TBD

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Abstract. This is the abstract.

1. Introduction

We are concerned with the PDE in two dimensions

$$\partial_t\omega + u\cdot\nabla\omega = 0 \tag{1.1}$$

with

$$u \coloneqq -\nabla^\perp(-\Delta)^{-1+\alpha}\omega \tag{1.2}$$

and $\alpha\in\left(0,\frac{1}{2}\right)$, where $(x_1,x_2)^\perp \coloneqq (-x_2,x_1)$ and $\nabla^\perp \coloneqq (-\partial_{x_2},\partial_{x_1})$.

Given $\omega\in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$, the velocity field $u(\omega)$ generated by ω as in (1.2) is given as

$$u(\omega;x) \coloneqq \int_{\mathbb{R}^2}\nabla^\perp K(x-y)\omega(y)\,dy,$$

where kernel $K\colon\mathbb{R}^2\to(0,\infty]$ is defined as

$$K(x) \coloneqq \frac{c_\alpha}{2\alpha\,|x|^{2\alpha}}$$

for some $c_\alpha>0$. Since ω is in $L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$, it can be easily seen that $u(\omega)$ is a well-defined $(1-2\alpha)$-Hölder continuous function $\mathbb{R}^2\to\mathbb{R}^2$.

More generally, when ω is a finite signed Borel measure on $\mathbb{R}^2$, (1.2) yields

$$u(\omega;x) \coloneqq \int_{\mathbb{R}^2}\nabla^\perp K(x-y)\,d\omega(y)$$

whenever the integral converges absolutely. (The reason for considering this general case is merely because of convenience in certain aspects of developing the theory, and we will not be concerned with the well-posedness of (1.1)–(1.2) in this general setting.) When ω is an L^1 function, we identify it with the finite signed Borel measure it defines through integration with respect to the Lebesgue measure, so that the interpretations of $u(\omega)$ in both ways are consistent. Note that for any measure-preserving homeomorphism $\Phi\colon\mathbb{R}^2\to\mathbb{R}^2$, this correspondence between L^1 functions and finite signed Borel measures identifies the function $\omega\circ\Phi^{-1}$ with the pushforward measure $\Phi_*\omega$.

Our first result is local well-posedness of (1.1)–(1.2) within a class of $\omega\in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ admitting a decomposition of the form

$$\omega(x) = \int_{\mathcal{L}}\mathbb{1}_{\Omega^\lambda}(x)\,d\theta(\lambda), \tag{1.3}$$

where $\mathcal{L}$ is a measurable space (whose σ-algebra is not explicitly written), θ is a σ-finite signed measure on $\mathcal{L}$, Ω is a set in the product σ-algebra of $\mathbb{R}^2 \times \mathcal{L}$, and $\Omega^\lambda \subseteq \mathbb{R}^2$ is the λ-section of Ω for each $\lambda \in \mathcal{L}$. Here, $\mathcal{L}$ is meant to be the range of ω and Ω^λ each superlevel and sublevel set of ω^+ and ω^-, respectively, so that $\partial\Omega^\lambda$ is a level set of ω and (1.3) is the layer cake representation of ω. In this reason, we call a choice of $(\mathcal{L}, \theta, \Omega)$, or simply Ω, a *generalized layer cake representation* of ω.

It turns out that Ω^λ's being superlevel or sublevel sets of ω is not strictly necessary for the well-posedness theory we develop. Furthermore, this abstract setting allows a simple setup for studying H^2 regularity of level sets, which our second result is mainly about, that encompasses situations where some level sets of ω may consist of multiple (or even infinitely many) disjoint curves. In these reasons, we state and prove our well-posedness result in terms of an arbitrarily given generalized layer cake representation, rather than the standard layer cake representation.

In addition to merely having a decomposition (1.3), we impose a regularity condition

$$L(\Omega) \coloneqq \sup_{x \in \mathbb{R}^2} \int_{\mathcal{L}} \frac{d|\theta|(\lambda)}{d(x, \partial\Omega^\lambda)^{2\alpha}} < \infty \tag{1.4}$$

where $|\theta|$ is the variation of θ. Note that a standard measure theory argument shows joint measurability of

$$d(x, \partial\Omega^\lambda) = \max\{d(x, \Omega^\lambda), d(x, \mathbb{R}^2 \setminus \Omega^\lambda)\}$$

in (x, λ), so $L(\Omega)$ is always well-defined (with the convention $\frac{1}{0} = \infty$ and $\infty \cdot 0 = 0$). The quantity $L(\Omega)$ arises in the context of estimating the growth of H^2 norm of a boundary curve $\partial\Omega^\lambda$ (Lemma 3.7), which was the motivation for us to study the well-posedness problem under the condition (1.4).

It turns out that any ω admitting Ω with (1.4) must be 2α-Hölder continuous. In fact, this is the best possible L^∞-type regularity condition (1.4) implies because there exists ω satisfying (1.4) whose modulus of continuity $\rho(\delta)$ is in between constant multiples of $\min\{\delta^{2\alpha}, 1\}$. In the below (Lemma 2.1) we show that (1.4) implies Lipschitz continuity of $u(\omega)$, which does not follow from mere 2α-Hölder continuity in general. This is one of the key components that lead to our well-posedness result.

On the other hand, investigating the distributional derivative of $u(\omega)$ shows that the most general condition on ω *in terms of* ρ that ensures Lipschitz continuity of $u(\omega)$ is

$$\int_0^1 \frac{\min\{\rho(\delta), 1\}}{\delta^{1+2\alpha}}\, d\delta < \infty. \tag{1.5}$$

It can be shown that (1.5) in fact implies (1.4) when Ω^λ's are chosen to be superlevel sets of ω^+ and sublevel sets of ω^-. Since (1.5) does not hold for $\rho(\delta) = \min\{\delta^{2\alpha}, 1\}$, we see that (1.4) is strictly more general than any assumption on the modulus of continuity of ω from which Lipschitz continuity of $u(\omega)$ is guaranteed.

Since any ω in the class of functions we consider always generates Lipschitz velocity field, it is natural to consider the following notion of solutions to (1.1)–(1.2).

Definition 1.1. Let $\omega^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. A *(transport?) solution* to (1.1)–(1.2) on a time interval $I \ni 0$ with the initial data ω^0 is a function $\omega\colon I \to L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ given as $\omega^t := \omega \circ (\Phi^t)^{-1}$, where $\Phi \in C\left(I; C(\mathbb{R}^2; \mathbb{R}^2)\right)$ (with respect to the extended metric $d(F, G) := \|F - G\|_{L^\infty}$) satisfies the initial value problem

$$\begin{cases} \partial_t \Phi^t(x) = u(\omega^t; \Phi^t(x)), \\ \quad \Phi^0(x) = x \end{cases} \tag{1.6}$$

for all $x \in \mathbb{R}^2$ where the time derivative is one-sided at any end-point of I, and each Φ^t is a measure-preserving homeomorphism. We call Φ the *flow map* associated to ω.

Remark. It is easy to show that any (transport?) solution ω is a weak solution to (1.1)–(1.2) in the sense that

$$\frac{d}{dt} \int_{\mathbb{R}^2} \varphi(x)\, \omega^t(x)\, dx = \int_{\mathbb{R}^2} \big(u(\omega^t; x) \cdot \nabla\varphi(x)\big)\, \omega^t(x)\, dx$$

holds for all $\varphi \in C^1(\mathbb{R}^2)$.

Given a measure-preserving homeomorphism $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$, it can be easily seen that $\Phi_*\Omega := \{(\Phi(x), \lambda) \in \mathbb{R}^2 \times \mathcal{L}\colon (x, \lambda) \in \Omega\}$ is a generalized layer cake representation of $\omega \circ \Phi^{-1}$ and

$$\omega(\Phi^{-1}(x)) = \int_{\mathcal{L}} \mathbb{1}_{\Phi(\Omega^\lambda)}(x)\, d\theta(\lambda)$$

for $x \in \mathbb{R}^2$. Also,

$$\begin{aligned} L(\Phi_*\Omega) &= \sup_{x \in \mathbb{R}^2} \int_{\mathcal{L}} \frac{d|\theta|(\lambda)}{d(x, \partial\Phi(\Omega^\lambda))^{2\alpha}} = \sup_{x \in \mathbb{R}^2} \int_{\mathcal{L}} \frac{d|\theta|(\lambda)}{d(x, \Phi(\partial\Omega^\lambda))^{2\alpha}} \\ &= \sup_{x \in \mathbb{R}^2} \int_{\mathcal{L}} \frac{d|\theta|(\lambda)}{d(\Phi(x), \Phi(\partial\Omega^\lambda))^{2\alpha}} \le \left\|\Phi^{-1}\right\|_{\dot{C}^{0,1}}^{2\alpha} L(\Omega). \end{aligned} \tag{1.7}$$

Now we state our first main result.

Theorem 1.2. *Let $\omega^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ admit a generalized layer cake representation $(\mathcal{L}, \theta, \Omega)$ such that $L(\Omega) < \infty$. Then there is a unique maximal (transport?) solution $\omega\colon I \to L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ to (1.1)–(1.2) with the initial data ω^0 and the associated flow map $\Phi \in C\left(I; C(\mathbb{R}^2; \mathbb{R}^2)\right)$ such that $\sup_{t \in J} L(\Phi^t_*\Omega) < \infty$ for all compact subinterval $J \subseteq I$. Also, $\sup_{t \in J} \max\left\{\|\Phi^t\|_{\dot{C}^{0,1}}, \|(\Phi^t)^{-1}\|_{\dot{C}^{0,1}}\right\} < \infty$ holds for all compact subinterval $J \subseteq I$. Furthermore, I is open and if T is an endpoint of I, then either T is infinite or $\lim_{t \to T} L(\Phi^t_*\Omega) = \infty$.*

Our second result shows that in the setting of Theorem 1.2, if each $\partial\Omega^\lambda$ is an H^2 curve and certain additional assumptions are satisfied, then these properties must be retained by the solution provided by Theorem 1.2, at least for a short time. To state it precisely, let us give the following definitions. We refer to [1] for notions related to the space of closed plane curves ($\mathrm{CC}(\mathbb{R}^2)$, $\mathrm{PSC}(\mathbb{R}^2)$, $\mathrm{im}(\,\cdot\,)$, $\ell(\,\cdot\,)$, $\|\cdot\|_{\dot{H}^2}$ and $\Delta(\,\cdot\,,\,\cdot\,)$).

Definition 1.3. Let $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Then a generalized layer cake representation $(\mathcal{L}, \theta, \Omega)$ of ω is said to be *composed of simple closed curves* if for each $\lambda \in \mathcal{L}$, Ω^λ is a bounded open set and $\partial\Omega^\lambda = \operatorname{im}(z^\lambda)$ for some $z^\lambda \in \mathrm{PSC}(\mathbb{R}^2)$. In this case, we define

$$Q(\Omega) := \sup_{\lambda\in\mathcal{L}} \ell(z^\lambda) \left\|z^\lambda\right\|_{\dot{H}^2}^2, \quad R(\Omega) := \sup_{\lambda\in\mathcal{L}} \ell(z^\lambda)^{1/2} \overline{\int_{\mathcal{L}}} \frac{d|\theta|(\lambda')}{\ell(z^{\lambda'})^{1/2}\Delta(z^\lambda, z^{\lambda'})^{2\alpha}}.$$

Here, $\overline{\int_{\mathcal{L}}} f(\lambda)\, d|\theta|(\lambda)$ for any function $f\colon \mathcal{L} \to [0,\infty]$ is the *upper Lebesgue integral* of f; that is,

$$\overline{\int_{\mathcal{L}}} f(\lambda)\, d|\theta|(\lambda) := \inf_g \int_{\mathcal{L}} g(\lambda)\, d|\theta|(\lambda)$$

where $g\colon \mathcal{L} \to [0,\infty]$ ranges over all measurable functions bounded below by f.

For our second result, we will impose in addition to $L(\Omega) < \infty$ that Ω is composed of simple closed curves and $Q(\Omega), R(\Omega) < \infty$. The condition $Q(\Omega) < \infty$ ensures a form of *scaling-invariant* uniform H^2 regularity of z^λ's. (Recall that for $a > 0$ and $\gamma \in \mathrm{CC}(\mathbb{R}^2)$, $\|a\gamma\|_{\dot{H}^2}^2 = \frac{1}{a}\|\gamma\|_{\dot{H}^2}^2$ while $\ell(a\gamma) = a\ell(\gamma)$.) The condition $R(\Omega) < \infty$ on the other hand controls how densely z^λ's of *different scales* can be packed together. More specifically, it prevents too many small level sets to be placed near a large level set. Interpreting z^λ's as level sets of ω, this in effect rules out too sharp "pinched tops/bottoms". (More elaboration of why this is needed?)

Next we state our second main result.

Theorem 1.4. *Let $\omega^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ admit a generalized layer cake representation $(\mathcal{L}, \theta, \Omega)$ composed of simple closed curves such that $L(\Omega), Q(\Omega), R(\Omega) < \infty$. Let $\omega\colon I \to L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ be the (transport?) solution to (1.1)–(1.2) with the initial data ω^0 and the associated flow map $\Phi \in C\left(I; C(\mathbb{R}^2;\mathbb{R}^2)\right)$ provided by Theorem 1.2. Then $\Phi^t_*\Omega$ is composed of simple closed curves for each $t \in I$, $\sup_{t\in J} R(\Phi^t_*\Omega) < \infty$ for each compact subinterval $J \subseteq I$, and the set*

$$\left\{t \in I\colon Q(\Phi^t_*\Omega) < \infty\right\}$$

is nonempty and open.

Consider some smooth even $\chi\colon \mathbb{R} \to \mathbb{R}$ such that $0 \le \chi \le 1$, $\chi \equiv 1$ on $\mathbb{R} \setminus (-1,1)$, and $0 \notin \operatorname{supp}\chi$. For each $\varepsilon > 0$, let $K_\varepsilon(x) := \chi\left(\frac{|x|}{\varepsilon}\right) K(x)$. Note that for any $n \in \mathbb{Z}_{\ge 0}$ and $\beta \ge 0$ there is $C_{\alpha,n,\beta}$ that only depends on α, n, β such that the norm of the n-linear form $D^n K_\varepsilon(x)$ is bounded by $\frac{C_{\alpha,n,\beta}}{\varepsilon^\beta |x|^{n-\beta+2\alpha}}$. For any finite signed Borel measure ω on $\mathbb{R}^2$ we now define the mollified velocity field

$$u_\varepsilon(\omega; x) := \int_{\mathbb{R}^2} \nabla^\perp K_\varepsilon(x-y)\, d\omega(y)$$

for $x \in \mathbb{R}^2$. Since $\nabla^\perp K_\varepsilon$ is a smooth function whose all derivatives vanish at infinity, this integral is always well-defined and $u_\varepsilon(\omega)$ is a smooth function such that

$$\left\|D^k(u_\varepsilon(\omega))\right\|_{L^\infty} \le \left\|D^k(\nabla^\perp K_\varepsilon)\right\|_{L^\infty} \|\omega\|_{\mathrm{TV}} \tag{1.8}$$

for all $k \in \mathbb{Z}_{\ge 0}$.

2. Proof of Theorem 1.2

We start with some estimates on the velocity field $u(\omega)$ in terms of $L(\Omega)$, where $(\mathcal{L}, \theta, \Omega)$ is a generalized layer cake representation of ω. Unless specified otherwise, all constants C_α in the proofs can change from one inequality to another and depend only on α.

Lemma 2.1. *There is C_α such that for any $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ with a generalized layer cake representation $(\mathcal{L}, \theta, \Omega)$, $\varepsilon > 0$ and $x \in \mathbb{R}^2$,*

$$|D(u_\varepsilon(\omega))(x)| \le C_\alpha \int_{\mathcal{L}} \frac{d|\theta|(\lambda)}{d(x, \partial\Omega^\lambda)^{2\alpha}}.$$

Therefore, with this C_α,

$$\|u(\omega)\|_{\dot{C}^{0,1}} \le C_\alpha L(\Omega)$$

holds for any such ω.

