbb{R}^2$ to $\mathbb{R}^2$). That is, $\Phi^t_\varepsilon := \mathrm{Id} + \Psi^t_\varepsilon$ is the flow map generated by the vector field u_ε that is in turn generated by the measure $\Phi^t_{\varepsilon *}\omega^0$. We will next show that (2.2) is globally well-posed in $BC(\mathbb{R}^2;\mathbb{R}^2)$.

Lemma 2.4. *For each $F \in BC(\mathbb{R}^2;\mathbb{R}^2)$, let*

$$\mathcal{F}(F) := u_\varepsilon((\mathrm{Id} + F)_*\omega^0) \circ (\mathrm{Id} + F).$$

Then $\mathcal{F}\colon BC(\mathbb{R}^2;\mathbb{R}^2) \to BC(\mathbb{R}^2;\mathbb{R}^2)$ is well-defined and Lipschitz continuous.

Proof. Clearly $\mathcal{F}(F) \in C(\mathbb{R}^2;\mathbb{R}^2)$ for any $F \in BC(\mathbb{R}^2;\mathbb{R}^2)$. For any $F_1, F_2 \in BC(\mathbb{R}^2;\mathbb{R}^2)$ and $x \in \mathbb{R}^2$ we see that $(\mathcal{F}(F_1) - \mathcal{F}(F_2))(x)$ equals

$$\begin{aligned}&\int_{\mathbb{R}^2} \nabla^\perp K_\varepsilon(x + F_1(x) - y)\, d(\mathrm{Id} + F_1)_*\omega^0(y) - \int_{\mathbb{R}^2} \nabla^\perp K_\varepsilon(x + F_2(x) - y)\, d(\mathrm{Id} + F_2)_*\omega^0(y) \\ &= \int_{\mathbb{R}^2} \left[\nabla^\perp K_\varepsilon(x - y + F_1(x) - F_1(y)) - \nabla^\perp K_\varepsilon(x - y + F_2(x) - F_2(y))\right] \omega^0(y)\, dy,\end{aligned}$$

so

$$\|\mathcal{F}(F_1) - \mathcal{F}(F_2)\|_{L^\infty} \leq 2\left\|D(\nabla^\perp K_\varepsilon)\right\|_{L^\infty} \left\|\omega^0\right\|_{L^1} \|F_1 - F_2\|_{L^\infty}.$$

Since $\mathcal{F}(0) = u_\varepsilon(\omega^0)$ is bounded, both claims follow from this. □

Therefore (2.2) is globally well-posed, and we let $\Phi^t_\varepsilon := \mathrm{Id} + \Psi^t_\varepsilon$ and $\omega^t_\varepsilon := \Phi^t_{\varepsilon *}\omega^0$, with the latter being for now only a finite signed Borel measure. We will next show that Φ^t_ε is in fact a measure-preserving homeomorphism, which will mean that ω^t_ε is an $L^1 \cap L^\infty$ function.

Clearly the ODE

$$\partial_t G^t = u_\varepsilon(\omega^t_\varepsilon) \circ (\mathrm{Id} + G^t) \tag{2.3}$$

is globally well-posed in $BC(\mathbb{R}^2;\mathbb{R}^2)$ for any initial data at any initial time. For each $t_0, t_1 \in \mathbb{R}$, let $\Theta^{t_0,t}_\varepsilon$ be the unique solution to (2.3) with initial data $\Theta^{t_0,t_0}_\varepsilon := 0$ at time $t = t_0$, and consider $G^t := \Theta^{t_0,t_1}_\varepsilon + \Theta^{t_1,t}_\varepsilon \circ (\mathrm{Id} + \Theta^{t_0,t_1}_\varepsilon)$. Then G^t solves (2.3) and $G^{t_1} = \Theta^{t_0,t_1}_\varepsilon$, so uniqueness of the solution with the initial data $\Theta^{t_0,t_1}_\varepsilon$ at time $t = t_1$ shows that

$$\mathrm{Id} + \Theta^{t_0,t}_\varepsilon = \mathrm{Id} + G^t = (\mathrm{Id} + \Theta^{t_1,t}_\varepsilon) \circ (\mathrm{Id} + \Theta^{t_0,t_1}_\varepsilon)$$

for all $t \in \mathbb{R}$. Letting $t := t_0$ shows that $(\mathrm{Id} + \Theta_\varepsilon^{t_1,t_0}) \circ (\mathrm{Id} + \Theta_\varepsilon^{t_0,t_1}) = \mathrm{Id}$, so we conclude that each $\mathrm{Id} + \Theta_\varepsilon^{t_0,t}$ is a homeomorphism. Then so is $\Phi_\varepsilon^t = \mathrm{Id} + \Theta_\varepsilon^{0,t}$.

Letting $BC^1(\mathbb{R}^2;\mathbb{R}^2)$ be the space of bounded C^1 functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ with bounded first derivatives, we see that for any $F \in BC^1(\mathbb{R}^2;\mathbb{R}^2)$ and $x, h \in \mathbb{R}^2$ we have

$$D(u_\varepsilon(\omega_\varepsilon^t) \circ (\mathrm{Id} + F))(x)h = \int_{\mathbb{R}^2} D(\nabla^\perp K_\varepsilon)(x + F(x) - y)(h + DF(x)h)\, d\omega_\varepsilon^t(y).$$

Therefore $F \mapsto u_\varepsilon(\omega_\varepsilon^t) \circ (\mathrm{Id} + F)$ is locally Lipschitz on $BC^1(\mathbb{R}^2;\mathbb{R}^2)$, so (2.3) is locally well-posed there. But since for any $F \in BC^1(\mathbb{R}^2;\mathbb{R}^2)$ we have

$$\left\| D(u_\varepsilon(\omega_\varepsilon^t) \circ (\mathrm{Id} + F)) \right\|_{L^\infty} \le \left\| D(\nabla^\perp K_\varepsilon) \right\|_{L^\infty} \left\| \omega^0 \right\|_{L^1} \left\| \mathrm{Id} + DF \right\|_{L^\infty},$$

a Grönwall-type argument shows that the C^1 norm of any solution to (2.3) can grow no faster than exponentially. Therefore (2.3) is even globally well-posed in $BC^1(\mathbb{R}^2;\mathbb{R}^2)$, and so $\Theta_\varepsilon^{t_0,t} \in BC^1(\mathbb{R}^2;\mathbb{R}^2)$ for all $t \in \mathbb{R}$. This and $\nabla \cdot u_\varepsilon(\omega_\varepsilon^t) \equiv 0$ now show that the map $\mathrm{Id} + \Theta_\varepsilon^{t_0,t}$ is measure-preserving. Then $\omega_\varepsilon^t = \omega^0 \circ (\Phi_\varepsilon^t)^{-1} \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $\Phi_{\varepsilon *}^t \Omega$ is its generalized layer cake representation.

Similarly, with $BC^2(\mathbb{R}^2;\mathbb{R}^2)$ the space of bounded C^2 functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ with bounded first and second derivatives, for each $F \in BC^2(\mathbb{R}^2;\mathbb{R}^2)$ and $x, h_1, h_2 \in \mathbb{R}^2$ we have

$$\begin{aligned} &D^2(u_\varepsilon(\omega_\varepsilon^t) \circ (\mathrm{Id} + F))(x)(h_1, h_2) \\ &\quad = \int_{\mathbb{R}^2} D^2(\nabla^\perp K_\varepsilon)(x + F(x) - y)(h_1 + DF(x)h_1, h_2 + DF(x)h_2)\, \omega_\varepsilon^t(y)\, dy \\ &\qquad + \int_{\mathbb{R}^2} D(\nabla^\perp K_\varepsilon)(x + F(x) - y) \left(D^2F(x)(h_1, h_2) \right) \omega_\varepsilon^t(y)\, dy \end{aligned}$$

and

$$\begin{aligned} \left\| D^2(u_\varepsilon(\omega_\varepsilon^t) \circ (\mathrm{Id} + F)) \right\|_{L^\infty} \le{}& \left\| D^2(\nabla^\perp K_\varepsilon) \right\|_{L^\infty} \left\| \omega^0 \right\|_{L^1} \left\| \mathrm{Id} + DF \right\|_{L^\infty}^2 \\ &+ \left\| D(\nabla^\perp K_\varepsilon) \right\|_{L^\infty} \left\| \omega^0 \right\|_{L^1} \left\| D^2F \right\|_{L^\infty}. \end{aligned}$$

Another Grönwall-type argument and the time-exponential bound on the C^1 norms of solutions to (2.3) now shows that (2.3) is globally well-posed in $BC^2(\mathbb{R}^2;\mathbb{R}^2)$, which we will use in Section 3. One can continue and inductively show that (2.3) is globally well-posed in $BC^k(\mathbb{R}^2;\mathbb{R}^2)$ for all $k \in \mathbb{N}$ (then each Φ_ε^t is a diffeomorphism), but we will not need this here.

Next we derive an ε-independent estimate on the growth of $L_\theta(\Phi_{\varepsilon *}^t \Omega)$.

Lemma 2.5. *$\left\| D^k(u_\varepsilon(\omega_\varepsilon^t)) \right\|_{L^\infty}$ is continuous in t for all $k \in \mathbb{Z}_{\ge 0}$, and*

$$\begin{aligned} &\left| \Theta_\varepsilon^{t_0,t_1}(x) - \Theta_\varepsilon^{t_0,t_1}(y) - \Theta_\varepsilon^{t_0,t_2}(x) + \Theta_\varepsilon^{t_0,t_2}(y) \right| \\ &\qquad \le \left(\exp\left(\left| \int_{t_2}^{t_1} \left\| u_\varepsilon(\omega_\varepsilon^\tau) \right\|_{\dot{C}^{0,1}} d\tau \right| \right) - 1 \right) \left| x + \Theta_\varepsilon^{t_0,t_2}(x) - y - \Theta_\varepsilon^{t_0,t_2}(y) \right| \end{aligned} \tag{2.4}$$

holds for all $x, y \in \mathbb{R}^2$ and $t_0, t_1, t_2 \in \mathbb{R}$.

Proof. Change of variables yields

$$\left\| D^k(u_\varepsilon(\omega_\varepsilon^{t_1})) - D^k(u_\varepsilon(\omega_\varepsilon^{t_2})) \right\|_{L^\infty} \le \left\| D^{k+1}(\nabla^\perp K_\varepsilon) \right\|_{L^\infty} \left\| \omega^0 \right\|_{L^1} \left\| \Phi_\varepsilon^{t_1} - \Phi_\varepsilon^{t_2} \right\|_{L^\infty}$$

for any $(k, t_1, t_2) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^2$. This shows the first claim, and in particular that $\|u_\varepsilon(\omega_\varepsilon^t)\|_{\dot{C}^{0,1}}$ is continuous in t.

Next, letting $x' := x + \Theta_\varepsilon^{t_0,t_2}(x)$, we see that

$$\Theta_\varepsilon^{t_0,t_1}(x) = x' + \Theta_\varepsilon^{t_2,t_1}(x') - x = \Theta_\varepsilon^{t_2,t_1}(x') + \Theta_\varepsilon^{t_0,t_2}(x).$$

So with $y' := y + \Theta_\varepsilon^{t_0,t_2}(y)$, the left-hand side of (2.4) is just $|\Theta_\varepsilon^{t_2,t_1}(x') - \Theta_\varepsilon^{t_2,t_1}(y')|$, while the last factor is $|x' - y'|$. The result now follows from the definition of $\Theta_\varepsilon^{t_2,t}$. $\square$

Proposition 2.6. *$L_\theta(\Phi_{\varepsilon*}^t\Omega)$ is continuous in t and*

$$\max\left\{\partial_t^+ L_\theta(\Phi_{\varepsilon*}^t\Omega), -\partial_{t-} L_\theta(\Phi_{\varepsilon*}^t\Omega)\right\} \leq \left\|u_\varepsilon(\omega_\varepsilon^t)\right\|_{\dot{C}^{0,1}} L_\theta(\Phi_{\varepsilon*}^t\Omega).$$

Proof. Fix any $t_1, t_2 \in \mathbb{R}$, $x \in \mathbb{R}^2$, $\lambda \in \mathcal{L}$, and $\eta > 0$. Pick $y \in \partial\Omega^\lambda$ such that

$$\left|\Phi_\varepsilon^{t_1}(x) - \Phi_\varepsilon^{t_1}(y)\right| \leq d(\Phi_\varepsilon^{t_1}(x), \partial\Phi_\varepsilon^{t_1}(\Omega^\lambda)) + \eta.$$

