WELL-POSEDNESS FOR LOW REGULARITY SOLUTIONS TO THE G-SQG EQUATION WITH H^2 LEVEL SETS

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ABSTRACT. We show that the generalized SQG equation is locally well-posed in spaces of low regularity solutions (essentially Hölder continuous with Hölder exponents depending on the equation parameter $\alpha \in (0, \frac{1}{2})$) that have H^2 level sets. Moreover, for $\alpha \leq \frac{1}{6}$ and initial data satisfying additional hypotheses we show that the corresponding solutions can stop existing only when their level sets lose H^2-regularity, and hence not just due to level set collision or pile up.

1. INTRODUCTION

The *generalized surface quasi-geostrophic equation (g-SQG)* is the active scalar PDE

$$\partial_t \theta + u(\theta) \cdot \nabla \theta = 0 \tag{1.1}$$

on $I \times \mathbb{R}^2$ for some open time interval I, where the transporting velocity is given by

$$u(\theta) := -\nabla^\perp (-\Delta)^{-1+\alpha} \theta \tag{1.2}$$

with $\alpha \in \left(0, \frac{1}{2}\right)$, and we denote $\nabla^\perp := (-\partial_{x_2}, \partial_{x_1})$. When $\alpha = 0$ and $\alpha = \frac{1}{2}$, this becomes the *Euler* and *surface quasi-geostrophic equation*, respectively, with $\theta = \nabla^\perp \cdot u(\theta)$ being the vorticity of $u(\theta)$ in the former case. Denoting $(x_1, x_2)^\perp := (-x_2, x_1)$, another form of (1.2) is

$$u(\theta^t; x) := c_\alpha \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} \theta^t(y) \, dy = \int_{\mathbb{R}^2} \nabla^\perp K(x-y) \, \theta^t(y) \, dy \tag{1.3}$$

when $\theta^t := \theta(t, \cdot) \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ at any time $t \in I$, where $c_\alpha > 0$ is some constant and

$$K(x) := -\frac{c_\alpha}{2\alpha |x|^{2\alpha}}.$$

The g-SQG equation first appeared in physics and atmospheric science literature (see [17, 18]), with the first mathematical studies of it, including local regularity in sufficiently smooth spaces of functions, being undertaken in [3, 5]. Note that since (1.1) is a transport PDE, it can in principle also be solved in low regularity spaces in both weak [16] and Lagrangian (i.e., transport) sense. The latter notion of solutions typically also guarantees the former, and we will employ it here, too (see Definition 1.2 below).

While $u(\theta^t)$ is clearly a $(1-2\alpha)$-Hölder continuous vector field when $\theta^t \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, this is not sufficient to obtain well-posedness for the PDE in this space. However, if also $\theta^t \in C^\beta(\mathbb{R}^2)$ for some $\beta > 2\alpha$, then $u(\theta^t)$ is Lipschitz and this easily yields local well-posedness in $L^1(\mathbb{R}^2) \cap C^\beta(\mathbb{R}^2)$ for any $\beta > 2\alpha$ [4]. We note that recently, local well-posedness [11,19] and

finite time singularity formation [19] were proved for (1.1) on the half-plane in appropriate spaces of continuous functions.

On the other hand, (1.1)–(1.2) is also locally well-posed in the space of (spatially discontinuous) *patch solutions* of the form

$$\theta^t = \sum_{n=1}^{N} \mu_n \mathbb{1}_{\Theta_n^t}$$

at any time t, with some constants $\mu_n \in \mathbb{R} \setminus \{0\}$ and bounded *patches* $\Theta_n^t \subseteq \mathbb{R}^2$ that are transported by the velocity $u(\theta)$, provided all the initial patch boundaries $\partial \Theta_n^0$ are disjoint and sufficiently smooth simple closed curves [6, 7]. The reason for this is that the dynamic of these solutions is fully determined by $u(\theta)$ restricted to the patch boundary curves, and as long as these are smooth enough at all times t, the same is true for the velocity restricted to them (because near each patch boundary, θ^t is constant in the tangential direction). In addition, blow-up criteria [8, 9, 12] and ill-posedness results [13] were recently obtained for the g-SQG patch problem, along with local well-posedness and finite time singularity results on the half-plane [7, 14, 15] and when some of the patches are allowed to touch [10] (see below).

Since patch boundaries are also the boundaries of the level sets of patch solutions, the above explanation of existence of local regularity results for the g-SQG patch problem suggests a natural question: *Is there a local well-posedness theory for (1.1)–(1.2) in spaces of low regularity continuous solutions with sufficiently regular level sets?*

In our main result, Theorem 1.5 below, we answer this in the affirmative in the space of functions whose level sets (or their boundaries) are simple closed H^2 curves and they belong to $L^1(\mathbb{R}^2) \cap C_\rho(\mathbb{R}^2)$ for some increasing continuous $\rho : [0, \infty) \to [0, \infty)$ satisfying $\rho(0) = 0$ and

$$\int_0^1 \frac{\rho(s)}{s^{1+2\alpha}} \, ds < \infty. \tag{1.4}$$

Here $f \in C_\rho(\mathbb{R}^2)$ whenever f has modulus of continuity $a\rho$ for some $a > 0$, where the latter means that $|f(x) - f(y)| \le a\rho(|x - y|)$ for all $x, y \in \mathbb{R}^2$. For instance, we can take $\rho(s) := s^{2\alpha} \max\{-\ln s, 1\}^{-p}$ with $p > 1$ and then $\bigcup_{\beta > 2\alpha} C^\beta(\mathbb{R}^2) \subseteq C_\rho(\mathbb{R}^2)$.

Note that since the solutions in Theorem 1.5 are essentially no better than functions from $L^1(\mathbb{R}^2) \cap C^{2\alpha}(\mathbb{R}^2)$ at each fixed time t, the corresponding velocities $u(\theta^t)$ are essentially no better than Lipschitz. Nevertheless, the result implies that the non-linear dynamic still preserves H^2-regularity of level sets, which means that the velocities indeed have a higher degree of regularity when restricted to any of the (infinitely many) individual level sets of the solution.

Moreover, when $\alpha \le \frac{1}{6}$ and the initial datum satisfies appropriate hypotheses, we also show that the corresponding solution can be uniquely continued as long as H^2-regularity of the level sets is preserved. That is, any finite time singularity development must be accompanied by a loss of H^2-regularity of solution level sets, and in particular cannot be caused by a "pile up" or collision of level sets while these remain regular.

The first step in the proof of Theorem 1.5 is showing that (1.1)–(1.2) is locally well-posed in $L^1(\mathbb{R}^2) \cap C_\rho(\mathbb{R}^2)$ provided the modulus ρ is as above, regardless of the geometry of the

level sets of solutions. This follows from Theorem 1.3 below, which is then also a marginal improvement of the abovementioned result from [4].

However, based on our discussion above, one might hope to have local well-posedness for solutions with significantly lower degree of regularity when only their level sets are sufficiently regular. Nevertheless, this question remains open, the reason for which can be seen from the proofs in our recent work [10]. In it we considered patch solutions for (1.1)–(1.2) with possibly touching H^2 patches and obtained local well-posedness as long as only exterior touches of patches Θ_n^t and $\Theta_{n'}^t$ with $\mu_n \mu_{n'} < 0$ and interior touches of patches with $\mu_n \mu_{n'} > 0$ are allowed. In particular, one can consider arbitrary initial configurations of H^2 patches satisfying $\Theta_1^0 \subseteq \Theta_2^0 \subseteq \cdots \subseteq \Theta_N^0$ with all $\mu_n > 0$, which can be used to approximate (as $N \to \infty$) general, even discontinuous, functions whose each level set is a simple closed H^2 curve. The problem is, however, that in [10] we were only able to control the rate of change of $\sum_{n=1}^N \mu_n \|\partial \Theta_n^t\|_{H^2}^2$ rather than of the individual H^2 norms in this sum, which only yields an N-dependent bound on each H^2 norm. This is why we have to stay within $C_\rho(\mathbb{R}^2)$ for solutions taking infinitely many values, and in fact the velocities $u(\theta^t)$ of all solutions considered here will be Lipschitz at any fixed time t from their intervals of existence (see Lemma 2.1 below).

Let us now turn to the level-set view of the dynamic of (1.1). To work directly with level sets of some $\theta \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, one can use its *layer cake decomposition*

$$\theta(x) = \int_0^{\sup \theta} \mathbb{1}_{\{\theta > \lambda\}}(x) \, d\lambda - \int_{\inf \theta}^0 \mathbb{1}_{\{\theta < \lambda\}}(x) \, d\lambda. \tag{1.5}$$

We will in fact use a generalization of this concept, where the relevant λ-dependent domains need not be nested and hence need not be sub- and/or super-level sets of θ.

Definition 1.1. Let $\mathcal{L}$ be a measurable space (with some σ-algebra), let μ be a σ-finite signed measure on $\mathcal{L}$, and let $\Theta \subseteq \mathbb{R}^2 \times \mathcal{L}$ be a set in the product σ-algebra (i.e., the σ-algebra generated by measurable rectangles) with finite measure with respect to the product of the Lebesgue measure on $\mathbb{R}^2$ and the total variation $|\mu|$ of μ on $\mathcal{L}$. For any $\lambda \in \mathcal{L}$, let $\Theta^\lambda := \{x \in \mathbb{R}^2 : (x, \lambda) \in \Theta\}$ be the λ-section of Θ. If for each $x \in \mathbb{R}^2$ we have

$$\theta(x) = \int_{\mathcal{L}} \mathbb{1}_{\Theta^\lambda}(x) \, d\mu(\lambda), \tag{1.6}$$

then (Θ, μ) is a *generalized layer cake representation* of θ.

Remark. 1. Since the measurable space $\mathcal{L}$ is implicitly given by Θ and μ, we will suppress it in the notation (Θ, μ) but will always denote it by $\mathcal{L}$.

2. We require Θ to be in the product σ-algebra (instead of allowing it to be any set in its completion with respect to the product measure) in order to ensure measurability of (1.10) below. This is general enough to cover the case of the layer cake decomposition, which can be seen by dyadically dividing the second component of Θ from (1.7) below.

Hence (1.5) shows that the layer cake decomposition of any $\theta \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ is also its generalized layer cake representation with $\mathcal{L} = (\inf \theta, \sup \theta)$, $d\mu(\lambda) = \mathrm{sgn}(\lambda) d\lambda$, and

$$\Theta = \left\{ (x, \lambda) \in \mathbb{R}^2 \times [0, \sup \theta) : \theta(x) > \lambda \right\} \cup \left\{ (x, \lambda) \in \mathbb{R}^2 \times (\inf \theta, 0) : \theta(x) < \lambda \right\}. \tag{1.7}$$

Besides (1.6) being more general than (1.5), another reason for our usage of it is that it can be a more convenient way to represent functions whose level sets have multiple connected components, and in particular are multiple disjoint H^2 curves.

Generalized layer cake representations obviously commute with measure-preserving homeomorphisms $\Phi \in C(\mathbb{R}^2; \mathbb{R}^2)$ in the sense that if (Θ, μ) is a generalized layer cake representation of θ, then $(\Phi_*\Theta, \mu)$ with $\Phi_*\Theta := \{(\Phi(x), \lambda) \colon (x, \lambda) \in \Theta\}$ (and so $(\Phi_*\Theta)^\lambda = \Phi(\Theta^\lambda)$) is a generalized layer cake representation of $\Phi_*\theta := \theta \circ \Phi^{-1}$. This, and the fact that our $u(\theta^t)$ will all be Lipschitz, makes the following notion of solutions particularly suited to our approach.

Definition 1.2. A *Lagrangian solution* to (1.1)–(1.2) on an open time interval $I \ni 0$ with initial datum $\theta^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ is any $\theta \colon I \to L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ given by $\theta^t := \Phi_*^t \theta$ for each $t \in I$, where $\Phi \in C\left(I; C(\mathbb{R}^2; \mathbb{R}^2)\right)$ solves the initial value problem

$$\partial_t \Phi^t = u(\theta^t) \circ \Phi^t \qquad \text{and} \qquad \Phi^0 = \mathrm{Id} \tag{1.8}$$

on I, with Φ^t being a measure-preserving homeomorphism for each $t \in I$.

Remarks. 1. Here $C(A; B)$ will always refer to the space of all continuous $f : A \to B$, not just the bounded ones. Although we may then have $\|f\|_{L^\infty} = \infty$, it still makes sense to consider continuity of Φ in time with respect to the metric $d(f, g) := \|f - g\|_{L^\infty}$ on $C(\mathbb{R}^2; \mathbb{R}^2)$.

2. We will call the above Φ the *flow map* of θ. Since $\nabla^\perp$ in (1.2) implies that $\nabla \cdot u(\theta^t) \equiv 0$, transport by u preserves the Lebesgue measure and hence so must each Φ^t.

3. It is easy to show that any Lagrangian solution θ to (1.1)–(1.2) is also a weak solution in the sense that for all $\varphi \in C_c^1(\mathbb{R}^2)$ and $t \in I$ we have

$$\frac{d}{dt} \int_{\mathbb{R}^2} \varphi(x)\, \theta^t(x)\, dx = \int_{\mathbb{R}^2} \left[u(\theta^t; x) \cdot \nabla\varphi(x)\right] \theta^t(x)\, dx.$$

To obtain a general local well-posedness result for (1.1)–(1.2), we will also need to add a hypothesis that guarantees $u(\theta^0)$ to be Lipschitz. In terms of (1.6), this will be that

$$L_\mu(\Theta) := \sup_{x \in \mathbb{R}^2} \int_{\mathcal{L}} \frac{d|\mu|(\lambda)}{d(x, \partial\Theta^\lambda)^{2\alpha}} < \infty \tag{1.9}$$

holds for some generalized layer cake representation (Θ, μ) of θ^0 (with $\frac{1}{0} := \infty$). Note that a standard measure theory argument shows joint measurability of

$$d(x, \partial\Theta^\lambda) = \max\{d(x, \Theta^\lambda), d(x, \mathbb{R}^2 \setminus \Theta^\lambda)\} \tag{1.10}$$

in (x, λ), so $L_\mu(\Theta)$ is always well-defined and Lemma 2.1 shows that $u(\theta^0)$ is then Lipschitz. We note that $L_\mu(\Phi_*^t\Theta)$ will also be used to control the growth of H^2 norms of the boundary curves $\partial((\Phi_*^t\Theta)^\lambda) = \Phi^t(\partial\Theta^\lambda)$ in the proof of Theorem 1.5 (see Lemma 3.7 below), and that for any generalized layer cake representation (Θ, μ) and any measure-preserving homeomorphism $\Phi \in C(\mathbb{R}^2; \mathbb{R}^2)$ we have

$$L_\mu(\Phi_*\Theta) = \sup_{x \in \mathbb{R}^2} \int_{\mathcal{L}} \frac{d|\mu|(\lambda)}{d(\Phi(x), \Phi(\partial\Theta^\lambda))^{2\alpha}} \leq \left\|\Phi^{-1}\right\|_{\dot{C}^{0,1}}^{2\alpha} L_\mu(\Theta). \tag{1.11}$$

Moreover, when $\theta \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ has a modulus of continuity ρ that satisfies (1.4) (which includes $\rho(s) := s^{2\alpha} \max\{-\ln s, 1\}^{-p}$ with any $p > 1$, when $\bigcup_{\beta > 2\alpha} C^\beta(\mathbb{R}^2) \subseteq C_\rho(\mathbb{R}^2)$), then (1.9) also holds when (Θ, μ) is given by the layer cake decomposition of θ from (1.5). Indeed, with $\kappa(\lambda) := \inf\{s \geq 0 \colon \rho(s) \geq \lambda\}$ we have $\kappa^{-1}((a, b]) = (\rho(a), \rho(b)]$ for any $0 \leq a < b$, so a change of variables and subsequent integration by parts show that

$$L_\mu(\Theta) \leq 2 \int_0^{\sup\theta^0 - \inf\theta^0} \frac{d\lambda}{\kappa(\lambda)^{2\alpha}} = 2 \int_0^{\kappa(\sup\theta^0 - \inf\theta^0)} \frac{d\kappa_* m(s)}{s^{2\alpha}} = 2 \int_0^{\kappa(\sup\theta^0 - \inf\theta^0)} \frac{d\rho(s)}{s^{2\alpha}} < \infty,$$

where m is the 1-dimensional Lebesgue measure and the last integral is a Lebesgue-Stieltjes integral. On the other hand, for any $x, y \in \mathbb{R}^2$ we have

$$|\theta(x) - \theta(y)| \leq \int_{\mathcal{L}} |\mathbb{1}_{\Theta^\lambda}(x) - \mathbb{1}_{\Theta^\lambda}(y)| \, d|\mu|(\lambda) \leq \int_{\mathcal{L}} \frac{|x-y|^{2\alpha}}{d(x, \partial\Theta^\lambda)^{2\alpha}} \, d|\mu|(\lambda) \leq L_\mu(\Theta) \, |x-y|^{2\alpha}$$

because the first integrand is nonzero only when $d(x, \partial\Theta^\lambda) \leq |x - y|$, and so $\|\theta\|_{\dot{C}^{0,2\alpha}} \leq L_\mu(\Theta)$ holds for any generalized layer cake representation (Θ, μ) of θ. This all shows that (1.9) always implies $\theta \in C^{2\alpha}(\mathbb{R}^2)$ (with the latter not guaranteeing Lipschitzness of $u(\theta)$) but it is also very close to being equivalent to $\theta \in C^{2\alpha}(\mathbb{R}^2)$. In fact, for any $a > 0$, the layer cake decomposition of $a \sum_{n \geq 1} 3^{1-2\alpha n}[1 - 3^n|x - (2^{1-n}, 0)|]_+$ satisfies (1.9) but all its moduli of continuity ρ have $\rho(s) \geq as^{2\alpha}$ for all $s \in [0, 1]$.