Proof. For each $\varepsilon > 0$ and $x, h \in \mathbb{R}^2$, oddness of $\nabla^\perp K_\varepsilon$ shows

$$u_\varepsilon(\omega; x+h) - u_\varepsilon(\omega; x) = \int_{\mathbb{R}^2} \left(\nabla^\perp K_\varepsilon(x+h-y) - \nabla^\perp K_\varepsilon(x-y)\right)(\omega(y) - \omega(x))\, dy,$$

so

$$\begin{aligned} D(u_\varepsilon(\omega))(x)h &= \int_{\mathbb{R}^2} D(\nabla^\perp K_\varepsilon)(x-y)h(\omega(y) - \omega(x))\, dy \\ &= \int_{\mathbb{R}^2} \int_{\mathcal{L}} D(\nabla^\perp K_\varepsilon)(x-y)h\left(\mathbb{1}_{\Omega^\lambda}(y) - \mathbb{1}_{\Omega^\lambda}(x)\right) d\theta(\lambda)\, dy \end{aligned}$$

follows, because the difference between $u_\varepsilon(\omega; x+h) - u_\varepsilon(\omega; x)$ and the right-hand side of the first equality of the above is $O\left(|h|^2\right)$. Note that whenever $\mathbb{1}_{\Omega^\lambda}(y) - \mathbb{1}_{\Omega^\lambda}(x) \neq 0$, $|x-y| \ge d(x, \partial\Omega^\lambda)$ must hold, which shows

$$|D(u_\varepsilon(\omega))(x)| \le \int_{\mathcal{L}} \int_{|x-y| \ge d(x, \partial\Omega^\lambda)} \frac{C_\alpha}{|x-y|^{2+2\alpha}}\, dy\, d|\theta|(\lambda) \le \int_{\mathcal{L}} \frac{C_\alpha}{d(x, \partial\Omega^\lambda)^{2\alpha}} d|\theta|(\lambda),$$

thus the first claim is shown. Then the second claim follows by taking the supremum over $x \in \mathbb{R}^2$ and letting $\varepsilon \to 0^+$. $\square$

Lemma 2.2. *There is C_α such that for any $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ with a generalized layer cake representation $(\mathcal{L}, \theta, \Omega)$ and measure-preserving homeomorphisms $\Phi_1, \Phi_2 \colon \mathbb{R}^2 \to \mathbb{R}^2$,*

$$\|u(\Phi_{1*}\omega) - u(\Phi_{2*}\omega)\|_{L^\infty} \le C_\alpha (L(\Phi_{1*}\Omega) + L(\Phi_{2*}\Omega))\, \|\Phi_1 - \Phi_2\|_{L^\infty}.$$

Proof. Let $d := \|\Phi_1 - \Phi_2\|_{L^\infty}$ and $\omega_i := \omega \circ \Phi_i^{-1}$ for $i = 1, 2$. Then for given $x \in \mathbb{R}^2$, $u(\Phi_{1*}\omega; x) - u(\Phi_{2*}\omega; x)$ is equal to the sum of the following two:

$$\begin{aligned} I_1 &:= \int_{|x-y| \le 2d} \nabla^\perp K(x-y)(\omega_1(y) - \omega_2(y))\, dy \\ I_2 &:= \int_{|x-y| > 2d} \nabla^\perp K(x-y)(\omega_1(y) - \omega_2(y))\, dy. \end{aligned}$$

Estimate for I_1. By oddness of $\nabla^\perp K$, we have

$$I_1 = \int_{|x-y|\le 2d} \nabla^\perp K(x-y)(\omega_1(y)-\omega_1(x))\,dy - \int_{|x-y|\le 2d} \nabla^\perp K(x-y)(\omega_2(y)-\omega_2(x))\,dy,$$

and by symmetry, it is enough to estimate

$$I_3 \coloneqq \int_{|x-y|\le 2d} \frac{|\omega_1(y)-\omega_1(x)|}{|x-y|^{1+2\alpha}}\,dy \le \int_{\mathcal{L}} \int_{|x-y|\le 2d} \frac{\left|\mathbb{1}_{\Phi_1(\Omega^\lambda)}(y) - \mathbb{1}_{\Phi_1(\Omega^\lambda)}(x)\right|}{|x-y|^{1+2\alpha}}\,dy\,d\lambda.$$

In the same way as in the proof of Lemma 2.1, $|x-y| \ge d(x, \partial\Phi_1(\Omega^\lambda))$ must hold whenever the numerator of the integrand in the right-hand side is nonzero, which shows

$$I_3 \le \int_{\mathcal{L}} \int_{|x-y|\le 2d} \frac{1}{|x-y|\cdot d(x,\partial\Phi_1(\Omega^\lambda))^{2\alpha}}\,dy\,d|\theta|(\lambda) \le 4\pi L(\Phi_{1*}\Omega) d.$$

Then repeating the same argument for ω_2 yields

$$|I_1| \le C_\alpha (L(\Phi_{1*}\Omega) + L(\Phi_{2*}\Omega))d.$$

Estimate for I_2. For each $R > 2d$, let

$$I_2^R \coloneqq \int_{2d<|x-y|\le R} \nabla^\perp K(x-y)(\omega_1(y)-\omega_2(y))\,dy,$$

then $I_2 = \lim_{R\to\infty} I_2^R$. Fix $R > 2d$, then

$$\begin{aligned}
I_2^R &= \int_{2d<|x-y|\le R} \nabla^\perp K(x-y)(\omega_1(y)-\omega_1(x))\,dy \\
&\qquad - \int_{2d<|x-y|\le R} \nabla^\perp K(x-y)(\omega_2(y)-\omega_1(x))\,dy \\
&= \int_{2d<|x-\Phi_1(y)|\le R} \nabla^\perp K(x-\Phi_1(y))(\omega(y)-\omega_1(x))\,dy \\
&\qquad - \int_{2d<|x-\Phi_2(y)|\le R} \nabla^\perp K(x-\Phi_2(y))(\omega(y)-\omega_1(x))\,dy \\
&= \int_{|x-\Phi_1(y)|,|x-\Phi_2(y)|\in(2d,R]} \left(\nabla^\perp K(x-\Phi_1(y)) - \nabla^\perp K(x-\Phi_2(y))\right)(\omega(y)-\omega_1(x))\,dy \\
&\qquad + \int_{|x-\Phi_2(y)|\le 2d<|x-\Phi_1(y)|\le R} \nabla^\perp K(x-\Phi_1(y))(\omega(y)-\omega_1(x))\,dy \\
&\qquad - \int_{|x-\Phi_1(y)|\le 2d<|x-\Phi_2(y)|\le R} \nabla^\perp K(x-\Phi_2(y))(\omega(y)-\omega_1(x))\,dy \\
&\qquad + \int_{2d<|x-\Phi_1(y)|\le R<|x-\Phi_2(y)|} \nabla^\perp K(x-\Phi_1(y))(\omega(y)-\omega_1(x))\,dy \\
&\qquad - \int_{2d<|x-\Phi_2(y)|\le R<|x-\Phi_1(y)|} \nabla^\perp K(x-\Phi_2(y))(\omega(y)-\omega_1(x))\,dy.
\end{aligned}$$

Let us call the terms in the right-hand side as I_n for $n = 4,5,6,7,8$, respectively.

To estimate I_4, note that for any y in the domain of integration,

$$\min_{\eta\in[0,1]} |x-(1-\eta)\Phi_1(y)-\eta\Phi_2(y)| \geq |x-\Phi_1(y)|-d \geq \frac{1}{2}|x-\Phi_1(y)|$$

holds, so the mean value theorem shows

$$\left|\nabla^\perp K(x-\Phi_1(y))-\nabla^\perp K(x-\Phi_2(y))\right| \leq \frac{C_\alpha d}{|x-\Phi_1(y)|^{2+2\alpha}},$$

thus applying the change of variables formula yields

$$|I_4| \leq \int_{|x-y|>2d} \frac{C_\alpha d\,|\omega_1(y)-\omega_1(x)|}{|x-y|^{2+2\alpha}}\,dy \leq \int_{\mathcal{L}}\int_{|x-y|>2d} \frac{C_\alpha d\left|\mathbb{1}_{\Phi_1(\Omega^\lambda)}(y)-\mathbb{1}_{\Phi_1(\Omega^\lambda)}(x)\right|}{|x-y|^{2+2\alpha}}\,dy\,d|\theta|(\lambda).$$

Again, $|x-y| \geq d(x,\partial\Phi_1(\Omega^\lambda))$ must hold whenever the numerator of the integrand in the right-hand side is nonzero, which shows

$$|I_4| \leq \int_{\mathcal{L}}\int_{|x-y|\geq d(x,\partial\Phi_1(\Omega^\lambda))} \frac{C_\alpha d}{|x-y|^{2+2\alpha}}\,dy\,d|\theta|(\lambda) \leq C_\alpha L(\Phi_{1*}\Omega)d.$$

For I_5, note that for any y in the domain of integration,

$$|x-\Phi_1(y)| \leq |x-\Phi_2(y)|+d \leq 3d$$

holds, so applying the change of variables formula yields

$$|I_5| \leq \int_{|x-y|\leq 3d} \frac{C_\alpha\,|\omega_1(y)-\omega_1(x)|}{|x-y|^{1+2\alpha}}\,dy,$$

so the same argument as in the estimate for I_3 shows $|I_5| \leq C_\alpha L(\Phi_{1*}\Omega)d$.

Similarly, for I_6, again the change of variables formula shows

$$|I_6| \leq \int_{|x-\Phi_1(y)|\leq 2d} \frac{C_\alpha\,|\omega(y)-\omega_1(x)|}{|x-\Phi_1(y)|^{1+2\alpha}}\,dy \leq \int_{|x-y|\leq 2d} \frac{C_\alpha\,|\omega_1(y)-\omega_1(x)|}{|x-y|^{1+2\alpha}}\,dy,$$

so we obtain $|I_6| \leq C_\alpha L(\Phi_{1*}\Omega)d$ in the same way as in the estimate of I_3.

To estimate I_7, note that for any y in the domain of integration,

$$|x-\Phi_1(y)| \geq |x-\Phi_2(y)|-d > R-d,$$

so the change of variables formula yields

$$|I_7| \leq \int_{R-d<|x-y|\leq R} \frac{C_\alpha\,\|\omega\|_{L^\infty}}{|x-y|^{1+2\alpha}}\,dy = 2\pi C_\alpha\,\|\omega\|_{L^\infty} R^{1-2\alpha}\int_{1-\frac{d}{R}}^{1}\frac{dr}{r^{2\alpha}} \leq \frac{2^{1+2\alpha}\pi C_\alpha\,\|\omega\|_{L^\infty} d}{R^{2\alpha}},$$

where the last inequality is because $\frac{d}{R}<\frac{1}{2}$. In the same way, I_8 can be estimated by $\frac{C_\alpha\|\omega\|_{L^\infty}d}{R^{2\alpha}}$.

Therefore, collecting the estimates for I_4 through I_8 and letting $R\to\infty$ shows $|I_2| \leq C_\alpha L(\Phi_{1*}\Omega)d$. Then since $x\in\mathbb{R}^2$ is arbitrary, collecting the estimates for I_1 and I_2 shows the desired conclusion. $\square$

Lemma 2.3. *There is C_α such that for any $\omega\in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ and $\varepsilon>0$,*

$$\|u(\omega)-u_\varepsilon(\omega)\|_{L^\infty} \leq C_\alpha\,\|\omega\|_{L^\infty}\,\varepsilon^{1-2\alpha}.$$

Proof. By definition of K_ε, we have $\nabla^\perp K_\varepsilon(x) = \nabla^\perp K(x)$ if $|x| \geq \varepsilon$, so for any $x \in \mathbb{R}^2$,

$$\begin{aligned} |u(\omega;x) - u_\varepsilon(\omega;x)| &\leq \int_{|x-y|\leq\varepsilon} \left|\nabla^\perp K(x-y) - \nabla^\perp K_\varepsilon(x-y)\right| \|\omega\|_{L^\infty} \, dy \\ &\leq \int_{|x-y|\leq\varepsilon} \frac{C_\alpha \|\omega\|_{L^\infty}}{|x-y|^{1+2\alpha}} \, dy = C_\alpha \|\omega\|_{L^\infty} \varepsilon^{1-2\alpha}. \end{aligned}$$

□

Now, let initial data $\omega^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ admitting a generalized layer cake representation $(\mathcal{L}, \theta, \Omega)$ with $L(\Omega) < \infty$ be given.

Fix $\varepsilon > 0$ and consider the ODE

$$\begin{cases} \partial_t \Psi_\varepsilon^t(x) = u_\varepsilon((\mathrm{Id} + \Psi_\varepsilon^t)_* \omega^0; x + \Psi_\varepsilon^t(x)), \\ \Psi_\varepsilon^0(x) = 0. \end{cases} \tag{2.1}$$

That is, $\Phi_\varepsilon^t := \mathrm{Id} + \Psi_\varepsilon^t$ is the flow map generated by the vector field u_ε which in turn is generated by the measure ω^0 transported by Φ_ε^t. We show that this ODE is globally well-posed in the Banach space $C_b(\mathbb{R}^2;\mathbb{R}^2)$ of bounded continuous functions $\mathbb{R}^2 \to \mathbb{R}^2$.

Lemma 2.4. *For each $F \in C_b(\mathbb{R}^2;\mathbb{R}^2)$, define*

$$\mathcal{F}(F)(x) := u_\varepsilon((\mathrm{Id} + F)_* \omega^0; x + F(x))$$

for each $x \in \mathbb{R}^2$, then $\mathcal{F} \colon C_b(\mathbb{R}^2;\mathbb{R}^2) \to C_b(\mathbb{R}^2;\mathbb{R}^2)$ is well-defined and is Lipschitz continuous.

Proof. Clearly, $\mathcal{F}(F)$ is a continuous function $\mathbb{R}^2 \to \mathbb{R}^2$ for any $F \in C_b(\mathbb{R}^2;\mathbb{R}^2)$. Given $F_1, F_2 \in C_b(\mathbb{R}^2;\mathbb{R}^2)$ and $x \in \mathbb{R}^2$,

$$\begin{aligned} &(\mathcal{F}(F_1) - \mathcal{F}(F_2))(x) \\ &= \int_{\mathbb{R}^2} \nabla^\perp K_\varepsilon(x + F_1(x) - y) \, d(\mathrm{Id} + F_1)_* \omega^0(y) - \int_{\mathbb{R}^2} \nabla^\perp K_\varepsilon(x + F_2(x) - y) \, d(\mathrm{Id} + F_2)_* \omega^0(y) \\ &= \int_{\mathbb{R}^2} \left(\nabla^\perp K_\varepsilon(x - y + F_1(x) - F_1(y)) - \nabla^\perp K_\varepsilon(x - y + F_2(x) - F_2(y))\right) \omega^0(y) \, dy, \end{aligned}$$

so

$$\|\mathcal{F}(F_1) - \mathcal{F}(F_2)\|_{L^\infty} \leq 2 \left\|D(\nabla^\perp K_\varepsilon)\right\|_{L^\infty} \left\|\omega^0\right\|_{L^1} \|F_1 - F_2\|_{L^\infty}.$$

Since $\mathcal{F}(0) = u_\varepsilon(\omega^0)$ is bounded, all claims follow. □

Therefore, (2.1) is globally well-posed. Then with $\Phi_\varepsilon^t := \mathrm{Id} + \Psi_\varepsilon^t$ and $\omega_\varepsilon^t := \Phi_{\varepsilon *}^t \omega^0$ (which for now we only regard as a finite signed Borel measure rather than an L^1 function), we can now easily see that the ODE

$$\partial_t G^t(x) = u_\varepsilon(\omega_\varepsilon^t; x + G^t(x)) \tag{2.2}$$

is globally well-posed in $C_b(\mathbb{R}^2;\mathbb{R}^2)$ for any initial data at any initial time. For each $t_0 \in \mathbb{R}$, let $\Theta_\varepsilon^{t_0,t}$ be the unique solution to (2.2) with the initial data $\Theta_\varepsilon^{t_0,t_0} = 0$, and consider $G^t := \Theta_\varepsilon^{t_0,t_1} + \Theta_\varepsilon^{t_1,t} \circ (\mathrm{Id} + \Theta_\varepsilon^{t_0,t_1})$, then it can be easily seen that G solves (2.2) and it satisfies $G^{t_1} = \Theta_\varepsilon^{t_0,t_1}$, so the uniqueness of the solution with the initial data $\Theta_\varepsilon^{t_0,t_1}$ at $t = t_1$ shows that

$$\mathrm{Id} + \Theta_\varepsilon^{t_0,t} = \mathrm{Id} + G^t = (\mathrm{Id} + \Theta_\varepsilon^{t_1,t}) \circ (\mathrm{Id} + \Theta_\varepsilon^{t_0,t_1})$$

holds for all $t \in \mathbb{R}$. Hence, letting $t = t_0$ shows $(\mathrm{Id} + \Theta_\varepsilon^{t_1,t_0}) \circ (\mathrm{Id} + \Theta_\varepsilon^{t_0,t_1}) = \mathrm{Id}$, so we conclude that each $\mathrm{Id} + \Theta_\varepsilon^{t_0,t}$, and in particular $\Phi_\varepsilon^t = \mathrm{Id} + \Theta_\varepsilon^{0,t}$, is a homeomorphism.