Then Lemma 2.5 and the inequality $\left|\frac{1}{a^{2\alpha}} - \frac{1}{b^{2\alpha}}\right| \leq \frac{|a-b|}{ab^{2\alpha}}$ for $a, b > 0$ show that

$$\begin{aligned}
\frac{1}{\left(d(\Phi_\varepsilon^{t_1}(x), \partial\Phi_\varepsilon^{t_1}(\Omega^\lambda)) + 2\eta\right)^{2\alpha}} &- \frac{1}{\left(d(\Phi_\varepsilon^{t_2}(x), \partial\Phi_\varepsilon^{t_2}(\Omega^\lambda)) + \eta\right)^{2\alpha}} \\
&\leq \frac{1}{\left(\left|\Phi_\varepsilon^{t_1}(x) - \Phi_\varepsilon^{t_1}(y)\right| + \eta\right)^{2\alpha}} - \frac{1}{\left(\left|\Phi_\varepsilon^{t_2}(x) - \Phi_\varepsilon^{t_2}(y)\right| + \eta\right)^{2\alpha}} \\
&\leq \frac{|\Phi_\varepsilon^{t_1}(x) - \Phi_\varepsilon^{t_1}(y) - \Phi_\varepsilon^{t_2}(x) + \Phi_\varepsilon^{t_2}(y)|}{\left(\left|\Phi_\varepsilon^{t_1}(x) - \Phi_\varepsilon^{t_1}(y)\right| + \eta\right)\left(\left|\Phi_\varepsilon^{t_2}(x) - \Phi_\varepsilon^{t_2}(y)\right| + \eta\right)^{2\alpha}} \\
&\leq \frac{\exp\left(\left|\int_{t_2}^{t_1} \|u_\varepsilon(\omega_\varepsilon^\tau)\|_{\dot{C}^{0,1}}\, d\tau\right|\right) - 1}{(d(\Phi_\varepsilon^{t_2}(x), \partial\Phi_\varepsilon^{t_2}(\Omega^\lambda)) + \eta)^{2\alpha}},
\end{aligned} \tag{2.5}$$

so letting $\eta \to 0^+$, integrating over λ, and then taking supremum over $x \in \mathbb{R}^2$ shows

$$L_\theta(\Phi_{\varepsilon*}^{t_1}\Omega) \leq \exp\left(\left|\int_{t_2}^{t_1} \|u_\varepsilon(\omega_\varepsilon^\tau)\|_{\dot{C}^{0,1}}\, d\tau\right|\right) L_\theta(\Phi_{\varepsilon*}^{t_2}\Omega).$$

Since $t_1, t_2 \in \mathbb{R}$ were arbitrary, both claims follow from this. $\square$

From Lemma 2.1, Proposition 2.6, and a Grönwall-type argument we now obtain the following result.

Corollary 2.7. *With C_α from Lemma 2.1, for all $t \in \mathbb{R}$ we have*

$$\max\left\{\partial_t^+ L_\theta(\Phi_{\varepsilon*}^t\Omega), -\partial_{t-} L_\theta(\Phi_{\varepsilon*}^t\Omega)\right\} \leq C_\alpha L_\theta(\Phi_{\varepsilon*}^t\Omega)^2.$$

In particular, for all $t \in (-\frac{1}{C_\alpha L_\theta(\Omega)}, \frac{1}{C_\alpha L_\theta(\Omega)})$ we have

$$L_\theta(\Phi_{\varepsilon*}^t\Omega) \leq \frac{L_\theta(\Omega)}{1 - C_\alpha L_\theta(\Omega)\,|t|}.$$

Let $T_0 := \frac{1}{2C_\alpha L_\theta(\Omega)}$ with C_α from Lemma 2.1.

Proposition 2.8. *There is* $\Psi := \lim_{\varepsilon\to 0}\Psi_\varepsilon \in C\left([-T_0,T_0]; BC(\mathbb{R}^2;\mathbb{R}^2)\right)$*, and* $\Phi^t := \mathrm{Id} + \Psi^t$ *is a measure-preserving homeomorphism for each* $t\in[-T_0,T_0]$ *that solves* (1.6)*. Moreover, for each* $t\in[-T_0,T_0]$ *we have*

$$L_\theta(\Phi^t_*\Omega) \le \sup_{\varepsilon>0} L_\theta(\Phi^t_{\varepsilon *}\Omega) \le 2L_\theta(\Omega)$$

and

$$\max\left\{\left\|\Phi^t\right\|_{\dot C^{0,1}}, \left\|(\Phi^t)^{-1}\right\|_{\dot C^{0,1}}\right\} \le \sup_{\varepsilon>0}\max\left\{\left\|\Phi^t_\varepsilon\right\|_{\dot C^{0,1}}, \left\|(\Phi^t_\varepsilon)^{-1}\right\|_{\dot C^{0,1}}\right\} \le e^{2C_\alpha L_\theta(\Omega)|t|}.$$

Proof. Corollary 2.7 shows that

$$M := \sup_{\varepsilon>0}\sup_{t\in[-T_0,T_0]} L_\theta(\Phi^t_{\varepsilon *}\Omega) \in [L_\theta(\Omega), 2L_\theta(\Omega)].$$

We may assume that $L_\theta(\Omega) > 0$ because otherwise $\omega^0 \equiv 0$ and the result follows trivially. Fix any $t_0\in[-T_0,T_0]$ and pick any $t\in[-T_0,T_0]$, $\varepsilon>0$, and $\varepsilon'\in(0,\varepsilon)$. Then Lemmas 2.1, 2.2, and 2.3 show that

$$\begin{aligned}
&\left\|u_\varepsilon(\omega^t_\varepsilon)\circ(\mathrm{Id}+\Theta^{t_0,t}_\varepsilon) - u_{\varepsilon'}(\omega^t_{\varepsilon'})\circ(\mathrm{Id}+\Theta^{t_0,t}_{\varepsilon'})\right\|_{L^\infty}\\
&\quad\le \left\|u_\varepsilon(\omega^t_\varepsilon)\right\|_{\dot C^{0,1}}\left\|\Theta^{t_0,t}_\varepsilon - \Theta^{t_0,t}_{\varepsilon'}\right\|_{L^\infty} + \left\|u_\varepsilon(\omega^t_\varepsilon) - u(\omega^t_\varepsilon)\right\|_{L^\infty}\\
&\qquad\quad + \left\|u(\omega^t_\varepsilon) - u(\omega^t_{\varepsilon'})\right\|_{L^\infty} + \left\|u(\omega^t_{\varepsilon'}) - u_{\varepsilon'}(\omega^t_{\varepsilon'})\right\|_{L^\infty}\\
&\quad\le C_\alpha M\left\|\Theta^{t_0,t}_\varepsilon - \Theta^{t_0,t}_{\varepsilon'}\right\|_{L^\infty} + C_\alpha M\left\|\Phi^t_\varepsilon - \Phi^t_{\varepsilon'}\right\|_{L^\infty} + C_\alpha\left\|\omega^0\right\|_{L^\infty}\varepsilon^{1-2\alpha}
\end{aligned}\tag{2.6}$$

where C_α (which we now fix for the rest of the proof) is two times the maximum of all the C_α's appearing in those lemmas. Integrating (2.6) between any $t_1,t_2\in[-T_0,T_0]$ yields

$$\begin{aligned}
&\left\|\Theta^{t_0,t_1}_\varepsilon - \Theta^{t_0,t_1}_{\varepsilon'} - \Theta^{t_0,t_2}_\varepsilon + \Theta^{t_0,t_2}_{\varepsilon'}\right\|_{L^\infty}\\
&\quad\le C_\alpha M\left|\int_{t_2}^{t_1}\left\|\Theta^{t_0,\tau}_\varepsilon - \Theta^{t_0,\tau}_{\varepsilon'}\right\|_{L^\infty}d\tau\right| + C_\alpha M\left|\int_{t_2}^{t_1}\left\|\Phi^\tau_\varepsilon - \Phi^\tau_{\varepsilon'}\right\|_{L^\infty}d\tau\right|\\
&\qquad\quad + C_\alpha\left\|\omega^0\right\|_{L^\infty}|t_1-t_2|\,\varepsilon^{1-2\alpha}.
\end{aligned}\tag{2.7}$$

In particular, taking $t_0=0$, dividing by $|t_1-t_2|$, and letting $t_1\to t_2^\pm$ shows that

$$\max\left\{\partial_t^+\left\|\Psi^t_\varepsilon - \Psi^t_{\varepsilon'}\right\|_{L^\infty}, -\partial_{t-}\left\|\Psi^t_\varepsilon - \Psi^t_{\varepsilon'}\right\|_{L^\infty}\right\} \le 2C_\alpha M\left\|\Psi^t_\varepsilon - \Psi^t_{\varepsilon'}\right\|_{L^\infty} + C_\alpha\left\|\omega^0\right\|_{L^\infty}\varepsilon^{1-2\alpha}$$

for each $t\in[-T_0,T_0]$, and then a Grönwall-type argument yields

$$\left\|\Phi^t_\varepsilon - \Phi^t_{\varepsilon'}\right\|_{L^\infty} = \left\|\Psi^t_\varepsilon - \Psi^t_{\varepsilon'}\right\|_{L^\infty} \le \frac{\|\omega^0\|_{L^\infty}}{2M}(e^{2C_\alpha M|t|}-1)\varepsilon^{1-2\alpha}.$$

Applying this inequality to (2.7), dividing by $|t_1-t_2|$ and then sending $t_1\to t_2^\pm$ shows that

$$\begin{aligned}
&\max\left\{\partial_t^+\left\|\Theta^{t_0,t}_\varepsilon - \Theta^{t_0,t}_{\varepsilon'}\right\|_{L^\infty}, -\partial_{t-}\left\|\Theta^{t_0,t}_\varepsilon - \Theta^{t_0,t}_{\varepsilon'}\right\|_{L^\infty}\right\}\\
&\qquad\qquad\le C_\alpha M\left\|\Theta^{t_0,t}_\varepsilon - \Theta^{t_0,t}_{\varepsilon'}\right\|_{L^\infty} + C_\alpha\left\|\omega^0\right\|_{L^\infty}e^{2C_\alpha M|t|}\varepsilon^{1-2\alpha}
\end{aligned}$$

for all $t\in[-T_0,T_0]$, so a Grönwall-type argument yields

$$\left\|\Theta^{t_0,t}_\varepsilon - \Theta^{t_0,t}_{\varepsilon'}\right\|_{L^\infty} \le \frac{\|\omega^0\|_{L^\infty}e^{2C_\alpha MT_0}\varepsilon^{1-2\alpha}}{M}\left(e^{C_\alpha M|t-t_0|}-1\right).$$

Therefore, $\Theta_\varepsilon^{t_0,\cdot}$ converges uniformly to some $\Theta^{t_0,\cdot} : [-T_0, T_0] \to BC(\mathbb{R}^2;\mathbb{R}^2)$ as $\varepsilon \to 0$.

Let $\Psi^t := \Theta^{0,t}$. Since $(\mathrm{Id} + \Theta_\varepsilon^{t_0,t_1}) \circ (\mathrm{Id} + \Theta_\varepsilon^{t_1,t_0}) = \mathrm{Id}$ for all $t_0, t_1 \in [-T_0, T_0]$ and $\varepsilon > 0$, sending $\varepsilon \to 0$ shows that $(\mathrm{Id} + \Theta^{t_0,t_1}) \circ (\mathrm{Id} + \Theta^{t_1,t_0}) = \mathrm{Id}$. In particular, $\Phi^t := \mathrm{Id} + \Psi^t$ is a homeomorphism whose inverse is $\mathrm{Id} + \Theta^{t,0}$. Also, Lemma 2.5 and the definition of C_α show that $\|\mathrm{Id} + \Theta_\varepsilon^{t_0,t}\|_{\dot{C}^{0,1}} \le e^{C_\alpha M|t-t_0|}$ for all $t_0, t \in [-T_0, T_0]$ and $\varepsilon > 0$, thus $\max\{\|\Phi^t\|_{\dot{C}^{0,1}}, \|(\Phi^t)^{-1}\|_{\dot{C}^{0,1}}\} \le e^{C_\alpha M|t|}$ holds for all $t \in [-T_0, T_0]$. By Fatou's lemma we also have $L_\theta(\Phi^t_*\Omega) \le \liminf_{\varepsilon\to 0} L_\theta(\Phi^t_{\varepsilon*}\Omega) \le M$ for each $t \in [-T_0, T_0]$. And since each Φ_ε^t is measure-preserving, their uniform limit Φ^t is also such because for any open set $U \subseteq \mathbb{R}^2$ we have that $\mathbb{1}_U \circ \Phi_\varepsilon^t \to \mathbb{1}_U \circ \Phi^t$ pointwise as $\varepsilon \to 0$.