We are now ready to state our general well-posedness result.

Theorem 1.3. *Assume that $\theta^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ admits a generalized layer cake representation (Θ, μ) with $L_\mu(\Theta) < \infty$. Then there is an open interval $I \ni 0$ and a Lagrangian solution θ to (1.1)–(1.2) on I with $\alpha \in (0, \frac{1}{2})$ and initial data θ^0, with the associated flow map $\Phi \in C(I; C(\mathbb{R}^2; \mathbb{R}^2))$ such that $\sup_{t \in J} L_\mu(\Phi_*^t \Theta) < \infty$ for any compact interval $J \subseteq I$. Let I be the maximal such interval. Then the solution θ is unique and independent of the choice of (Θ, μ), for any compact interval $J \subseteq I$ we have $\sup_{t \in J} \max\{\|\Phi^t\|_{\dot{C}^{0,1}}, \|(\Phi^t)^{-1}\|_{\dot{C}^{0,1}}\} < \infty$, and for any endpoint T of I we have either $|T| = \infty$ or $\lim_{t \to T} L_\mu(\Phi_*^t \Theta) = \infty$.*

Remark. As mentioned above, the hypothesis is satisfied whenever $\theta^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ has modulus of continuity ρ satisfying (1.4).

We next turn to our main result, which in particular shows that if, in the setting of Theorem 1.3, each $\partial\Theta^\lambda$ is an H^2 curve and certain additional assumptions are satisfied, then the solution from Theorem 1.3 retains these properties on some open time interval $I' \ni 0$. To state it precisely, let $\mathrm{CC}(\mathbb{R}^2)$ be the space of equivalence classes of planar closed curves from $C(\mathbb{T}; \mathbb{R}^2)$ with respect to the equivalence relation given by the *Fréchet pseudometric*

$$d_{\mathrm{F}}(\tilde{\gamma}_1, \tilde{\gamma}_2) := \inf_\phi \|\tilde{\gamma}_1 - \tilde{\gamma}_2 \circ \phi\|_{L^\infty(\mathbb{T})},$$

where the infimum is taken over all orientation-preserving homeomorphisms $\phi \colon \mathbb{T} \to \mathbb{T}$ (and $\mathbb{T}$ is $[0, 1]$ with 0 and 1 identified). Any $\tilde{\gamma} \in C(\mathbb{T}; \mathbb{R}^2)$ belonging to some equivalence class $\gamma \in \mathrm{CC}(\mathbb{R}^2)$ (which we also call a *closed curve*) will be called a *representative* of γ, and we denote $\mathrm{im}(\gamma) := \mathrm{im}(\tilde{\gamma})$ and let the *length* $\ell(\gamma)$ of γ be the total variation of $\tilde{\gamma}$. If $\ell(\gamma) < \infty$, then γ is rectifiable and its *arclength parametrization* is any $\tilde{\gamma} \in C(\ell(\gamma)\mathbb{T}; \mathbb{R}^2)$ such that

$|\partial_s \tilde{\gamma}(s)| = 1$ for almost all $s \in \ell(\gamma)\mathbb{T}$ and $\tilde{\gamma}(\ell(\gamma) \cdot)$ is a representative of γ. We then also denote by $\|\gamma\|_{\dot{H}^2} := \|\tilde{\gamma}\|_{\dot{H}^2(\ell(\gamma)\mathbb{T})}$ the L^2 norm of the curvature of γ (if it is finite, then γ is an H^2 *curve*). Finally, we let $\Delta(\gamma_1, \gamma_2) := d(\mathrm{im}(\gamma_1), \mathrm{im}(\gamma_2))$ for any $\gamma_1, \gamma_2 \in \mathrm{CC}(\mathbb{R}^2)$ (with d the distance of sets), and let $\mathrm{PSC}(\mathbb{R}^2) \subseteq \mathrm{CC}(\mathbb{R}^2)$ be the set of all positively oriented simple closed curves in $\mathbb{R}^2$. All these definitions appear in [10].

We can now define the quantities relevant to studying solutions with H^2 level sets.

Definition 1.4. Let (Θ, μ) be a generalized layer cake representation $\theta \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. If for each $\lambda \in \mathcal{L}$ there is $z^\lambda \in \mathrm{PSC}(\mathbb{R}^2)$ such that Θ^λ is a bounded open set with $\partial\Theta^\lambda = \mathrm{im}(z^\lambda)$, then (Θ, μ) (or just Θ) is *composed of simple closed curves*, and we let

$$R_\mu(\Theta) := \sup_{\lambda \in \mathcal{L}} \ell(z^\lambda)^{1/2} \int_{\mathcal{L}} \frac{d|\mu|(\lambda')}{\ell(z^{\lambda'})^{1/2} \Delta(z^\lambda, z^{\lambda'})^{2\alpha}} \qquad \text{and} \qquad Q(\Theta) := \sup_{\lambda \in \mathcal{L}} \ell(z^\lambda) \left\| z^\lambda \right\|_{\dot{H}^2}^2.$$

We also denote $\ell(\partial\Theta^\lambda) := \ell(z^\lambda)$ and $\left\| \partial\Theta^\lambda \right\|_{\dot{H}^2} := \left\| z^\lambda \right\|_{\dot{H}^2}$.

Remark. Note that $\Delta(z^\lambda, z^{\lambda'}) = \inf_{x \in \mathbb{Q}^2} \left(d(x, \partial\Theta^\lambda) + d(x, \partial\Theta^{\lambda'}) \right)$ is measurable in λ' since $d(x, \partial\Theta^{\lambda'})$ is jointly measurable in (x, λ'). Also, a standard measure theory argument shows that $\lambda' \mapsto \Theta^{\lambda'}$ is measurable with respect to the topology on the set of all measurable subsets of $\mathbb{R}^2$ given by the family of pseudometrics $(A, B) \mapsto |(A \triangle B) \cap K|$ for all (fixed) compacts $K \subseteq \mathbb{R}^2$. Then lower semi-continuity of the perimeter functional (in the sense of Caccioppoli) with respect to this topology (see [1, Proposition 3.38]) shows measurability of $\lambda' \mapsto \ell(z^{\lambda'})$ ([1, Proposition 3.62] and [2, Theorem I] show that $\ell(z^{\lambda'})$ is the perimeter of $\Theta^{\lambda'}$). Hence, $R_\mu(\Theta)$ is well-defined.

We will next add to $L_\mu(\Theta) < \infty$ the hypothesis that Θ is composed of simple closed curves and $R_\mu(\Theta), Q(\Theta) < \infty$. Assuming $Q(\Theta) < \infty$ ensures a form of scaling-invariant uniform H^2 regularity of the z^λ because for any $\gamma \in \mathrm{CC}(\mathbb{R}^2)$ and $a > 0$ we have $\|a\gamma\|_{\dot{H}^2}^2 = \frac{1}{a} \|\gamma\|_{\dot{H}^2}^2$ and $\ell(a\gamma) = a\ell(\gamma)$. Hypothesis $R_\mu(\Theta) < \infty$ is a version of $L_\mu(\Theta) < \infty$ but with individual points x replaced by whole curves z^λ, and it also controls how densely z^λ of different scales can be packed together. More specifically, it prevents too many "small" curves $z^{\lambda'}$ to be close to some "large" z^λ. When (Θ, μ) is given by the layer cake decomposition of some θ, so the z^λ are (the boundaries of) the level sets of θ, this assumption in effect rules out bumps with "too sharp" tops or bottoms. For instance, if (Θ, μ) is the layer cake decomposition of a radial bump $\theta(x) := \left[1 - |x|^\beta \right]_+$ for some $\beta \in (0, 1]$, then $R_\mu(\Theta) < \infty$ if and only if $\beta > \max\left\{ \frac{1}{2}, 2\alpha \right\}$. On the other hand, if instead $\theta(x) := [1 - |x|]_+^\beta$, then $R_\mu(\Theta) < \infty$ if and only if $\beta > 2\alpha$. Note that in both cases $L_\mu(\Theta) < \infty$ if and only if $\beta > 2\alpha$.

We can now finally state our main result. Its part (ii) proves well-posedness for low regularity solutions to (1.1) with H^2 level sets, while its part (iii) shows that for $\alpha \leq \frac{1}{6}$ and initial conditions satisfying some additional hypotheses, the corresponding solutions can stop existing only when their level sets lose H^2-regularity, not just when they collide or pile up.

Theorem 1.5. *Let $\theta^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ admit a generalized layer cake representation (Θ, μ) composed of simple closed curves such that $L_\mu(\Theta), R_\mu(\Theta), Q(\Theta) < \infty$, and let $\theta \colon I \to L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ be the Lagrangian solution to (1.1)–(1.2) with $\alpha \in (0, \frac{1}{2})$ from Theorem 1.3.*

(i) Then $\Phi_*^t\Theta$ is composed of simple closed curves for each $t \in I$, and $\sup_{t\in J} R_\mu(\Phi_*^t\Theta) < \infty$ for each compact interval $J \subseteq I$.

(ii) There is an open interval $I' \subseteq I$ containing 0 such that $\sup_{t\in J} Q(\Phi_*^t\Theta) < \infty$ for each compact interval $J \subseteq I'$. Let I' be the maximal such interval and T its endpoint. Then either $|T| = \infty$ or $\lim_{t\to T} L_\mu(\Phi_*^t\Theta) = \infty$ or $\lim_{t\to T} Q(\Phi_*^t\Theta) = \infty$.

(iii) If also $\alpha \in \left(0, \frac{1}{6}\right]$, $|\mu|(\mathcal{L}) < \infty$, $\int_{\mathcal{L}} \ell(\partial\Theta^\lambda)\, d|\mu|(\lambda) < \infty$, and $\inf_{\lambda\in\mathcal{L}} |\Theta^\lambda| > 0$, then for any endpoint T of I' from (ii) we have either $|T| = \infty$ or $\limsup_{t\to T} Q(\Phi_*^t\Theta) = \infty$.

Remark. Note that the last hypothesis in (iii) excludes initial data that are bumps with rounded tops/bottoms (instead, the latter must be flat), and Φ^t being measure preserving then also implies $\inf_{(t,\lambda)\in I'\times\mathcal{L}} \mathrm{diam}(\Phi^t\Theta^\lambda) > 0$. When also $\sup_{\lambda\in\mathcal{L}} \mathrm{diam}(\Theta^\lambda) < \infty$, then (4.2) below shows that $\sup_{(t,\lambda)\in[0,T)\times\mathcal{L}} \mathrm{diam}(\Phi^t\Theta^\lambda) < \infty$ when T in (iii) is finite. So in that case the claim of (iii) implies

$$\limsup_{t\to T} \sup_{\lambda\in\mathcal{L}} \left\|\Phi^t(\partial\Theta^\lambda)\right\|_{\dot{H}^2} = \infty$$

(while the diameters of $\Phi^t\Theta^\lambda$ remain uniformly bounded above and below by positive constants), which means true loss of H^2-regularity of the level sets at time T.

2. Proof of Theorem 1.3

All constants C_α below can change from one inequality to another, but they always only depend on α. We will construct solutions to (1.1)–(1.2) as limits of solutions to a family of similar problems with smooth velocities

$$u_\varepsilon(\theta^t; x) := \int_{\mathbb{R}^2} \nabla^\perp K_\varepsilon(x - y)\, \theta^t(y) dy, \tag{2.1}$$

where $K_\varepsilon(x) := \chi(\varepsilon^{-1}|x|)K(x)$ for any $\varepsilon > 0$ and $x \in \mathbb{R}^2$, with $\chi \in C^\infty(\mathbb{R})$ is even and satisfying $\mathbb{1}_{\mathbb{R}\setminus(-1,1)} \leq \chi \leq \mathbb{1}_{\mathbb{R}\setminus(-1/2,1/2)}$. Note that for any $n \geq 0$, there is $C_{\alpha,n}$ that only depends on α, n such that the norm of the n-linear form $D^n K_\varepsilon(x)$ is bounded by $\frac{C_{\alpha,n}}{\max\{|x|,\varepsilon\}^{n+2\alpha}}$. In particular, this norm is a bounded function of $x \in \mathbb{R}^2$ for any $\varepsilon > 0$, and we clearly have

$$\left\|D^n(u_\varepsilon(\theta^t))\right\|_{L^\infty} \leq \left\|D^n(\nabla^\perp K_\varepsilon)\right\|_{L^\infty} \left\|\theta^t\right\|_{L^1}. \tag{2.2}$$

Let us start with some estimates on the velocity fields from (1.3) and (2.1) in terms of $L_\mu(\Theta)$ for some generalized layer cake representation (Θ, μ) of θ (in these, we drop t from the notation for convenience).