On the other hand, let $C_b^1(\mathbb{R}^2;\mathbb{R}^2)$ be the Banach space of bounded C^1-functions $\mathbb{R}^2 \to \mathbb{R}^2$ with bounded derivatives, then for $F \in C_b^1(\mathbb{R}^2;\mathbb{R}^2)$ and $x, h \in \mathbb{R}^2$,

$$D(u_\varepsilon(\omega_\varepsilon^t) \circ (\mathrm{Id} + F))(x)h = \int_{\mathbb{R}^2} D(\nabla^\perp K_\varepsilon)(x + F(x) - y)(h + DF(x)h)\, d\omega_\varepsilon^t(y).$$

Then it can be easily seen that $F \mapsto u_\varepsilon(\omega_\varepsilon^t) \circ (\mathrm{Id} + F)$ is locally Lipschitz on $C_b^1(\mathbb{R}^2;\mathbb{R}^2)$, so (2.2) is locally well-posed in $C_b^1(\mathbb{R}^2;\mathbb{R}^2)$. In fact it is globally well-posed in that space, because

$$\left\|D(u_\varepsilon(\omega_\varepsilon^t) \circ (\mathrm{Id} + F))\right\|_{L^\infty} \le \left\|D(\nabla^\perp K_\varepsilon)\right\|_{L^\infty} \left\|\omega^0\right\|_{L^1} \left\|\mathrm{Id} + DF\right\|_{L^\infty}$$

for any $F \in C_b^1(\mathbb{R}^2;\mathbb{R}^2)$, which together with a Grönwall-type argument shows that the C^1 norm of the solution to (2.2) can grow only at most exponentially. Hence, $\Theta_\varepsilon^{t_0,t}$ is in $C_b^1(\mathbb{R}^2;\mathbb{R}^2)$ for all $t \in \mathbb{R}$, and since the vector field $u_\varepsilon(\omega_\varepsilon^t)$ is divergence-free, it follows that each $\mathrm{Id} + \Theta_\varepsilon^{t_0,t}$ is measure-preserving. Therefore, the measure ω_ε^t corresponds to the L^1 function $\omega^0 \circ (\Phi_\varepsilon^t)^{-1}$, of which $\Phi_{\varepsilon*}^t\Omega$ is a generalized layer cake representation.

Remark. Similarly, let $C_b^2(\mathbb{R}^2;\mathbb{R}^2)$ be the Banach space of bounded C^2-functions $\mathbb{R}^2 \to \mathbb{R}^2$ with bounded first and second derivatives, then for $F \in C_b^2(\mathbb{R}^2;\mathbb{R}^2)$ and $x, h_1, h_2 \in \mathbb{R}^2$,

$$\begin{aligned} &D^2(u_\varepsilon(\omega_\varepsilon^t) \circ (\mathrm{Id} + F))(x)(h_1, h_2) \\ &\quad = \int_{\mathbb{R}^2} D^2(\nabla^\perp K_\varepsilon)(x + F(x) - y)(h_1 + DF(x)h_1, h_2 + DF(x)h_2)\, d\omega_\varepsilon^t(y) \\ &\qquad + \int_{\mathbb{R}^2} D(\nabla^\perp K_\varepsilon)(x + F(x) - y)\left(D^2F(x)(h_1, h_2)\right) d\omega_\varepsilon^t(y) \end{aligned}$$

and

$$\begin{aligned} \left\|D^2(u_\varepsilon(\omega_\varepsilon^t) \circ (\mathrm{Id} + F))\right\|_{L^\infty} \le{}& \left\|D^2(\nabla^\perp K_\varepsilon)\right\|_{L^\infty} \left\|\omega^0\right\|_{L^1} \left\|\mathrm{Id} + DF\right\|_{L^\infty}^2 \\ &+ \left\|D(\nabla^\perp K_\varepsilon)\right\|_{L^\infty} \left\|D^2F\right\|_{L^\infty} \end{aligned}$$

hold. Therefore, at most exponential growth of the C^1 norm with another Grönwall-type argument show that (2.2) is globally well-posed in $C_b^2(\mathbb{R}^2;\mathbb{R}^2)$. This fact will be used in Section 3. Repeating a similar argument inductively for higher derivatives shows that (2.2) is globally well-posed in every $C_b^k(\mathbb{R}^2;\mathbb{R}^2)$ so each Φ_ε^t is a diffeomorphism, but we do not need that fact.

Now, we derive an ε-independent estimate on the growth of $L(\Phi_{\varepsilon*}^t\Omega)$.

Lemma 2.5. $\left\|D^k(u_\varepsilon(\omega_\varepsilon^t))\right\|_{L^\infty}$ *is continuous in t for all $k \in \mathbb{Z}_{\ge 0}$ and*

$$\begin{aligned} &\left|\Theta_\varepsilon^{t_0,t_1}(x) - \Theta_\varepsilon^{t_0,t_1}(y) - \Theta_\varepsilon^{t_0,t_2}(x) + \Theta_\varepsilon^{t_0,t_2}(y)\right| \\ &\qquad \le \left(\exp\left(\left|\int_{t_2}^{t_1} \|u_\varepsilon(\omega_\varepsilon^\tau)\|_{\dot{C}^{0,1}}\, d\tau\right|\right) - 1\right) \left|x + \Theta_\varepsilon^{t_0,t_2}(x) - y - \Theta_\varepsilon^{t_0,t_2}(y)\right| \end{aligned}$$

holds for all $x, y \in \mathbb{R}^2$ and $t_0, t_1, t_2 \in \mathbb{R}$.

Proof. We first note that the change of variables formula yields

$$\left\|D^k(u_\varepsilon(\omega_\varepsilon^{t_1})) - D^k(u_\varepsilon(\omega_\varepsilon^{t_2}))\right\|_{L^\infty} \le \left\|D^{k+1}(\nabla^\perp K_\varepsilon)\right\|_{L^\infty} \left\|\omega^0\right\|_{L^1} \left\|\Phi_\varepsilon^{t_1} - \Phi_\varepsilon^{t_2}\right\|_{L^\infty}$$

for any $(k, t_1, t_2) \in \mathbb{Z}_{\ge 0} \times \mathbb{R}^2$, which shows the first claim and in particular that $\|u_\varepsilon(\omega_\varepsilon^t)\|_{\dot C^{0,1}}$ is continuous in t. Then for any $x, y \in \mathbb{R}^2$ and $t_0, t_1, t_2 \in \mathbb{R}$, we have

$$\begin{aligned}
&\left|\left|x + \Theta_\varepsilon^{t_0,t_1}(x) - y - \Theta_\varepsilon^{t_0,t_1}(y)\right| - \left|x + \Theta_\varepsilon^{t_0,t_2}(x) - y - \Theta_\varepsilon^{t_0,t_2}(y)\right|\right| \\
&\qquad \le \left|\Theta_\varepsilon^{t_0,t_1}(x) - \Theta_\varepsilon^{t_0,t_1}(y) - \Theta_\varepsilon^{t_0,t_2}(x) + \Theta_\varepsilon^{t_0,t_2}(y)\right| \\
&\qquad \le \left|\int_{t_2}^{t_1} \left|u_\varepsilon(\omega_\varepsilon^\tau; \mathrm{Id} + \Theta_\varepsilon^{t_0,\tau}(x)) - u_\varepsilon(\omega_\varepsilon^\tau; \mathrm{Id} + \Theta_\varepsilon^{t_0,\tau}(y))\right| \, d\tau\right| \qquad (2.3) \\
&\qquad \le \left|\int_{t_2}^{t_1} \|u_\varepsilon(\omega_\varepsilon^\tau)\|_{\dot C^{0,1}} \left|x + \Theta_\varepsilon^{t_0,\tau}(x) - y - \Theta_\varepsilon^{t_0,\tau}(y)\right| \, d\tau\right|,
\end{aligned}$$

so dividing by $|t_1 - t_2|$, letting $t_1 \to t_2$ and then applying a Grönwall-type argument shows

$$\left|x + \Theta_\varepsilon^{t_0,t_1}(x) - y - \Theta_\varepsilon^{t_0,t_1}(y)\right| \le \exp\left(\left|\int_{t_2}^{t_1} \|u_\varepsilon(\omega_\varepsilon^\tau)\|_{\dot C^{0,1}} \, d\tau\right|\right) \left|x + \Theta_\varepsilon^{t_0,t_2}(x) - y - \Theta_\varepsilon^{t_0,t_2}(y)\right|.$$

Then replacing (τ, t_1) by (τ', τ) in the above inequality and applying it to (2.3) shows the claim. □

Proposition 2.6. *$L(\Phi_{\varepsilon*}^t \Omega)$ is continuous in t and*

$$\max\left\{\partial_t^+ L(\Phi_{\varepsilon*}^t \Omega), -\partial_{t-} L(\Phi_{\varepsilon*}^t \Omega)\right\} \le \left\|u_\varepsilon(\omega_\varepsilon^t)\right\|_{\dot C^{0,1}} L(\Phi_{\varepsilon*}^t \Omega)$$

holds.

Proof. Fix $t_1, t_2 \in \mathbb{R}$, $x \in \mathbb{R}^2$, $\lambda \in \mathcal{L}$ and $\eta > 0$. Then there exists $y \in \partial\Omega^\lambda$ such that

$$d(\Phi_\varepsilon^{t_1}(x), \partial\Phi_\varepsilon^{t_1}(\Omega^\lambda)) + \eta \ge \left|\Phi_\varepsilon^{t_1}(x) - \Phi_\varepsilon^{t_1}(y)\right|.$$

Then Lemma 2.5 and the inequality $\left|\frac{1}{a^{2\alpha}} - \frac{1}{b^{2\alpha}}\right| \le \frac{|a-b|}{ab^{2\alpha}}$ for $a, b > 0$ show

$$\begin{aligned}
&\frac{1}{\left(d(\Phi_\varepsilon^{t_1}(x), \partial\Phi_\varepsilon^{t_1}(\Omega^\lambda)) + 2\eta\right)^{2\alpha}} - \frac{1}{\left(d(\Phi_\varepsilon^{t_2}(x), \partial\Phi_\varepsilon^{t_2}(\Omega^\lambda)) + \eta\right)^{2\alpha}} \\
&\qquad \le \frac{1}{\left(\left|\Phi_\varepsilon^{t_1}(x) - \Phi_\varepsilon^{t_1}(y)\right| + \eta\right)^{2\alpha}} - \frac{1}{\left(\left|\Phi_\varepsilon^{t_2}(x) - \Phi_\varepsilon^{t_2}(y)\right| + \eta\right)^{2\alpha}} \\
&\qquad \le \frac{|\Phi_\varepsilon^{t_1}(x) - \Phi_\varepsilon^{t_1}(y) - \Phi_\varepsilon^{t_2}(x) + \Phi_\varepsilon^{t_2}(y)|}{\left(\left|\Phi_\varepsilon^{t_1}(x) - \Phi_\varepsilon^{t_1}(y)\right| + \eta\right)\left(\left|\Phi_\varepsilon^{t_2}(x) - \Phi_\varepsilon^{t_2}(y)\right| + \eta\right)^{2\alpha}} \qquad (2.4) \\
&\qquad \le \frac{\exp\left(\left|\int_{t_2}^{t_1} \|u_\varepsilon(\omega_\varepsilon^\tau)\|_{\dot C^{0,1}} \, d\tau\right|\right) - 1}{(d(\Phi_\varepsilon^{t_2}(x), \partial\Phi_\varepsilon^{t_2}(\Omega^\lambda)) + \eta)^{2\alpha}},
\end{aligned}$$

so letting $\eta \to 0^+$, integrating over λ, and then taking supremum over x shows

$$L(\Phi_{\varepsilon*}^{t_1}\Omega) \le \exp\left(\left|\int_{t_2}^{t_1} \|u_\varepsilon(\omega_\varepsilon^\tau)\|_{\dot C^{0,1}} \, d\tau\right|\right) L(\Phi_{\varepsilon*}^{t_2}\Omega).$$

Since $t_1, t_2 \in \mathbb{R}$ are arbitrary, all claims follow immediately. □

Corollary 2.7. *With C_α from Lemma 2.1,*

$$\max\left\{\partial_t^+ L(\Phi^t_{\varepsilon*}\Omega), -\partial_{t-} L(\Phi^t_{\varepsilon*}\Omega)\right\} \le C_\alpha L(\Phi^t_{\varepsilon*}\Omega)^2$$

holds for any $t \in \mathbb{R}$. Also, whenever $|t| < \frac{1}{C_\alpha L(\Omega)}$,

$$L(\Phi^t_{\varepsilon*}\Omega) \le \frac{L(\Omega)}{1 - C_\alpha L(\Omega)\,|t|}$$

holds.

Proof. Since the second claim of Lemma 2.1 also holds with u_ε in place of u, Proposition 2.6 shows the first claim, and the second claim follows from the first claim and a Grönwall-type argument. □

Let $T_0 \coloneqq \frac{1}{2C_\alpha L(\Omega)}$ with C_α from Lemma 2.1.

Proposition 2.8. *When $\varepsilon \to 0^+$, $\Psi_\varepsilon \in C\left([-T_0, T_0]; C_b(\mathbb{R}^2;\mathbb{R}^2)\right)$ converges uniformly to some $\Psi \in C\left([-T_0, T_0]; C_b(\mathbb{R}^2;\mathbb{R}^2)\right)$. Also, $\Phi^t \coloneqq \mathrm{Id} + \Psi^t$ is a measure-preserving homeomorphism for each $t \in [-T_0, T_0]$ and Φ solves* (1.6)*. Furthermore,*

$$L(\Phi^t_*\Omega) \le \sup_{\varepsilon>0} L(\Phi^t_{\varepsilon*}\Omega) \le 2L(\Omega)$$

and

$$\max\left\{\left\|\Phi^t\right\|_{\dot C^{0,1}}, \left\|(\Phi^t)^{-1}\right\|_{\dot C^{0,1}}\right\} \le \sup_{\varepsilon>0}\max\left\{\left\|\Phi^t_\varepsilon\right\|_{\dot C^{0,1}}, \left\|(\Phi^t_\varepsilon)^{-1}\right\|_{\dot C^{0,1}}\right\} \le e^{2C_\alpha L(\Omega)|t|}$$

hold for each $t \in [-T_0, T_0]$.