It remains to show that Φ^t satisfies (1.6), that is, with $\omega^t := \omega^0 \circ (\Phi^t)^{-1}$ we have

$$\Phi^t = \mathrm{Id} + \int_0^t u(\omega^\tau) \circ \Phi^\tau \, d\tau \tag{2.8}$$

for each $t \in [-T_0, T_0]$. Taking $t_0 = 0$ and letting $\varepsilon' \to 0^+$ in (2.6) yields

$$\left\|u_\varepsilon(\omega_\varepsilon^t) \circ \Phi_\varepsilon^t - u(\omega^t) \circ \Phi^t\right\|_{L^\infty} \le 2C_\alpha M \left\|\Phi_\varepsilon^t - \Phi^t\right\|_{L^\infty} + C_\alpha \left\|\omega^0\right\|_{L^\infty} \varepsilon^{1-2\alpha},$$

which shows that the right-hand side of

$$\Phi_\varepsilon^t = \mathrm{Id} + \int_0^t u(\omega_\varepsilon^\tau) \circ \Phi_\varepsilon^\tau \, d\tau$$

converges uniformly to the right-hand side of (2.8) as $\varepsilon \to 0$. This now proves (1.6). $\square$

Proposition 2.9. *Let (Ω_1, θ_1), (Ω_2, θ_2) be generalized layer cake representations of ω^0 and $\Phi_1, \Phi_2 \in C(I; C(\mathbb{R}^2;\mathbb{R}^2))$ be solutions to (1.6) on a compact interval $I \ni 0$ that are both measure-preserving homeomorphisms and $\sup_{t\in I} \max_{i\in\{1,2\}} L_{\theta_i}(\Phi^t_{i*}\Omega_i) < \infty$. Then $\Phi_1 = \Phi_2$.*

Proof. Let

$$M := \sup_{t\in I} \max_{i=1,2} L_{\theta_i}(\Phi^t_{i*}\Omega_i).$$

Then Lemmas 2.1 and 2.2 show that

$$\begin{aligned}\left\|u(\Phi^t_{1*}\omega^0) \circ \Phi_1^t - u(\Phi^t_{2*}\omega^0) \circ \Phi_2^t\right\|_{L^\infty} &\le \left\|u(\Phi^t_{1*}\omega^0)\right\|_{\dot{C}^{0,1}} \left\|\Phi_1^t - \Phi_2^t\right\|_{L^\infty} + \left\|u(\Phi^t_{1*}\omega^0) - u(\Phi^t_{2*}\omega^0)\right\|_{L^\infty} \\ &\le C_\alpha M \left\|\Phi_1^t - \Phi_2^t\right\|_{L^\infty}\end{aligned}$$

with some C_α, which together with continuity of $\|\Phi_1^t - \Phi_2^t\|_{L^\infty}$ in t yields

$$\max\left\{\partial_t^+ \left\|\Phi_1^t - \Phi_2^t\right\|_{L^\infty}, -\partial_{t-} \left\|\Phi_1^t - \Phi_2^t\right\|_{L^\infty}\right\} \le C_\alpha M \left\|\Phi_1^t - \Phi_2^t\right\|_{L^\infty}.$$

A Grönwall-type argument finishes the proof. $\square$

Combining Propositions 2.8 and 2.9 with (1.7), the latter showing that the time spans of maximal solutions for any two generalized layer cake representations of ω^0 must coincide (recall that $\sup_{t\in J} \|(\Phi^t)^{-1}\|_{\dot{C}^{0,1}} < \infty$ for any compact interval $J \subseteq I$), now yields Theorem 1.2.

3. Proof of Theorem 1.4

Again, all constants C_α below can change from one inequality to another, but they always only depend on α. Suppose that the initial data $\omega^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ admits a generalized layer cake representation (Ω, θ) composed of simple closed curves such that $L_\theta(\Omega), Q(\Omega), R_\theta(\Omega) < \infty$. For each $\lambda \in \mathcal{L}$, let $z^{0,\lambda} \in \mathrm{PSC}(\mathbb{R}^2)$ be such that $\partial\Omega^\lambda = \mathrm{im}(z^{0,\lambda})$.

Let $\omega \colon I \to L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ be the Lagrangian solution to (1.1)–(1.2) from Theorem 1.2, with initial data ω^0 and flow map $\Phi \in C\left(I; C(\mathbb{R}^2;\mathbb{R}^2)\right)$. Then since Φ^t is a homeomorphism for each $t \in I$, it follows that $\Phi^t_*\Omega$ is composed of simple closed curves and we denote these $z^{t,\lambda} := \Phi^t \circ z^{0,\lambda} \in \mathrm{PSC}(\mathbb{R}^2)$, where $\Phi^t \circ z^{0,\lambda} \in \mathrm{CC}(\mathbb{R}^2)$ is the curve whose representative is $\Phi^t \circ \tilde{z}^{0,\lambda}$ whenever $\tilde{z}^{0,\lambda}$ is a representative of $z^{0,\lambda}$ (since $\left\{z^{t,\lambda}\right\}_{t\in I}$ is clearly a connected subset of $\mathrm{CC}(\mathbb{R}^2)$, [1, Lemma B.4] shows that each $z^{t,\lambda}$ is positively oriented).

Fix any $\varepsilon > 0$ and recall that $\Phi^t_\varepsilon := \mathrm{Id} + \Psi^t_\varepsilon$, where Ψ^t_ε is the solution to (2.2). For each $(t,\lambda) \in \mathbb{R}\times\mathcal{L}$ let $z^{t,\lambda}_\varepsilon := \Phi^t_\varepsilon \circ z^{0,\lambda}$ and $\omega^t_\varepsilon := \omega^0 \circ (\Phi^t_\varepsilon)^{-1}$, then fix any arclength parametrization of $z^{t,\lambda}_\varepsilon$ (we denote it again $z^{t,\lambda}_\varepsilon(\cdot)$) and for $s \in [0, \ell(z^{t,\lambda}_\varepsilon)]$ define (suppressing ε in most of this notation)

- $\ell^{t,\lambda} := \ell(z^{t,\lambda}_\varepsilon)$,
- $\mathbf{T}^{t,\lambda}(s) := \partial_s z^{t,\lambda}_\varepsilon(s)$,
- $\mathbf{N}^{t,\lambda}(s) := \mathbf{T}^{t,\lambda}(s)^\perp$,
- $\kappa^{t,\lambda}(s) := \partial_s^2 z^{t,\lambda}_\varepsilon(s) \cdot \mathbf{N}^{t,\lambda}(s)$,
- $\Delta^{t,\lambda,\lambda'} := \Delta(z^{t,\lambda}_\varepsilon, z^{t,\lambda'}_\varepsilon)$,
- $u^{t,\lambda}_\varepsilon(s) := u_\varepsilon(\omega^t_\varepsilon; z^{t,\lambda}_\varepsilon(s))$.

Proposition 2.8 shows that $\lim_{\varepsilon\to 0} z^{t,\lambda}_\varepsilon = z^{t,\lambda}$ in $\mathrm{CC}(\mathbb{R}^2)$, and as noted in [1, Section 4],

$$\partial_s^2 z^{t,\lambda}_\varepsilon(s) = \partial_s \mathbf{T}^{t,\lambda}(s) = \kappa^{t,\lambda}(s)\mathbf{N}^{t,\lambda}(s) \qquad \text{and} \qquad \partial_s \mathbf{N}^{t,\lambda}(s) = -\kappa^{t,\lambda}(s)\mathbf{T}^{t,\lambda}(s)$$

holds as well. Then the argument in [1, Lemma 4.1] also applies here, and we obtain

$$\begin{aligned}\partial_t \left\|z^{t,\lambda}_\varepsilon\right\|^2_{\dot{H}^2} = -3 \int_{\ell^{t,\lambda}\mathbb{T}} \kappa^{t,\lambda}(s)^2 \left(\partial_s u^{t,\lambda}_\varepsilon(s) \cdot \mathbf{T}^{t,\lambda}(s)\right) ds \\ + 2 \int_{\ell^{t,\lambda}\mathbb{T}} \kappa^{t,\lambda}(s) \left(\partial_s^2 u^{t,\lambda}_\varepsilon(s) \cdot \mathbf{N}^{t,\lambda}(s)\right) ds.\end{aligned} \tag{3.1}$$

We can similarly compute the time derivative of $\ell^{t,\lambda}$.

Lemma 3.1. *For any $(t,\lambda) \in \mathbb{R}\times\mathcal{L}$ we have*

$$\partial_t \ell^{t,\lambda} = \int_{\ell^{t,\lambda}\mathbb{T}} \partial_s u^{t,\lambda}_\varepsilon(s) \cdot \mathbf{T}^{t,\lambda}(s)\, ds.$$

Proof. Fix some constant-speed parametrization $\tilde{z}^{t,\lambda}_\varepsilon \colon \mathbb{T} \to \mathbb{R}^2$ of $z^{t,\lambda}_\varepsilon$, and for each $h \in \mathbb{R}$ let $\tilde{z}^{t+h,\lambda}_\varepsilon := \Phi^{t+h}_\varepsilon \circ (\Phi^t_\varepsilon)^{-1} \circ \tilde{z}^{t,\lambda}_\varepsilon$. Since (2.3) is globally well-posed in $BC^2(\mathbb{R}^2;\mathbb{R}^2)$ (see the paragraph before Lemma 2.5), it easily follows that $\tilde{z}^{t,\lambda}_\varepsilon \in H^2(\mathbb{T};\mathbb{R}^2)$ and

$$\tilde{z}^{t+h,\lambda}_\varepsilon = \tilde{z}^{t,\lambda}_\varepsilon + \int_t^{t+h} u_\varepsilon(\omega^\tau_\varepsilon) \circ \tilde{z}^{\tau,\lambda}_\varepsilon \, d\tau \tag{3.2}$$

holds for all $h \in \mathbb{R}$ (in $H^2(\mathbb{T};\mathbb{R}^2)$). Since $\left|\partial_\xi \tilde{z}_\varepsilon^{t,\lambda}(\xi)\right| = \ell^{t,\lambda} > 0$, for any $(h,\xi) \in \mathbb{R}\times\mathbb{T}$ we get

$$\begin{aligned}&\left|\partial_\xi \tilde{z}_\varepsilon^{t+h,\lambda}(\xi)\right| - \left|\partial_\xi \tilde{z}_\varepsilon^{t,\lambda}(\xi)\right| \\ &\quad = \frac{2\int_t^{t+h} \frac{d}{d\xi} u_\varepsilon(\omega_\varepsilon^\tau; \tilde{z}_\varepsilon^{\tau,\lambda}(\xi)) \cdot \partial_\xi \tilde{z}_\varepsilon^{t,\lambda}(\xi)\, d\tau}{\left|\partial_\xi \tilde{z}_\varepsilon^{t+h,\lambda}(\xi)\right| + \left|\partial_\xi \tilde{z}_\varepsilon^{t,\lambda}(\xi)\right|} + \frac{\left|\int_t^{t+h} \frac{d}{d\xi} u_\varepsilon(\omega_\varepsilon^\tau; \tilde{z}_\varepsilon^{\tau,\lambda}(\xi))\, d\tau\right|^2}{\left|\partial_\xi \tilde{z}_\varepsilon^{t+h,\lambda}(\xi)\right| + \left|\partial_\xi \tilde{z}_\varepsilon^{t,\lambda}(\xi)\right|}.\end{aligned}$$

Since $\|D(u_\varepsilon(\omega_\varepsilon^\tau))\|_{L^\infty}$ is continuous in τ by Lemma 2.5, integrating in ξ, dividing by h, and then letting $h \to 0$ shows that

$$\partial_t \ell^{t,\lambda} = \int_{\mathbb{T}} \frac{d}{d\xi} u_\varepsilon(\omega_\varepsilon^t; \tilde{z}_\varepsilon^{t,\lambda}(\xi)) \cdot \frac{\partial_\xi \tilde{z}_\varepsilon^{t,\lambda}(\xi)}{\left|\partial_\xi \tilde{z}_\varepsilon^{t,\lambda}(\xi)\right|}\, d\xi.$$

The claim now follows by a change of variables. □

Corollary 3.2. *With C_α from Lemma 2.1, for any $(t,\lambda) \in I \times \mathcal{L}$ and all small enough $h \in \mathbb{R}$ we have*

$$e^{-3C_\alpha L_\theta(\Phi_*^t\Omega)|h|}\ell(z^{t,\lambda}) \le \ell(z^{t+h,\lambda}) \le e^{3C_\alpha L_\theta(\Phi_*^t\Omega)|h|}\ell(z^{t,\lambda}).$$

Proof. Lemmas 2.1, 3.1 and Corollary 2.7 show that for any $|t| \le \frac{1}{2C_\alpha L_\theta(\Omega)}$ we have

$$\left|\partial_t \ell^{t,\lambda}\right| \le \left\|u_\varepsilon(\omega_\varepsilon^t)\right\|_{\dot{C}^{0,1}} \ell^{t,\lambda} \le 2C_\alpha L_\theta(\Omega)\ell^{t,\lambda} \tag{3.3}$$

(such t must be in I because I is maximal). Since $\ell\colon \mathrm{CC}(\mathbb{R}^2) \to [0,\infty]$ is lower semicontinuous (by definition) and $\lim_{\varepsilon\to 0} z_\varepsilon^{t,\lambda} = z^{t,\lambda}$ in $\mathrm{CC}(\mathbb{R}^2)$, a Grönwall-type argument shows that

$$\ell(z^{t,\lambda}) \le e^{2C_\alpha L_\theta(\Omega)|t|}\ell(z^{0,\lambda}) \tag{3.4}$$

holds when $|t| \le \frac{1}{2C_\alpha L_\theta(\Omega)}$.