Lemma 2.1. *There is C_α such that for any $\theta \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ with a generalized layer cake representation (Θ, μ), and for any $\varepsilon > 0$ and $x \in \mathbb{R}^2$, we have*

$$|D(u_\varepsilon(\theta))(x)| \leq C_\alpha \int_{\mathcal{L}} \frac{d|\mu|(\lambda)}{d(x, \partial\Theta^\lambda)^{2\alpha}}.$$

Therefore,

$$\|u_\varepsilon(\theta)\|_{\dot{C}^{0,1}} \leq C_\alpha L_\mu(\Theta) \qquad and \qquad \|u(\theta)\|_{\dot{C}^{0,1}} \leq C_\alpha L_\mu(\Theta).$$

Proof. For each $\varepsilon > 0$ and $x, h \in \mathbb{R}^2$, oddness of $\nabla^\perp K_\varepsilon$ shows that

$$u_\varepsilon(\theta; x + h) - u_\varepsilon(\theta; x) = \int_{\mathbb{R}^2} \left(\nabla^\perp K_\varepsilon(x + h - y) - \nabla^\perp K_\varepsilon(x - y) \right) (\theta(y) - \theta(x)) \, dy.$$

Replacing h by sh with $s \in \mathbb{R}$, and then taking $s \to 0$ yields

$$D(u_\varepsilon(\theta))(x)h = \int_{\mathbb{R}^2} D(\nabla^\perp K_\varepsilon)(x - y)h \, (\theta(y) - \theta(x)) \, dy$$

$$= \int_{\mathbb{R}^2} \int_{\mathcal{L}} D(\nabla^\perp K_\varepsilon)(x - y)h \, (\mathbb{1}_{\Theta^\lambda}(y) - \mathbb{1}_{\Theta^\lambda}(x)) \, d\mu(\lambda) \, dy.$$

Note that $\mathbb{1}_{\Theta^\lambda}(y) - \mathbb{1}_{\Theta^\lambda}(x) \neq 0$ implies $|x - y| \geq d(x, \partial\Theta^\lambda)$, so

$$|D(u_\varepsilon(\theta))(x)| \leq \int_{\mathcal{L}} \int_{|x-y| \geq d(x,\partial\Theta^\lambda)} \frac{C_\alpha}{|x - y|^{2+2\alpha}} \, dy \, d|\mu|(\lambda) \leq \int_{\mathcal{L}} \frac{C_\alpha}{d(x, \partial\Theta^\lambda)^{2\alpha}} d|\mu|(\lambda).$$

This proves the first and second claims, and the third follows by taking $\varepsilon \to 0^+$. $\qquad\square$

Lemma 2.2. *There is C_α such that for any $\theta \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ with generalized layer cake representations (Θ_i, μ_i) and any measure-preserving homeomorphisms $\Phi_i \colon \mathbb{R}^2 \to \mathbb{R}^2$ $(i = 1, 2)$,*

$$\|u(\Phi_{1*}\theta) - u(\Phi_{2*}\theta)\|_{L^\infty} \leq C_\alpha \left(L_{\mu_1}(\Phi_{1*}\Theta_1) + L_{\mu_2}(\Phi_{2*}\Theta_2) \right) \|\Phi_1 - \Phi_2\|_{L^\infty}.$$

Proof. Let $\mathcal{L}_i$ the measurable space associated to (Θ_i, μ_i) and $\theta_i := \theta \circ \Phi_i^{-1}$ for $i = 1, 2$. Let $d := \|\Phi_1 - \Phi_2\|_{L^\infty}$ and fix any $x \in \mathbb{R}^2$. Then $u(\Phi_{1*}\theta; x) - u(\Phi_{2*}\theta; x)$ is the sum of

$$I_1 := \int_{|x-y| \leq 2d} \nabla^\perp K(x - y)(\theta_1(y) - \theta_2(y)) \, dy,$$

$$I_2 := \int_{|x-y| > 2d} \nabla^\perp K(x - y)(\theta_1(y) - \theta_2(y)) \, dy.$$

Estimate for I_1. By oddness of $\nabla^\perp K$, we have

$$I_1 = \int_{|x-y| \leq 2d} \nabla^\perp K(x - y)(\theta_1(y) - \theta_1(x)) \, dy - \int_{|x-y| \leq 2d} \nabla^\perp K(x - y)(\theta_2(y) - \theta_2(x)) \, dy.$$

Since

$$I_3 := \int_{|x-y| \leq 2d} \frac{|\theta_1(y) - \theta_1(x)|}{|x - y|^{1+2\alpha}} \, dy \leq \int_{\mathcal{L}_1} \int_{|x-y| \leq 2d} \frac{\left| \mathbb{1}_{\Phi_1(\Theta_1^\lambda)}(y) - \mathbb{1}_{\Phi_1(\Theta_1^\lambda)}(x) \right|}{|x - y|^{1+2\alpha}} \, dy \, d|\mu_1|(\lambda)$$

and (as in the proof of Lemma 2.1) we have $|x - y| \geq d(x, \partial\Phi_1(\Theta_1^\lambda))$ whenever the last integrand is nonzero, we see that

$$I_3 \leq \int_{\mathcal{L}_1} \int_{|x-y| \leq 2d} \frac{1}{|x - y| \, d(x, \partial\Phi_1(\Theta_1^\lambda))^{2\alpha}} \, dy \, d|\mu_1|(\lambda) \leq 4\pi L_{\mu_1}(\Phi_{1*}\Theta_1)d.$$

The same argument for θ_2 in place of θ_1 now yields

$$|I_1| \leq C_\alpha \left(L_{\mu_1}(\Phi_{1*}\Theta_1) + L_{\mu_2}(\Phi_{2*}\Theta_2) \right) d. \tag{2.3}$$

Estimate for I_2. For each $R > 2d$ let

$$I_2^R := \int_{2d < |x-y| \leq R} \nabla^\perp K(x-y)(\theta_1(y) - \theta_2(y))\, dy,$$

so that $I_2 = \lim_{R \to \infty} I_2^R$. Fix $R > 2d$, and then Φ_i being measure-preserving yields

$$I_2^R = \int_{2d < |x-y| \leq R} \nabla^\perp K(x-y)(\theta_1(y) - \theta_1(x))\, dy$$

$$- \int_{2d < |x-y| \leq R} \nabla^\perp K(x-y)(\theta_2(y) - \theta_1(x))\, dy$$

$$= \int_{2d < |x-\Phi_1(y)| \leq R} \nabla^\perp K(x - \Phi_1(y))(\theta(y) - \theta_1(x))\, dy$$

$$- \int_{2d < |x-\Phi_2(y)| \leq R} \nabla^\perp K(x - \Phi_2(y))(\theta(y) - \theta_1(x))\, dy$$

$$= \int_{|x-\Phi_1(y)|,|x-\Phi_2(y)| \in (2d,R]} \left[\nabla^\perp K(x - \Phi_1(y)) - \nabla^\perp K(x - \Phi_2(y)) \right] (\theta(y) - \theta_1(x))\, dy$$

$$+ \int_{|x-\Phi_2(y)| \leq 2d < |x-\Phi_1(y)| \leq R} \nabla^\perp K(x - \Phi_1(y))(\theta(y) - \theta_1(x))\, dy$$

$$- \int_{|x-\Phi_1(y)| \leq 2d < |x-\Phi_2(y)| \leq R} \nabla^\perp K(x - \Phi_2(y))(\theta(y) - \theta_1(x))\, dy$$

$$+ \int_{2d < |x-\Phi_1(y)| \leq R < |x-\Phi_2(y)|} \nabla^\perp K(x - \Phi_1(y))(\theta(y) - \theta_1(x))\, dy$$

$$- \int_{2d < |x-\Phi_2(y)| \leq R < |x-\Phi_1(y)|} \nabla^\perp K(x - \Phi_2(y))(\theta(y) - \theta_1(x))\, dy.$$

Let us denote the integrals on the right-hand side I_4, I_5, I_6, I_7, I_8 (in the order of appearance).

To estimate I_4, note that for any y in the domain of integration we have

$$\min_{\eta \in [0,1]} |x - (1-\eta)\Phi_1(y) - \eta\Phi_2(y)| \geq |x - \Phi_1(y)| - d \geq \frac{|x - \Phi_1(y)|}{2},$$

so the mean value theorem shows that

$$\left| \nabla^\perp K(x - \Phi_1(y)) - \nabla^\perp K(x - \Phi_2(y)) \right| \leq \frac{C_\alpha d}{|x - \Phi_1(y)|^{2+2\alpha}}.$$

The change of variables formula now yields

$$|I_4| \leq \int_{|x-y|>2d} \frac{C_\alpha d\, |\theta_1(y) - \theta_1(x)|}{|x - y|^{2+2\alpha}}\, dy$$

$$\leq \int_{\mathcal{L}_1} \int_{|x-y|>2d} \frac{C_\alpha d \left| \mathbb{1}_{\Phi_1(\Theta_1^\lambda)}(y) - \mathbb{1}_{\Phi_1(\Theta_1^\lambda)}(x) \right|}{|x - y|^{2+2\alpha}}\, dy\, d|\mu_1|(\lambda).$$

Again, $|x - y| \geq d(x, \partial\Phi_1(\Theta_1^\lambda))$ holds whenever the last integrand is nonzero, so

$$|I_4| \leq \int_{\mathcal{L}_1} \int_{|x-y| \geq d(x, \partial\Phi_1(\Theta_1^\lambda))} \frac{C_\alpha d}{|x - y|^{2+2\alpha}} \, dy \, d|\mu_1|(\lambda) \leq C_\alpha L_{\mu_1}(\Phi_{1*}\Theta_1)d.$$

For I_5, note that for any y in the domain of integration we have

$$|x - \Phi_1(y)| \leq |x - \Phi_2(y)| + d \leq 3d.$$

By applying again change of variables we obtain

$$|I_5| \leq \int_{|x-y| \leq 3d} \frac{C_\alpha \, |\theta_1(y) - \theta_1(x)|}{|x - y|^{1+2\alpha}} \, dy,$$

so the same argument as in the estimate for I_3 shows $|I_5| \leq C_\alpha L_{\mu_1}(\Phi_{1*}\Theta_1)d$. And clearly

$$|I_6| \leq \int_{|x-\Phi_1(y)| \leq 2d} \frac{C_\alpha \, |\theta(y) - \theta_1(x)|}{|x - \Phi_1(y)|^{1+2\alpha}} \, dy = \int_{|x-y| \leq 2d} \frac{C_\alpha \, |\theta_1(y) - \theta_1(x)|}{|x - y|^{1+2\alpha}} \, dy,$$

so again $|I_6| \leq C_\alpha L_{\mu_1}(\Phi_{1*}\Theta_1)d$.

To estimate I_7, note that for any y in the domain of integration we have

$$|x - \Phi_1(y)| \geq |x - \Phi_2(y)| - d > R - d,$$

so the change of variables formula yields

$$|I_7| \leq \int_{R-d < |x-y| \leq R} \frac{C_\alpha \, \|\theta\|_{L^\infty}}{|x - y|^{1+2\alpha}} \, dy \leq \frac{C_\alpha \, \|\theta\|_{L^\infty} \, d}{R^{2\alpha}}$$

because $R > 2d$. In the same way we also obtain $|I_8| \leq \frac{C_\alpha \|\theta\|_{L^\infty} d}{R^{2\alpha}}$.

Collecting the above estimates and letting $R \to \infty$, we see that $|I_2| \leq C_\alpha L_{\mu_1}(\Phi_{1*}\Theta_1)d$. This and (2.3) now hold uniformly in $x \in \mathbb{R}^2$, finishing the proof. $\square$

Lemma 2.3. *There is C_α such that for any $\theta \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $\varepsilon > 0$ we have*

$$\|u(\theta) - u_\varepsilon(\theta)\|_{L^\infty} \leq C_\alpha \, \|\theta\|_{L^\infty} \, \varepsilon^{1-2\alpha}.$$

Proof. Since $\nabla^\perp K_\varepsilon(x) = \nabla^\perp K(x)$ when $|x| \geq \varepsilon$, for any $x \in \mathbb{R}^2$ we have

$$|u(\theta; x) - u_\varepsilon(\theta; x)| \leq \int_{|x-y| \leq \varepsilon} \left|\nabla^\perp K(x - y) - \nabla^\perp K_\varepsilon(x - y)\right| \|\theta\|_{L^\infty} \, dy$$

$$\leq \int_{|x-y| \leq \varepsilon} \frac{C_\alpha \, \|\theta\|_{L^\infty}}{|x - y|^{1+2\alpha}} \, dy = C_\alpha \, \|\theta\|_{L^\infty} \, \varepsilon^{1-2\alpha}.$$

$\square$

Now, fix any initial datum $\theta^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ admitting a generalized layer cake representation (Θ, μ) with $L_\mu(\Theta) < \infty$. Fix any $\varepsilon > 0$ and consider the ODE

$$\partial_t \Psi_\varepsilon^t = u_\varepsilon((\mathrm{Id} + \Psi_\varepsilon^t)_* \theta^0) \circ (\mathrm{Id} + \Psi_\varepsilon^t) \qquad \text{and} \qquad \Psi_\varepsilon^0 \equiv 0 \qquad (2.4)$$

with $\Psi_\varepsilon^t \in BC(\mathbb{R}^2; \mathbb{R}^2)$ (the space of bounded continuous functions from $\mathbb{R}^2$ to $\mathbb{R}^2$). That is, $\Phi_\varepsilon^t := \mathrm{Id} + \Psi_\varepsilon^t$ is the flow map generated by the vector field $u_\varepsilon(\Phi_{\varepsilon*}^t \theta^0)$. However, until we show that Φ_ε^t is a measure-preserving homeomorphism, $\Phi_{\varepsilon*}^t \theta^0$ will be the pushforward

of the measure $\theta^0(y)dy$ by Φ_ε^t (which is $(\theta^0 \circ (\Phi_\varepsilon^t)^{-1})(y)dy$ when Φ_ε^t is a measure-preserving homeomorphism) and we replace (2.1) by

$$u_\varepsilon(\nu; x) := \int_{\mathbb{R}^2} \nabla^\perp K_\varepsilon(x - y)\, d\nu(y)$$

for a finite signed Borel measure ν on $\mathbb{R}^2$. We will next show that given any θ^0 as above, (2.4) is globally well-posed in $BC(\mathbb{R}^2; \mathbb{R}^2)$.

Lemma 2.4. *For each $F \in BC(\mathbb{R}^2; \mathbb{R}^2)$, let*

$$\mathcal{F}(F) := u_\varepsilon((\mathrm{Id} + F)_* \theta^0) \circ (\mathrm{Id} + F).$$

Then $\mathcal{F}: BC(\mathbb{R}^2; \mathbb{R}^2) \to BC(\mathbb{R}^2; \mathbb{R}^2)$ is well-defined and Lipschitz continuous.

Proof. Clearly $\mathcal{F}(F) \in C(\mathbb{R}^2; \mathbb{R}^2)$ for any $F \in BC(\mathbb{R}^2; \mathbb{R}^2)$. For any $F_1, F_2 \in BC(\mathbb{R}^2; \mathbb{R}^2)$ and $x \in \mathbb{R}^2$ we see that $(\mathcal{F}(F_1) - \mathcal{F}(F_2))(x)$ equals

$$\int_{\mathbb{R}^2} \nabla^\perp K_\varepsilon(x + F_1(x) - y)\, d(\mathrm{Id} + F_1)_* \theta^0(y) - \int_{\mathbb{R}^2} \nabla^\perp K_\varepsilon(x + F_2(x) - y)\, d(\mathrm{Id} + F_2)_* \theta^0(y)$$

$$= \int_{\mathbb{R}^2} \left[\nabla^\perp K_\varepsilon(x - y + F_1(x) - F_1(y)) - \nabla^\perp K_\varepsilon(x - y + F_2(x) - F_2(y)) \right] \theta^0(y)\, dy,$$

so

$$\|\mathcal{F}(F_1) - \mathcal{F}(F_2)\|_{L^\infty} \le 2 \left\| D(\nabla^\perp K_\varepsilon) \right\|_{L^\infty} \left\| \theta^0 \right\|_{L^1} \|F_1 - F_2\|_{L^\infty}.$$

Since $\mathcal{F}(0) = u_\varepsilon(\theta^0)$ is bounded, both claims follow from this. $\qquad\square$

Lemma 2.4 shows that (2.4) is globally well-posed, and we let $\Phi_\varepsilon^t := \mathrm{Id} + \Psi_\varepsilon^t$ and $\theta_\varepsilon^t := \Phi_{\varepsilon*}^t \theta^0$, with the latter being for now only a finite signed Borel measure. We will next show that Φ_ε^t is in fact a measure-preserving homeomorphism, which will mean that $\theta_\varepsilon^t = \theta^0 \circ (\Phi_\varepsilon^t)^{-1}$ and it is also an $L^1 \cap L^\infty$ function.