Proof. Corollary 2.7 shows

$$M \coloneqq \sup_{\varepsilon>0}\sup_{t\in[-T_0,T_0]} L(\Phi^t_{\varepsilon*}\Omega) \in [L(\Omega), 2L(\Omega)] \subseteq [0,\infty).$$

We may assume $L(\Omega) > 0$ because otherwise $\omega^0 \equiv 0$ so the conclusion follows trivially. Fix $t_0 \in [-T_0, T_0]$, then for any $t \in [-T_0, T_0]$, $\varepsilon > 0$ and $\varepsilon' \in (0, \varepsilon)$, Lemmas 2.1, 2.2 and 2.3 show

$$\begin{aligned}
&\left\|u_\varepsilon(\omega^t_\varepsilon)\circ(\mathrm{Id}+\Theta^{t_0,t}_\varepsilon) - u_{\varepsilon'}(\omega^t_{\varepsilon'})\circ(\mathrm{Id}+\Theta^{t_0,t}_{\varepsilon'})\right\|_{L^\infty}\\
&\quad\le \left\|u_\varepsilon(\omega^t_\varepsilon)\right\|_{\dot C^{0,1}}\left\|\Theta^{t_0,t}_\varepsilon - \Theta^{t_0,t}_{\varepsilon'}\right\|_{L^\infty} + \left\|u(\omega^t_\varepsilon) - u(\omega^t_{\varepsilon'})\right\|_{L^\infty}\\
&\qquad\qquad + \left\|u(\omega^t_\varepsilon) - u_\varepsilon(\omega^t_\varepsilon)\right\|_{L^\infty} + \left\|u(\omega^t_{\varepsilon'}) - u_{\varepsilon'}(\omega^t_{\varepsilon'})\right\|_{L^\infty}\\
&\quad\le C_\alpha M\left\|\Theta^{t_0,t}_\varepsilon - \Theta^{t_0,t}_{\varepsilon'}\right\|_{L^\infty} + C_\alpha M\left\|\Phi^t_\varepsilon - \Phi^t_{\varepsilon'}\right\|_{L^\infty} + C_\alpha\left\|\omega^0\right\|_{L^\infty}\varepsilon^{1-2\alpha}
\end{aligned}\tag{2.5}$$

where C_α (which we now fix for the rest of the proof) is two times the maximum of all C_α's appearing in those lemmas. Then for any $t_1, t_2 \in [-T_0, T_0]$, integrating (2.5) from t_2 to t_1 shows

$$\begin{aligned}
&\left\|\Theta^{t_0,t_1}_\varepsilon - \Theta^{t_0,t_1}_{\varepsilon'} - \Theta^{t_0,t_2}_\varepsilon + \Theta^{t_0,t_2}_{\varepsilon'}\right\|_{L^\infty}\\
&\quad\le C_\alpha M\left|\int_{t_2}^{t_1}\left\|\Theta^{t_0,\tau}_\varepsilon - \Theta^{t_0,\tau}_{\varepsilon'}\right\|_{L^\infty}d\tau\right| + C_\alpha M\left|\int_{t_2}^{t_1}\left\|\Phi^\tau_\varepsilon - \Phi^\tau_{\varepsilon'}\right\|_{L^\infty}d\tau\right|\\
&\qquad\qquad + C_\alpha\left\|\omega^0\right\|_{L^\infty}|t_1 - t_2|\,\varepsilon^{1-2\alpha}.
\end{aligned}\tag{2.6}$$

In particular, taking $t_0 = 0$, dividing by $|t_1 - t_2|$ and letting $t_1 \to t_2^{\pm}$ shows

$$\max\left\{\partial_t^+ \left\|\Psi_\varepsilon^t - \Psi_{\varepsilon'}^t\right\|_{L^\infty}, -\partial_{t-} \left\|\Psi_\varepsilon^t - \Psi_{\varepsilon'}^t\right\|_{L^\infty}\right\} \le 2C_\alpha M \left\|\Psi_\varepsilon^t - \Psi_{\varepsilon'}^t\right\|_{L^\infty} + C_\alpha \left\|\omega^0\right\|_{L^\infty} \varepsilon^{1-2\alpha}$$

for each $t \in [-T_0, T_0]$, so a Grönwall-type argument shows

$$\left\|\Phi_\varepsilon^t - \Phi_{\varepsilon'}^t\right\|_{L^\infty} = \left\|\Psi_\varepsilon^t - \Psi_{\varepsilon'}^t\right\|_{L^\infty} \le \frac{\|\omega^0\|_{L^\infty}}{2M}(e^{2C_\alpha M|t|} - 1)\varepsilon^{1-2\alpha}.$$

Applying this inequality to (2.6) with $C_\alpha M$'s replaced by $2C_\alpha M$'s, dividing by $|t_1 - t_2|$ and then sending $t_1 \to t_2^+$ shows

$$\partial_t^+ \left\|\Theta_\varepsilon^{t_0,t} - \Theta_{\varepsilon'}^{t_0,t}\right\|_{L^\infty} \le 2C_\alpha M \left\|\Theta_\varepsilon^{t_0,t} - \Theta_{\varepsilon'}^{t_0,t}\right\|_{L^\infty} + C_\alpha \left\|\omega^0\right\|_{L^\infty} e^{2C_\alpha M t}\varepsilon^{1-2\alpha}$$

for $t \ge 0$, and we can also obtain a similar bound for $-\partial_{t-} \left\|\Theta_\varepsilon^{t_0,t} - \Theta_{\varepsilon'}^{t_0,t}\right\|_{L^\infty}$ and $t \le 0$, so a Grönwall-type argument shows

$$\left\|\Theta_\varepsilon^{t_0,t} - \Theta_{\varepsilon'}^{t_0,t}\right\|_{L^\infty} \le \frac{\|\omega^0\|_{L^\infty} e^{2C_\alpha M T_0}\varepsilon^{1-2\alpha}}{2M}\left(e^{2C_\alpha M|t-t_0|} - 1\right).$$

Therefore, $\Theta_\varepsilon^{t_0,\cdot}$ converges uniformly to some $\Theta^{t_0,\cdot} : [-T_0, T_0] \to C_b(\mathbb{R}^2;\mathbb{R}^2)$ as $\varepsilon \to 0^+$. We let $\Psi^t := \Theta^{0,t}$.

Since $(\mathrm{Id}+\Theta_\varepsilon^{t_0,t_1})\circ(\mathrm{Id}+\Theta_\varepsilon^{t_1,t_0}) = \mathrm{Id}$ holds for all $t_0, t_1 \in [-T_0, T_0]$ and $\varepsilon > 0$, sending $\varepsilon \to 0^+$ shows $(\mathrm{Id}+\Theta^{t_0,t_1})\circ(\mathrm{Id}+\Theta^{t_1,t_0}) = \mathrm{Id}$, so in particular $\Phi^t := \mathrm{Id}+\Psi^t$ is a homeomorphism whose inverse is $\mathrm{Id}+\Theta^{t,0}$. Also, Lemma 2.5 and the definition of C_α show $\|\mathrm{Id} + \Theta_\varepsilon^{t_0,t}\|_{\dot{C}^{0,1}} \le e^{C_\alpha M|t-t_0|}$ for all $t_0, t \in [-T_0, T_0]$ and $\varepsilon > 0$, thus $\max\{\|\Phi^t\|_{\dot{C}^{0,1}}, \|(\Phi^t)^{-1}\|_{\dot{C}^{0,1}}\} \le e^{C_\alpha M|t|}$ holds for all $t \in [-T_0, T_0]$. By Fatou's lemma, we also have $L(\Phi_*^t\Omega) \le \liminf_{\varepsilon\to 0^+} L(\Phi_{\varepsilon *}^t\Omega) \le M$ for each $t \in [-T_0, T_0]$. Furthermore, since each Φ_ε^t is measure-preserving, its uniform limit Φ^t is also measure-preserving, because $\mathbb{1}_U \circ \Phi_\varepsilon^t$ converges pointwise to $\mathbb{1}_U \circ \Phi^t$ as $\varepsilon \to 0^+$ for any open set $U \subseteq \mathbb{R}^2$.

It remains to show that Φ^t satisfies (1.6), or equivalently, that with $\omega^t := \omega^0 \circ (\Phi^t)^{-1}$,

$$\Phi^t(x) = x + \int_0^t u(\omega^\tau; \Phi^\tau(x))\, d\tau \tag{2.7}$$

holds for each $t \in [-T_0, T_0]$ and $x \in \mathbb{R}^2$. Note that taking $t_0 = 0$ and letting $\varepsilon' \to 0^+$ in (2.5) yields

$$\left\|u_\varepsilon(\omega_\varepsilon^t) \circ \Phi_\varepsilon^t - u(\omega^t) \circ \Phi^t\right\|_{L^\infty} \le 2C_\alpha M \left\|\Phi_\varepsilon^t - \Phi^t\right\|_{L^\infty} + C_\alpha \left\|\omega^0\right\|_{L^\infty} \varepsilon^{1-2\alpha},$$

which shows that the right-hand side of

$$\Phi_\varepsilon^t(x) = x + \int_0^t u(\omega_\varepsilon^\tau; \Phi_\varepsilon^\tau(x))\, d\tau$$

converges to the right-hand side of (2.7) when $\varepsilon \to 0^+$, thus proving (1.6). $\square$

Proposition 2.9. *Let I be a compact interval containing 0 and $\Phi, \Pi \in C(I; C(\mathbb{R}^2;\mathbb{R}^2))$ be solutions to (1.6) such that $\sup_{t\in I}\max\{L(\Phi_*^t\Omega), L(\Pi_*^t\Omega)\} < \infty$. Then $\Phi = \Pi$.*

Proof. Define

$$M := \sup_{t\in I} \max\left\{L(\Phi_*^t\Omega), L(\Pi_*^t\Omega)\right\}.$$

Then Lemmas 2.1 and 2.2 show

$$\begin{aligned}\left\|u(\Phi_*^t\omega^0)\circ\Phi^t - u(\Pi_*^t\omega^0)\circ\Pi^t\right\|_{L^\infty} &\le \left\|u(\Phi_*^t\omega^0)\right\|_{\dot{C}^{0,1}}\left\|\Phi^t-\Pi^t\right\|_{L^\infty} + \left\|u(\Phi_*^t\omega^0)-u(\Pi_*^t\omega^0)\right\|_{L^\infty} \\ &\le C_\alpha M\left\|\Phi^t-\Pi^t\right\|_{L^\infty}\end{aligned}$$

for some C_α, which together with continuity of $\|\Phi^t-\Pi^t\|_{L^\infty}$ in t shows

$$\max\left\{\partial_t^+\left\|\Phi^t-\Pi^t\right\|_{L^\infty}, -\partial_{t-}\left\|\Phi^t-\Pi^t\right\|_{L^\infty}\right\} \le C_\alpha M\left\|\Phi^t-\Pi^t\right\|_{L^\infty}.$$

Therefore, a Grönwall-type argument proves the claim. □

Combining Propositions 2.8 and 2.9 then shows Theorem 1.2.

3. Proof of Theorem 1.4

Suppose that the initial data $\omega^0 \in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ admits a generalized layer cake representation $(\mathcal{L},\theta,\Omega)$ composed of simple closed curves such that $L(\Omega), Q(\Omega), R(\Omega) < \infty$. For each $\lambda\in\mathcal{L}$, let $z^{0,\lambda}\in \mathrm{PSC}(\mathbb{R}^2)$ be such that $\partial\Omega^\lambda = \mathrm{im}(z^{0,\lambda})$.

Let $\omega\colon I\to L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ be the (transport?) solution to (1.1)–(1.2) with the initial data ω^0 and the associated flow map $\Phi\in C\left(I; C(\mathbb{R}^2;\mathbb{R}^2)\right)$ provided by Theorem 1.2. Then since each Φ^t is a homeomorphism, let $z^{t,\lambda} := \Phi^t\circ z^{0,\lambda}$ (i.e., the equivalence class in $\mathrm{CC}(\mathbb{R}^2)$ of $\Phi^t\circ\tilde{z}^{0,\lambda}$ for any representative $\tilde{z}^{0,\lambda}$ of $z^{0,\lambda}$) then $\Phi_*^t\Omega$ is clearly composed of simple closed curves. (Since $\left\{z^{t,\lambda}\right\}_{t\in I}$ is a connected subset of $\mathrm{CC}(\mathbb{R}^2)$, [1, Lemma B.4] shows that each $z^{t,\lambda}$ is positively oriented.)

Fix $\varepsilon>0$ and recall $\Phi_\varepsilon^t := \mathrm{Id} + \Psi_\varepsilon^t$ where Ψ_ε^t is the solution to (2.1). Let $z_\varepsilon^{t,\lambda} := \Phi_\varepsilon^t\circ z^{0,\lambda}$ and $\omega_\varepsilon^t := \omega^0\circ(\Phi_\varepsilon^t)^{-1}$ for each $t\in\mathbb{R}$. For given $(t,\lambda)\in\mathbb{R}\times\mathcal{L}$, take any constant-speed parametrization $\tilde{z}_\varepsilon^{t,\lambda}\colon\mathbb{T}\to\mathbb{R}^2$ of $z_\varepsilon^{t,\lambda}$, and let $\tilde{z}_\varepsilon^{t+h,\lambda} := \Phi_\varepsilon^{t+h}\circ(\Phi_\varepsilon^t)^{-1}\circ\tilde{z}_\varepsilon^{t,\lambda}$ for each $h\in\mathbb{R}$. Then since (2.2) is globally well-posed in $C_b^2(\mathbb{R}^2;\mathbb{R}^2)$ (see the remark before Lemma 2.5), it easily follows that $\tilde{z}_\varepsilon^{t,\lambda}\in H^2(\mathbb{T};\mathbb{R}^2)$ and

$$\tilde{z}_\varepsilon^{t+h,\lambda} = \tilde{z}_\varepsilon^{t,\lambda} + \int_t^{t+h} u_\varepsilon(\omega_\varepsilon^\tau)\circ\tilde{z}_\varepsilon^{\tau,\lambda}\,d\tau \tag{3.1}$$

holds for all $h\in\mathbb{R}$, where both sides are in $H^2(\mathbb{T};\mathbb{R}^2)$ and the integral is taken with respect to the norm of $H^2(\mathbb{T};\mathbb{R}^2)$.

For each $t\in\mathbb{R}$, define

- $\ell^{t,\lambda} := \ell(z_\varepsilon^{t,\lambda})$,
- $\mathbf{T}^{t,\lambda}(s) := \partial_s z_\varepsilon^{t,\lambda}(s)$,
- $\mathbf{N}^{t,\lambda}(s) := \mathbf{T}^{t,\lambda}(s)^\perp$,
- $\kappa^{t,\lambda}(s) := \partial_s^2 z_\varepsilon^{t,\lambda}(s)\cdot\mathbf{N}^{t,\lambda}(s)$,
- $u_\varepsilon^{t,\lambda}(s) := u_\varepsilon(\omega_\varepsilon^t; z_\varepsilon^{t,\lambda}(s))$, and
- $\Delta^{t,\lambda,\lambda'} := \Delta(z_\varepsilon^{t,\lambda}, z_\varepsilon^{t,\lambda'})$,

for an arbitrary arclength parametrization of $z_\varepsilon^{t,\lambda}$, which we then fix. As noted in [1, Section 4], we have

$$\partial_s^2 z_\varepsilon^{t,\lambda}(s) = \partial_s \mathbf{T}^{t,\lambda}(s) = \kappa^{t,\lambda}(s)\mathbf{N}^{t,\lambda}(s) \qquad \text{and} \qquad \partial_s \mathbf{N}^{t,\lambda}(s) = -\kappa^{t,\lambda}(s)\mathbf{T}^{t,\lambda}(s).$$

Then a similar argument as in [1, Lemma 4.1] applies, so that

$$\begin{aligned}\partial_t \left\| z_\varepsilon^{t,\lambda} \right\|_{\dot{H}^2}^2 = -3 \int_{\ell^{t,\lambda}\mathbb{T}} \kappa^{t,\lambda}(s)^2 \left(\partial_s u_\varepsilon^{t,\lambda}(s) \cdot \mathbf{T}^{t,\lambda}(s)\right) ds \\ + 2 \int_{\ell^{t,\lambda}\mathbb{T}} \kappa^{t,\lambda}(s) \left(\partial_s^2 u_\varepsilon^{t,\lambda}(s) \cdot \mathbf{N}^{t,\lambda}(s)\right) ds.\end{aligned} \tag{3.2}$$

In a similar manner, we can derive the formula for the time derivative of $\ell^{t,\lambda}$.