Next fix any $t_0 \in I$ with $|t_0| \le \frac{1}{4C_\alpha L_\theta(\Omega)}$. Considering ω^{t_0} as the initial condition for (1.1)–(1.2) at time t_0 and then repeating the proof of (3.4) shows that

$$\ell(z^{t,\lambda}) \le e^{2C_\alpha L_\theta(\Phi_*^{t_0}\Omega)|t-t_0|}\ell(z^{t_0,\lambda})$$

whenever $|t - t_0| \le \frac{1}{2C_\alpha L_\theta(\Phi_*^{t_0}\Omega)}$. We do not yet know whether (3.2) holds in $H^2(\mathbb{T};\mathbb{R}^2)$ without all the ε because Φ^t is only Lipschitz, but all the involved curves are rectifiable and (3.2) without ε does hold in $C^{0,1}(\mathbb{T};\mathbb{R}^2)$. Since $L_\theta(\Phi_*^{t_0}\Omega) \le \frac{4}{3}L_\theta(\Omega)$ by Corollary 2.7, we see that

$$\ell(z^{0,\lambda}) \le e^{3C_\alpha L_\theta(\Omega)|t_0|}\ell(z^{t_0,\lambda}),$$

and so

$$e^{-3C_\alpha L_\theta(\Omega)|t|}\ell(z^{0,\lambda}) \le \ell(z^{t,\lambda}) \le e^{3C_\alpha L_\theta(\Omega)|t|}\ell(z^{0,\lambda})$$

holds whenever $|t| \le \frac{1}{4C_\alpha L_\theta(\Omega)}$. Applying this with (ω^t, t, h) in place of $(\omega^0, 0, t)$, for any $t \in I$, now shows the claim. □

Proposition 3.3. *$R_\theta(\Phi_*^t\Omega)$ is continuous in t. In particular, for any compact interval $J \subseteq I$ we have*

$$\sup_{t\in J} R_\theta(\Phi_*^t\Omega) < \infty.$$

Proof. Take any compact interval $J \subseteq I$ containing 0. With T_0 as in Proposition 2.8, Lemmas 2.1, 2.5 and Corollary 2.7 show that

$$\left|\Phi_\varepsilon^{t_1}(x) - \Phi_\varepsilon^{t_1}(y) - \Phi_\varepsilon^{t_2}(x) + \Phi_\varepsilon^{t_2}(y)\right| \leq (e^{2C_\alpha L_\theta(\Omega)|t_1-t_2|} - 1)\left|\Phi_\varepsilon^{t_1}(x) - \Phi_\varepsilon^{t_1}(y)\right| \tag{3.5}$$

for all $\varepsilon > 0$, $t_1, t_2 \in [-T_0, T_0]$ and $x, y \in \mathbb{R}^2$. Hence, taking $\varepsilon \to 0$ shows that (3.5) continues to hold with Φ in place of Φ_ε. Then applying the same argument to (ω^t, t) in place of $(\omega^0, 0)$ shows that each $t \in J$ has a neighborhood such that

$$\left|\Phi^{t_1}(x) - \Phi^{t_1}(y) - \Phi^{t_2}(x) + \Phi^{t_2}(y)\right| \leq (e^{2C_\alpha M|t_1-t_2|} - 1)\left|\Phi^{t_1}(x) - \Phi^{t_1}(y)\right| \tag{3.6}$$

holds for any $t_1, t_2 \in J$ in that neighborhood, where

$$M := \sup_{t\in J} L_\theta(\Phi_*^t\Omega) < \infty.$$

Since J is an interval, it follows that (3.6) in fact holds for all $t_1, t_2 \in J$.

Fix $t_1, t_2 \in J$, $\lambda, \lambda' \in \mathcal{L}$, and $\eta > 0$. Then there are $x \in \operatorname{im}(z^{0,\lambda})$ and $y \in \operatorname{im}(z^{0,\lambda'})$ such that $\Delta(z^{t_1,\lambda}, z^{t_1,\lambda'}) = |\Phi^{t_1}(x) - \Phi^{t_1}(y)|$. A similar argument as in (2.5) now shows that

$$\begin{aligned}
&\frac{1}{\left(\Delta(z^{t_1,\lambda}, z^{t_1,\lambda'}) + \eta\right)^{2\alpha}} - \frac{1}{\left(\Delta(z^{t_2,\lambda}, z^{t_2,\lambda'}) + \eta\right)^{2\alpha}} \\
&\qquad \leq \frac{|\Phi^{t_1}(x) - \Phi^{t_1}(y) - \Phi^{t_2}(x) + \Phi^{t_2}(y)|}{\left(|\Phi^{t_1}(x) - \Phi^{t_1}(y)| + \eta\right)\left(|\Phi^{t_2}(x) - \Phi^{t_2}(y)| + \eta\right)^{2\alpha}} \\
&\qquad \leq \frac{e^{2C_\alpha M|t_1-t_2|} - 1}{(\Delta(z^{t_2,\lambda}, z^{t_2,\lambda'}) + \eta)^{2\alpha}},
\end{aligned}$$

thus

$$\frac{1}{\left(\Delta(z^{t_1,\lambda}, z^{t_1,\lambda'}) + \eta\right)^{2\alpha}} \leq \frac{e^{2C_\alpha M|t_1-t_2|}}{\Delta(z^{t_2,\lambda}, z^{t_2,\lambda'})^{2\alpha}}. \tag{3.7}$$

Corollary 3.2 yields

$$e^{-3C_\alpha M|t_1-t_2|}\ell(z^{t_2,\lambda}) \leq \ell(z^{t_1,\lambda}) \leq e^{3C_\alpha M|t_1-t_2|}\ell(z^{t_2,\lambda})$$

for any $t_1, t_2 \in J$ because J is an interval, and so we get

$$\frac{\ell(z^{t_1,\lambda})^{1/2}}{\ell(z^{t_1,\lambda'})^{1/2}\left(\Delta(z^{t_1,\lambda}, z^{t_1,\lambda'}) + \eta\right)^{2\alpha}} \leq \frac{e^{5C_\alpha M|t_1-t_2|}\ell(z^{t_2,\lambda})^{1/2}}{\ell(z^{t_2,\lambda'})^{1/2}\Delta(z^{t_2,\lambda}, z^{t_2,\lambda'})^{2\alpha}}. \tag{3.8}$$

Letting $\eta \to 0^+$, taking the upper Lebesgue integral with respect to λ', and then taking the supremum over $\lambda \in \mathcal{L}$ shows that

$$R_\theta(\Phi_*^{t_1}\Omega) \leq e^{5C_\alpha M|t_1-t_2|}R_\theta(\Phi_*^{t_2}\Omega). \tag{3.9}$$

Since $t_1, t_2 \in J$ were arbitrary, the claim follows. $\square$

Next, we show that $Q(\Phi_*^t\Omega) < \infty$ for all small enough t. Since $\ell(\cdot)$ and $\|\cdot\|_{\dot{H}^2}$ are both lower semicontinuous on $\mathrm{CC}(\mathbb{R}^2)$ (the latter by [1, Corollary B.3]), so is $\ell(\cdot)\|\cdot\|_{\dot{H}^2}^2$ and so it is enough to establish a uniform-in-(ε, λ) bound on $\ell(z_\varepsilon^{t,\lambda})\left\|z_\varepsilon^{t,\lambda}\right\|_{\dot{H}^2}^2$. We do this by estimating the right-hand side of (3.1) in Lemma 3.7 below. Since the resulting estimate will involve $R_\theta(\Phi_{\varepsilon*}^t\Omega)$, we first extend a bound on $R_\theta(\Phi_*^t\Omega)$ from the proof of Proposition 3.3 to $\varepsilon > 0$.

Lemma 3.4. *$R_\theta(\Phi^t_{\varepsilon*}\Omega)$ is continuous in t, and for any $t \in [-T_0, T_0]$ we have*

$$R_\theta(\Phi^t_{\varepsilon*}\Omega) \le e^{4C_\alpha L_\theta(\Omega)|t|} R_\theta(\Omega) \le 8 R_\theta(\Omega), \tag{3.10}$$

with C_α from Lemma 2.1 and T_0 from Proposition 2.8.

Proof. We have (3.3) for $t \in [-T_0, T_0]$, so a Grönwall-type argument shows that

$$e^{-2C_\alpha L_\theta(\Omega)|t_1-t_2|} \ell(z_\varepsilon^{t_2,\lambda}) \le \ell(z_\varepsilon^{t_1,\lambda}) \le e^{2C_\alpha L_\theta(\Omega)|t_1-t_2|} \ell(z_\varepsilon^{t_2,\lambda})$$

for any $t_1, t_2 \in [-T_0, T_0]$ and $\lambda \in \mathcal{L}$. Since (3.5) holds, a similar argument as in (3.7) shows

$$\frac{1}{\left(\Delta(z_\varepsilon^{t_1,\lambda}, z_\varepsilon^{t_1,\lambda'}) + \eta\right)^{2\alpha}} \le \frac{e^{2C_\alpha L_\theta(\Omega)|t_1-t_2|}}{\Delta(z_\varepsilon^{t_2,\lambda}, z_\varepsilon^{t_2,\lambda'})^{2\alpha}}$$

for any $t_1, t_2 \in [-T_0, T_0]$, $\lambda, \lambda' \in \mathcal{L}$ and $\eta > 0$. Hence we see that (3.8), and thus also (3.9), continue to hold with $(z_\varepsilon, 4C_\alpha L_\theta(\Omega), \Phi_\varepsilon)$ in place of $(z, 5C_\alpha M, \Phi)$. This now shows both claims (the second inequality in (3.10) follows by the definition of T_0 and $e^2 \le 8$). $\square$

To estimate the right-hand side of (3.1), we will use the following lemmas.

Lemma 3.5. *There is C_α such that for any $\beta \in (0, 1]$, any $C^{1,\beta}$ closed curve $\gamma\colon \ell\mathbb{T} \to \mathbb{R}^2$ parametrized by arclength, and any $x \in \mathbb{R}^2$ we have*

$$\int_{\ell\mathbb{T}} \frac{ds}{|x-\gamma(s)|^{1+2\alpha}} \le C_\alpha \frac{\ell \|\gamma\|_{\dot{C}^{1,\beta}}^{1/\beta}}{d(x, \mathrm{im}(\gamma))^{2\alpha}}.$$

Proof. Let $d := \frac{1}{4}\|\gamma\|_{\dot{C}^{1,\beta}}^{-1/\beta}$ and $\Delta := d(x, \mathrm{im}(\gamma))$. Then [1, Lemma A.2] shows that

$$\begin{aligned}
\int_{\ell\mathbb{T}} \frac{ds}{|x-\gamma(s)|^{1+2\alpha}} &\le \frac{\ell}{4d}\left(\int_{|s|\le\Delta} \frac{ds}{\Delta^{1+2\alpha}} + \int_{\Delta\le|s|\le 2d} \frac{ds}{|s/2|^{1+2\alpha}}\right) + \frac{1}{\Delta^{2\alpha}}\int_{\ell\mathbb{T}} \frac{ds}{d} \\
&\le \frac{\ell}{2d\Delta^{2\alpha}} + \frac{\ell}{2^{1-2\alpha}\alpha d\Delta^{2\alpha}} + \frac{\ell}{d\Delta^{2\alpha}} = C_\alpha \frac{\ell\|\gamma\|_{\dot{C}^{1,\beta}}^{1/\beta}}{\Delta^{2\alpha}}.
\end{aligned}$$

$\square$

Lemma 3.6. *For any $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ with a generalized layer cake representation (Ω, θ) such that $L_\theta(\Omega) < \infty$, and for any $\varepsilon > 0$ and $x, h_1, h_2 \in \mathbb{R}^2$, we have*

$$\begin{aligned}
D^2(u_\varepsilon(\omega))(x)(h_1, h_2) &= \int_{\mathcal{L}}\int_{\mathbb{R}^2} D^2(\nabla^\perp K_\varepsilon)(x-y)(h_1, h_2)\left(\mathbb{1}_{\Omega^\lambda}(y) - \mathbb{1}_{\Omega^\lambda}(x)\right) dy\, d\theta(\lambda) \\
&= \int_{\mathcal{L}}\int_{\Omega^\lambda} D^2(\nabla^\perp K_\varepsilon)(x-y)(h_1, h_2)\, dy\, d\theta(\lambda).
\end{aligned}$$