Clearly the ODE

$$\partial_t G^t = u_\varepsilon(\theta_\varepsilon^t) \circ (\mathrm{Id} + G^t) \tag{2.5}$$

is globally well-posed in $BC(\mathbb{R}^2; \mathbb{R}^2)$ for any initial data at any initial time. For each $t_0, t_1 \in \mathbb{R}$, let $\Gamma_\varepsilon^{t_0, t}$ be the unique solution to (2.5) with initial data $\Gamma_\varepsilon^{t_0, t_0} := 0$ at time $t = t_0$, and consider $G^t := \Gamma_\varepsilon^{t_0, t_1} + \Gamma_\varepsilon^{t_1, t} \circ (\mathrm{Id} + \Gamma_\varepsilon^{t_0, t_1})$. Then G^t solves (2.5) and $G^{t_1} = \Gamma_\varepsilon^{t_0, t_1}$, so uniqueness of the solution with the initial data $\Gamma_\varepsilon^{t_0, t_1}$ at time $t = t_1$ shows that

$$\mathrm{Id} + \Gamma_\varepsilon^{t_0, t} = \mathrm{Id} + G^t = (\mathrm{Id} + \Gamma_\varepsilon^{t_1, t}) \circ (\mathrm{Id} + \Gamma_\varepsilon^{t_0, t_1})$$

for all $t \in \mathbb{R}$. Letting $t := t_0$ shows that $(\mathrm{Id} + \Gamma_\varepsilon^{t_1, t_0}) \circ (\mathrm{Id} + \Gamma_\varepsilon^{t_0, t_1}) = \mathrm{Id}$, so we conclude that each $\mathrm{Id} + \Gamma_\varepsilon^{t_0, t}$ is a homeomorphism. Then so is $\Phi_\varepsilon^t = \mathrm{Id} + \Gamma_\varepsilon^{0, t}$.

Letting $BC^1(\mathbb{R}^2; \mathbb{R}^2)$ be the space of bounded C^1 functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ with bounded first derivatives, we see that for any $F \in BC^1(\mathbb{R}^2; \mathbb{R}^2)$ and $x, h \in \mathbb{R}^2$ we have

$$D(u_\varepsilon(\theta_\varepsilon^t) \circ (\mathrm{Id} + F))(x)h = \int_{\mathbb{R}^2} D(\nabla^\perp K_\varepsilon)(x + F(x) - y)(h + DF(x)h)\, d\theta_\varepsilon^t(y).$$

Therefore $F \mapsto u_\varepsilon(\theta_\varepsilon^t) \circ (\mathrm{Id} + F)$ is locally Lipschitz on $BC^1(\mathbb{R}^2; \mathbb{R}^2)$, so (2.5) is locally well-posed there. But since for any $F \in BC^1(\mathbb{R}^2; \mathbb{R}^2)$ we have

$$\left\| D(u_\varepsilon(\theta_\varepsilon^t) \circ (\mathrm{Id} + F)) \right\|_{L^\infty} \le \left\| D(\nabla^\perp K_\varepsilon) \right\|_{L^\infty} \left\| \theta^0 \right\|_{L^1} \left\| \mathrm{Id} + DF \right\|_{L^\infty},$$

a Grönwall-type argument shows that the C^1 norm of any solution to (2.5) can grow no faster than exponentially. Therefore (2.5) is even globally well-posed in $BC^1(\mathbb{R}^2; \mathbb{R}^2)$, and so $\Gamma_\varepsilon^{t_0, t} \in BC^1(\mathbb{R}^2; \mathbb{R}^2)$ for all $t \in \mathbb{R}$. This and $\nabla \cdot u_\varepsilon(\theta_\varepsilon^t) \equiv 0$ now show that the map $\mathrm{Id} + \Gamma_\varepsilon^{t_0, t}$ is measure-preserving. Then $\theta_\varepsilon^t = \theta^0 \circ (\Phi_\varepsilon^t)^{-1} \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $\Phi_{\varepsilon *}^t \Theta$ is its generalized layer cake representation.

Similarly, with $BC^2(\mathbb{R}^2; \mathbb{R}^2)$ the space of bounded C^2 functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ with bounded first and second derivatives, for each $F \in BC^2(\mathbb{R}^2; \mathbb{R}^2)$ and $x, h_1, h_2 \in \mathbb{R}^2$ we have

$$\begin{aligned}
D^2(u_\varepsilon&(\theta_\varepsilon^t) \circ (\mathrm{Id} + F))(x)(h_1, h_2) \\
&= \int_{\mathbb{R}^2} D^2(\nabla^\perp K_\varepsilon)(x + F(x) - y)(h_1 + DF(x)h_1, h_2 + DF(x)h_2) \, \theta_\varepsilon^t(y) \, dy \\
&\quad + \int_{\mathbb{R}^2} D(\nabla^\perp K_\varepsilon)(x + F(x) - y) \left(D^2 F(x)(h_1, h_2) \right) \theta_\varepsilon^t(y) \, dy
\end{aligned}$$

and

$$\begin{aligned}
\left\| D^2(u_\varepsilon(\theta_\varepsilon^t) \circ (\mathrm{Id} + F)) \right\|_{L^\infty} &\le \left\| D^2(\nabla^\perp K_\varepsilon) \right\|_{L^\infty} \left\| \theta^0 \right\|_{L^1} \left\| \mathrm{Id} + DF \right\|_{L^\infty}^2 \\
&\quad + \left\| D(\nabla^\perp K_\varepsilon) \right\|_{L^\infty} \left\| \theta^0 \right\|_{L^1} \left\| D^2 F \right\|_{L^\infty}.
\end{aligned}$$

Another Grönwall-type argument and the time-exponential bound on the C^1 norms of solutions to (2.5) now shows that (2.5) is globally well-posed in $BC^2(\mathbb{R}^2; \mathbb{R}^2)$, which we will use in Section 3. One can continue and inductively show that (2.5) is globally well-posed in $BC^k(\mathbb{R}^2; \mathbb{R}^2)$ for all $k \in \mathbb{N}$ (then each Φ_ε^t is a diffeomorphism), but we will not need this here.

Next we derive an ε-independent estimate on the growth of $L_\mu(\Phi_{\varepsilon *}^t \Theta)$.

Lemma 2.5. $\left\| D^k(u_\varepsilon(\theta_\varepsilon^t)) \right\|_{L^\infty}$ *is continuous in t for all $k \in \mathbb{Z}_{\ge 0}$, and*

$$\begin{aligned}
\left| \Gamma_\varepsilon^{t_0, t_1}(x) \right. &- \Gamma_\varepsilon^{t_0, t_1}(y) - \Gamma_\varepsilon^{t_0, t_2}(x) + \left. \Gamma_\varepsilon^{t_0, t_2}(y) \right| \\
&\le \left(\exp\left(\left| \int_{t_2}^{t_1} \| u_\varepsilon(\theta_\varepsilon^\tau) \|_{\dot{C}^{0,1}} \, d\tau \right| \right) - 1 \right) \left| x + \Gamma_\varepsilon^{t_0, t_2}(x) - y - \Gamma_\varepsilon^{t_0, t_2}(y) \right| \qquad (2.6)
\end{aligned}$$

holds for all $x, y \in \mathbb{R}^2$ and $t_0, t_1, t_2 \in \mathbb{R}$.

Proof. Change of variables yields

$$\left\| D^k(u_\varepsilon(\theta_\varepsilon^{t_1})) - D^k(u_\varepsilon(\theta_\varepsilon^{t_2})) \right\|_{L^\infty} \le \left\| D^{k+1}(\nabla^\perp K_\varepsilon) \right\|_{L^\infty} \left\| \theta^0 \right\|_{L^1} \left\| \Phi_\varepsilon^{t_1} - \Phi_\varepsilon^{t_2} \right\|_{L^\infty}$$

for any $(k, t_1, t_2) \in \mathbb{Z}_{\ge 0} \times \mathbb{R}^2$. This shows the first claim, and in particular that $\| u_\varepsilon(\theta_\varepsilon^t) \|_{\dot{C}^{0,1}}$ is continuous in t.

Next, letting $x' := x + \Gamma_\varepsilon^{t_0, t_2}(x)$, we see that

$$\Gamma_\varepsilon^{t_0, t_1}(x) = x' + \Gamma_\varepsilon^{t_2, t_1}(x') - x = \Gamma_\varepsilon^{t_2, t_1}(x') + \Gamma_\varepsilon^{t_0, t_2}(x).$$

So with $y' := y + \Gamma_\varepsilon^{t_0, t_2}(y)$, the left-hand side of (2.6) is just $|\Gamma_\varepsilon^{t_2, t_1}(x') - \Gamma_\varepsilon^{t_2, t_1}(y')|$, while the last factor is $|x' - y'|$. The result now follows from the definition of $\Gamma_\varepsilon^{t_2, t}$. $\qquad \square$

Next, recall that the *upper-right* and *lower-left Dini derivatives* of $f : (a,b) \to \mathbb{R}$ are

$$\partial_t^+ f(t) := \limsup_{h \to 0^+} \frac{f(t+h) - f(t)}{h} \qquad \text{and} \qquad \partial_{t-} f(t) := \liminf_{h \to 0^-} \frac{f(t+h) - f(t)}{h}.$$

Proposition 2.6. $L_\mu(\Phi_{\varepsilon*}^t \Theta)$ *is continuous in t and*

$$\max\left\{\partial_t^+ L_\mu(\Phi_{\varepsilon*}^t \Theta), -\partial_{t-} L_\mu(\Phi_{\varepsilon*}^t \Theta)\right\} \leq \left\|u_\varepsilon(\theta_\varepsilon^t)\right\|_{\dot{C}^{0,1}} L_\mu(\Phi_{\varepsilon*}^t \Theta).$$

Proof. Fix any $t_1, t_2 \in \mathbb{R}$, $x \in \mathbb{R}^2$, $\lambda \in \mathcal{L}$, and $\eta > 0$. Pick $y \in \partial\Theta^\lambda$ such that

$$\left|\Phi_\varepsilon^{t_1}(x) - \Phi_\varepsilon^{t_1}(y)\right| \leq d(\Phi_\varepsilon^{t_1}(x), \partial\Phi_\varepsilon^{t_1}(\Theta^\lambda)) + \eta.$$

Then Lemma 2.5 and the inequality $\left|\frac{1}{a^{2\alpha}} - \frac{1}{b^{2\alpha}}\right| \leq \frac{|a-b|}{ab^{2\alpha}}$ for $a, b > 0$ show that

$$\frac{1}{\left(d(\Phi_\varepsilon^{t_1}(x), \partial\Phi_\varepsilon^{t_1}(\Theta^\lambda)) + 2\eta\right)^{2\alpha}} - \frac{1}{\left(d(\Phi_\varepsilon^{t_2}(x), \partial\Phi_\varepsilon^{t_2}(\Theta^\lambda)) + \eta\right)^{2\alpha}}$$

$$\leq \frac{1}{\left(\left|\Phi_\varepsilon^{t_1}(x) - \Phi_\varepsilon^{t_1}(y)\right| + \eta\right)^{2\alpha}} - \frac{1}{\left(\left|\Phi_\varepsilon^{t_2}(x) - \Phi_\varepsilon^{t_2}(y)\right| + \eta\right)^{2\alpha}}$$

$$\leq \frac{\left|\Phi_\varepsilon^{t_1}(x) - \Phi_\varepsilon^{t_1}(y) - \Phi_\varepsilon^{t_2}(x) + \Phi_\varepsilon^{t_2}(y)\right|}{\left(\left|\Phi_\varepsilon^{t_1}(x) - \Phi_\varepsilon^{t_1}(y)\right| + \eta\right)\left(\left|\Phi_\varepsilon^{t_2}(x) - \Phi_\varepsilon^{t_2}(y)\right| + \eta\right)^{2\alpha}} \qquad (2.7)$$

$$\leq \frac{\exp\left(\left|\int_{t_2}^{t_1} \|u_\varepsilon(\theta_\varepsilon^\tau)\|_{\dot{C}^{0,1}}\, d\tau\right|\right) - 1}{(d(\Phi_\varepsilon^{t_2}(x), \partial\Phi_\varepsilon^{t_2}(\Theta^\lambda)) + \eta)^{2\alpha}},$$

so letting $\eta \to 0^+$, integrating over λ, and then taking supremum over $x \in \mathbb{R}^2$ shows

$$L_\mu(\Phi_{\varepsilon*}^{t_1}\Theta) \leq \exp\left(\left|\int_{t_2}^{t_1} \|u_\varepsilon(\theta_\varepsilon^\tau)\|_{\dot{C}^{0,1}}\, d\tau\right|\right) L_\mu(\Phi_{\varepsilon*}^{t_2}\Theta).$$

Since $t_1, t_2 \in \mathbb{R}$ were arbitrary, both claims follow from this. $\qquad\qquad\square$

From Lemma 2.1, Proposition 2.6, and a Grönwall-type argument we now obtain the following result.

Corollary 2.7. *With C_α from Lemma 2.1, for all $t \in \mathbb{R}$ we have*

$$\max\left\{\partial_t^+ L_\mu(\Phi_{\varepsilon*}^t \Theta), -\partial_{t-} L_\mu(\Phi_{\varepsilon*}^t \Theta)\right\} \leq C_\alpha L_\mu(\Phi_{\varepsilon*}^t \Theta)^2.$$

In particular, for all $t \in \left(-\frac{1}{C_\alpha L_\mu(\Theta)}, \frac{1}{C_\alpha L_\mu(\Theta)}\right)$ we have

$$L_\mu(\Phi_{\varepsilon*}^t \Theta) \leq \frac{L_\mu(\Theta)}{1 - C_\alpha L_\mu(\Theta)\, |t|}.$$

Proposition 2.8. *Let $T_0 := \frac{1}{2C_\alpha L_\mu(\Theta)}$, with C_α from Lemma 2.1. There is $\Psi := \lim_{\varepsilon \to 0} \Psi_\varepsilon \in C\left([-T_0, T_0]; BC(\mathbb{R}^2; \mathbb{R}^2)\right)$, and $\Phi^t := \mathrm{Id} + \Psi^t$ is a measure-preserving homeomorphism for each $t \in [-T_0, T_0]$ that solves (1.8). Moreover, for each $t \in [-T_0, T_0]$ we have*

$$L_\mu(\Phi_*^t \Theta) \leq \sup_{\varepsilon > 0} L_\mu(\Phi_{\varepsilon*}^t \Theta) \leq 2L_\mu(\Theta)$$

and

$$\max\left\{\left\|\Phi^t\right\|_{\dot{C}^{0,1}},\left\|(\Phi^t)^{-1}\right\|_{\dot{C}^{0,1}}\right\}\le\sup_{\varepsilon>0}\max\left\{\left\|\Phi_\varepsilon^t\right\|_{\dot{C}^{0,1}},\left\|(\Phi_\varepsilon^t)^{-1}\right\|_{\dot{C}^{0,1}}\right\}\le e^{2C_\alpha L_\mu(\Theta)|t|}.$$

Proof. Corollary 2.7 shows that

$$M:=\sup_{\varepsilon>0}\sup_{t\in[-T_0,T_0]}L_\mu(\Phi_{\varepsilon*}^t\Theta)\in[L_\mu(\Theta),2L_\mu(\Theta)].$$