Lemma 3.1. *For any $(t,\lambda) \in \mathbb{R} \times \mathcal{L}$,*

$$\partial_t \ell^{t,\lambda} = \int_{\ell^{t,\lambda}\mathbb{T}} \partial_s u_\varepsilon^{t,\lambda}(s) \cdot \mathbf{T}^{t,\lambda}(s)\, ds.$$

Proof. Let $(t,\lambda) \in \mathbb{R} \times \mathcal{L}$, fix a constant-speed parametrization $\tilde{z}^{t,\lambda} \colon \mathbb{T} \to \mathbb{R}^2$ of $z^{t,\lambda}$ and define $\tilde{z}^{\tau,\lambda}$ for each $\tau \in \mathbb{R}$ as in (3.1). Since $\left|\partial_\xi \tilde{z}_\varepsilon^{t,\lambda}(\xi)\right| = \ell(z^{t,\lambda})^{-1} \neq 0$, for any $(h,\xi) \in \mathbb{R} \times \mathbb{T}$, (3.1) shows

$$\begin{aligned}&\left|\partial_\xi \tilde{z}_\varepsilon^{t+h,\lambda}(\xi)\right| - \left|\partial_\xi \tilde{z}_\varepsilon^{t,\lambda}(\xi)\right| \\ &\quad = \frac{2\int_t^{t+h} \partial_\xi u_\varepsilon(\omega_\varepsilon^\tau; \tilde{z}_\varepsilon^{\tau,\lambda}(\xi)) \cdot \partial_\xi \tilde{z}_\varepsilon^{t,\lambda}(\xi)\, d\tau}{\left|\partial_\xi \tilde{z}_\varepsilon^{t+h,\lambda}(\xi)\right| + \left|\partial_\xi \tilde{z}_\varepsilon^{t,\lambda}(\xi)\right|} + \frac{\left|\int_t^{t+h} \partial_\xi u_\varepsilon(\omega_\varepsilon^\tau; \tilde{z}_\varepsilon^{\tau,\lambda}(\xi))\, d\tau\right|^2}{\left|\partial_\xi \tilde{z}_\varepsilon^{t+h,\lambda}(\xi)\right| + \left|\partial_\xi \tilde{z}_\varepsilon^{t,\lambda}(\xi)\right|}.\end{aligned}$$

Since $\left\|D(u_\varepsilon(\omega_\varepsilon^\tau))\right\|_{L^\infty}$ is continuous in τ by Lemma 2.5, integrating over ξ, dividing by h and then letting $h \to 0$ shows

$$\partial_t \ell^{t,\lambda} = \int_{\mathbb{T}} \partial_\xi u_\varepsilon(\omega_\varepsilon^t; \tilde{z}_\varepsilon^{t,\lambda}(\xi)) \cdot \frac{\partial_\xi \tilde{z}_\varepsilon^{t,\lambda}(\xi)}{\left|\partial_\xi \tilde{z}_\varepsilon^{t,\lambda}(\xi)\right|}\, d\xi.$$

Then the claim follows by change of variables. $\square$

Corollary 3.2. *With C_α from Lemma 2.1, for any $(t,\lambda) \in I \times \mathcal{L}$,*

$$e^{-3C_\alpha L(\Phi_*^t\Omega)|h|}\ell(z^{t,\lambda}) \le \ell(z^{t+h,\lambda}) \le e^{3C_\alpha L(\Phi_*^t\Omega)|h|}\ell(z^{t,\lambda})$$

holds for any small enough $h \in \mathbb{R}$.

Proof. Lemmas 2.1, 3.1 and Corollary 2.7 show

$$\left|\partial_t \ell(z_\varepsilon^{t,\lambda})\right| \le \left\|u_\varepsilon(\omega_\varepsilon^t)\right\|_{\dot{C}^{0,1}} \ell(z_\varepsilon^{t,\lambda}) \le 2C_\alpha L(\Omega)\ell(z_\varepsilon^{t,\lambda}) \tag{3.3}$$

for any $t \in \mathbb{R}$ with $|t| \le \frac{1}{2C_\alpha L(\Omega)}$ (such t must be in I by maximality of I). Since $\ell\colon \mathrm{CC}(\mathbb{R}^2) \to [0,\infty]$ is lower semicontinuous (need proof?) and $z_\varepsilon^{t,\lambda} \to z^{t,\lambda}$ in $\mathrm{CC}(\mathbb{R}^2)$ as $\varepsilon \to 0^+$, a Grönwall-type argument shows that

$$\ell(z^{t,\lambda}) \le e^{2C_\alpha L(\Omega)|t|}\ell(z^{0,\lambda}) \tag{3.4}$$

holds for such t.

For any $t_0 \in I$ with $|t_0| \leq \frac{1}{4C_\alpha L(\Omega)}$, considering ω^{t_0} as the initial condition to (1.1)–(1.2) at $t = t_0$ and then repeating the proof of (3.4) shows that (3.4) continues to hold with $(z^{t_0,\lambda}, \Phi_*^{t_0}\Omega, |t - t_0|)$ in place of $(z^{0,\lambda}, \Omega, |t|)$, for $t \in I$ with $|t - t_0| \leq \frac{1}{2C_\alpha L(\Phi_*^{t_0}\Omega)}$. Even though we do not yet know if (3.1) continues to hold in $H^2(\mathbb{T};\mathbb{R}^2)$, since Φ^{t_0} is Lipschitz, all involved curves are rectifiable and (3.1) holds at least in $C^{0,1}(\mathbb{T};\mathbb{R}^2)$, and that is enough for further arguments.

Since $L(\Phi_*^{t_0}\Omega) \leq \frac{4}{3}L(\Omega)$ by Corollary 2.7, we can substitute 0 to t which shows

$$\ell(z^{0,\lambda}) \leq e^{3C_\alpha L(\Omega)|t_0|}\ell(z^{t_0,\lambda}).$$

Therefore,

$$e^{-3C_\alpha L(\Omega)|t|}\ell(z^{0,\lambda}) \leq \ell(z^{t,\lambda}) \leq e^{3C_\alpha L(\Omega)|t|}\ell(z^{0,\lambda})$$

holds for any $t \in \mathbb{R}$ with $|t| \leq \frac{1}{4C_\alpha L(\Omega)}$. Applying this inequality to (ω^t, t) in place of $(\omega^0, 0)$ for $t \in I$ then shows the claim. □

Proposition 3.3. *$R(\Phi_*^t\Omega)$ is continuous in t, so in particular for any compact subinterval $J \subseteq I$,*

$$\sup_{t\in J} R(\Phi_*^t\Omega) < \infty.$$

Proof. Take any compact subinterval $J \subseteq I$ containing 0. With T_0 as in Proposition 2.8, Lemmas 2.1, 2.5 and Corollary 2.7 show

$$\left|\Phi_\varepsilon^{t_1}(x) - \Phi_\varepsilon^{t_1}(y) - \Phi_\varepsilon^{t_2}(x) + \Phi_\varepsilon^{t_2}(y)\right| \leq (e^{2C_\alpha L(\Omega)|t_1-t_2|} - 1)\left|\Phi_\varepsilon^{t_1}(x) - \Phi_\varepsilon^{t_1}(y)\right| \tag{3.5}$$

for all $\varepsilon > 0$, $t_1, t_2 \in [-T_0, T_0]$ and $x, y \in \mathbb{R}^2$. Hence, letting $\varepsilon \to 0^+$ shows that the above continues to hold with Φ in place of Φ_ε. Then applying the same argument to (ω^t, t) in place of $(\omega^0, 0)$ shows that each $t \in J$ admits a neighborhood such that

$$\left|\Phi^{t_1}(x) - \Phi^{t_1}(y) - \Phi^{t_2}(x) + \Phi^{t_2}(y)\right| \leq (e^{2C_\alpha M|t_1-t_2|} - 1)\left|\Phi^{t_1}(x) - \Phi^{t_1}(y)\right| \tag{3.6}$$

holds for any $t_1, t_2 \in J$ in that neighborhood, where

$$M \coloneqq \sup_{t\in J} L(\Phi_{t*}\Omega) < \infty.$$

By compactness, we then conclude that (3.6) in fact holds for all $t_1, t_2 \in J$.

Fix $t_1, t_2 \in J$, $\lambda, \lambda' \in \mathcal{L}$, and $\eta > 0$. Then there are $x \in \operatorname{im}(z^{0,\lambda})$ and $y \in \operatorname{im}(z^{0,\lambda'})$ such that $\Delta(z^{t_1,\lambda}, z^{t_1,\lambda'}) = |\Phi^{t_1}(x) - \Phi^{t_1}(y)|$. Then a similar argument as in (2.4) shows

$$\begin{aligned}
&\frac{1}{\left(\Delta(z^{t_1,\lambda}, z^{t_1,\lambda'}) + \eta\right)^{2\alpha}} - \frac{1}{\left(\Delta(z^{t_2,\lambda}, z^{t_2,\lambda'}) + \eta\right)^{2\alpha}} \\
&\qquad \leq \frac{|\Phi^{t_1}(x) - \Phi^{t_1}(y) - \Phi^{t_2}(x) + \Phi^{t_2}(y)|}{\left(|\Phi^{t_1}(x) - \Phi^{t_1}(y)| + \eta\right)\left(|\Phi^{t_2}(x) - \Phi^{t_2}(y)| + \eta\right)^{2\alpha}} \\
&\qquad \leq \frac{e^{2C_\alpha M|t_1-t_2|} - 1}{(\Delta(z^{t_2,\lambda}, z^{t_2,\lambda'}) + \eta)^{2\alpha}},
\end{aligned}$$

thus

$$\frac{1}{\left(\Delta(z^{t_1,\lambda}, z^{t_1,\lambda'}) + \eta\right)^{2\alpha}} \le \frac{e^{2C_\alpha M|t_1-t_2|}}{\Delta(z^{t_2,\lambda}, z^{t_2,\lambda'})^{2\alpha}}. \tag{3.7}$$

Again by compactness, Corollary 3.2 shows

$$e^{-3C_\alpha M|t_1-t_2|}\ell(z^{t_2,\lambda}) \le \ell(z^{t_1,\lambda}) \le e^{3C_\alpha M|t_1-t_2|}\ell(z^{t_2,\lambda})$$

for any $t_1, t_2 \in J$, thus we get

$$\frac{\ell(z^{t_1,\lambda})^{1/2}}{\ell(z^{t_1,\lambda'})^{1/2}\left(\Delta(z^{t_1,\lambda}, z^{t_1,\lambda'}) + \eta\right)^{2\alpha}} \le \frac{e^{5C_\alpha M|t_1-t_2|}\ell(z^{t_2,\lambda})^{1/2}}{\ell(z^{t_2,\lambda'})^{1/2}\Delta(z^{t_2,\lambda}, z^{t_2,\lambda'})^{2\alpha}}. \tag{3.8}$$

Therefore, letting $\eta \to 0^+$, taking upper Lebesgue integral over λ', and then taking supremum over λ shows

$$R(\Phi_*^{t_1}\Omega) \le e^{5C_\alpha M|t_1-t_2|}R(\Phi_*^{t_2}\Omega). \tag{3.9}$$

Since $t_1, t_2 \in J$, the claim follows. □

Next, we show that $Q(\Phi_*^t\Omega) < \infty$ for t small enough. Since the functional $\gamma \mapsto \ell(\gamma)\left\|\gamma\right\|_{\dot{H}^2}^2$ on $\mathrm{CC}(\mathbb{R}^2)$ is lower semicontinuous, it is enough to establish a uniform-in-(ε, λ) bound on $\ell(z_\varepsilon^{t,\lambda})\left\|z_\varepsilon^{t,\lambda}\right\|_{\dot{H}^2}^2$. We do this by estimating the right-hand side of (3.2) in Lemma 3.7 below. Since the resulting estimate will involve $R(\Phi_{\varepsilon*}^t\Omega)$, we first state a relevant bound on it which we already have obtained in the proof of Proposition 3.3.

Lemma 3.4. *$R(\Phi_{\varepsilon*}^t\Omega)$ is continuous in t, and with C_α from Lemma 2.1 and T_0 from Proposition 2.8,*

$$R(\Phi_{\varepsilon*}^t\Omega) \le e^{4C_\alpha L(\Omega)|t|}R(\Omega) \le 8R(\Omega) \tag{3.10}$$

for any $t \in [-T_0, T_0]$.

Proof. We have (3.3) for $t \in [-T_0, T_0]$, so a Grönwall-type argument shows

$$e^{-2C_\alpha L(\Omega)|t_1-t_2|}\ell(z_\varepsilon^{t_2,\lambda}) \le \ell(z_\varepsilon^{t_1,\lambda}) \le e^{2C_\alpha L(\Omega)|t_1-t_2|}\ell(z_\varepsilon^{t_2,\lambda})$$

for any $t_1, t_2 \in [-T_0, T_0]$ and $\lambda \in \mathcal{L}$. Also, since (3.5) holds, a similar argument as in (3.7) shows

$$\frac{1}{\left(\Delta(z_\varepsilon^{t_1,\lambda}, z_\varepsilon^{t_1,\lambda'}) + \eta\right)^{2\alpha}} \le \frac{e^{2C_\alpha L(\Omega)|t_1-t_2|}}{\Delta(z_\varepsilon^{t_2,\lambda}, z_\varepsilon^{t_2,\lambda'})^{2\alpha}}$$

for any $t_1, t_2 \in [-T_0, T_0]$, $\lambda, \lambda' \in \mathcal{L}$ and $\eta > 0$. Hence, we see that (3.8), and thus (3.9) as well, continue to hold with $(z_\varepsilon, 4C_\alpha L(\Omega), \Phi_\varepsilon)$ in place of $(z, 5C_\alpha M, \Phi)$, which then shows all the claims. (The second inequality of (3.10) follows by the definition of T_0 and the inequality $e^2 \le 8$.) □

To estimate the right-hand side of (3.2), we need the following lemmas.

Lemma 3.5. *There is C_α such that for any $C^{1,\beta}$ closed curve $\gamma\colon \ell\mathbb{T} \to \mathbb{R}^2$ parametrized by arclength and any $x \in \mathbb{R}^2$ we have*

$$\int_{\ell\mathbb{T}} \frac{ds}{\left|x - \gamma(s)\right|^{1+2\alpha}} \le C_\alpha \frac{\ell\left\|\gamma\right\|_{\dot{C}^{1,\beta}}^{1/\beta}}{d(x, \mathrm{im}(\gamma))^{2\alpha}}.$$

Proof. Let $d \coloneqq \frac{1}{4}\|\gamma\|_{\dot{C}^{1,\beta}}^{-1/\beta}$ and $\Delta \coloneqq d(x, \mathrm{im}(\gamma))$. Then [1, Lemma A.2] shows that

$$\int_{\ell\mathbb{T}} \frac{ds}{|x-\gamma(s)|^{1+2\alpha}} \leq \frac{\ell}{4d}\left(\int_{|s|\leq\Delta} \frac{ds}{\Delta^{1+2\alpha}} + \int_{\Delta\leq|s|\leq 2d} \frac{ds}{|s/2|^{1+2\alpha}}\right) + \frac{1}{\Delta^{2\alpha}}\int_{\ell\mathbb{T}} \frac{ds}{d}$$
$$\leq \frac{\ell}{2d\Delta^{2\alpha}} + \frac{\ell}{2^{1-2\alpha}\alpha d\Delta^{2\alpha}} + \frac{\ell}{d\Delta^{2\alpha}} = C_\alpha \frac{\ell\|\gamma\|_{\dot{C}^{1,\beta}}^{-1/\beta}}{\Delta^{2\alpha}}.$$

□

Lemma 3.6. *For any $\omega \in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ with a generalized layer cake representation $(\mathcal{L},\theta,\Omega)$ such that $L(\Omega)<\infty$, $\varepsilon>0$ and $x,h_1,h_2\in\mathbb{R}^2$,*

$$D^2(u_\varepsilon(\omega))(x)(h_1,h_2) = \int_{\mathcal{L}}\int_{\mathbb{R}^2} D^2(\nabla^\perp K_\varepsilon)(x-y)(h_1,h_2)\left(\mathbb{1}_{\Omega^\lambda}(y)-\mathbb{1}_{\Omega^\lambda}(x)\right)dy\,d\theta(\lambda)$$
$$= \int_{\mathcal{L}}\int_{\Omega^\lambda} D^2(\nabla^\perp K_\varepsilon)(x-y)(h_1,h_2)\,dy\,d\theta(\lambda).$$

Furthermore, there is C_α that depends only on α such that

$$\int_{\mathcal{L}}\int_{\mathbb{R}^2} \left|D^2(\nabla^\perp K_\varepsilon)(x-y)\right|\left|\mathbb{1}_{\Omega^\lambda}(y)-\mathbb{1}_{\Omega^\lambda}(x)\right| d|\theta|(\lambda)\,dy \leq \frac{C_\alpha}{\varepsilon}\int_{\mathcal{L}} \frac{d|\theta|(\lambda)}{d(x,\partial\Omega^\lambda)^{2\alpha}}.$$