Moreover, there is C_α such that

$$\int_{\mathcal{L}}\int_{\mathbb{R}^2} \left|D^2(\nabla^\perp K_\varepsilon)(x-y)\right| |\mathbb{1}_{\Omega^\lambda}(y) - \mathbb{1}_{\Omega^\lambda}(x)|\, d|\theta|(\lambda)\, dy \le \frac{C_\alpha}{\varepsilon}\int_{\mathcal{L}} \frac{d|\theta|(\lambda)}{d(x, \partial\Omega^\lambda)^{2\alpha}}.$$

Proof. Oddness of $D^2(\nabla^\perp K_\varepsilon)$ shows that

$$D^2(u_\varepsilon(\omega))(x)(h_1,h_2) = \int_{\mathbb{R}^2} D^2(\nabla^\perp K_\varepsilon)(x-y)(h_1,h_2)(\omega(y)-\omega(x))\,dy$$
$$= \int_{\mathbb{R}^2}\int_{\mathcal{L}} D^2(\nabla^\perp K_\varepsilon)(x-y)(h_1,h_2)\,(\mathbb{1}_{\Omega^\lambda}(y)-\mathbb{1}_{\Omega^\lambda}(x))\,d\theta(\lambda)\,dy.$$

Then proceeding as in Lemma 2.1 and using $\left|D^2(\nabla^\perp K_\varepsilon)(x-y)\right| \le \frac{C_\alpha}{\varepsilon|x-y|^{2+2\alpha}}$ in place of $\left|D(\nabla^\perp K_\varepsilon)(x-y)\right| \le \frac{C_\alpha}{|x-y|^{2+2\alpha}}$ proves the second claim. Fubini's theorem now yields the first equality of the first claim, and the second one follows by oddness of $D^2(\nabla^\perp K_\varepsilon)$. □

Lemma 3.7. *There is C_α such that for each $(t,\lambda)\in\mathbb{R}\times\mathcal{L}$ and $\varepsilon>0$ we have*

$$\left|\partial_t \left\|z_\varepsilon^{t,\lambda}\right\|_{\dot H^2}^2\right| \le C_\alpha(L_\theta(\Phi_{\varepsilon*}^t\Omega)+R_\theta(\Phi_{\varepsilon*}^t\Omega))\,Q(\Phi_{\varepsilon*}^t\Omega)\left\|z_\varepsilon^{t,\lambda}\right\|_{\dot H^2}^2 \tag{3.11}$$

Proof. With $z_\varepsilon^{t,\lambda}(\cdot)$ being the previously fixed arclength parametrization of $z_\varepsilon^{t,\lambda}$, we have

$$\partial_s^2 u_\varepsilon^{t,\lambda}(s) = D^2(u_\varepsilon(\omega_\varepsilon^t))(z_\varepsilon^{t,\lambda}(s))(\mathbf{T}^{t,\lambda}(s),\mathbf{T}^{t,\lambda}(s)) + \kappa^{t,\lambda}(s)D(u_\varepsilon(\omega_\varepsilon^t))(z_\varepsilon^{t,\lambda}(s))(\mathbf{N}^{t,\lambda}(s)).$$

Hence (3.1) yields

$$\begin{aligned}\partial_t \left\|z_\varepsilon^{t,\lambda}\right\|_{\dot H^2}^2 &= \int_{\ell^{t,\lambda}\mathbb{T}} \kappa^{t,\lambda}(s)^2\left(2D(u_\varepsilon(\omega_\varepsilon^t))(z_\varepsilon^{t,\lambda}(s))(\mathbf{N}^{t,\lambda}(s))\cdot\mathbf{N}^{t,\lambda}(s) - 3\,\partial_s u_\varepsilon^{t,\lambda}(s)\cdot\mathbf{T}^{t,\lambda}(s)\right)ds\\ &\quad + 2\int_{\ell^{t,\lambda}\mathbb{T}} \kappa^{t,\lambda}(s)\left(D^2(u_\varepsilon(\omega_\varepsilon^t))(z_\varepsilon^{t,\lambda}(s))(\mathbf{T}^{t,\lambda}(s),\mathbf{T}^{t,\lambda}(s))\cdot\mathbf{N}^{t,\lambda}(s)\right)ds.\end{aligned} \tag{3.12}$$

Lemma 2.1 shows that the absolute value of the first integral is bounded by

$$5\left\|u_\varepsilon(\omega_\varepsilon^t)\right\|_{\dot C^{0,1}}\left\|z_\varepsilon^{t,\lambda}\right\|_{\dot H^2}^2 \le C_\alpha L_\theta(\Phi_{\varepsilon*}^t\Omega)\left\|z_\varepsilon^{t,\lambda}\right\|_{\dot H^2}^2 \le C_\alpha L_\theta(\Phi_{\varepsilon*}^t\Omega)Q(\Phi_{\varepsilon*}^t\Omega)\left\|z_\varepsilon^{t,\lambda}\right\|_{\dot H^2}^2,$$

where the second inequality holds by $Q(\Phi_{\varepsilon*}^t\Omega)\ge 4$, which in turn follows from [1, Lemma A.1]. Hence, it remains to estimate the second integral, which we denote G_1. We will suppress t from the notation for the sake of simplicity because it will be fixed in the arguments below.

Since $L_\theta(\Phi_{\varepsilon*}\Omega)<\infty$, Lemma 3.6 and Green's theorem show that

$$G_1 = -\int_{\mathcal{L}}\int_{\ell^\lambda\mathbb{T}\times\ell^{\lambda'}\mathbb{T}} \kappa^\lambda(s)D^2K_\varepsilon(z_\varepsilon^\lambda(s)-z_\varepsilon^{\lambda'}(s'))(\mathbf{T}^\lambda(s),\mathbf{T}^\lambda(s))(\mathbf{T}^{\lambda'}(s')\cdot\mathbf{N}^\lambda(s))\,ds'\,ds\,d\theta(\lambda').$$

From

$$\mathbf{T}^\lambda(s) = (\mathbf{T}^{\lambda'}(s')\cdot\mathbf{T}^\lambda(s))\mathbf{T}^{\lambda'}(s') + (\mathbf{N}^{\lambda'}(s')\cdot\mathbf{T}^\lambda(s))\mathbf{N}^{\lambda'}(s')$$

and $\mathbf{N}^{\lambda'}(s')\cdot\mathbf{T}^\lambda(s) = -\mathbf{T}^{\lambda'}(s')\cdot\mathbf{N}^\lambda(s)$ it now follows that $|G_1|$ is bounded by the sum of

$$\begin{aligned}G_2 := \overline{\int_{\mathcal{L}}}\Bigg|\int_{\ell^\lambda\mathbb{T}\times\ell^{\lambda'}\mathbb{T}} &\kappa^\lambda(s)D^2K_\varepsilon(z_\varepsilon^\lambda(s)-z_\varepsilon^{\lambda'}(s'))(\mathbf{T}^\lambda(s),\mathbf{T}^{\lambda'}(s'))\\ &(\mathbf{T}^{\lambda'}(s')\cdot\mathbf{N}^\lambda(s))(\mathbf{T}^{\lambda'}(s')\cdot\mathbf{T}^\lambda(s))\,ds'\,ds\Bigg|\,d|\theta|(\lambda'),\end{aligned}$$

$$G_3 := \overline{\int_{\mathcal{L}}}\left|\int_{\ell^\lambda\mathbb{T}\times\ell^{\lambda'}\mathbb{T}} \kappa^\lambda(s)D^2K_\varepsilon(z_\varepsilon^\lambda(s)-z_\varepsilon^{\lambda'}(s'))(\mathbf{T}^\lambda(s),\mathbf{N}^{\lambda'}(s'))(\mathbf{T}^{\lambda'}(s')\cdot\mathbf{N}^\lambda(s))^2\,ds'\,ds\right|d|\theta|(\lambda'),$$

which we estimate separately next.

Estimate for G_2. Since

$$\frac{\partial}{\partial s'}\Big(DK_\varepsilon(z_\varepsilon^\lambda(s)-z_\varepsilon^{\lambda'}(s'))(\mathbf{T}^\lambda(s))(\mathbf{T}^{\lambda'}(s')\cdot\mathbf{N}^\lambda(s))(\mathbf{T}^{\lambda'}(s')\cdot\mathbf{T}^\lambda(s))\Big)$$
$$=-D^2K_\varepsilon(z_\varepsilon^\lambda(s)-z_\varepsilon^{\lambda'}(s'))(\mathbf{T}^\lambda(s),\mathbf{T}^{\lambda'}(s'))(\mathbf{T}^{\lambda'}(s')\cdot\mathbf{N}^\lambda(s))(\mathbf{T}^{\lambda'}(s')\cdot\mathbf{T}^\lambda(s))$$
$$+\kappa^{\lambda'}(s')DK_\varepsilon(z_\varepsilon^\lambda(s)-z_\varepsilon^{\lambda'}(s'))\Big((\mathbf{T}^{\lambda'}(s')\cdot\mathbf{T}^\lambda(s))^2-(\mathbf{T}^{\lambda'}(s')\cdot\mathbf{N}^\lambda(s))^2\Big),$$

we see that

$$G_2\le C_\alpha\overline{\int_{\mathcal{L}}}\int_{\ell^\lambda\mathbb{T}\times\ell^{\lambda'}\mathbb{T}}\frac{\left|\kappa^\lambda(s)\kappa^{\lambda'}(s')\right|}{\left|z_\varepsilon^\lambda(s)-z_\varepsilon^{\lambda'}(s')\right|^{1+2\alpha}}\,ds'\,ds\,d|\theta|(\lambda').$$

By the Schwarz inequality, the inner integral of the right-hand side is bounded by

$$\left(\int_{\ell^\lambda\mathbb{T}}\kappa^\lambda(s)^2\int_{\ell^{\lambda'}\mathbb{T}}\frac{ds'}{\left|z_\varepsilon^\lambda(s)-z_\varepsilon^{\lambda'}(s')\right|^{1+2\alpha}}\,ds\right)^{1/2}\left(\int_{\ell^{\lambda'}\mathbb{T}}\kappa^{\lambda'}(s')^2\int_{\ell^\lambda\mathbb{T}}\frac{ds}{\left|z_\varepsilon^\lambda(s)-z_\varepsilon^{\lambda'}(s')\right|^{1+2\alpha}}\,ds'\right)^{1/2},$$

and Lemma 3.5 with $\beta=\frac12$ shows that this is bounded by

$$C_\alpha\left(\left\|z_\varepsilon^\lambda\right\|_{\dot H^2}^2\frac{\ell^{\lambda'}\left\|z_\varepsilon^{\lambda'}\right\|_{\dot H^2}^2}{(\Delta^{\lambda,\lambda'})^{2\alpha}}\right)^{1/2}\left(\left\|z_\varepsilon^{\lambda'}\right\|_{\dot H^2}^2\frac{\ell^{\lambda}\left\|z_\varepsilon^{\lambda}\right\|_{\dot H^2}^2}{(\Delta^{\lambda,\lambda'})^{2\alpha}}\right)^{1/2}=C_\alpha\left\|z_\varepsilon^\lambda\right\|_{\dot H^2}^2\frac{\ell^{\lambda'}\left\|z_\varepsilon^{\lambda'}\right\|_{\dot H^2}^2(\ell^\lambda)^{1/2}}{(\ell^{\lambda'})^{1/2}(\Delta^{\lambda,\lambda'})^{2\alpha}}.$$

Therefore

$$|G_2|\le C_\alpha R_\theta(\Phi_{\varepsilon*}\Omega)Q(\Phi_{\varepsilon*}\Omega)\left\|z_\varepsilon^\lambda\right\|_{\dot H^2}^2.$$