We may assume that $L_\mu(\Theta)>0$ because otherwise $\theta^0\equiv 0$ and the result follows trivially. Fix any $t_0\in[-T_0,T_0]$ and pick any $t\in[-T_0,T_0]$, $\varepsilon>0$, and $\varepsilon'\in(0,\varepsilon)$. Then Lemmas 2.1, 2.2, and 2.3 show that

$$\begin{aligned}
\left\|u_\varepsilon(\theta_\varepsilon^t)\right.&\left.\circ(\operatorname{Id}+\Gamma_\varepsilon^{t_0,t})-u_{\varepsilon'}(\theta_{\varepsilon'}^t)\circ(\operatorname{Id}+\Gamma_{\varepsilon'}^{t_0,t})\right\|_{L^\infty}\\
&\le\left\|u_\varepsilon(\theta_\varepsilon^t)\right\|_{\dot{C}^{0,1}}\left\|\Gamma_\varepsilon^{t_0,t}-\Gamma_{\varepsilon'}^{t_0,t}\right\|_{L^\infty}+\left\|u_\varepsilon(\theta_\varepsilon^t)-u(\theta_\varepsilon^t)\right\|_{L^\infty}\\
&\quad+\left\|u(\theta_\varepsilon^t)-u(\theta_{\varepsilon'}^t)\right\|_{L^\infty}+\left\|u(\theta_{\varepsilon'}^t)-u_{\varepsilon'}(\theta_{\varepsilon'}^t)\right\|_{L^\infty}\\
&\le C_\alpha M\left\|\Gamma_\varepsilon^{t_0,t}-\Gamma_{\varepsilon'}^{t_0,t}\right\|_{L^\infty}+C_\alpha M\left\|\Phi_\varepsilon^t-\Phi_{\varepsilon'}^t\right\|_{L^\infty}+C_\alpha\left\|\theta^0\right\|_{L^\infty}\varepsilon^{1-2\alpha}
\end{aligned}\tag{2.8}$$

where C_α (which we now fix for the rest of the proof) is two times the maximum of all the C_α's appearing in those lemmas. Integrating (2.8) between any $t_1,t_2\in[-T_0,T_0]$ yields

$$\begin{aligned}
\left\|\Gamma_\varepsilon^{t_0,t_1}\right.&\left.-\Gamma_{\varepsilon'}^{t_0,t_1}-\Gamma_\varepsilon^{t_0,t_2}+\Gamma_{\varepsilon'}^{t_0,t_2}\right\|_{L^\infty}\\
&\le C_\alpha M\left|\int_{t_2}^{t_1}\left\|\Gamma_\varepsilon^{t_0,\tau}-\Gamma_{\varepsilon'}^{t_0,\tau}\right\|_{L^\infty}d\tau\right|+C_\alpha M\left|\int_{t_2}^{t_1}\left\|\Phi_\varepsilon^\tau-\Phi_{\varepsilon'}^\tau\right\|_{L^\infty}d\tau\right|\\
&\quad+C_\alpha\left\|\theta^0\right\|_{L^\infty}|t_1-t_2|\,\varepsilon^{1-2\alpha}.
\end{aligned}\tag{2.9}$$

In particular, taking $t_0=0$, dividing by $|t_1-t_2|$, and letting $t_1\to t_2^\pm$ shows that

$$\max\left\{\partial_t^+\left\|\Psi_\varepsilon^t-\Psi_{\varepsilon'}^t\right\|_{L^\infty},-\partial_{t-}\left\|\Psi_\varepsilon^t-\Psi_{\varepsilon'}^t\right\|_{L^\infty}\right\}\le 2C_\alpha M\left\|\Psi_\varepsilon^t-\Psi_{\varepsilon'}^t\right\|_{L^\infty}+C_\alpha\left\|\theta^0\right\|_{L^\infty}\varepsilon^{1-2\alpha}$$

for each $t\in[-T_0,T_0]$, and then a Grönwall-type argument yields

$$\left\|\Phi_\varepsilon^t-\Phi_{\varepsilon'}^t\right\|_{L^\infty}=\left\|\Psi_\varepsilon^t-\Psi_{\varepsilon'}^t\right\|_{L^\infty}\le\frac{\left\|\theta^0\right\|_{L^\infty}}{2M}(e^{2C_\alpha M|t|}-1)\varepsilon^{1-2\alpha}.$$

Applying this inequality to (2.9), dividing by $|t_1-t_2|$ and then sending $t_1\to t_2^\pm$ shows that

$$\begin{aligned}
\max\left\{\partial_t^+\left\|\Gamma_\varepsilon^{t_0,t}-\Gamma_{\varepsilon'}^{t_0,t}\right\|_{L^\infty}\right.&\left.,-\partial_{t-}\left\|\Gamma_\varepsilon^{t_0,t}-\Gamma_{\varepsilon'}^{t_0,t}\right\|_{L^\infty}\right\}\\
&\le C_\alpha M\left\|\Gamma_\varepsilon^{t_0,t}-\Gamma_{\varepsilon'}^{t_0,t}\right\|_{L^\infty}+C_\alpha\left\|\theta^0\right\|_{L^\infty}e^{2C_\alpha M|t|}\varepsilon^{1-2\alpha}
\end{aligned}$$

for all $t\in[-T_0,T_0]$, so a Grönwall-type argument yields

$$\left\|\Gamma_\varepsilon^{t_0,t}-\Gamma_{\varepsilon'}^{t_0,t}\right\|_{L^\infty}\le\frac{\left\|\theta^0\right\|_{L^\infty}e^{2C_\alpha MT_0}\varepsilon^{1-2\alpha}}{M}\left(e^{C_\alpha M|t-t_0|}-1\right).$$

Therefore, $\Gamma_\varepsilon^{t_0,\cdot}$ converges uniformly to some $\Gamma^{t_0,\cdot}:[-T_0,T_0]\to BC(\mathbb{R}^2;\mathbb{R}^2)$ as $\varepsilon\to 0$.

Let $\Psi^t:=\Gamma^{0,t}$. Since $(\operatorname{Id}+\Gamma_\varepsilon^{t_0,t_1})\circ(\operatorname{Id}+\Gamma_\varepsilon^{t_1,t_0})=\operatorname{Id}$ for all $t_0,t_1\in[-T_0,T_0]$ and $\varepsilon>0$, sending $\varepsilon\to 0$ shows that $(\operatorname{Id}+\Gamma^{t_0,t_1})\circ(\operatorname{Id}+\Gamma^{t_1,t_0})=\operatorname{Id}$. In particular, $\Phi^t:=\operatorname{Id}+\Psi^t$ is a homeomorphism whose inverse is $\operatorname{Id}+\Gamma^{t,0}$. Also, Lemma 2.5 and the definition of C_α show that $\|\operatorname{Id}+\Gamma_\varepsilon^{t_0,t}\|_{\dot{C}^{0,1}}\le e^{C_\alpha M|t-t_0|}$ for all $t_0,t\in[-T_0,T_0]$ and $\varepsilon>0$, thus

$\max\left\{\|\Phi^t\|_{\dot{C}^{0,1}}, \|(\Phi^t)^{-1}\|_{\dot{C}^{0,1}}\right\} \le e^{C_\alpha M|t|}$ holds for all $t \in [-T_0, T_0]$. By Fatou's lemma we also have $L_\mu(\Phi^t_* \Theta) \le \liminf_{\varepsilon \to 0} L_\mu(\Phi^t_{\varepsilon *}\Theta) \le M$ for each $t \in [-T_0, T_0]$. And since each Φ^t_ε is measure-preserving, their uniform limit Φ^t is also such because for any open set $U \subseteq \mathbb{R}^2$ we have that $\mathbb{1}_U \circ \Phi^t_\varepsilon \to \mathbb{1}_U \circ \Phi^t$ pointwise as $\varepsilon \to 0$.

It remains to show that Φ^t satisfies (1.8), that is, with $\theta^t := \theta^0 \circ (\Phi^t)^{-1}$ we have

$$\Phi^t = \mathrm{Id} + \int_0^t u(\theta^\tau) \circ \Phi^\tau \, d\tau \tag{2.10}$$

for each $t \in [-T_0, T_0]$. Taking $t_0 = 0$ and letting $\varepsilon' \to 0^+$ in (2.8) yields

$$\left\|u_\varepsilon(\theta^t_\varepsilon) \circ \Phi^t_\varepsilon - u(\theta^t) \circ \Phi^t\right\|_{L^\infty} \le 2C_\alpha M \left\|\Phi^t_\varepsilon - \Phi^t\right\|_{L^\infty} + C_\alpha \left\|\theta^0\right\|_{L^\infty} \varepsilon^{1-2\alpha},$$

which shows that the right-hand side of

$$\Phi^t_\varepsilon = \mathrm{Id} + \int_0^t u(\theta^\tau_\varepsilon) \circ \Phi^\tau_\varepsilon \, d\tau$$

converges uniformly to the right-hand side of (2.10) as $\varepsilon \to 0$. This now proves (1.8). $\qquad\square$

Proposition 2.9. *Let* (Θ_1, μ_1), (Θ_2, μ_2) *be generalized layer cake representations of* θ^0 *and* $\Phi_1, \Phi_2 \in C\left(I; C(\mathbb{R}^2; \mathbb{R}^2)\right)$ *be solutions to* (1.8) *on a compact interval* $I \ni 0$ *that are both measure-preserving homeomorphisms and* $\sup_{t \in I} \max_{i \in \{1,2\}} L_{\mu_i}(\Phi^t_{i*}\Theta_i) < \infty$. *Then* $\Phi_1 = \Phi_2$.

Proof. Let

$$M := \sup_{t \in I} \max_{i=1,2} L_{\mu_i}(\Phi^t_{i*}\Theta_i).$$

Then Lemmas 2.1 and 2.2 show that

$$\left\|u(\Phi^t_{1*}\theta^0) \circ \Phi^t_1 - u(\Phi^t_{2*}\theta^0) \circ \Phi^t_2\right\|_{L^\infty} \le \left\|u(\Phi^t_{1*}\theta^0)\right\|_{\dot{C}^{0,1}} \left\|\Phi^t_1 - \Phi^t_2\right\|_{L^\infty} + \left\|u(\Phi^t_{1*}\theta^0) - u(\Phi^t_{2*}\theta^0)\right\|_{L^\infty}$$
$$\le C_\alpha M \left\|\Phi^t_1 - \Phi^t_2\right\|_{L^\infty}$$

with some C_α, which together with continuity of $\|\Phi^t_1 - \Phi^t_2\|_{L^\infty}$ in t yields

$$\max\left\{\partial^+_t \left\|\Phi^t_1 - \Phi^t_2\right\|_{L^\infty}, -\partial_{t-} \left\|\Phi^t_1 - \Phi^t_2\right\|_{L^\infty}\right\} \le C_\alpha M \left\|\Phi^t_1 - \Phi^t_2\right\|_{L^\infty}.$$

A Grönwall-type argument finishes the proof. $\qquad\square$

Combining Propositions 2.8 and 2.9 with (1.11), the latter showing that the time spans of maximal solutions for any two generalized layer cake representations of θ^0 must coincide (recall that $\sup_{t \in J} \|(\Phi^t)^{-1}\|_{\dot{C}^{0,1}} < \infty$ for any compact interval $J \subseteq I$), now yields Theorem 1.3.

3. Proof of Theorem 1.5(i)–(ii)

Again, all constants C_α below can change from one inequality to another, but they always only depend on α. Fix any θ^0 satisfying the hypotheses and for each $\lambda \in \mathcal{L}$, let $z^{0,\lambda} \in \mathrm{PSC}(\mathbb{R}^2)$ be such that $\partial\Theta^\lambda = \mathrm{im}(z^{0,\lambda})$.

Let $\theta \colon I \to L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ be the Lagrangian solution to (1.1)–(1.2) from Theorem 1.3, with initial data θ^0 and flow map $\Phi \in C\left(I; C(\mathbb{R}^2; \mathbb{R}^2)\right)$, and consider $T_0 := \frac{1}{2C_\alpha L_\mu(\Theta)}$ as in Proposition 2.8 (note that $[-T_0, T_0] \subseteq I$ because I is maximal). Then since Φ^t is a homeomorphism for each $t \in I$, it follows that $\Phi^t_* \Theta$ is composed of simple closed curves (this

proves the first claim in Theorem 1.5(i)). We denote these $z^{t,\lambda} := \Phi^t \circ z^{0,\lambda} \in \mathrm{PSC}(\mathbb{R}^2)$, where $\Phi^t \circ z^{0,\lambda} \in \mathrm{CC}(\mathbb{R}^2)$ is the curve whose representative is $\Phi^t \circ \tilde{z}^{0,\lambda}$ whenever $\tilde{z}^{0,\lambda}$ is a representative of $z^{0,\lambda}$ (since $\{z^{t,\lambda}\}_{t \in I}$ is clearly a connected subset of $\mathrm{CC}(\mathbb{R}^2)$, [10, Lemma B.4] shows that each $z^{t,\lambda}$ is positively oriented).

Fix any $\varepsilon > 0$, and recall that $\Phi^t_\varepsilon := \mathrm{Id} + \Psi^t_\varepsilon$, where Ψ^t_ε is the solution to (2.4). For each $(t,\lambda) \in \mathbb{R} \times \mathcal{L}$ let $z^{t,\lambda}_\varepsilon := \Phi^t_\varepsilon \circ z^{0,\lambda}$ and $\theta^t_\varepsilon := \theta^0 \circ (\Phi^t_\varepsilon)^{-1}$, then fix any arclength parametrization of $z^{t,\lambda}_\varepsilon$ (we denote it again $z^{t,\lambda}_\varepsilon(\cdot)$) and for $s \in [0, \ell(z^{t,\lambda}_\varepsilon)]$ define

- $\ell^{t,\lambda}_\varepsilon := \ell(z^{t,\lambda}_\varepsilon)$,
- $\mathbf{T}^{t,\lambda}_\varepsilon(s) := \partial_s z^{t,\lambda}_\varepsilon(s)$,
- $\mathbf{N}^{t,\lambda}_\varepsilon(s) := \mathbf{T}^{t,\lambda}(s)^\perp$,
- $\kappa^{t,\lambda}_\varepsilon(s) := \partial^2_s z^{t,\lambda}_\varepsilon(s) \cdot \mathbf{N}^{t,\lambda}(s)$,
- $\Delta^{t,\lambda,\lambda'}_\varepsilon := \Delta(z^{t,\lambda}_\varepsilon, z^{t,\lambda'}_\varepsilon)$,
- $u^{t,\lambda}_\varepsilon(s) := u_\varepsilon(\theta^t_\varepsilon; z^{t,\lambda}_\varepsilon(s))$.

Proposition 2.8 shows that $\lim_{\varepsilon \to 0} z^{t,\lambda}_\varepsilon = z^{t,\lambda}$ in $\mathrm{CC}(\mathbb{R}^2)$, and as noted in [10, Section 4],

$$\partial^2_s z^{t,\lambda}_\varepsilon(s) = \partial_s \mathbf{T}^{t,\lambda}_\varepsilon(s) = \kappa^{t,\lambda}_\varepsilon(s) \mathbf{N}^{t,\lambda}_\varepsilon(s) \qquad \text{and} \qquad \partial_s \mathbf{N}^{t,\lambda}_\varepsilon(s) = -\kappa^{t,\lambda}_\varepsilon(s) \mathbf{T}^{t,\lambda}_\varepsilon(s)$$

holds as well. Then the argument in [10, Lemma 4.1] also applies here, and we obtain

$$\begin{aligned}
\partial_t \left\| z^{t,\lambda}_\varepsilon \right\|^2_{\dot{H}^2} = &-3 \int_{\ell^{t,\lambda}_\varepsilon \mathbb{T}} \kappa^{t,\lambda}_\varepsilon(s)^2 \left(\partial_s u^{t,\lambda}_\varepsilon(s) \cdot \mathbf{T}^{t,\lambda}_\varepsilon(s) \right) ds \\
&+ 2 \int_{\ell^{t,\lambda}_\varepsilon \mathbb{T}} \kappa^{t,\lambda}_\varepsilon(s) \left(\partial^2_s u^{t,\lambda}_\varepsilon(s) \cdot \mathbf{N}^{t,\lambda}_\varepsilon(s) \right) ds.
\end{aligned} \tag{3.1}$$

In the proof of this we fix some *constant-speed parametrization* $\tilde{z}^{t,\lambda}_\varepsilon \colon \mathbb{T} \to \mathbb{R}^2$ of $z^{t,\lambda}_\varepsilon$ (that is, $\tilde{\gamma}(\ell(z^{t,\lambda}_\varepsilon) \cdot)$ for some arclength parametrization $\tilde{\gamma}$ of $z^{t,\lambda}_\varepsilon$), and for each $h \in \mathbb{R}$ let $\tilde{z}^{t+h,\lambda}_\varepsilon := \Phi^{t+h}_\varepsilon \circ (\Phi^t_\varepsilon)^{-1} \circ \tilde{z}^{t,\lambda}_\varepsilon$. Since (2.5) is globally well-posed in $BC^2(\mathbb{R}^2; \mathbb{R}^2)$ (see the paragraph before Lemma 2.5), it easily follows that $\tilde{z}^{t,\lambda}_\varepsilon \in H^2(\mathbb{T}; \mathbb{R}^2)$ and

$$\tilde{z}^{t+h,\lambda}_\varepsilon = \tilde{z}^{t,\lambda}_\varepsilon + \int_t^{t+h} u_\varepsilon(\theta^\tau_\varepsilon) \circ \tilde{z}^{\tau,\lambda}_\varepsilon \, d\tau \tag{3.2}$$

holds for all $h \in \mathbb{R}$ (in $H^2(\mathbb{T}; \mathbb{R}^2)$).