Proof. For each $x,h_1,h_2\in\mathbb{R}^2$, oddness of $D^2(\nabla^\perp K_\varepsilon)$ shows

$$D^2(u_\varepsilon(\omega))(x)(h_1,h_2) = \int_{\mathbb{R}^2} D^2(\nabla^\perp K_\varepsilon)(x-y)(h_1,h_2)(\omega(y)-\omega(x))\,dy$$
$$= \int_{\mathbb{R}^2}\int_{\mathcal{L}} D^2(\nabla^\perp K_\varepsilon)(x-y)(h_1,h_2)\left(\mathbb{1}_{\Omega^\lambda}(y)-\mathbb{1}_{\Omega^\lambda}(x)\right)d\theta(\lambda)\,dy.$$

Then proceeding as in Lemma 2.1 with using the inequality $\left|D^2(\nabla^\perp K_\varepsilon)(x-y)\right| \leq \frac{C_\alpha}{\varepsilon|x-y|^{2+2\alpha}}$ in place of $\left|D(\nabla^\perp K_\varepsilon)(x-y)\right| \leq \frac{C_\alpha}{|x-y|^{2+2\alpha}}$ shows the second claim. Then Fubini's theorem shows the first equality of the first claim, and the second equality follows by oddness of $D^2(\nabla^\perp K_\varepsilon)$. □

Lemma 3.7. *There is C_α such that for each $(t,\lambda)\in\mathbb{R}\times\mathcal{L}$,*

$$\left|\partial_t \left\|z_\varepsilon^{t,\lambda}\right\|_{\dot{H}^2}^2\right| \leq C_\alpha(L(\Phi_{\varepsilon*}^t\Omega)+R(\Phi_{\varepsilon*}^t\Omega))Q(\Phi_{\varepsilon*}^t\Omega)\left\|z_\varepsilon^{t,\lambda}\right\|_{\dot{H}^2}^2 \tag{3.11}$$

Proof. All constants C_α in this proof depend only on α and can change from one inequality to another. With $z_\varepsilon^\lambda(\cdot)$ being the previously fixed arclength parametrization of z_ε^λ, we have

$$\partial_s^2 u_\varepsilon^{t,\lambda}(s) = D^2(u_\varepsilon(\omega_\varepsilon^t))(z_\varepsilon^{t,\lambda}(s))(\mathbf{T}^{t,\lambda}(s),\mathbf{T}^{t,\lambda}(s)) + \kappa^{t,\lambda}(s)D(u_\varepsilon(\omega_\varepsilon^t))(z_\varepsilon^{t,\lambda}(s))(\mathbf{N}^{t,\lambda}(s)),$$

so (3.2) shows

$$\partial_t\left\|z_\varepsilon^{t,\lambda}\right\|_{\dot{H}^2}^2 = \int_{\ell^{t,\lambda}\mathbb{T}} \kappa^{t,\lambda}(s)^2\left(2D(u_\varepsilon(\omega_\varepsilon^t))(z_\varepsilon^{t,\lambda}(s))(\mathbf{N}^{t,\lambda}(s))\cdot\mathbf{N}^{t,\lambda}(s) - 3\,\partial_s u_\varepsilon^{t,\lambda}(s)\cdot\mathbf{T}^{t,\lambda}(s)\right)ds$$
$$+ 2\int_{\ell^{t,\lambda}\mathbb{T}} \kappa^{t,\lambda}(s)\left(D^2(u_\varepsilon(\omega_\varepsilon^t))(z_\varepsilon^{t,\lambda}(s))(\mathbf{T}^{t,\lambda}(s),\mathbf{T}^{t,\lambda}(s))\cdot\mathbf{N}^{t,\lambda}(s)\right)ds. \tag{3.12}$$

Clearly, Lemma 2.1 shows the absolute value of the first integral is bounded by

$$5\left\|u_\varepsilon(\omega_\varepsilon^t)\right\|_{\dot C^{0,1}}\left\|z_\varepsilon^{t,\lambda}\right\|_{\dot H^2}^2 \le 5C_\alpha L(\Phi_{\varepsilon*}^t\Omega)\left\|z_\varepsilon^{t,\lambda}\right\|_{\dot H^2}^2 \le 5C_\alpha L(\Phi_{\varepsilon*}^t\Omega)Q(\Phi_{\varepsilon*}^t\Omega)\left\|z_\varepsilon^{t,\lambda}\right\|_{\dot H^2}^2$$

where the second inequality is because $Q(\Phi_{\varepsilon*}^t\Omega) \ge 4$ which follows from [1, Lemma A.1]. Hence, it remains to estimate the second integral, which we now denote as G_1. From now on, we will suppress t from the notation for sake of simplicity because we will only work with the snapshot at t of every involved time-dependent quantity.

Since $L(\Phi_{\varepsilon*}\Omega) < \infty$, Lemma 3.6 and Green's theorem show that

$$G_1 = -\int_{\mathcal L}\int_{\ell^\lambda\mathbb T\times\ell^{\lambda'}\mathbb T}\kappa^\lambda(s)D^2K_\varepsilon(z_\varepsilon^\lambda(s)-z_\varepsilon^{\lambda'}(s'))(\mathbf T^\lambda(s),\mathbf T^\lambda(s))(\mathbf T^{\lambda'}(s')\cdot\mathbf N^\lambda(s))\,ds'\,ds\,d\theta(\lambda').$$

Then using

$$\mathbf T^\lambda(s) = (\mathbf T^{\lambda'}(s')\cdot\mathbf T^\lambda(s))\mathbf T^{\lambda'}(s') + (\mathbf N^{\lambda'}(s')\cdot\mathbf T^\lambda(s))\mathbf N^{\lambda'}(s')$$

and $\mathbf N^{\lambda'}(s')\cdot\mathbf T^\lambda(s) = -\mathbf T^{\lambda'}(s')\cdot\mathbf N^\lambda(s)$, it follows that $|G_1|$ is bounded by the sum of the terms

$$\begin{aligned}G_2 \coloneqq \overline{\int_{\mathcal L}}\Bigg|\int_{\ell^\lambda\mathbb T\times\ell^{\lambda'}\mathbb T}&\kappa^\lambda(s)D^2K_\varepsilon(z_\varepsilon^\lambda(s)-z_\varepsilon^{\lambda'}(s'))(\mathbf T^\lambda(s),\mathbf T^{\lambda'}(s'))\\ &(\mathbf T^{\lambda'}(s')\cdot\mathbf N^\lambda(s))(\mathbf T^{\lambda'}(s')\cdot\mathbf T^\lambda(s))\,ds'\,ds\Bigg|\,d|\theta|(\lambda'),\end{aligned}$$

$$G_3 \coloneqq \overline{\int_{\mathcal L}}\Bigg|\int_{\ell^\lambda\mathbb T\times\ell^{\lambda'}\mathbb T}\kappa^\lambda(s)D^2K_\varepsilon(z_\varepsilon^\lambda(s)-z_\varepsilon^{\lambda'}(s'))(\mathbf T^\lambda(s),\mathbf N^{\lambda'}(s'))(\mathbf T^{\lambda'}(s')\cdot\mathbf N^\lambda(s))^2\,ds'\,ds\Bigg|\,d|\theta|(\lambda')$$

which we now estimate separately.

Estimate for G_2. Since

$$\begin{aligned}&\frac{\partial}{\partial s'}\Big(DK_\varepsilon(z_\varepsilon^\lambda(s)-z_\varepsilon^{\lambda'}(s'))(\mathbf T^\lambda(s))(\mathbf T^{\lambda'}(s')\cdot\mathbf N^\lambda(s))(\mathbf T^{\lambda'}(s')\cdot\mathbf T^\lambda(s))\Big)\\ &\quad= -D^2K_\varepsilon(z_\varepsilon^\lambda(s)-z_\varepsilon^{\lambda'}(s'))(\mathbf T^\lambda(s),\mathbf T^{\lambda'}(s'))(\mathbf T^{\lambda'}(s')\cdot\mathbf N^\lambda(s))(\mathbf T^{\lambda'}(s')\cdot\mathbf T^\lambda(s))\\ &\qquad+\kappa^{\lambda'}(s')DK_\varepsilon(z_\varepsilon^\lambda(s)-z_\varepsilon^{\lambda'}(s'))\Big((\mathbf T^{\lambda'}(s')\cdot\mathbf T^\lambda(s))^2-(\mathbf T^{\lambda'}(s')\cdot\mathbf N^\lambda(s))^2\Big),\end{aligned}$$

we see that

$$G_2 \le C_\alpha\overline{\int_{\mathcal L}}\int_{\ell^\lambda\mathbb T\times\ell^{\lambda'}\mathbb T}\frac{\left|\kappa^\lambda(s)\kappa^{\lambda'}(s')\right|}{\left|z_\varepsilon^\lambda(s)-z_\varepsilon^{\lambda'}(s')\right|^{1+2\alpha}}\,ds'\,ds\,d|\theta|(\lambda').$$

By Cauchy-Schwarz inequality, the inner integral of the right-hand side is bounded by

$$\left(\int_{\ell^\lambda\mathbb T}\kappa^\lambda(s)^2\int_{\ell^{\lambda'}\mathbb T}\frac{ds'}{\left|z_\varepsilon^\lambda(s)-z_\varepsilon^{\lambda'}(s')\right|^{1+2\alpha}}\,ds\right)^{1/2}\left(\int_{\ell^{\lambda'}\mathbb T}\kappa^{\lambda'}(s')^2\int_{\ell^\lambda\mathbb T}\frac{ds}{\left|z_\varepsilon^\lambda(s)-z_\varepsilon^{\lambda'}(s')\right|^{1+2\alpha}}\,ds'\right)^{1/2}$$

which, by Lemma 3.5, is bounded by

$$C_\alpha\left(\left\|z_\varepsilon^\lambda\right\|_{\dot H^2}^2\frac{\ell^{\lambda'}\left\|z_\varepsilon^{\lambda'}\right\|_{\dot H^2}^2}{(\Delta^{\lambda,\lambda'})^{2\alpha}}\right)^{1/2}\left(\left\|z_\varepsilon^{\lambda'}\right\|_{\dot H^2}^2\frac{\ell^\lambda\left\|z_\varepsilon^\lambda\right\|_{\dot H^2}^2}{(\Delta^{\lambda,\lambda'})^{2\alpha}}\right)^{1/2} = C_\alpha\left\|z_\varepsilon^\lambda\right\|_{\dot H^2}^2\cdot\frac{\ell^{\lambda'}\left\|z_\varepsilon^{\lambda'}\right\|_{\dot H^2}^2\cdot(\ell^\lambda)^{1/2}}{(\ell^{\lambda'})^{1/2}(\Delta^{\lambda,\lambda'})^{2\alpha}}.$$

Therefore, we obtain

$$|G_2| \leq C_\alpha R(\Phi_{\varepsilon*}\Omega)Q(\Phi_{\varepsilon*}\Omega)\left\|z_\varepsilon^\lambda\right\|_{\dot{H}^2}^2.$$

Estimate for G_3. We can assume $R(\Phi_{\varepsilon*}\Omega)<\infty$ in which case $\Delta(z_\varepsilon^\lambda, z_\varepsilon^{\lambda'})>0$ for $|\theta|$-almost every λ', thus we can apply [1, Lemma A.4] to conclude

$$G_3 \leq C_\alpha \overline{\int_{\mathcal{L}}}\int_{\ell^\lambda\mathbb{T}\times\ell^{\lambda'}\mathbb{T}} \frac{\left|\kappa^\lambda(s)\right|\left(\mathcal{M}\kappa^\lambda(s)+\mathcal{M}\kappa^{\lambda'}(s')\right)}{\left|z_\varepsilon^\lambda(s)-z_\varepsilon^{\lambda'}(s')\right|^{1+2\alpha}}\,ds'\,ds\,d|\theta|(\lambda')$$

where $\mathcal{M}$ is the maximal operator defined in [1, (A.2)]. Split the integrand into the sum of two terms with denominators $\left|\kappa^\lambda(s)\right|\mathcal{M}\kappa^\lambda(s)$ and $\left|\kappa^\lambda(s)\right|\mathcal{M}\kappa^{\lambda'}(s')$, respectively, then by the maximal inequality ([1, (A.3)]), we can apply the same argument as in the estimate for G_2 to bound the integral of the second term by $C_\alpha R(\Phi_{\varepsilon*}\Omega)Q(\Phi_{\varepsilon*}\Omega)\left\|z_\varepsilon^\lambda\right\|_{\dot{H}^2}$. For the first term, Lemma 3.5, Cauchy-Schwarz inequality and the maximal inequality show

$$\begin{aligned}
&\overline{\int_{\mathcal{L}}}\int_{\ell^\lambda\mathbb{T}\times\ell^{\lambda'}\mathbb{T}} \frac{\left|\kappa^\lambda(s)\right|\mathcal{M}\kappa^\lambda(s)}{\left|z_\varepsilon^\lambda(s)-z_\varepsilon^{\lambda'}(s')\right|^{1+2\alpha}}\,ds'\,ds\,d|\theta|(\lambda')\\
&\quad\leq \overline{\int_{\mathcal{L}}}\int_{\ell^\lambda\mathbb{T}}\left|\kappa^\lambda(s)\right|\mathcal{M}\kappa^\lambda(s)\frac{C_\alpha\ell^{\lambda'}\left\|z_\varepsilon^{\lambda'}\right\|_{\dot{H}^2}^2}{d(z_\varepsilon^\lambda(s),\operatorname{im}(z_\varepsilon^{\lambda'}))^{2\alpha}}\,ds\,d|\theta|(\lambda')\\
&\quad\leq C_\alpha Q(\Phi_{\varepsilon*}\Omega)\int_{\mathcal{L}}\int_{\ell^\lambda\mathbb{T}}\frac{\left|\kappa^\lambda(s)\right|\mathcal{M}\kappa^\lambda(s)}{d(z_\varepsilon^\lambda(s),\operatorname{im}(z_\varepsilon^{\lambda'}))^{2\alpha}}\,ds\,d|\theta|(\lambda')\\
&\quad\leq C_\alpha L(\Phi_{\varepsilon*}\Omega)Q(\Phi_{\varepsilon*}\Omega)\int_{\ell^\lambda\mathbb{T}}\left|\kappa^\lambda(s)\right|\mathcal{M}\kappa^\lambda(s)\,ds \leq C_\alpha L(\Phi_{\varepsilon*}\Omega)Q(\Phi_{\varepsilon*}\Omega)\left\|z_\varepsilon^\lambda\right\|_{\dot{H}^2}^2,
\end{aligned}$$

where the reason why we have the plain integral rather than the upper Lebesgue integral in the right-hand side of the second inequality is because the integrand is jointly measurable in (s,λ'). Aggregating the estimates for G_2 and G_3 now yields the desired conclusion. □

Lemmas 3.1 and 3.7 suggest that we may have an ε-independent estimate

$$\max\left\{\partial_t^+Q(\Phi_{\varepsilon*}^t\Omega), -\partial_{t-}Q(\Phi_{\varepsilon*}^t\Omega)\right\} \leq C_\alpha(L(\Phi_{\varepsilon*}^t\Omega)+R(\Phi_{\varepsilon*}^t\Omega))Q(\Phi_{\varepsilon*}^t\Omega)^2 \tag{3.13}$$

from which we can run a Grönwall-type argument. However, because of the factor $Q(\Phi_{\varepsilon*}^t\Omega)$ in the right-hand side of (3.11), (3.13) follows from these Lemmas only if we know a priori that $Q(\Phi_{\varepsilon*}^t\Omega)$ is upper semicontinuous (or locally bounded, which implies upper semicontinuity via (3.11)).