Estimate for G_3. We can assume $R_\theta(\Phi_{\varepsilon*}\Omega)<\infty$, in which case $\Delta(z_\varepsilon^\lambda,z_\varepsilon^{\lambda'})>0$ for $|\theta|$-almost all λ'. Thus we can apply [1, Lemma A.4] to conclude that

$$G_3\le C_\alpha\overline{\int_{\mathcal{L}}}\int_{\ell^\lambda\mathbb{T}\times\ell^{\lambda'}\mathbb{T}}\frac{\left|\kappa^\lambda(s)\right|\left(\mathcal{M}\kappa^\lambda(s)+\mathcal{M}\kappa^{\lambda'}(s')\right)}{\left|z_\varepsilon^\lambda(s)-z_\varepsilon^{\lambda'}(s')\right|^{1+2\alpha}}\,ds'\,ds\,d|\theta|(\lambda'),$$

where $\mathcal{M}$ is the maximal operator from (A.2) in [1]. Let us split the integrand into the sum of two terms with numerators $\left|\kappa^\lambda(s)\right|\mathcal{M}\kappa^\lambda(s)$ and $\left|\kappa^\lambda(s)\right|\mathcal{M}\kappa^{\lambda'}(s')$, respectively. Then the maximal inequality (see (A.3) in [1]) shows that the same argument as in the estimate for G_2 bounds the integral of the second term by $C_\alpha R_\theta(\Phi_{\varepsilon*}\Omega)Q(\Phi_{\varepsilon*}\Omega)\left\|z_\varepsilon^\lambda\right\|_{\dot H^2}^2$. As for the first term, Lemma 3.5 with $\beta=\frac12$, Schwarz inequality, and the maximal inequality show that

$$\overline{\int_{\mathcal{L}}}\int_{\ell^\lambda\mathbb{T}\times\ell^{\lambda'}\mathbb{T}}\frac{\left|\kappa^\lambda(s)\right|\mathcal{M}\kappa^\lambda(s)}{\left|z_\varepsilon^\lambda(s)-z_\varepsilon^{\lambda'}(s')\right|^{1+2\alpha}}\,ds'\,ds\,d|\theta|(\lambda')$$
$$\le\overline{\int_{\mathcal{L}}}\int_{\ell^\lambda\mathbb{T}}\left|\kappa^\lambda(s)\right|\mathcal{M}\kappa^\lambda(s)\frac{C_\alpha\ell^{\lambda'}\left\|z_\varepsilon^{\lambda'}\right\|_{\dot H^2}^2}{d(z_\varepsilon^\lambda(s),\mathrm{im}(z_\varepsilon^{\lambda'}))^{2\alpha}}\,ds\,d|\theta|(\lambda')$$
$$\le C_\alpha Q(\Phi_{\varepsilon*}\Omega)\int_{\mathcal{L}}\int_{\ell^\lambda\mathbb{T}}\frac{\left|\kappa^\lambda(s)\right|\mathcal{M}\kappa^\lambda(s)}{d(z_\varepsilon^\lambda(s),\mathrm{im}(z_\varepsilon^{\lambda'}))^{2\alpha}}\,ds\,d|\theta|(\lambda')$$

$$
\begin{aligned}
&\le C_\alpha L_\theta(\Phi_{\varepsilon*}\Omega) Q(\Phi_{\varepsilon*}\Omega) \int_{\ell^\lambda \mathbb{T}} \left|\kappa^\lambda(s)\right| \mathcal{M}\kappa^\lambda(s)\, ds \\
&\le C_\alpha L_\theta(\Phi_{\varepsilon*}\Omega) Q(\Phi_{\varepsilon*}\Omega) \left\|z_\varepsilon^\lambda\right\|_{\dot H^2}^2 ,
\end{aligned}
$$

where the regular (rather than upper) integral can be taken after the second inequality because the integrand is jointly measurable in (s, λ'). Aggregating the estimates for G_2 and G_3 now yields the desired conclusion. □

Lemmas 3.1 and 3.7 suggest that we may have an ε-independent estimate

$$
\max\left\{\partial_t^+ Q(\Phi_{\varepsilon*}^t\Omega), -\partial_{t-} Q(\Phi_{\varepsilon*}^t\Omega)\right\} \le C_\alpha (L_\theta(\Phi_{\varepsilon*}^t\Omega) + R_\theta(\Phi_{\varepsilon*}^t\Omega)) Q(\Phi_{\varepsilon*}^t\Omega)^2 \tag{3.13}
$$

from which we can run a Grönwall-type argument. However, because of the factor $Q(\Phi_{\varepsilon*}^t\Omega)$ in the right-hand side of (3.11), (3.13) follows from these Lemmas only if we know a priori that $Q(\Phi_{\varepsilon*}^t\Omega)$ is upper semicontinuous (or locally bounded, which implies upper semicontinuity via (3.11)).

One way to derive upper semicontinuity (or local boundedness) would be again through estimating the right-hand side of (3.1) in terms of $\left\|z_\varepsilon^{t,\lambda}\right\|_{\dot H^2}^2$. In contrast to (3.11), such an estimate can depend on ε while it must not have the factor $Q(\Phi_{\varepsilon*}^t\Omega)$. However, the second integral of (3.12) contains only one factor of $\kappa^{t,\lambda}(s)$, so the trivial bound we obtain by estimating $D^2(u_\varepsilon(\omega_\varepsilon^t))$ by its L^∞ norm will contain the L^1 norm of $\kappa^{t,\lambda}$, and in order to replace it by the L^2 norm we have to introduce an additional factor of $(\ell^{t,\lambda})^{1/2}$. This means that unless we impose an upper on $\ell^{t,\lambda}$, the resulting estimate will not be enough for concluding upper semicontinuity (nor local boundedness), because it does not rule out the possibility that over an arbitrarily small time interval and for an arbitrarily large constant M, some $\ell^{t,\lambda}\left\|z_\varepsilon^{t,\lambda}\right\|_{\dot H^2}^2$ with very large $\ell^{t,\lambda}$ grows larger than M. Another way would be to estimate the $\dot H^2$ norm of $\Phi_\varepsilon^t \circ \tilde z^{0,\lambda}$ where $\tilde z^{0,\lambda}\colon \mathbb{T} \to \mathbb{R}^2$ is a constant-speed parametrization of $z^{0,\lambda}$, but it runs into a similar issue.

To resolve this, we now introduce a sequence of approximations of ω_ε^t. Take an increasing sequence $\{\mathcal{L}_N'\}_{N=1}^\infty$ of measurable subsets of $\mathcal{L}$ such that $|\theta|(\mathcal{L}_N') < \infty$ for each $N \in \mathbb{N}$ and $\mathcal{L} = \bigcup_{N=1}^\infty \mathcal{L}_N'$. For each $N \in \mathbb{N}$, let

$$
\mathcal{L}_N := \left\{\lambda \in \mathcal{L}_N' \colon \ell(z^{0,\lambda}) \le N\right\}, \quad \Omega_N := \Omega \cap (\mathbb{R}^2 \times \mathcal{L}_N), \quad \omega_N^0(x) := \int_{\mathcal{L}_N} \mathbb{1}_{\Omega^\lambda}(x)\, d\theta(\lambda).
$$

Note that $\omega_N^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ because $\|\omega_N^0\|_{L^\infty} \le |\theta|(\mathcal{L}_N') < \infty$ and also $\|\omega_N^0\|_{L^1} \le \frac{N^2}{4\pi}|\theta|(\mathcal{L}_N') < \infty$ by the isoperimetric inequality. Clearly, $L_\theta(\Omega_N) \le L_\theta(\Omega) < \infty$.

Let $\Phi_{\varepsilon,N} \in C(\mathbb{R}; C(\mathbb{R}^2;\mathbb{R}^2))$ be the corresponding ε-mollified flow map, i.e., the identity map plus the solution to (2.2) with ω_N^0 in place of ω^0. Let $z_{\varepsilon,N}^{t,\lambda} := \Phi_{\varepsilon,N}^t \circ z^{0,\lambda}$ and $\omega_{\varepsilon,N}^t := \omega_N^0 \circ (\Phi_{\varepsilon,N}^t)^{-1}$.

Then (3.12), Cauchy-Schwarz inequality, the inequality $\ell(\gamma)\left\|\gamma\right\|_{\dot H^2}^2 \ge 4$ for any $\gamma \in \mathrm{CC}(\mathbb{R}^2)$ (which follows by [1, Lemma A.1]) and (1.8) show

$$
\left|\partial_t \left\|z_{\varepsilon,N}^{t,\lambda}\right\|_{\dot H^2}^2\right| \le 5\left\|u_\varepsilon(\omega_{\varepsilon,N}^t)\right\|_{\dot C^{0,1}} \left\|z_{\varepsilon,N}^{t,\lambda}\right\|_{\dot H^2}^2 + 2\left\|D^2(u_\varepsilon(\omega_{\varepsilon,N}^t))\right\|_{L^\infty} \ell(z_{\varepsilon,N}^{t,\lambda})^{1/2} \left\|z_{\varepsilon,N}^{t,\lambda}\right\|_{\dot H^2}
$$

$$\leq \left(5\left\|D(\nabla^\perp K_\varepsilon)\right\|_{L^\infty} + \left\|D^2(\nabla^\perp K_\varepsilon)\right\|_{L^\infty} N\right)\left\|\omega_N^0\right\|_{L^1}\left\|z_{\varepsilon,N}^{t,\lambda}\right\|_{\dot{H}^2}^2,$$

which implies

$$\left\|z_{\varepsilon,N}^{t+h,\lambda}\right\|_{\dot{H}^2}^2 \leq e^{C|h|}\left\|z_{\varepsilon,N}^{t,\lambda}\right\|_{\dot{H}^2}^2$$

for each $(t,h,\lambda) \in \mathbb{R}^2 \times \mathcal{L}_N$, for some constant C that depends on $\alpha, \varepsilon, N, \|\omega_N^0\|_{L^1}$, but not on $\lambda \in \mathcal{L}_N$. Together with the inequality $\ell(z_{\varepsilon,N}^{t+h,\lambda}) \leq e^{C|h|}\ell(z_{\varepsilon,N}^{t,\lambda})$ for another C that depends on $\alpha, \varepsilon, \|\omega_N^0\|_{L^1}$, which follows from (3.3) (with $(\omega_{\varepsilon,N}, z_{\varepsilon,N})$ in place of $(\omega_\varepsilon, z_\varepsilon)$) and (1.8), this shows

$$Q(\Phi_{\varepsilon,N*}^{t+h}\Omega) \leq e^{C|h|}Q(\Phi_{\varepsilon,N*}^{t}\Omega),$$

from which we conclude upper semicontinuity of $Q(\Phi_{\varepsilon,N*}^{t}\Omega)$ in t.

From Lemmas 3.1, 3.4, 3.7, Corollary 2.7, and (3.3) with $(\omega_{\varepsilon,N}, \Omega_N, z_{\varepsilon,N}, \Phi_{\varepsilon,N})$ in place of $(\omega_\varepsilon, \Omega, z_\varepsilon, \Phi_\varepsilon)$, and the inequalities $L_\theta(\Omega_N) \leq L_\theta(\Omega)$, $R_\theta(\Omega_N) \leq R_\theta(\Omega)$, and $Q(\Phi_{\varepsilon,N*}^{t}\Omega) \geq 4$, we see that

$$\begin{aligned}
\ell(z_{\varepsilon,N}^{t+h,\lambda})\left\|z_{\varepsilon,N}^{t+h,\lambda}\right\|_{\dot{H}^2}^2 - Q(\Phi_{\varepsilon,N*}^{t}\Omega) &\leq \ell(z_{\varepsilon,N}^{t+h,\lambda})\left\|z_{\varepsilon,N}^{t+h,\lambda}\right\|_{\dot{H}^2}^2 - \ell(z_{\varepsilon,N}^{t,\lambda})\left\|z_{\varepsilon,N}^{t,\lambda}\right\|_{\dot{H}^2}^2 \\
&\leq C_\alpha\left|\int_t^{t+h}(L_\theta(\Omega)+R_\theta(\Omega))Q(\Phi_{\varepsilon,N*}^{\tau}\Omega)^2\,d\tau\right|
\end{aligned}$$

holds for each $(t, t+h, \lambda) \in [-T_0, T_0]^2 \times \mathcal{L}_N$, for some C_α that depends only on α. By upper semicontinuity of $Q(\Phi_{\varepsilon,N*}^{t}\Omega)$, taking supremum over $\lambda \in \mathcal{L}_N$, dividing by $|h|$ and then letting $h \to 0$ yields

$$\max\left\{\partial_t^+ Q(\Phi_{\varepsilon,N*}^{t}\Omega), -\partial_{t-} Q(\Phi_{\varepsilon,N*}^{t}\Omega)\right\} \leq C_\alpha(L_\theta(\Omega)+R_\theta(\Omega))Q(\Phi_{\varepsilon,N*}^{t}\Omega)^2,$$

which we then use with a Grönwall-type argument to conclude

$$Q(\Phi_{\varepsilon,N*}^{t}\Omega) \leq \frac{Q(\Omega_N)}{1 - C_\alpha(L_\theta(\Omega)+R_\theta(\Omega))Q(\Omega_N)\,|t|} \leq \frac{Q(\Omega)}{1 - C_\alpha(L_\theta(\Omega)+R_\theta(\Omega))Q(\Omega)\,|t|} \tag{3.14}$$

for any t with $|t| < \frac{1}{C_\alpha(L_\theta(\Omega)+R_\theta(\Omega))Q(\Omega)}$, where the second inequality is because $Q(\Omega_N) \leq Q(\Omega)$.