Remark. Below we will also consider the above setup with initial data being (θ^{t_0}, z^{t_0}) for some $t_0 \in I$ instead of (θ^0, z^0). Since we do not yet know whether the $z^{t_0,\lambda}$ are H^2 curves when $t_0 \neq 0$, we cannot define $\kappa^{t,\lambda}_\varepsilon$ for these initial data and also cannot yet claim (3.2) to hold in $H^2(\mathbb{T}; \mathbb{R}^2)$. However, since Φ^{t_0} is Lipschitz by Proposition 2.8, so are the $z^{t_0,\lambda}$ and then (3.2) holds in $C^{0,1}(\mathbb{T}; \mathbb{R}^2)$. We will use this in the following results, up to Lemma 3.4.

Lemma 3.1. *With C_α from Lemma 2.1, for any $(t,\lambda) \in [-T_0, T_0] \times \mathcal{L}$ and $\varepsilon > 0$ we have*

$$\left| \partial_t \ell^{t,\lambda}_\varepsilon \right| \leq 2 C_\alpha L_\mu(\Theta) \ell^{t,\lambda}_\varepsilon \tag{3.3}$$

Proof. With $\tilde{z}^{t,\lambda}_\varepsilon$ as above, we have $\left|\partial_\xi \tilde{z}^{t,\lambda}_\varepsilon(\xi)\right| = \ell^{t,\lambda}_\varepsilon > 0$ for any $(t,\lambda) \in \mathbb{R} \times \mathcal{L}$. Then for any $(h,\xi) \in \mathbb{R} \times \mathbb{T}$ we get

$$\left|\partial_\xi \tilde{z}^{t+h,\lambda}_\varepsilon(\xi)\right| - \left|\partial_\xi \tilde{z}^{t,\lambda}_\varepsilon(\xi)\right|$$

$$= \frac{2 \int_t^{t+h} \frac{d}{d\xi} u_\varepsilon(\theta^\tau_\varepsilon; \tilde{z}^{\tau,\lambda}_\varepsilon(\xi)) \cdot \partial_\xi \tilde{z}^{t,\lambda}_\varepsilon(\xi)\, d\tau}{\left|\partial_\xi \tilde{z}^{t+h,\lambda}_\varepsilon(\xi)\right| + \left|\partial_\xi \tilde{z}^{t,\lambda}_\varepsilon(\xi)\right|} + \frac{\left|\int_t^{t+h} \frac{d}{d\xi} u_\varepsilon(\theta^\tau_\varepsilon; \tilde{z}^{\tau,\lambda}_\varepsilon(\xi))\, d\tau\right|^2}{\left|\partial_\xi \tilde{z}^{t+h,\lambda}_\varepsilon(\xi)\right| + \left|\partial_\xi \tilde{z}^{t,\lambda}_\varepsilon(\xi)\right|}.$$

Since $\|D(u_\varepsilon(\theta^\tau_\varepsilon))\|_{L^\infty}$ is continuous in τ by Lemma 2.5, integrating in ξ, dividing by h, and then letting $h \to 0$ shows that

$$\partial_t \ell^{t,\lambda}_\varepsilon = \int_{\mathbb{T}} \frac{d}{d\xi} u_\varepsilon(\theta^t_\varepsilon; \tilde{z}^{t,\lambda}_\varepsilon(\xi)) \cdot \frac{\partial_\xi \tilde{z}^{t,\lambda}_\varepsilon(\xi)}{\left|\partial_\xi \tilde{z}^{t,\lambda}_\varepsilon(\xi)\right|}\, d\xi = \int_{\ell^{t,\lambda}_\varepsilon \mathbb{T}} \partial_s u^{t,\lambda}_\varepsilon(s) \cdot \mathbf{T}^{t,\lambda}_\varepsilon(s)\, ds.$$

The result now follows from Lemma 2.1 and Corollary 2.7. $\qquad\square$

Lemma 3.2. *With C_α from Lemma 2.1, for any $(t,\lambda) \in I \times \mathcal{L}$ and all small enough $h \in \mathbb{R}$ we have*

$$e^{-3C_\alpha L_\mu(\Phi^t_* \Theta)|h|} \ell(z^{t,\lambda}) \le \ell(z^{t+h,\lambda}) \le e^{3C_\alpha L_\mu(\Phi^t_* \Theta)|h|} \ell(z^{t,\lambda}).$$

Proof. Since $\ell\colon \mathrm{CC}(\mathbb{R}^2) \to [0,\infty]$ is lower semi-continuous (by definition) and $\lim_{\varepsilon \to 0} z^{t,\lambda}_\varepsilon = z^{t,\lambda}$ in $\mathrm{CC}(\mathbb{R}^2)$, a Grönwall-type argument applied to (3.3) shows that for $|t| \le T_0$ we have

$$\ell(z^{t,\lambda}) \le e^{2C_\alpha L_\mu(\Theta)|t|} \ell(z^{0,\lambda}). \tag{3.4}$$

Next fix any $t_0 \in I$ with $|t_0| \le \frac{T_0}{2}$. Repeating the proof of (3.4) with $z^{t_0,\lambda}$ in place of $z^{0,\lambda}$ (recall the remark after (3.2)) shows that

$$\ell(z^{t,\lambda}) \le e^{2C_\alpha L_\mu(\Phi^{t_0}_* \Theta)|t-t_0|} \ell(z^{t_0,\lambda})$$

whenever $|t - t_0| \le \frac{1}{2C_\alpha L_\mu(\Phi^{t_0}_* \Theta)}$. Since $L_\mu(\Phi^{t_0}_* \Theta) \le \frac{4}{3} L_\mu(\Theta)$ by Corollary 2.7, we see that

$$\ell(z^{0,\lambda}) \le e^{3C_\alpha L_\mu(\Theta)|t_0|} \ell(z^{t_0,\lambda}),$$

and so

$$e^{-3C_\alpha L_\mu(\Theta)|t|} \ell(z^{0,\lambda}) \le \ell(z^{t,\lambda}) \le e^{3C_\alpha L_\mu(\Theta)|t|} \ell(z^{0,\lambda})$$

holds whenever $|t| \le \frac{T_0}{2}$. Applying this with (θ^t, t, h) in place of $(\theta^0, 0, t)$, for any $t \in I$, now shows the claim. $\qquad\square$

Proposition 3.3. *$R_\mu(\Phi^t_* \Theta)$ is continuous in t. In particular, for any compact interval $J \subseteq I$ we have*

$$\sup_{t \in J} R_\mu(\Phi^t_* \Theta) < \infty.$$

Proof. Take any compact interval $J \subseteq I$ containing 0. Lemmas 2.1, 2.5 and Corollary 2.7 show that

$$\left|\Phi^{t_1}_\varepsilon(x) - \Phi^{t_1}_\varepsilon(y) - \Phi^{t_2}_\varepsilon(x) + \Phi^{t_2}_\varepsilon(y)\right| \le \left(e^{2C_\alpha L_\mu(\Theta)|t_1 - t_2|} - 1\right) \left|\Phi^{t_1}_\varepsilon(x) - \Phi^{t_1}_\varepsilon(y)\right| \tag{3.5}$$

for all $\varepsilon > 0$, $t_1, t_2 \in [-T_0, T_0]$ and $x, y \in \mathbb{R}^2$. Hence, taking $\varepsilon \to 0$ shows that (3.5) continues to hold with Φ in place of Φ_ε. Applying the same argument with $\Phi_*^t \Theta$ in place of Θ (recall the remark after (3.2)), shows that each $t \in J$ has a neighborhood such that

$$\left| \Phi^{t_1}(x) - \Phi^{t_1}(y) - \Phi^{t_2}(x) + \Phi^{t_2}(y) \right| \leq (e^{2C_\alpha M |t_1 - t_2|} - 1) \left| \Phi^{t_1}(x) - \Phi^{t_1}(y) \right| \qquad (3.6)$$

holds for any $t_1, t_2 \in J$ in that neighborhood, where

$$M := \sup_{t \in J} L_\mu(\Phi_*^t \Theta) < \infty.$$

Since J is an interval, it follows that (3.6) in fact holds for all $t_1, t_2 \in J$.

Fix $t_1, t_2 \in J$, $\lambda, \lambda' \in \mathcal{L}$, and $\eta > 0$. Then there are $x \in \mathrm{im}(z^{0,\lambda})$ and $y \in \mathrm{im}(z^{0,\lambda'})$ such that $\Delta(z^{t_1,\lambda}, z^{t_1,\lambda'}) = |\Phi^{t_1}(x) - \Phi^{t_1}(y)|$. A similar argument as in (2.7) now shows that

$$\frac{1}{(\Delta(z^{t_1,\lambda}, z^{t_1,\lambda'}) + \eta)^{2\alpha}} - \frac{1}{(\Delta(z^{t_2,\lambda}, z^{t_2,\lambda'}) + \eta)^{2\alpha}}$$
$$\leq \frac{|\Phi^{t_1}(x) - \Phi^{t_1}(y) - \Phi^{t_2}(x) + \Phi^{t_2}(y)|}{(|\Phi^{t_1}(x) - \Phi^{t_1}(y)| + \eta)(|\Phi^{t_2}(x) - \Phi^{t_2}(y)| + \eta)^{2\alpha}}$$
$$\leq \frac{e^{2C_\alpha M |t_1 - t_2|} - 1}{(\Delta(z^{t_2,\lambda}, z^{t_2,\lambda'}) + \eta)^{2\alpha}},$$

thus

$$\frac{1}{(\Delta(z^{t_1,\lambda}, z^{t_1,\lambda'}) + \eta)^{2\alpha}} \leq \frac{e^{2C_\alpha M |t_1 - t_2|}}{\Delta(z^{t_2,\lambda}, z^{t_2,\lambda'})^{2\alpha}}. \qquad (3.7)$$

Lemma 3.2 yields

$$e^{-3C_\alpha M |t_1 - t_2|} \ell(z^{t_2,\lambda}) \leq \ell(z^{t_1,\lambda}) \leq e^{3C_\alpha M |t_1 - t_2|} \ell(z^{t_2,\lambda})$$

for any $t_1, t_2 \in J$ because J is an interval, and so we get

$$\frac{\ell(z^{t_1,\lambda})^{1/2}}{\ell(z^{t_1,\lambda'})^{1/2} (\Delta(z^{t_1,\lambda}, z^{t_1,\lambda'}) + \eta)^{2\alpha}} \leq \frac{e^{5C_\alpha M |t_1 - t_2|} \ell(z^{t_2,\lambda})^{1/2}}{\ell(z^{t_2,\lambda'})^{1/2} \Delta(z^{t_2,\lambda}, z^{t_2,\lambda'})^{2\alpha}}. \qquad (3.8)$$

Letting $\eta \to 0^+$, integrating in λ', and then taking the supremum over $\lambda \in \mathcal{L}$ shows that

$$R_\mu(\Phi_*^{t_1} \Theta) \leq e^{5C_\alpha M |t_1 - t_2|} R_\mu(\Phi_*^{t_2} \Theta). \qquad (3.9)$$

Since $t_1, t_2 \in J$ were arbitrary, the claim follows. $\square$

This proves the second claim in Theorem 1.5(i). Our proof of Theorem 1.5(ii) will use the following extension to $R_\mu(\Phi_{\varepsilon*}^t \Theta)$ of the bound on $R_\mu(\Phi_*^t \Theta)$ from the last proof.

Lemma 3.4. $R_\mu(\Phi_{\varepsilon*}^t \Theta)$ *is continuous in t for each $\varepsilon > 0$, and*

$$R_\mu(\Phi_{\varepsilon*}^t \Theta) \leq e^{4C_\alpha L_\mu(\Theta) |t|} R_\mu(\Theta) \leq 8 R_\mu(\Theta) \qquad (3.10)$$

holds for any $t \in [-T_0, T_0]$, with C_α from Lemma 2.1.

Proof. We have (3.3) for $t \in [-T_0, T_0]$, so a Grönwall-type argument shows that

$$e^{-2C_\alpha L_\mu(\Theta)|t_1-t_2|} \ell_\varepsilon^{t_2,\lambda} \leq \ell_\varepsilon^{t_1,\lambda} \leq e^{2C_\alpha L_\mu(\Theta)|t_1-t_2|} \ell_\varepsilon^{t_2,\lambda}$$

for any $t_1, t_2 \in [-T_0, T_0]$ and $\lambda \in \mathcal{L}$. Since (3.5) holds, a similar argument as in (3.7) shows

$$\frac{1}{\left(\Delta_\varepsilon^{t_1,\lambda,\lambda'} + \eta\right)^{2\alpha}} \leq \frac{e^{2C_\alpha L_\mu(\Theta)|t_1-t_2|}}{(\Delta_\varepsilon^{t_2,\lambda,\lambda'})^{2\alpha}}$$

for any $t_1, t_2 \in [-T_0, T_0]$, $\lambda, \lambda' \in \mathcal{L}$ and $\eta > 0$. Hence we see that (3.8), and thus also (3.9), continue to hold with $(z_\varepsilon, 4C_\alpha L_\mu(\Theta), \Phi_\varepsilon)$ in place of $(z, 5C_\alpha M, \Phi)$. This now shows both claims (the second inequality in (3.10) follows by the definition of T_0 and $e^2 \leq 8$). $\qquad \square$

We now turn to Theorem 1.5(ii), which we prove by using Lemma 3.1 and by estimating the right-hand side of (3.1) in lemmas below.