One way to derive upper semicontinuity (or local boundedness) would be again through estimating the right-hand side of (3.2) in terms of $\left\|z_\varepsilon^{t,\lambda}\right\|_{\dot{H}^2}^2$. In contrast to (3.11), such an estimate can depend on ε while it must not have the factor $Q(\Phi_{\varepsilon*}^t\Omega)$. However, the second integral of (3.12) contains only one factor of $\kappa^{t,\lambda}(s)$, so the trivial bound we obtain by estimating $D^2(u_\varepsilon(\omega_\varepsilon^t))$ by its L^∞ norm will contain the L^1 norm of $\kappa^{t,\lambda}$, and in order to replace it by the L^2 norm we have to introduce an additional factor of $(\ell^{t,\lambda})^{1/2}$. This means that unless we impose an upper on $\ell^{t,\lambda}$, the resulting estimate will not be enough for concluding upper semicontinuity (nor local boundedness), because it does not rule out the possibility that over an arbitrarily small time interval and for an arbitrarily large constant

M, some $\ell^{t,\lambda}\left\|z_\varepsilon^{t,\lambda}\right\|_{\dot H^2}^2$ with very large $\ell^{t,\lambda}$ grows larger than M. Another way would be to estimate the $\dot H^2$ norm of $\Phi_\varepsilon^t\circ\tilde z^{0,\lambda}$ where $\tilde z^{0,\lambda}\colon\mathbb{T}\to\mathbb{R}^2$ is a constant-speed parametrization of $z^{0,\lambda}$, but it runs into a similar issue.

To resolve this, we now introduce a sequence of approximations of ω_ε^t. Take an increasing sequence $\{\mathcal{L}'_N\}_{N=1}^\infty$ of measurable subsets of $\mathcal{L}$ such that $|\theta|(\mathcal{L}'_N)<\infty$ for each $N\in\mathbb{N}$ and $\mathcal{L}=\bigcup_{N=1}^\infty\mathcal{L}'_N$. For each $N\in\mathbb{N}$, let

$$\mathcal{L}_N:=\left\{\lambda\in\mathcal{L}'_N\colon\ell(z^{0,\lambda})\le N\right\},\quad \Omega_N:=\Omega\cap(\mathbb{R}^2\times\mathcal{L}_N),\quad \omega_N^0(x):=\int_{\mathcal{L}_N}\mathbb{1}_{\Omega^\lambda}(x)\,d\theta(\lambda).$$

Note that $\omega_N^0\in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ because $\|\omega_N^0\|_{L^\infty}\le|\theta|(\mathcal{L}'_N)<\infty$ and also $\|\omega_N^0\|_{L^1}\le\frac{N^2}{4\pi}|\theta|(\mathcal{L}'_N)<\infty$ by the isoperimetric inequality. Clearly, $L(\Omega_N)\le L(\Omega)<\infty$.

Let $\Phi_{\varepsilon,N}\in C(\mathbb{R};C(\mathbb{R}^2;\mathbb{R}^2))$ be the corresponding ε-mollified flow map, i.e., the identity map plus the solution to (2.1) with ω_N^0 in place of ω^0. Let $z_{\varepsilon,N}^{t,\lambda}:=\Phi_{\varepsilon,N}^t\circ z^{0,\lambda}$ and $\omega_{\varepsilon,N}^t:=\omega_N^0\circ(\Phi_{\varepsilon,N}^t)^{-1}$.

Then (3.12), Cauchy-Schwarz inequality, the inequality $\ell(\gamma)\,\|\gamma\|_{\dot H^2}^2\ge 4$ for any $\gamma\in\mathrm{CC}(\mathbb{R}^2)$ (which follows by [1, Lemma A.1]) and (1.8) show

$$\begin{aligned}\left|\partial_t\left\|z_{\varepsilon,N}^{t,\lambda}\right\|_{\dot H^2}^2\right|&\le 5\left\|u_\varepsilon(\omega_{\varepsilon,N}^t)\right\|_{\dot C^{0,1}}\left\|z_{\varepsilon,N}^{t,\lambda}\right\|_{\dot H^2}^2+2\left\|D^2(u_\varepsilon(\omega_{\varepsilon,N}^t))\right\|_{L^\infty}\ell(z_{\varepsilon,N}^{t,\lambda})^{1/2}\left\|z_{\varepsilon,N}^{t,\lambda}\right\|_{\dot H^2}\\&\le\left(5\left\|D(\nabla^\perp K_\varepsilon)\right\|_{L^\infty}+\left\|D^2(\nabla^\perp K_\varepsilon)\right\|_{L^\infty}N\right)\left\|\omega_N^0\right\|_{L^1}\left\|z_{\varepsilon,N}^{t,\lambda}\right\|_{\dot H^2}^2,\end{aligned}$$

which implies

$$\left\|z_{\varepsilon,N}^{t+h,\lambda}\right\|_{\dot H^2}^2\le e^{C|h|}\left\|z_{\varepsilon,N}^{t,\lambda}\right\|_{\dot H^2}^2$$

for each $(t,h,\lambda)\in\mathbb{R}^2\times\mathcal{L}_N$, for some constant C that depends on $\alpha,\varepsilon,N,\|\omega_N^0\|_{L^1}$, but not on $\lambda\in\mathcal{L}_N$. Together with the inequality $\ell(z_{\varepsilon,N}^{t+h,\lambda})\le e^{C|h|}\ell(z_{\varepsilon,N}^{t,\lambda})$ for another C that depends on $\alpha,\varepsilon,\|\omega_N^0\|_{L^1}$, which follows from (3.3) (with $(\omega_{\varepsilon,N},z_{\varepsilon,N})$ in place of $(\omega_\varepsilon,z_\varepsilon)$) and (1.8), this shows

$$Q(\Phi_{\varepsilon,N*}^{t+h}\Omega)\le e^{C|h|}Q(\Phi_{\varepsilon,N*}^t\Omega),$$

from which we conclude upper semicontinuity of $Q(\Phi_{\varepsilon,N*}^t\Omega)$ in t.

From Lemmas 3.1, 3.4, 3.7, Corollary 2.7, and (3.3) with $(\omega_{\varepsilon,N},\Omega_N,z_{\varepsilon,N},\Phi_{\varepsilon,N})$ in place of $(\omega_\varepsilon,\Omega,z_\varepsilon,\Phi_\varepsilon)$, and the inequalities $L(\Omega_N)\le L(\Omega)$, $R(\Omega_N)\le R(\Omega)$, and $Q(\Phi_{\varepsilon,N*}^t\Omega)\ge 4$, we see that

$$\begin{aligned}\ell(z_{\varepsilon,N}^{t+h,\lambda})\left\|z_{\varepsilon,N}^{t+h,\lambda}\right\|_{\dot H^2}^2-Q(\Phi_{\varepsilon,N*}^t\Omega)&\le\ell(z_{\varepsilon,N}^{t+h,\lambda})\left\|z_{\varepsilon,N}^{t+h,\lambda}\right\|_{\dot H^2}^2-\ell(z_{\varepsilon,N}^{t,\lambda})\left\|z_{\varepsilon,N}^{t,\lambda}\right\|_{\dot H^2}^2\\&\le C_\alpha\left|\int_t^{t+h}(L(\Omega)+R(\Omega))Q(\Phi_{\varepsilon,N*}^\tau\Omega)^2\,d\tau\right|\end{aligned}$$

holds for each $(t,t+h,\lambda)\in[-T_0,T_0]^2\times\mathcal{L}_N$, for some C_α that depends only on α. By upper semicontinuity of $Q(\Phi_{\varepsilon,N*}^t\Omega)$, taking supremum over $\lambda\in\mathcal{L}_N$, dividing by $|h|$ and then letting

$h \to 0$ yields

$$\max\left\{\partial_t^+ Q(\Phi^t_{\varepsilon,N*}\Omega), -\partial_{t-} Q(\Phi^t_{\varepsilon,N*}\Omega)\right\} \le C_\alpha(L(\Omega)+R(\Omega))Q(\Phi^t_{\varepsilon,N*}\Omega)^2,$$

which we then use with a Grönwall-type argument to conclude

$$Q(\Phi^t_{\varepsilon,N*}\Omega) \le \frac{Q(\Omega_N)}{1-C_\alpha(L(\Omega)+R(\Omega))Q(\Omega_N)\,|t|} \le \frac{Q(\Omega)}{1-C_\alpha(L(\Omega)+R(\Omega))Q(\Omega)\,|t|} \tag{3.14}$$

for any t with $|t| < \frac{1}{C_\alpha(L(\Omega)+R(\Omega))Q(\Omega)}$, where the second inequality is because $Q(\Omega_N) \le Q(\Omega)$.

In order to turn (3.14) into a bound on $Q(\Phi^t_*\Omega)$, we need to know if $z_{\varepsilon,N}$ converges to z_ε when $N \to \infty$. This indeed happens if $\omega^0_N \to \omega^0$ in L^1 (which also implies the convergence in L^∞ since $\|\omega^0_N\|_{\dot{C}^{0,2\alpha}}$ is uniformly bounded), but our assumptions on Ω alone do not seem to imply the L^1 convergence. Note that if Ω^λ's for λ's in the positive part of θ are disjoint from those for λ's in the negative part of θ, in particular if Ω^λ's are the suprelevel sets and sublevel sets of $(\omega^0)^+$ and $(\omega^0)^-$, respectively, then L^1 convergence does follow. However, we are allowing these "positive level sets" and "negative level sets" to sit on top of each other, possibly creating "ripples" of the graph of ω^0.

Even though we do not have the L^1 convergence, we can still show that the difference between $z_{\varepsilon,N}$ and z_ε is almost constant, although that constant may vary as N varies, as the following lemma shows.

Lemma 3.8. *Let T_0 be as in Proposition 2.8. Then for any $R \ge 0$ and $x_0 \in \mathbb{R}^2$,*

$$\lim_{N\to\infty} \sup_{t\in[-T_0,T_0]} \left\| \left(\Phi^t_{\varepsilon,N} - \Phi^t_\varepsilon - \Phi^t_{\varepsilon,N}(x_0) + \Phi^t_\varepsilon(x_0)\right)\big|_{B_R(x_0)} \right\|_{L^\infty} = 0.$$

Once this is shown, for each $\lambda \in \mathcal{L}$, we pick any constant-speed parametrization $\tilde{z}^{0,\lambda}\colon \mathbb{T} \to \mathbb{R}^2$ of $z^{0,\lambda}$, then

$$\begin{aligned}
&\left\|\Phi^t_{\varepsilon,N}\circ\tilde{z}^{0,\lambda} - \Phi^t_\varepsilon\circ\tilde{z}^{0,\lambda} - \Phi^t_{\varepsilon,N}\circ\tilde{z}^{0,\lambda}(0) + \Phi^t_\varepsilon\circ\tilde{z}^{0,\lambda}(0)\right\|_{L^\infty} \\
&\quad\le \left\| \left(\Phi^t_{\varepsilon,N} - \Phi^t_\varepsilon - \Phi^t_{\varepsilon,N}(\tilde{z}^{0,\lambda}(0)) + \Phi^t_\varepsilon(\tilde{z}^{0,\lambda}(0))\right)\big|_{B_{\ell(z^{0,\lambda})/2}(\tilde{z}^{0,\lambda}(0))} \right\|_{L^\infty} \to 0
\end{aligned}$$

as $N \to \infty$, for each $t \in \mathbb{R}$. This shows that $w^{t,\lambda}_{\varepsilon,N} := z^{t,\lambda}_{\varepsilon,N} - \Phi^t_{\varepsilon,N}\circ\tilde{z}^{0,\lambda}(0) + \Phi^t_\varepsilon\circ\tilde{z}^{0,\lambda}(0) \to z^{t,\lambda}_\varepsilon$ in $\mathrm{CC}(\mathbb{R}^2)$ as $N \to \infty$. Since $w^{t,\lambda}_{\varepsilon,N}$ is just a spatial translation of $z^{t,\lambda}_{\varepsilon,N}$, we have $\ell(w^{t,\lambda}_{\varepsilon,N}) = \ell(z^{t,\lambda}_{\varepsilon,N})$ and $\left\|w^{t,\lambda}_{\varepsilon,N}\right\|_{\dot{H}^2} = \left\|z^{t,\lambda}_{\varepsilon,N}\right\|_{\dot{H}^2}$, so the lower semicontinuity of the functional $\gamma \mapsto \ell(\gamma)\,\|\gamma\|^2_{\dot{H}^2}$ on $\mathrm{CC}(\mathbb{R}^2)$ shows

$$\begin{aligned}
\ell(z^{t,\lambda}_\varepsilon)\left\|z^{t,\lambda}_\varepsilon\right\|^2_{\dot{H}^2} &\le \liminf_{N\to\infty} \ell(w^{t,\lambda}_{\varepsilon,N})\left\|w^{t,\lambda}_{\varepsilon,N}\right\|^2_{\dot{H}^2} = \liminf_{N\to\infty} \ell(z^{t,\lambda}_{\varepsilon,N})\left\|z^{t,\lambda}_{\varepsilon,N}\right\|^2_{\dot{H}^2} \\
&\le \frac{Q(\Omega)}{1-C_\alpha(L(\Omega)+R(\Omega))Q(\Omega)\,|t|}
\end{aligned}$$

for any t with $|t| < \frac{1}{C_\alpha(L(\Omega)+R(\Omega))Q(\Omega)}$. Then, again by lower semicontinuity of $\gamma \mapsto \ell(\gamma)\left\|\gamma\right\|_{\dot{H}^2}^2$, letting $\varepsilon \to 0^+$ yields

$$\ell(z^{t,\lambda})\left\|z^{t,\lambda}\right\|_{\dot{H}^2}^2 \le \frac{Q(\Omega)}{1 - C_\alpha(L(\Omega)+R(\Omega))Q(\Omega)\,|t|},$$

so taking supremum over λ finally gives

$$Q(\Phi_*^t\Omega) \le \frac{Q(\Omega)}{1 - C_\alpha(L(\Omega)+R(\Omega))Q(\Omega)\,|t|},$$

from which Theorem 1.4 follows.

Now it remains to prove Lemma 3.8.

Proof of Lemma 3.8. In this proof, all constants written as C with subscripts can change from one inequality to another, and they depend only on the indicated variables; e.g., C_α depends only on α and $C_{\alpha,\varepsilon}$ depends only on α and ε.

First, note that the inequality $L(\Omega_N) \le L(\Omega)$ and Proposition 2.8 show

$$\sup_{t\in[-T_0,T_0]} \max\left\{\left\|\Phi_\varepsilon^t\right\|_{\dot{C}^{0,1}}, \left\|(\Phi_\varepsilon^t)^{-1}\right\|_{\dot{C}^{0,1}}, \left\|\Phi_{\varepsilon,N}^t\right\|_{\dot{C}^{0,1}}, \left\|(\Phi_{\varepsilon,N}^t)^{-1}\right\|_{\dot{C}^{0,1}}\right\} \le 3. \tag{3.15}$$

Let $\delta > 0$ be given, then since $\omega^0 \in L^1(\mathbb{R}^2)$, we can find $R_\delta \ge R$ such that

$$\int_{\mathbb{R}^2\setminus B_{R_\delta}(x_0)} \left|\omega^0(y)\right| dy \le \delta. \tag{3.16}$$

For given $t \in [-T_0, T_0]$ and $x \in B_{R_\delta}(x_0)$ we can write

$$\partial_t\left(D\Phi_{\varepsilon,N}^t(x) - D\Phi_\varepsilon^t(x)\right) = D(u_\varepsilon(\omega_{\varepsilon,N}^t))(\Phi_{\varepsilon,N}^t(x))D\Phi_{\varepsilon,N}^t(x) - D(u_\varepsilon(\omega_\varepsilon^t))(\Phi_\varepsilon^t(x))D\Phi_\varepsilon^t(x)$$

as the sum of the terms

$$\begin{aligned}
V_1 &\coloneqq D(u_\varepsilon(\omega_{\varepsilon,N}^t))(\Phi_{\varepsilon,N}^t(x))\left(D\Phi_{\varepsilon,N}^t(x) - D\Phi_\varepsilon^t(x)\right),\\
V_2 &\coloneqq \left[D(u_\varepsilon(\Phi_{\varepsilon,N*}^t\omega_N^0)) - D(u_\varepsilon(\Phi_{\varepsilon,N*}^t\omega^0))\right](\Phi_{\varepsilon,N}^t(x))D\Phi_\varepsilon^t(x),\\
V_3 &\coloneqq \left[D(u_\varepsilon(\Phi_{\varepsilon,N*}^t\omega^0))(\Phi_{\varepsilon,N}^t(x)) - D(u_\varepsilon(\Phi_{\varepsilon*}^t\omega^0))(\Phi_\varepsilon^t(x))\right]D\Phi_\varepsilon^t(x),
\end{aligned}$$

which we estimate separately.