In order to turn (3.14) into a bound on $Q(\Phi_*^t\Omega)$, we need to know if $z_{\varepsilon,N}$ converges to z_ε when $N \to \infty$. This indeed happens if $\omega_N^0 \to \omega^0$ in L^1 (which also implies the convergence in L^∞ since $\|\omega_N^0\|_{\dot{C}^{0,2\alpha}}$ is uniformly bounded), but our assumptions on Ω alone do not seem to imply the L^1 convergence. Note that if Ω^λ's for λ's in the positive part of θ are disjoint from those for λ's in the negative part of θ, in particular if Ω^λ's are the super-level sets of $(\omega^0)^+$ and $(\omega^0)^-$, then L^1 convergence does follow. However, we are allowing these "positive super-level sets" and "negative super-level sets" to sit on top of each other, possibly creating "ripples" of the graph of ω^0.

Even though we do not have the L^1 convergence, we can still show that the difference between $z_{\varepsilon,N}$ and z_ε is almost constant, although that constant may vary as N varies, as the following lemma shows.

Lemma 3.8. *Let T_0 be as in Proposition 2.8. Then for any $R \geq 0$ and $x_0 \in \mathbb{R}^2$,*

$$\lim_{N\to\infty} \sup_{t\in[-T_0,T_0]} \left\| \left(\Phi^t_{\varepsilon,N} - \Phi^t_\varepsilon - \Phi^t_{\varepsilon,N}(x_0) + \Phi^t_\varepsilon(x_0)\right)\big|_{B_R(x_0)} \right\|_{L^\infty} = 0.$$

Once this is shown, for each $\lambda \in \mathcal{L}$, we pick any constant-speed parametrization $\tilde{z}^{0,\lambda}\colon \mathbb{T} \to \mathbb{R}^2$ of $z^{0,\lambda}$, then

$$\begin{aligned}
&\left\|\Phi^t_{\varepsilon,N} \circ \tilde{z}^{0,\lambda} - \Phi^t_\varepsilon \circ \tilde{z}^{0,\lambda} - \Phi^t_{\varepsilon,N} \circ \tilde{z}^{0,\lambda}(0) + \Phi^t_\varepsilon \circ \tilde{z}^{0,\lambda}(0)\right\|_{L^\infty} \\
&\quad\leq \left\| \left(\Phi^t_{\varepsilon,N} - \Phi^t_\varepsilon - \Phi^t_{\varepsilon,N}(\tilde{z}^{0,\lambda}(0)) + \Phi^t_\varepsilon(\tilde{z}^{0,\lambda}(0))\right)\big|_{B_{\ell(z^{0,\lambda})/2}(\tilde{z}^{0,\lambda}(0))} \right\|_{L^\infty} \to 0
\end{aligned}$$

as $N \to \infty$, for each $t \in \mathbb{R}$. This shows that $w^{t,\lambda}_{\varepsilon,N} := z^{t,\lambda}_{\varepsilon,N} - \Phi^t_{\varepsilon,N} \circ \tilde{z}^{0,\lambda}(0) + \Phi^t_\varepsilon \circ \tilde{z}^{0,\lambda}(0) \to z^{t,\lambda}_\varepsilon$ in $\mathrm{CC}(\mathbb{R}^2)$ as $N \to \infty$. Since $w^{t,\lambda}_{\varepsilon,N}$ is just a spatial translation of $z^{t,\lambda}_{\varepsilon,N}$, we have $\ell(w^{t,\lambda}_{\varepsilon,N}) = \ell(z^{t,\lambda}_{\varepsilon,N})$ and $\left\|w^{t,\lambda}_{\varepsilon,N}\right\|_{\dot{H}^2} = \left\|z^{t,\lambda}_{\varepsilon,N}\right\|_{\dot{H}^2}$, so the lower semicontinuity of the functional $\gamma \mapsto \ell(\gamma)\left\|\gamma\right\|^2_{\dot{H}^2}$ on $\mathrm{CC}(\mathbb{R}^2)$ shows

$$\begin{aligned}
\ell(z^{t,\lambda}_\varepsilon)\left\|z^{t,\lambda}_\varepsilon\right\|^2_{\dot{H}^2} &\leq \liminf_{N\to\infty} \ell(w^{t,\lambda}_{\varepsilon,N})\left\|w^{t,\lambda}_{\varepsilon,N}\right\|^2_{\dot{H}^2} = \liminf_{N\to\infty} \ell(z^{t,\lambda}_{\varepsilon,N})\left\|z^{t,\lambda}_{\varepsilon,N}\right\|^2_{\dot{H}^2} \\
&\leq \frac{Q(\Omega)}{1 - C_\alpha(L_\theta(\Omega) + R_\theta(\Omega))Q(\Omega)\left|t\right|}
\end{aligned}$$

for any t with $|t| < \frac{1}{C_\alpha(L_\theta(\Omega)+R_\theta(\Omega))Q(\Omega)}$. Then, again by lower semicontinuity of $\gamma \mapsto \ell(\gamma)\left\|\gamma\right\|^2_{\dot{H}^2}$, letting $\varepsilon \to 0^+$ yields

$$\ell(z^{t,\lambda})\left\|z^{t,\lambda}\right\|^2_{\dot{H}^2} \leq \frac{Q(\Omega)}{1 - C_\alpha(L_\theta(\Omega) + R_\theta(\Omega))Q(\Omega)\left|t\right|},$$

so taking supremum over λ finally gives

$$Q(\Phi^t_*\Omega) \leq \frac{Q(\Omega)}{1 - C_\alpha(L_\theta(\Omega) + R_\theta(\Omega))Q(\Omega)\left|t\right|},$$

from which Theorem 1.4 follows.

Now it remains to prove Lemma 3.8.

Proof of Lemma 3.8. In this proof, all constants written as C with subscripts can change from one inequality to another, and they depend only on the indicated variables; e.g., C_α depends only on α and $C_{\alpha,\varepsilon}$ depends only on α and ε.

First, note that the inequality $L_\theta(\Omega_N) \leq L_\theta(\Omega)$ and Proposition 2.8 show

$$\sup_{t\in[-T_0,T_0]} \max\left\{\left\|\Phi^t_\varepsilon\right\|_{\dot{C}^{0,1}}, \left\|(\Phi^t_\varepsilon)^{-1}\right\|_{\dot{C}^{0,1}}, \left\|\Phi^t_{\varepsilon,N}\right\|_{\dot{C}^{0,1}}, \left\|(\Phi^t_{\varepsilon,N})^{-1}\right\|_{\dot{C}^{0,1}}\right\} \leq 3. \tag{3.15}$$

Let $\delta > 0$ be given, then since $\omega^0 \in L^1(\mathbb{R}^2)$, we can find $R_\delta \geq R$ such that

$$\int_{\mathbb{R}^2\setminus B_{R_\delta}(x_0)} \left|\omega^0(y)\right| dy \leq \delta. \tag{3.16}$$

For given $t \in [-T_0, T_0]$ and $x \in B_{R_\delta}(x_0)$ we can write

$$\partial_t \left(D\Phi^t_{\varepsilon,N}(x) - D\Phi^t_\varepsilon(x) \right) = D(u_\varepsilon(\omega^t_{\varepsilon,N}))(\Phi^t_{\varepsilon,N}(x)) D\Phi^t_{\varepsilon,N}(x) - D(u_\varepsilon(\omega^t_\varepsilon))(\Phi^t_\varepsilon(x)) D\Phi^t_\varepsilon(x)$$

as the sum of the terms

$$\begin{aligned}
V_1 &\coloneqq D(u_\varepsilon(\omega^t_{\varepsilon,N}))(\Phi^t_{\varepsilon,N}(x)) \left(D\Phi^t_{\varepsilon,N}(x) - D\Phi^t_\varepsilon(x) \right), \\
V_2 &\coloneqq \left[D(u_\varepsilon(\Phi^t_{\varepsilon,N*}\omega^0_N)) - D(u_\varepsilon(\Phi^t_{\varepsilon,N*}\omega^0)) \right] (\Phi^t_{\varepsilon,N}(x)) D\Phi^t_\varepsilon(x), \\
V_3 &\coloneqq \left[D(u_\varepsilon(\Phi^t_{\varepsilon,N*}\omega^0))(\Phi^t_{\varepsilon,N}(x)) - D(u_\varepsilon(\Phi^t_{\varepsilon*}\omega^0))(\Phi^t_\varepsilon(x)) \right] D\Phi^t_\varepsilon(x),
\end{aligned}$$

which we estimate separately.

Estimate for V_1. Clearly, Lemma 2.1, (1.7), (3.15) and the inequality $L_\theta(\Omega_N) \le L_\theta(\Omega)$ show

$$\begin{aligned}
|V_1| &\le \left\| u_\varepsilon(\omega^t_{\varepsilon,N}) \right\|_{\dot{C}^{0,1}} \left| D\Phi^t_{\varepsilon,N}(x) - D\Phi^t_\varepsilon(x) \right| \le C_\alpha L_\theta(\Phi^t_{\varepsilon,N*}\Omega_N) \left| D\Phi^t_{\varepsilon,N}(x) - D\Phi^t_\varepsilon(x) \right| \\
&\le C_\alpha L_\theta(\Omega) \left\| \left(D\Phi^t_{\varepsilon,N} - D\Phi^t_\varepsilon \right)\big|_{B_{R_\delta}(x_0)} \right\|_{L^\infty}.
\end{aligned}$$

Estimate for V_2. Since

$$(\omega^0 - \omega^0_N)(x') = \int_{\mathcal{L}\setminus\mathcal{L}_N} \mathbb{1}_{\Omega^\lambda}(x')\, d\theta(\lambda)$$

is a generalized layer cake representation of $\omega^0 - \omega^0_N$, Lemma 2.1, the last inequality of (1.7) without the supremum, (3.15) and $L_\theta(\Omega) < \infty$ show

$$\begin{aligned}
\left| D\left(u_\varepsilon \left(\Phi^{t'}_{\varepsilon,N*}(\omega^0_N - \omega^0) \right) \right) (\Phi^{t'}_{\varepsilon,N}(x')) \right| &\le C_\alpha \int_{\mathcal{L}\setminus\mathcal{L}_N} \frac{d|\theta|(\lambda)}{d(\Phi^{t'}_{\varepsilon,N}(x'), \Phi^{t'}_{\varepsilon,N}(\partial\Omega^\lambda))^{2\alpha}} \\
&\le C_\alpha \left\| (\Phi^{t'}_{\varepsilon,N})^{-1} \right\|^{2\alpha}_{\dot{C}^{0,1}} \int_{\mathcal{L}\setminus\mathcal{L}_N} \frac{d|\theta|(\lambda)}{d(x', \partial\Omega^\lambda)^{2\alpha}} \to 0
\end{aligned}$$

uniformly in $t' \in [-T_0, T_0]$ as $N \to \infty$ for each $x' \in \mathbb{R}^2$. (Note that that we do not know if the convergence is uniform in x'.) On the other hand, Lemma 3.6, (1.7) and (3.15) show for any $t' \in [-T_0, T_0]$,

$$\begin{aligned}
\left\| D\left(D\left(u_\varepsilon \left(\Phi^{t'}_{\varepsilon,N*}(\omega^0_N - \omega^0) \right) \right) \circ \Phi^{t'}_{\varepsilon,N} \right) \right\|_{L^\infty} &\le \left\| D^2 \left(u_\varepsilon \left(\Phi^{t'}_{\varepsilon,N*}(\omega^0_N - \omega^0) \right) \right) \right\|_{L^\infty} \left\| D\Phi^{t'}_{\varepsilon,N} \right\|_{L^\infty} \\
&\le \frac{C_\alpha}{\varepsilon} \left\| \Phi^{t'}_{\varepsilon,N} \right\|_{\dot{C}^{0,1}} \sup_{x' \in \mathbb{R}^2} \int_{\mathcal{L}\setminus\mathcal{L}_N} \frac{d|\theta|(\lambda)}{d(x', \Phi^{t'}_{\varepsilon,N}(\partial\Omega^\lambda))^{2\alpha}} \\
&\le \frac{C_\alpha}{\varepsilon} \left\| \Phi^{t'}_{\varepsilon,N} \right\|_{\dot{C}^{0,1}} \left\| (\Phi^{t'}_{\varepsilon,N})^{-1} \right\|^{2\alpha}_{\dot{C}^{0,1}} L_\theta(\Omega) \\
&\le \frac{C_\alpha}{\varepsilon} L_\theta(\Omega).
\end{aligned}$$