Lemma 3.5. *There is C_α such that for any $\beta \in (0, 1]$, any $C^{1,\beta}$ closed curve $\gamma \colon \ell\mathbb{T} \to \mathbb{R}^2$ parametrized by arclength, and any $x \in \mathbb{R}^2$ we have*

$$\int_{\ell\mathbb{T}} \frac{ds}{|x - \gamma(s)|^{1+2\alpha}} \leq C_\alpha \frac{\ell \|\gamma\|_{\dot{C}^{1,\beta}}^{1/\beta}}{d(x, \operatorname{im}(\gamma))^{2\alpha}}.$$

Proof. Let $d := \frac{1}{4} \|\gamma\|_{\dot{C}^{1,\beta}}^{-1/\beta}$ and $\Delta := d(x, \operatorname{im}(\gamma))$. Then [10, Lemma A.2] shows that

$$\int_{\ell\mathbb{T}} \frac{ds}{|x - \gamma(s)|^{1+2\alpha}} \leq \frac{\ell}{4d} \left(\int_{|s|\leq\Delta} \frac{ds}{\Delta^{1+2\alpha}} + \int_{\Delta\leq|s|\leq 2d} \frac{ds}{|s/2|^{1+2\alpha}} \right) + \frac{1}{\Delta^{2\alpha}} \int_{\ell\mathbb{T}} \frac{ds}{d}$$

$$\leq \frac{\ell}{2d\Delta^{2\alpha}} + \frac{\ell}{2^{1-2\alpha}\alpha d\Delta^{2\alpha}} + \frac{\ell}{d\Delta^{2\alpha}} = C_\alpha \frac{\ell \|\gamma\|_{\dot{C}^{1,\beta}}^{1/\beta}}{\Delta^{2\alpha}}.$$

$\qquad \square$

Lemma 3.6. *For any $\theta \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ with a generalized layer cake representation (Θ, μ) such that $L_\mu(\Theta) < \infty$, and for any $\varepsilon > 0$ and $x, h_1, h_2 \in \mathbb{R}^2$, we have*

$$D^2(u_\varepsilon(\theta))(x)(h_1, h_2) = \int_\mathcal{L} \int_{\mathbb{R}^2} D^2(\nabla^\perp K_\varepsilon)(x-y)(h_1, h_2) \left(\mathbb{1}_{\Theta^\lambda}(y) - \mathbb{1}_{\Theta^\lambda}(x)\right) dy \, d\mu(\lambda)$$

$$= \int_\mathcal{L} \int_{\Theta^\lambda} D^2(\nabla^\perp K_\varepsilon)(x-y)(h_1, h_2) \, dy \, d\mu(\lambda).$$

Moreover, there is C_α such that

$$\int_\mathcal{L} \int_{\mathbb{R}^2} \left|D^2(\nabla^\perp K_\varepsilon)(x-y)\right| \left|\mathbb{1}_{\Theta^\lambda}(y) - \mathbb{1}_{\Theta^\lambda}(x)\right| d|\mu|(\lambda) \, dy \leq \frac{C_\alpha}{\varepsilon} \int_\mathcal{L} \frac{d|\mu|(\lambda)}{d(x, \partial\Theta^\lambda)^{2\alpha}}.$$

Proof. Oddness of $D^2(\nabla^\perp K_\varepsilon)$ shows that

$$D^2(u_\varepsilon(\theta))(x)(h_1, h_2) = \int_{\mathbb{R}^2} D^2(\nabla^\perp K_\varepsilon)(x-y)(h_1, h_2)(\theta(y) - \theta(x)) \, dy$$

$$= \int_{\mathbb{R}^2} \int_\mathcal{L} D^2(\nabla^\perp K_\varepsilon)(x-y)(h_1, h_2) \left(\mathbb{1}_{\Theta^\lambda}(y) - \mathbb{1}_{\Theta^\lambda}(x)\right) d\mu(\lambda) \, dy.$$

Then proceeding as in Lemma 2.1 and using $\left| D^2(\nabla^\perp K_\varepsilon)(x-y) \right| \leq \frac{C_\alpha}{\varepsilon|x-y|^{2+2\alpha}}$ in place of $\left| D(\nabla^\perp K_\varepsilon)(x-y) \right| \leq \frac{C_\alpha}{|x-y|^{2+2\alpha}}$ proves the second claim. Fubini's theorem now yields the first equality of the first claim, and the second one follows by oddness of $D^2(\nabla^\perp K_\varepsilon)$. $\qquad\square$

Lemma 3.7. *There is C_α such that for each $(t,\lambda) \in \mathbb{R} \times \mathcal{L}$ and $\varepsilon > 0$ we have*

$$\left| \partial_t \left\| z_\varepsilon^{t,\lambda} \right\|_{\dot{H}^2}^2 \right| \leq C_\alpha (L_\mu(\Phi_{\varepsilon*}^t \Theta) + R_\mu(\Phi_{\varepsilon*}^t \Theta)) \, Q(\Phi_{\varepsilon*}^t \Theta) \left\| z_\varepsilon^{t,\lambda} \right\|_{\dot{H}^2}^2 \tag{3.11}$$

Proof. With $z_\varepsilon^{t,\lambda}(\cdot)$ being the previously fixed arclength parametrization of $z_\varepsilon^{t,\lambda}$, we have

$$\partial_s^2 u_\varepsilon^{t,\lambda}(s) = D^2(u_\varepsilon(\theta_\varepsilon^t))(z_\varepsilon^{t,\lambda}(s))(\mathbf{T}_\varepsilon^{t,\lambda}(s), \mathbf{T}_\varepsilon^{t,\lambda}(s)) + \kappa_\varepsilon^{t,\lambda}(s) D(u_\varepsilon(\theta_\varepsilon^t))(z_\varepsilon^{t,\lambda}(s))(\mathbf{N}_\varepsilon^{t,\lambda}(s)).$$

Hence (3.1) yields

$$\partial_t \left\| z_\varepsilon^{t,\lambda} \right\|_{\dot{H}^2}^2 = \int_{\ell_\varepsilon^{t,\lambda}\mathbb{T}} \kappa_\varepsilon^{t,\lambda}(s)^2 \left[2 D(u_\varepsilon(\theta_\varepsilon^t))(z_\varepsilon^{t,\lambda}(s))(\mathbf{N}_\varepsilon^{t,\lambda}(s)) \cdot \mathbf{N}_\varepsilon^{t,\lambda}(s) - 3\, \partial_s u_\varepsilon^{t,\lambda}(s) \cdot \mathbf{T}_\varepsilon^{t,\lambda}(s) \right] ds$$

$$+ 2 \int_{\ell_\varepsilon^{t,\lambda}\mathbb{T}} \kappa_\varepsilon^{t,\lambda}(s) \left[D^2(u_\varepsilon(\theta_\varepsilon^t))(z_\varepsilon^{t,\lambda}(s))(\mathbf{T}_\varepsilon^{t,\lambda}(s), \mathbf{T}_\varepsilon^{t,\lambda}(s)) \cdot \mathbf{N}_\varepsilon^{t,\lambda}(s) \right] ds. \tag{3.12}$$

Lemma 2.1 shows that the absolute value of the first integral is bounded by

$$5 \left\| u_\varepsilon(\theta_\varepsilon^t) \right\|_{\dot{C}^{0,1}} \left\| z_\varepsilon^{t,\lambda} \right\|_{\dot{H}^2}^2 \leq C_\alpha L_\mu(\Phi_{\varepsilon*}^t \Theta) \left\| z_\varepsilon^{t,\lambda} \right\|_{\dot{H}^2}^2 \leq C_\alpha L_\mu(\Phi_{\varepsilon*}^t \Theta) Q(\Phi_{\varepsilon*}^t \Theta) \left\| z_\varepsilon^{t,\lambda} \right\|_{\dot{H}^2}^2 ,$$

where the second inequality holds by $Q(\Phi_{\varepsilon*}^t \Theta) \geq 4$, which in turn follows from [10, Lemma A.1]. Hence, it remains to estimate the second integral, which we denote G_1. We will suppress t from the notation for the sake of simplicity because it will be fixed in the arguments below.

Since $L_\mu(\Phi_{\varepsilon*}\Theta) < \infty$, Lemma 3.6, Fubini's theorem, and Green's theorem show that

$$G_1 = \int_{\ell_\varepsilon^\lambda\mathbb{T}} \int_{\mathcal{L}} \int_{\Phi_\varepsilon(\Theta^{\lambda'})} \kappa_\varepsilon^\lambda(s) \mathbf{N}_\varepsilon^\lambda(s) \cdot D^2(\nabla^\perp K_\varepsilon)(z_\varepsilon^\lambda(s) - y)(\mathbf{T}_\varepsilon^\lambda(s), \mathbf{T}_\varepsilon^\lambda(s)) \, dy \, d\mu(\lambda') \, ds$$

$$= \int_{\mathcal{L}} \int_{\ell_\varepsilon^\lambda\mathbb{T}} \int_{\Phi_\varepsilon(\Theta^{\lambda'})} \kappa_\varepsilon^\lambda(s) \mathbf{N}_\varepsilon^\lambda(s) \cdot D^2(\nabla^\perp K_\varepsilon)(z_\varepsilon^\lambda(s) - y)(\mathbf{T}_\varepsilon^\lambda(s), \mathbf{T}_\varepsilon^\lambda(s)) \, dy \, ds \, d\mu(\lambda')$$

$$= -\int_{\mathcal{L}} \int_{\ell_\varepsilon^\lambda\mathbb{T} \times \ell_\varepsilon^{\lambda'}\mathbb{T}} \kappa_\varepsilon^\lambda(s) D^2 K_\varepsilon(z_\varepsilon^\lambda(s) - z_\varepsilon^{\lambda'}(s'))(\mathbf{T}_\varepsilon^\lambda(s), \mathbf{T}_\varepsilon^\lambda(s))(\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{N}_\varepsilon^\lambda(s)) \, ds \, ds' d\mu(\lambda').$$

Note that the last integrand is jointly measurable in (s,s') but not necessarily in (s,s',λ') because the parametrizations $z_\varepsilon^{\lambda'}(\cdot)$ are chosen independently from each other. From

$$\mathbf{T}_\varepsilon^\lambda(s) = (\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{T}_\varepsilon^\lambda(s))\mathbf{T}_\varepsilon^{\lambda'}(s') + (\mathbf{N}_\varepsilon^{\lambda'}(s') \cdot \mathbf{T}_\varepsilon^\lambda(s))\mathbf{N}_\varepsilon^{\lambda'}(s')$$

and $\mathbf{N}_\varepsilon^{\lambda'}(s') \cdot \mathbf{T}_\varepsilon^\lambda(s) = -\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{N}_\varepsilon^\lambda(s)$ it now follows that $|G_1|$ is bounded by the sum of

$$G_2 := \overline{\int_{\mathcal{L}}} \left| \int_{\ell_\varepsilon^\lambda\mathbb{T} \times \ell_\varepsilon^{\lambda'}\mathbb{T}} \kappa_\varepsilon^\lambda(s) D^2 K_\varepsilon(z_\varepsilon^\lambda(s) - z_\varepsilon^{\lambda'}(s'))(\mathbf{T}_\varepsilon^\lambda(s), \mathbf{T}_\varepsilon^{\lambda'}(s')) \right.$$

$$\left. (\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{N}_\varepsilon^\lambda(s))(\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{T}_\varepsilon^\lambda(s)) \, ds \, ds' \right| d|\mu|(\lambda'),$$

$$G_3 := \overline{\int_{\mathcal{L}}} \left| \int_{\ell_\varepsilon^\lambda \mathbb{T} \times \ell_\varepsilon^{\lambda'} \mathbb{T}} \kappa_\varepsilon^\lambda(s) D^2 K_\varepsilon(z_\varepsilon^\lambda(s) - z_\varepsilon^{\lambda'}(s'))(\mathbf{T}_\varepsilon^\lambda(s), \mathbf{N}_\varepsilon^{\lambda'}(s'))(\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{N}_\varepsilon^\lambda(s))^2 \, ds \, ds' \right| d|\mu|(\lambda'),$$

which we estimate separately next. Here $\overline{\int_{\mathcal{L}}} f(\lambda') \, d|\mu|(\lambda')$ for $f \colon \mathcal{L} \to [0, \infty]$ is the upper Lebesgue integral

$$\overline{\int_{\mathcal{L}}} f(\lambda') \, d|\mu|(\lambda') := \inf_g \int_{\mathcal{L}} g(\lambda') \, d|\mu|(\lambda'),$$

where the inf ranges over all measurable $g \geq f$.

Estimate for G_2. Since

$$\frac{\partial}{\partial s'} \left(DK_\varepsilon(z_\varepsilon^\lambda(s) - z_\varepsilon^{\lambda'}(s'))(\mathbf{T}_\varepsilon^\lambda(s))(\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{N}_\varepsilon^\lambda(s))(\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{T}_\varepsilon^\lambda(s)) \right)$$

$$= -D^2 K_\varepsilon(z_\varepsilon^\lambda(s) - z_\varepsilon^{\lambda'}(s'))(\mathbf{T}_\varepsilon^\lambda(s), \mathbf{T}_\varepsilon^{\lambda'}(s'))(\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{N}_\varepsilon^\lambda(s))(\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{T}_\varepsilon^\lambda(s))$$

$$+ \kappa_\varepsilon^{\lambda'}(s') DK_\varepsilon(z_\varepsilon^\lambda(s) - z_\varepsilon^{\lambda'}(s')) \left((\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{T}_\varepsilon^\lambda(s))^2 - (\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{N}_\varepsilon^\lambda(s))^2 \right),$$

we see that

$$G_2 \leq C_\alpha \overline{\int_{\mathcal{L}}} \int_{\ell_\varepsilon^\lambda \mathbb{T} \times \ell_\varepsilon^{\lambda'} \mathbb{T}} \frac{\left| \kappa_\varepsilon^\lambda(s) \kappa_\varepsilon^{\lambda'}(s') \right|}{|z_\varepsilon^\lambda(s) - z_\varepsilon^{\lambda'}(s')|^{1+2\alpha}} \, ds \, ds' d|\mu|(\lambda').$$

By the Schwarz inequality, the inner integral of the right-hand side is bounded by

$$\left(\int_{\ell_\varepsilon^\lambda \mathbb{T}} \kappa_\varepsilon^\lambda(s)^2 \int_{\ell_\varepsilon^{\lambda'} \mathbb{T}} \frac{ds'}{|z_\varepsilon^\lambda(s) - z_\varepsilon^{\lambda'}(s')|^{1+2\alpha}} \, ds \right)^{1/2} \left(\int_{\ell_\varepsilon^{\lambda'} \mathbb{T}} \kappa_\varepsilon^{\lambda'}(s')^2 \int_{\ell_\varepsilon^\lambda \mathbb{T}} \frac{ds}{|z_\varepsilon^\lambda(s) - z_\varepsilon^{\lambda'}(s')|^{1+2\alpha}} \, ds' \right)^{1/2},$$

and Lemma 3.5 with $\beta = \frac{1}{2}$ shows that this is bounded by

$$C_\alpha \left(\|z_\varepsilon^\lambda\|_{\dot{H}^2}^2 \frac{\ell_\varepsilon^{\lambda'} \|z_\varepsilon^{\lambda'}\|_{\dot{H}^2}^2}{(\Delta_\varepsilon^{\lambda,\lambda'})^{2\alpha}} \right)^{1/2} \left(\|z_\varepsilon^{\lambda'}\|_{\dot{H}^2}^2 \frac{\ell_\varepsilon^\lambda \|z_\varepsilon^\lambda\|_{\dot{H}^2}^2}{(\Delta_\varepsilon^{\lambda,\lambda'})^{2\alpha}} \right)^{1/2} = C_\alpha \|z_\varepsilon^\lambda\|_{\dot{H}^2}^2 \frac{\ell_\varepsilon^{\lambda'} \|z_\varepsilon^{\lambda'}\|_{\dot{H}^2}^2 (\ell_\varepsilon^\lambda)^{1/2}}{(\ell_\varepsilon^{\lambda'})^{1/2} (\Delta_\varepsilon^{\lambda,\lambda'})^{2\alpha}}.$$

Therefore

$$|G_2| \leq C_\alpha R_\mu(\Phi_{\varepsilon*}\Theta) Q(\Phi_{\varepsilon*}\Theta) \|z_\varepsilon^\lambda\|_{\dot{H}^2}^2.$$