Estimate for V_1. Clearly, Lemma 2.1, (1.7), (3.15) and the inequality $L(\Omega_N) \le L(\Omega)$ show

$$\begin{aligned}
|V_1| &\le \left\|u_\varepsilon(\omega_{\varepsilon,N}^t)\right\|_{\dot{C}^{0,1}}\left|D\Phi_{\varepsilon,N}^t(x) - D\Phi_\varepsilon^t(x)\right| \le C_\alpha L(\Phi_{\varepsilon,N*}^t\Omega_N)\left|D\Phi_{\varepsilon,N}^t(x) - D\Phi_\varepsilon^t(x)\right|\\
&\le C_\alpha L(\Omega)\left\|\left(D\Phi_{\varepsilon,N}^t - D\Phi_\varepsilon^t\right)\big|_{B_{R_\delta}(x_0)}\right\|_{L^\infty}.
\end{aligned}$$

Estimate for V_2. Since

$$(\omega^0 - \omega_N^0)(x') = \int_{\mathcal{L}\setminus\mathcal{L}_N} \mathbb{1}_{\Omega^\lambda}(x')\,d\theta(\lambda)$$

is a generalized layer cake representation of $\omega^0-\omega^0_N$, Lemma 2.1, (1.7) and (3.16) show

$$\begin{aligned}\left|D\left(u_\varepsilon\left(\Phi^t_{\varepsilon,N*}(\omega^0_N-\omega^0)\right)\right)\left(\Phi^t_{\varepsilon,N}(x')\right)\right| &\le C_\alpha\int_{\mathcal{L}\setminus\mathcal{L}_N}\frac{d|\theta|(\lambda)}{d(\Phi^t_{\varepsilon,N}(x'),\Phi^t_{\varepsilon,N}(\partial\Omega^\lambda))^{2\alpha}}\\ &\le C_\alpha\left\|(\Phi^t_{\varepsilon,N})^{-1}\right\|^{2\alpha}_{\dot C^{0,1}}\int_{\mathcal{L}\setminus\mathcal{L}_N}\frac{d|\theta|(\lambda)}{d(x',\partial\Omega^\lambda)^{2\alpha}}\to 0\end{aligned}$$

as $N\to\infty$ for each $x'\in\mathbb{R}^2$. On the other hand, Lemma 3.6, (1.7) and (3.15) show

$$\begin{aligned}\left\|D\left(D\left(u_\varepsilon\left(\Phi^t_{\varepsilon,N*}(\omega^0_N-\omega^0)\right)\right)\circ\Phi^t_{\varepsilon,N}\right)\right\|_{L^\infty} &\le\left\|D^2\left(u_\varepsilon\left(\Phi^t_{\varepsilon,N*}(\omega^0_N-\omega^0)\right)\right)\right\|_{L^\infty}\left\|D\Phi^t_{\varepsilon,N}\right\|_{L^\infty}\\ &\le\frac{C_\alpha}{\varepsilon}\left\|\Phi^t_{\varepsilon,N}\right\|_{\dot C^{0,1}}\sup_{x'\in\mathbb{R}^2}\int_{\mathcal{L}\setminus\mathcal{L}_N}\frac{d|\theta|(\lambda)}{d(x',\Phi^t_{\varepsilon,N}(\partial\Omega^\lambda))^{2\alpha}}\\ &\le\frac{C_\alpha}{\varepsilon}\left\|\Phi^t_{\varepsilon,N}\right\|_{\dot C^{0,1}}\left\|(\Phi^t_{\varepsilon,N})^{-1}\right\|^{2\alpha}_{\dot C^{0,1}}L(\Omega)\\ &\le\frac{C_\alpha}{\varepsilon}L(\Omega).\end{aligned}$$

Therefore, Arzeà-Ascoli theorem shows that there is $N_0\in\mathbb{N}$ (depending on δ but not on t and x) such that $\left\|D\left(u_\varepsilon\left(\Phi^t_{\varepsilon,N*}(\omega^0_N-\omega^0)\right)\right)\circ\Phi^t_{\varepsilon,N}\big|_{B_{R_\delta}(x_0)}\right\|_{L^\infty}\le\delta$ whenever $N\ge N_0$. Then for such N, (3.15) shows

$$|V_2|\le\left|D\left(u_\varepsilon\left(\Phi^t_{\varepsilon,N*}(\omega^0_N-\omega^0)\right)\right)\left(\Phi^t_{\varepsilon,N}(x)\right)\right|\left|D\Phi^t_\varepsilon(x)\right|\le 3\delta.$$

Estimate of V_3. Expanding the definition of u_ε gives

$$V_3=\int_{\mathbb{R}^2}\left(\left(D(\nabla^\perp K_\varepsilon)(\Phi^t_{\varepsilon,N}(x)-\Phi^t_{\varepsilon,N}(y))-D(\nabla^\perp K_\varepsilon)(\Phi^t_\varepsilon(x)-\Phi^t_\varepsilon(y))\right)D\Phi^t_\varepsilon(x)\right)\omega^0(y)\,dy.$$

By (3.16), the integral over $\mathbb{R}^2\setminus B_{R_\delta}(x_0)$ is bounded by $2\left\|D(\nabla^\perp K_\varepsilon)\right\|_{L^\infty}\|\Phi^t_\varepsilon\|_{\dot C^{0,1}}\delta$. For $y\in B_{R_\delta}(x_0)$, the fundamental theorem of calculus shows that the integrand is bounded by

$$\begin{aligned}&\left\|\omega^0\right\|_{L^\infty}\left\|\Phi^t_\varepsilon\right\|_{\dot C^{0,1}}\int_0^1\left|D^2(\nabla^\perp K_\varepsilon)(\eta(\Phi^t_{\varepsilon,N}(x)-\Phi^t_{\varepsilon,N}(y))+(1-\eta)(\Phi^t_\varepsilon(x)-\Phi^t_\varepsilon(y)))\right|d\eta\\ &\qquad\cdot\left|\int_0^1\left(D\Phi^t_{\varepsilon,N}-D\Phi^t_\varepsilon\right)(\eta x+(1-\eta)y)\,d\eta\right||x-y|\,.\end{aligned}$$

The second integral is bounded by $\left\|\left(D\Phi^t_{\varepsilon,N}-D\Phi^t_\varepsilon\right)\big|_{B_{R_\delta}(x_0)}\right\|_{L^\infty}$, and the first integral is bounded by

$$\begin{aligned}&\min\left\{\left\|D^2(\nabla^\perp K_\varepsilon)\right\|_{L^\infty},\frac{C_\alpha}{\min\left\{\left|\Phi^t_{\varepsilon,N}(x)-\Phi^t_{\varepsilon,N}(y)\right|,\left|\Phi^t_\varepsilon(x)-\Phi^t_\varepsilon(y)\right|\right\}^{3+2\alpha}}\right\}\\ &\quad\le\min\left\{\left\|D^2(\nabla^\perp K_\varepsilon)\right\|_{L^\infty},\max\left\{\left\|(\Phi^t_{\varepsilon,N})^{-1}\right\|_{\dot C^{0,1}},\left\|(\Phi^t_\varepsilon)^{-1}\right\|_{\dot C^{0,1}}\right\}^{3+2\alpha}\frac{C_\alpha}{|x-y|^{3+2\alpha}}\right\}.\end{aligned}$$

Therefore, (3.15) shows

$$\begin{aligned}
|V_3| \leq{}& \left\|\omega^0\right\|_{L^\infty} \left\|\Phi_\varepsilon^t\right\|_{\dot{C}^{0,1}} \left\| \left(D\Phi_{\varepsilon,N}^t - D\Phi_\varepsilon^t\right)\big|_{B_{R_\delta}(x_0)} \right\|_{L^\infty} \\
&\cdot \Bigg(\left\|D^2(\nabla^\perp K_\varepsilon)\right\|_{L^\infty} \int_{|x-y|\leq 1} |x-y| \, dy \\
&\qquad + C_\alpha \max\left\{ \left\|(\Phi_{\varepsilon,N}^t)^{-1}\right\|_{\dot{C}^{0,1}}, \left\|(\Phi_\varepsilon^t)^{-1}\right\|_{\dot{C}^{0,1}} \right\}^{3+2\alpha} \int_{|x-y|>1} \frac{dy}{|x-y|^{2+2\alpha}} \Bigg) \\
&+ 2\left\|D(\nabla^\perp K_\varepsilon)\right\|_{L^\infty} \left\|\Phi_\varepsilon^t\right\|_{\dot{C}^{0,1}} \delta \\
\leq{}& C_{\alpha,\varepsilon} \left\|\omega^0\right\|_{L^\infty} \left\| \left(D\Phi_{\varepsilon,N}^t - D\Phi_\varepsilon^t\right)\big|_{B_{R_\delta}(x_0)} \right\|_{L^\infty} + C_{\alpha,\varepsilon}\delta.
\end{aligned}$$

Aggregating the estimates for V_1, V_2, V_3 now yields

$$\left|\partial_t \left(D\Phi_{\varepsilon,N}^t(x) - D\Phi_\varepsilon^t(x)\right)\right| \leq C_{\alpha,\varepsilon,L(\Omega),\|\omega^0\|_{L^\infty}} \left\| \left(D\Phi_{\varepsilon,N}^t - D\Phi_\varepsilon^t\right)\big|_{B_{R_\delta}(x_0)} \right\|_{L^\infty} + C_{\alpha,\varepsilon}\delta$$

for all $t \in [-T_0, T_0]$, $x \in B_{R_\delta}(x_0)$ and $N \geq N_0$. Then for any $t+h \in [-T_0, T_0]$, we have

$$\begin{aligned}
\left|\left(D\Phi_{\varepsilon,N}^{t+h}(x) - D\Phi_\varepsilon^{t+h}(x)\right)\right| \leq{}& \left|\left(D\Phi_{\varepsilon,N}^t(x) - D\Phi_\varepsilon^t(x)\right)\right| \\
&+ C_{\alpha,\varepsilon,L(\Omega),\|\omega^0\|_{L^\infty}} \int_t^{t+h} \left\| \left(D\Phi_{\varepsilon,N}^\tau - D\Phi_\varepsilon^\tau\right)\big|_{B_{R_\delta}(x_0)} \right\|_{L^\infty} d\tau \\
&+ C_{\alpha,\varepsilon}\delta\,|h|.
\end{aligned}$$

Since $D\Phi_\varepsilon^t$ and $D\Phi_{\varepsilon,N}^t$ are continuous in t, taking supremum over $x \in B_{R_\delta}(x_0)$, dividing by $|h|$ and sending $h \to 0$ shows

$$\begin{aligned}
&\max\left\{ \partial_t^+ \left\| \left(D\Phi_{\varepsilon,N}^t - D\Phi_\varepsilon^t\right)\big|_{B_{R_\delta}(x_0)} \right\|_{L^\infty}, -\partial_{t-} \left\| \left(D\Phi_{\varepsilon,N}^t - D\Phi_\varepsilon^t\right)\big|_{B_{R_\delta}(x_0)} \right\|_{L^\infty} \right\} \\
&\qquad \leq C_{\alpha,\varepsilon,L(\Omega),\|\omega^0\|_{L^\infty}} \left\| \left(D\Phi_{\varepsilon,N}^t - D\Phi_\varepsilon^t\right)\big|_{B_{R_\delta}(x_0)} \right\|_{L^\infty} + C_{\alpha,\varepsilon}\delta,
\end{aligned}$$

thus a Grönwall-type argument shows

$$\left\| \left(D\Phi_{\varepsilon,N}^t - D\Phi_\varepsilon^t\right)\big|_{B_{R_\delta}(x_0)} \right\|_{L^\infty} \leq \frac{\exp(C_{\alpha,\varepsilon,L(\Omega),\|\omega^0\|_{L^\infty}} |t|) - 1}{C_{\alpha,\varepsilon,L(\Omega),\|\omega^0\|_{L^\infty}}} C_{\alpha,\varepsilon}\delta$$

since $D\Phi_{\varepsilon,N}^0 = D\Phi_\varepsilon^0 = \mathrm{Id}$.

Therefore, for any $t \in [-T_0, T_0]$, $x \in B_R(x_0) \subseteq B_{R_\delta}(x_0)$ and $N \geq N_0$,

$$\left|\Phi_{\varepsilon,N}^t(x) - \Phi_\varepsilon^t(x) - \Phi_{\varepsilon,N}^t(x_0) + \Phi_\varepsilon^t(x_0)\right| \leq \frac{\exp(C_{\alpha,\varepsilon,L(\Omega),\|\omega^0\|_{L^\infty}} T_0) - 1}{C_{\alpha,\varepsilon,L(\Omega),\|\omega^0\|_{L^\infty}}} R C_{\alpha,\varepsilon}\delta,$$

thus

$$\limsup_{N\to\infty} \sup_{t\in[-T_0,T_0]} \left\| \left(\Phi_{\varepsilon,N}^t - \Phi_\varepsilon^t - \Phi_{\varepsilon,N}^t(x_0) + \Phi_\varepsilon^t(x_0)\right)\big|_{B_R(x_0)} \right\|_{L^\infty} \leq C_{\alpha,\varepsilon,L(\Omega),\|\omega^0\|_{L^\infty}} R\delta,$$

and since $\delta > 0$ is arbitrary, the claim follows. $\square$

Appendix A. Regularity of Functions with $L(\Omega) < \infty$

Is it ever possible to have a generalized layer cake representation that is better than the usual superlevel/sublevel sets?

Proposition A.1. *For any $\omega\colon \mathbb{R}^2 \to \mathbb{R}$ and a generalized layer cake representation $(\mathcal{L}, \theta, \Omega)$ of ω,*

$$\|\omega\|_{\dot{C}^{0,2\alpha}} \le L(\Omega)$$

holds.

Proof. Let $x, y \in \mathbb{R}^2$, $x \neq y$ be given. Then

$$|\omega(x) - \omega(y)| \le \int_{\mathcal{L}} |\mathbb{1}_{\Omega^\lambda}(x) - \mathbb{1}_{\Omega^\lambda}(y)| \, d|\theta|(\lambda)$$

holds. Since the integrand in the right-hand side is nonzero only if $|x-y| \ge d(x, \partial\Omega^\lambda)$, we obtain

$$|\omega(x) - \omega(y)| \le \int_{\mathcal{L}} \frac{|x-y|^{2\alpha}}{d(x, \partial\Omega^\lambda)^{2\alpha}} \, d|\theta|(\lambda) \le L(\Omega) \, |x-y|^{2\alpha} ,$$

thus $\|\omega\|_{\dot{C}^{0,2\alpha}} \le L(\Omega)$ follows. □

Proposition A.2. *Let $\omega\colon \mathbb{R}^2 \to \mathbb{R}$ be a uniformly continuous bounded function whose modulus of continuity ρ satisfies $\int_0^1 \frac{\min\{\rho(\delta), 1\}}{\delta^{1+2\alpha}} \, d\delta < \infty$. Let $\mathcal{L} \coloneqq [-\|\omega^-\|_{L^\infty}, \|\omega^+\|_{L^\infty}]$, θ the signed measure on $\mathcal{L}$ given as*

$$\theta(A) \coloneqq \int_A \operatorname{sgn}(\lambda) \, d\lambda,$$

and

$$\Omega \coloneqq \left\{(x, \lambda) \in \mathbb{R}^2 \times \left(0, \|\omega^+\|_{L^\infty}\right] : \omega(x) > \lambda\right\} \cup \left\{(x, \lambda) \in \mathbb{R}^2 \times \left[-\|\omega^-\|_{L^\infty}, 0\right) : \omega(x) < \lambda\right\}.$$

Then $(\mathcal{L}, \theta, \Omega)$ is a generalized layer cake representation of ω satisfying $L(\Omega) < \infty$.

Proof. TBD. □

Proposition A.3. *There exist $\omega\colon \mathbb{R}^2 \to \mathbb{R}$ and a generalized layer cake representation $(\mathcal{L}, \theta, \Omega)$ of ω such that $L(\Omega) < \infty$ and*

$$\min\left\{\delta^{2\alpha}, 1\right\} \le \rho(\delta) \le 2 \min\left\{\delta^{2\alpha}, 1\right\}$$

hold for all $\delta \ge 0$ where ρ is the modulus of continuity of ω.

Proof. TBD. □

References

[1] J. Jeon and A. Zlatoš, *Well-Posedness and Finite Time Singularity for Touching g-SQG Patches on the Plane*, preprint.

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