Therefore, Arzelà-Ascoli theorem applied to the collection

$$\left\{ x' \mapsto \left(t' \mapsto D\left(u_\varepsilon \left(\Phi^{t'}_{\varepsilon,N*}(\omega^0_N - \omega^0) \right) \right) (\Phi^{t'}_{\varepsilon,N}(x')) \right) \right\}_{N \in \mathbb{N}}$$

shows that there is $N_0 \in \mathbb{N}$ (depending on δ) such that

$$\sup_{x' \in B_{R_\delta}(x_0)} \sup_{t' \in [-T_0, T_0]} D\left(u_\varepsilon\left(\Phi^{t'}_{\varepsilon,N*}(\omega^0_N - \omega^0)\right)\right)(\Phi^{t'}_{\varepsilon,N}(x')) \le \delta$$

whenever $N \ge N_0$. Then for such N, (3.15) shows

$$|V_2| \le \left|D\left(u_\varepsilon\left(\Phi^t_{\varepsilon,N*}(\omega^0_N - \omega^0)\right)\right)(\Phi^t_{\varepsilon,N}(x))\right| \left|D\Phi^t_\varepsilon(x)\right| \le 3\delta.$$

Estimate of V_3. Expanding the definition of u_ε gives

$$V_3 = \int_{\mathbb{R}^2} \left(\left(D(\nabla^\perp K_\varepsilon)(\Phi^t_{\varepsilon,N}(x) - \Phi^t_{\varepsilon,N}(y)) - D(\nabla^\perp K_\varepsilon)(\Phi^t_\varepsilon(x) - \Phi^t_\varepsilon(y))\right) D\Phi^t_\varepsilon(x)\right) \omega^0(y)\, dy.$$

By (3.16), the integral over $\mathbb{R}^2 \setminus B_{R_\delta}(x_0)$ is bounded by $2\left\|D(\nabla^\perp K_\varepsilon)\right\|_{L^\infty} \|\Phi^t_\varepsilon\|_{\dot{C}^{0,1}}\, \delta$. For $y \in B_{R_\delta}(x_0)$, the fundamental theorem of calculus shows that the integrand is bounded by

$$\begin{aligned}\left\|\omega^0\right\|_{L^\infty} \left\|\Phi^t_\varepsilon\right\|_{\dot{C}^{0,1}} &\int_0^1 \left|D^2(\nabla^\perp K_\varepsilon)(\eta(\Phi^t_{\varepsilon,N}(x) - \Phi^t_{\varepsilon,N}(y)) + (1-\eta)(\Phi^t_\varepsilon(x) - \Phi^t_\varepsilon(y)))\right| d\eta \\ &\cdot \left|\int_0^1 \left(D\Phi^t_{\varepsilon,N} - D\Phi^t_\varepsilon\right)(\eta x + (1-\eta)y)\, d\eta\right| |x-y|\,.\end{aligned}$$

The second integral is bounded by $\left\|\left(D\Phi^t_{\varepsilon,N} - D\Phi^t_\varepsilon\right)\big|_{B_{R_\delta}(x_0)}\right\|_{L^\infty}$, and the first integral is bounded by

$$\begin{aligned}&\min\left\{\left\|D^2(\nabla^\perp K_\varepsilon)\right\|_{L^\infty}, \frac{C_\alpha}{\min\left\{\left|\Phi^t_{\varepsilon,N}(x) - \Phi^t_{\varepsilon,N}(y)\right|, \left|\Phi^t_\varepsilon(x) - \Phi^t_\varepsilon(y)\right|\right\}^{3+2\alpha}}\right\} \\ &\quad \le \min\left\{\left\|D^2(\nabla^\perp K_\varepsilon)\right\|_{L^\infty}, \max\left\{\left\|(\Phi^t_{\varepsilon,N})^{-1}\right\|_{\dot{C}^{0,1}}, \left\|(\Phi^t_\varepsilon)^{-1}\right\|_{\dot{C}^{0,1}}\right\}^{3+2\alpha} \frac{C_\alpha}{|x-y|^{3+2\alpha}}\right\}.\end{aligned}$$

Therefore, (3.15) shows

$$\begin{aligned}|V_3| \le\ &\left\|\omega^0\right\|_{L^\infty} \left\|\Phi^t_\varepsilon\right\|_{\dot{C}^{0,1}} \left\|\left(D\Phi^t_{\varepsilon,N} - D\Phi^t_\varepsilon\right)\big|_{B_{R_\delta}(x_0)}\right\|_{L^\infty} \\ &\quad\cdot \Bigg(\left\|D^2(\nabla^\perp K_\varepsilon)\right\|_{L^\infty} \int_{|x-y|\le 1} |x-y|\ dy \\ &\qquad + C_\alpha \max\left\{\left\|(\Phi^t_{\varepsilon,N})^{-1}\right\|_{\dot{C}^{0,1}}, \left\|(\Phi^t_\varepsilon)^{-1}\right\|_{\dot{C}^{0,1}}\right\}^{3+2\alpha} \int_{|x-y|>1} \frac{dy}{|x-y|^{2+2\alpha}}\Bigg) \\ &+ 2\left\|D(\nabla^\perp K_\varepsilon)\right\|_{L^\infty} \left\|\Phi^t_\varepsilon\right\|_{\dot{C}^{0,1}}\, \delta \\ \le\ & C_{\alpha,\varepsilon} \left\|\omega^0\right\|_{L^\infty} \left\|\left(D\Phi^t_{\varepsilon,N} - D\Phi^t_\varepsilon\right)\big|_{B_{R_\delta}(x_0)}\right\|_{L^\infty} + C_{\alpha,\varepsilon}\delta.\end{aligned}$$

Aggregating the estimates for V_1, V_2, V_3 now yields

$$\left|\partial_t\left(D\Phi^t_{\varepsilon,N}(x) - D\Phi^t_\varepsilon(x)\right)\right| \le C_{\alpha,\varepsilon,L_\theta(\Omega),\|\omega^0\|_{L^\infty}} \left\|\left(D\Phi^t_{\varepsilon,N} - D\Phi^t_\varepsilon\right)\big|_{B_{R_\delta}(x_0)}\right\|_{L^\infty} + C_{\alpha,\varepsilon}\delta$$

for all $t \in [-T_0, T_0]$, $x \in B_{R_\delta}(x_0)$ and $N \geq N_0$. Then for any $t+h \in [-T_0, T_0]$, we have

$$\begin{aligned}\left|\left(D\Phi_{\varepsilon,N}^{t+h}(x) - D\Phi_{\varepsilon}^{t+h}(x)\right)\right| &\leq \left|\left(D\Phi_{\varepsilon,N}^{t}(x) - D\Phi_{\varepsilon}^{t}(x)\right)\right| \\ &\quad + C_{\alpha,\varepsilon,L_\theta(\Omega),\|\omega^0\|_{L^\infty}} \int_t^{t+h} \left\| \left(D\Phi_{\varepsilon,N}^{\tau} - D\Phi_{\varepsilon}^{\tau}\right)\big|_{B_{R_\delta}(x_0)} \right\|_{L^\infty} d\tau \\ &\quad + C_{\alpha,\varepsilon}\delta\,|h|\,.\end{aligned}$$

Since $D\Phi_\varepsilon^t$ and $D\Phi_{\varepsilon,N}^t$ are continuous in t, taking supremum over $x \in B_{R_\delta}(x_0)$, dividing by $|h|$ and sending $h \to 0$ shows

$$\begin{aligned}&\max\left\{\partial_t^+ \left\| \left(D\Phi_{\varepsilon,N}^{t} - D\Phi_{\varepsilon}^{t}\right)\big|_{B_{R_\delta}(x_0)} \right\|_{L^\infty}, -\partial_{t-} \left\| \left(D\Phi_{\varepsilon,N}^{t} - D\Phi_{\varepsilon}^{t}\right)\big|_{B_{R_\delta}(x_0)} \right\|_{L^\infty}\right\} \\ &\leq C_{\alpha,\varepsilon,L_\theta(\Omega),\|\omega^0\|_{L^\infty}} \left\| \left(D\Phi_{\varepsilon,N}^{t} - D\Phi_{\varepsilon}^{t}\right)\big|_{B_{R_\delta}(x_0)} \right\|_{L^\infty} + C_{\alpha,\varepsilon}\delta,\end{aligned}$$

thus a Grönwall-type argument shows

$$\left\| \left(D\Phi_{\varepsilon,N}^{t} - D\Phi_{\varepsilon}^{t}\right)\big|_{B_{R_\delta}(x_0)} \right\|_{L^\infty} \leq \frac{\exp(C_{\alpha,\varepsilon,L_\theta(\Omega),\|\omega^0\|_{L^\infty}}\,|t|) - 1}{C_{\alpha,\varepsilon,L_\theta(\Omega),\|\omega^0\|_{L^\infty}}} C_{\alpha,\varepsilon}\delta$$

since $D\Phi_{\varepsilon,N}^0 = D\Phi_\varepsilon^0 = \mathrm{Id}$.

Therefore, for any $t \in [-T_0, T_0]$, $x \in B_R(x_0) \subseteq B_{R_\delta}(x_0)$ and $N \geq N_0$,

$$\left|\Phi_{\varepsilon,N}^t(x) - \Phi_\varepsilon^t(x) - \Phi_{\varepsilon,N}^t(x_0) + \Phi_\varepsilon^t(x_0)\right| \leq \frac{\exp(C_{\alpha,\varepsilon,L_\theta(\Omega),\|\omega^0\|_{L^\infty}} T_0) - 1}{C_{\alpha,\varepsilon,L_\theta(\Omega),\|\omega^0\|_{L^\infty}}} R C_{\alpha,\varepsilon}\delta,$$

thus

$$\limsup_{N\to\infty} \sup_{t\in[-T_0,T_0]} \left\| \left(\Phi_{\varepsilon,N}^t - \Phi_\varepsilon^t - \Phi_{\varepsilon,N}^t(x_0) + \Phi_\varepsilon^t(x_0)\right)\big|_{B_R(x_0)} \right\|_{L^\infty} \leq C_{\alpha,\varepsilon,L_\theta(\Omega),\|\omega^0\|_{L^\infty}} R\delta,$$

and since $\delta > 0$ is arbitrary, the claim follows. □

Appendix A. Regularity of Functions with $L_\theta(\Omega) < \infty$

Is it ever possible to have a generalized layer cake representation that is better than the usual super-level sets?

Proposition A.1. *For any $\omega\colon \mathbb{R}^2 \to \mathbb{R}$ and a generalized layer cake representation (Ω, θ) of ω,*

$$\|\omega\|_{\dot{C}^{0,2\alpha}} \leq L_\theta(\Omega)$$

holds.

Proof. Let $x, y \in \mathbb{R}^2$, $x \neq y$ be given. Then

$$|\omega(x) - \omega(y)| \leq \int_{\mathcal{L}} |\mathbb{1}_{\Omega^\lambda}(x) - \mathbb{1}_{\Omega^\lambda}(y)|\; d|\theta|(\lambda)$$

holds. Since the integrand in the right-hand side is nonzero only if $|x-y| \geq d(x, \partial\Omega^\lambda)$, we obtain

$$|\omega(x) - \omega(y)| \leq \int_{\mathcal{L}} \frac{|x-y|^{2\alpha}}{d(x,\partial\Omega^\lambda)^{2\alpha}}\, d|\theta|(\lambda) \leq L_\theta(\Omega)\,|x-y|^{2\alpha}\,,$$

thus $\|\omega\|_{\dot{C}^{0,2\alpha}} \le L_\theta(\Omega)$ follows. □

Proposition A.2. *Let* $\omega \colon \mathbb{R}^2 \to \mathbb{R}$ *be a uniformly continuous bounded function whose modulus of continuity* ρ *satisfies* $\int_0^1 \frac{\min\{\rho(\delta),1\}}{\delta^{1+2\alpha}}\, d\delta < \infty$*. Let* $\mathcal{L} \coloneqq [\inf \omega, \sup \omega]$*,* θ *the signed measure on* $\mathcal{L}$ *given as*

$$\theta(A) \coloneqq \int_A \operatorname{sgn}(\lambda)\, d\lambda,$$

and

$$\Omega \coloneqq \left\{(x,\lambda) \in \mathbb{R}^2 \times (0, \sup \omega] : \omega(x) > \lambda\right\} \cup \left\{(x,\lambda) \in \mathbb{R}^2 \times [\inf \omega, 0) : \omega(x) < \lambda\right\}.$$

Then (Ω, θ) *is a generalized layer cake representation of* ω *satisfying* $L_\theta(\Omega) < \infty$*.*

Proof. TBD. □

Proposition A.3. *There exist* $\omega \colon \mathbb{R}^2 \to \mathbb{R}$ *and a generalized layer cake representation* (Ω, θ) *of* ω *such that* $L_\theta(\Omega) < \infty$ *and*

$$\min\left\{\delta^{2\alpha}, 1\right\} \le \rho(\delta) \le 2 \min\left\{\delta^{2\alpha}, 1\right\}$$

hold for all $\delta \ge 0$ *where* ρ *is the modulus of continuity of* ω*.*

Proof. TBD. □

References

[1] J. Jeon and A. Zlatoš, *Well-Posedness and Finite Time Singularity for Touching g-SQG Patches on the Plane*, preprint.

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