Estimate for G_3. We can assume $R_\mu(\Phi_{\varepsilon*}\Theta) < \infty$, in which case $\Delta_\varepsilon^{\lambda,\lambda'} > 0$ for $|\mu|$-almost all λ'. Thus we can apply [10, Lemma A.4] to conclude that

$$G_3 \leq C_\alpha \overline{\int_{\mathcal{L}}} \int_{\ell_\varepsilon^\lambda \mathbb{T} \times \ell_\varepsilon^{\lambda'} \mathbb{T}} \frac{|\kappa_\varepsilon^\lambda(s)| \left(\mathcal{M}\kappa_\varepsilon^\lambda(s) + \mathcal{M}\kappa_\varepsilon^{\lambda'}(s') \right)}{|z_\varepsilon^\lambda(s) - z_\varepsilon^{\lambda'}(s')|^{1+2\alpha}} \, ds \, ds' d|\mu|(\lambda'), \tag{3.13}$$

where $\mathcal{M}$ is the maximal operator given by

$$\mathcal{M}f(s) := \max \left\{ \sup_{h \in \left(0, \frac{\ell}{2}\right]} \frac{1}{h} \int_s^{s+h} |f(s')| \, ds', \ \sup_{h \in \left(0, \frac{\ell}{2}\right]} \frac{1}{h} \int_{s-h}^s |f(s')| \, ds' \right\}$$

for $\ell \in (0, \infty)$ and $f \in L^1(\ell\mathbb{T})$. Let us split the integrand of (3.13) into the sum of terms with numerators $\left|\kappa_\varepsilon^\lambda(s)\right| \mathcal{M}\kappa_\varepsilon^\lambda(s)$ and $\left|\kappa_\varepsilon^\lambda(s)\right| \mathcal{M}\kappa_\varepsilon^{\lambda'}(s')$. Then the maximal inequality

$$\|\mathcal{M}f\|_{L^2} \le C \|f\|_{L^2},$$

which holds for all $\ell \in (0, \infty)$ and $f \in L^2(\ell\mathbb{T})$ with some universal constant C, shows that the same argument as in the estimate for G_2 bounds the integral of the second term by $C_\alpha R_\mu(\Phi_{\varepsilon*}\Theta)Q(\Phi_{\varepsilon*}\Theta) \left\|z_\varepsilon^\lambda\right\|_{\dot{H}^2}^2$. As for the first term, Lemma 3.5 with $\beta = \frac{1}{2}$, Schwarz inequality, and the maximal inequality show that

$$\overline{\int}_\mathcal{L} \int_{\ell_\varepsilon^\lambda\mathbb{T}\times\ell_\varepsilon^{\lambda'}\mathbb{T}} \frac{\left|\kappa_\varepsilon^\lambda(s)\right| \mathcal{M}\kappa_\varepsilon^\lambda(s)}{\left|z_\varepsilon^\lambda(s) - z_\varepsilon^{\lambda'}(s')\right|^{1+2\alpha}}\, ds\, ds'd|\mu|(\lambda')$$

$$\le \overline{\int}_\mathcal{L} \int_{\ell_\varepsilon^\lambda\mathbb{T}} \left|\kappa_\varepsilon^\lambda(s)\right| \mathcal{M}\kappa_\varepsilon^\lambda(s) \frac{C_\alpha \ell_\varepsilon^{\lambda'} \left\|z_\varepsilon^{\lambda'}\right\|_{\dot{H}^2}^2}{d(z_\varepsilon^\lambda(s), \mathrm{im}(z_\varepsilon^{\lambda'}))^{2\alpha}}\, ds\, d|\mu|(\lambda')$$

$$\le C_\alpha Q(\Phi_{\varepsilon*}\Theta) \int_\mathcal{L} \int_{\ell_\varepsilon^\lambda\mathbb{T}} \frac{\left|\kappa_\varepsilon^\lambda(s)\right| \mathcal{M}\kappa_\varepsilon^\lambda(s)}{d(z_\varepsilon^\lambda(s), \mathrm{im}(z_\varepsilon^{\lambda'}))^{2\alpha}}\, ds\, d|\mu|(\lambda')$$

$$\le C_\alpha L_\mu(\Phi_{\varepsilon*}\Theta)Q(\Phi_{\varepsilon*}\Theta) \int_{\ell_\varepsilon^\lambda\mathbb{T}} \left|\kappa_\varepsilon^\lambda(s)\right| \mathcal{M}\kappa_\varepsilon^\lambda(s)\, ds$$

$$\le C_\alpha L_\mu(\Phi_{\varepsilon*}\Theta)Q(\Phi_{\varepsilon*}\Theta) \left\|z_\varepsilon^\lambda\right\|_{\dot{H}^2}^2.$$

Here, the regular (rather than upper) integral can be taken after the second inequality because the integrand is jointly measurable in (s, λ') (recall from (1.10) that $d(x, \mathrm{im}(z_\varepsilon^{\lambda'})) = d(x, \Phi_\varepsilon(\partial\Theta^{\lambda'}))$ is jointly measurable in (x, λ')), which was used in the following step when applying Fubini's theorem. Aggregating the estimates for G_2 and G_3 now yields the desired conclusion. $\qquad\square$

Lemmas 3.1 and 3.7 suggest existence of an ε-independent estimate

$$\max\left\{\partial_t^+ Q(\Phi_{\varepsilon*}^t\Theta), -\partial_{t-} Q(\Phi_{\varepsilon*}^t\Theta)\right\} \le C_\alpha(L_\mu(\Phi_{\varepsilon*}^t\Theta) + R_\mu(\Phi_{\varepsilon*}^t\Theta))Q(\Phi_{\varepsilon*}^t\Theta)^2, \qquad (3.14)$$

so that a Grönwall-type argument can be used to show that $Q(\Phi_*^t\Theta)$ is finite for all t near 0. However, to apply such an argument, we first need to prove that $Q(\Phi_{\varepsilon*}^t\Theta)$ is bounded on some (ε-independent) time interval $(-t_0, t_0)$, even if the bound depends on ε, or that $Q(\Phi_{\varepsilon*}^t\Theta)$ is upper semi-continuous for each $\varepsilon > 0$. For instance, (3.11) would become useless if $Q(\Phi_{\varepsilon*}^t\Theta) = \infty$ for all small $\varepsilon, |t| > 0$, and none of the above a priori excludes this.

It turns out that this problem can be overcome by estimating the second integral in (3.12) via the Schwarz inequality, provided $\sup_{\lambda\in\mathcal{L}} \ell(z^{0,\lambda}) < \infty$ (using also Lemma 3.1). Moreover, we will now show that one can similarly deal with general θ^0 considered in this section via a sequence of approximations satisfying this property. Take an increasing sequence $\{\mathcal{L}_N'\}_{N=1}^\infty$ of measurable subsets of $\mathcal{L}$ such that $|\mu|(\mathcal{L}_N') < \infty$ for each $N \in \mathbb{N}$ and $\mathcal{L} = \bigcup_{N=1}^\infty \mathcal{L}_N'$. For θ^0 as above and any $N \in \mathbb{N}$, let

$$\mathcal{L}_N := \left\{\lambda \in \mathcal{L}_N' : \ell(z^{0,\lambda}) \le N\right\}, \quad \Theta_N := \Theta \cap (\mathbb{R}^2 \times \mathcal{L}_N), \quad \theta_N^0(x) := \int_{\mathcal{L}_N} \mathbb{1}_{\Theta^\lambda}(x)\, d\mu(\lambda).$$

Then $\theta_N^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ because $\|\theta_N^0\|_{L^\infty} \le |\mu|(\mathcal{L}'_N)$ and $\|\theta_N^0\|_{L^1} \le \frac{N^2}{4\pi}|\mu|(\mathcal{L}'_N)$ (the latter by the isoperimetric inequality), and clearly also $L_\mu(\Theta_N) \le L_\mu(\Theta)$. Let $\Phi_{\varepsilon,N} \in C_{\text{loc}}(\mathbb{R}; C(\mathbb{R}^2; \mathbb{R}^2))$ be the corresponding ε-mollified flow map, that is, the identity map plus the solution to (2.4) with θ_N^0 in place of θ^0. Let $z_{\varepsilon,N}^{t,\lambda} := \Phi_{\varepsilon,N}^t \circ z^{0,\lambda}$ and $\theta_{\varepsilon,N}^t := \theta_N^0 \circ (\Phi_{\varepsilon,N}^t)^{-1}$.

Then (3.12), the Schwarz inequality, $\ell(\gamma)\|\gamma\|_{\dot{H}^2}^2 \ge 4$ for any $\gamma \in \text{CC}(\mathbb{R}^2)$ (see [10, Lemma A.1]), Lemma 3.1, and (2.2) show that

$$\left| \partial_t \left\| z_{\varepsilon,N}^{t,\lambda} \right\|_{\dot{H}^2}^2 \right| \le 5 \left\| u_\varepsilon(\theta_{\varepsilon,N}^t) \right\|_{\dot{C}^{0,1}} \left\| z_{\varepsilon,N}^{t,\lambda} \right\|_{\dot{H}^2}^2 + 2 \left\| D^2(u_\varepsilon(\theta_{\varepsilon,N}^t)) \right\|_{L^\infty} \ell(z_{\varepsilon,N}^{t,\lambda})^{1/2} \left\| z_{\varepsilon,N}^{t,\lambda} \right\|_{\dot{H}^2}$$

$$\le \left(5 \left\| D(\nabla^\perp K_\varepsilon) \right\|_{L^\infty} + C_\alpha L_\mu(\Theta) N \left\| D^2(\nabla^\perp K_\varepsilon) \right\|_{L^\infty} \right) \left\| \theta_N^0 \right\|_{L^1} \left\| z_{\varepsilon,N}^{t,\lambda} \right\|_{\dot{H}^2}^2$$

for $(t, \lambda) \in [-T_0, T_0] \times \mathcal{L}_N$. This implies

$$\left\| z_{\varepsilon,N}^{t+h,\lambda} \right\|_{\dot{H}^2}^2 \le e^{C|h|} \left\| z_{\varepsilon,N}^{t,\lambda} \right\|_{\dot{H}^2}^2$$

whenever $(t, t+h, \lambda) \in [-T_0, T_0]^2 \times \mathcal{L}_N$, with some C depending on $\alpha, \varepsilon, N, \Theta, \mu$. This and Lemma 3.1 now yield

$$Q(\Phi_{\varepsilon,N*}^{t+h}\Theta_N) \le e^{C|h|} Q(\Phi_{\varepsilon,N*}^t \Theta_N)$$

for the same (t, h, λ), with a new C depending again only on $\alpha, \varepsilon, N, \Theta, \mu$, from which we conclude the $Q(\Phi_{\varepsilon,N*}^t \Theta_N)$ is upper semi-continuous in $t \in [-T_0, T_0]$ for all N, ε (this then clearly extends to all $t \in \mathbb{R}$).

Lemmas 3.1, 3.4, 3.7, and Corollary 2.7 with $(\theta_{\varepsilon,N}, \Theta_N, z_{\varepsilon,N}, \Phi_{\varepsilon,N})$ in place of $(\theta_\varepsilon, \Theta, z_\varepsilon, \Phi_\varepsilon)$, and inequalities $L_\mu(\Theta_N) \le L_\mu(\Theta)$, $R_\mu(\Theta_N) \le R_\mu(\Theta)$, and $Q(\Phi_{\varepsilon,N*}^t \Theta_N) \ge 4$, show that

$$\ell(z_{\varepsilon,N}^{t+h,\lambda}) \left\| z_{\varepsilon,N}^{t+h,\lambda} \right\|_{\dot{H}^2}^2 - Q(\Phi_{\varepsilon,N*}^t \Theta_N) \le \ell(z_{\varepsilon,N}^{t+h,\lambda}) \left\| z_{\varepsilon,N}^{t+h,\lambda} \right\|_{\dot{H}^2}^2 - \ell(z_{\varepsilon,N}^{t,\lambda}) \left\| z_{\varepsilon,N}^{t,\lambda} \right\|_{\dot{H}^2}^2$$

$$\le C_\alpha \left| \int_t^{t+h} (L_\mu(\Theta) + R_\mu(\Theta)) Q(\Phi_{\varepsilon,N*}^\tau \Theta_N)^2 \, d\tau \right|$$

holds whenever $(t, t+h, \lambda) \in [-T_0, T_0]^2 \times \mathcal{L}_N$ (with C_α depending only on α, as always). Using upper semi-continuity of $Q(\Phi_{\varepsilon,N*}^t \Theta_N)$ in t, taking the supremum over $\lambda \in \mathcal{L}_N$, dividing by $|h|$, and letting $h \to 0$ now yields

$$\max \left\{ \partial_t^+ Q(\Phi_{\varepsilon,N*}^t \Theta_N), -\partial_{t-} Q(\Phi_{\varepsilon,N*}^t \Theta_N) \right\} \le C_\alpha (L_\mu(\Theta) + R_\mu(\Theta)) Q(\Phi_{\varepsilon,N*}^t \Theta_N)^2,$$

so via a Grönwall-type argument we conclude that

$$Q(\Phi_{\varepsilon,N*}^t \Theta_N) \le \frac{Q(\Theta_N)}{1 - C_\alpha(L_\mu(\Theta) + R_\mu(\Theta))Q(\Theta_N)\,|t|} \le \frac{Q(\Theta)}{1 - C_\alpha(L_\mu(\Theta) + R_\mu(\Theta))Q(\Theta)\,|t|} \tag{3.15}$$

whenever $|t| < T_1 := \frac{1}{C_\alpha(L_\mu(\Theta)+R_\mu(\Theta))Q(\Theta)}$. We let this C_α be no smaller than the one from Lemma 2.1 (which was used to define T_0), so that $T_1 \in (0, T_0]$.

The following result will allow us to turn (3.15) into a bound on $Q(\Phi_*^t \Theta)$.

Lemma 3.8. *With T_1 as above, for each $\varepsilon > 0$ we have*

$$\lim_{N \to \infty} \sup_{t \in [-T_1, T_1]} \left\| \Phi_{\varepsilon, N}^t - \Phi_\varepsilon^t \right\|_{L^\infty} = 0.$$

Proof. Since $L_\mu(\Theta_N) \le L_\mu(\Theta)$ for all N, the last claim in Proposition 2.8 and (1.11) show that

$$M := \sup_{(t,N) \in [-T_1, T_1] \times \mathbb{N}} \max \left\{ L_\mu(\Phi_{\varepsilon, N*}^t \theta_N^0), L_\mu(\Phi_{\varepsilon*}^t \theta_N^0) \right\} \le e^{2\alpha} L_\mu(\Theta_N) \le 3 L_\mu(\Theta).$$

Then Lemmas 2.1 and 2.2 show for any $t \in [-T_1, T_1]$ that

$$
\begin{aligned}
\left\| u_\varepsilon(\Phi_{\varepsilon, N*}^t \theta_N^0) \circ \Phi_{\varepsilon, N}^t - u_\varepsilon(\Phi_{\varepsilon*}^t \theta^0) \circ \Phi_\varepsilon^t \right\|_{L^\infty} & \\
\le \left\| u_\varepsilon(\Phi_{\varepsilon, N*}^t \theta_N^0) \right\|_{\dot{C}^{0,1}} \left\| \Phi_{\varepsilon, N}^t - \Phi_\varepsilon^t \right\|_{L^\infty} & + \left\| u_\varepsilon(\Phi_{\varepsilon, N*}^t \theta_N^0) - u_\varepsilon(\Phi_{\varepsilon*}^t \theta_N^0) \right\|_{L^\infty} \\
+ \left\| u_\varepsilon(\Phi_{\varepsilon*}^t \theta_N^0) - u_\varepsilon(\Phi_{\varepsilon*}^t \theta^0) \right\|_{L^\infty} & \\
\le C_\alpha M \left\| \Phi_{\varepsilon, N}^t - \Phi_\varepsilon^t \right\|_{L^\infty} & + \left\| u_\varepsilon(\Phi_{\varepsilon*}^t \theta_N^0) - u_\varepsilon(\Phi_{\varepsilon*}^t \theta^0) \right\|_{L^\infty}
\end{aligned}
\tag{3.16}
$$

for some C_α. We see that $\theta_N^0 \to \theta^0$ in L^1 because $\Theta \subseteq \mathbb{R}^2 \times \mathcal{L}$ has finite measure (with the Lebesgue measure on $\mathbb{R}^2$ and $|\mu|$ on $\mathcal{L}$). So for any $\eta > 0$ there is t-independent $N_\eta \in \mathbb{N}$ such that for all $N \ge N_\eta$ we have

$$
\begin{aligned}
\left\| u_\varepsilon(\Phi_{\varepsilon*}^t \theta_N^0) - u_\varepsilon(\Phi_{\varepsilon*}^t \theta^0) \right\|_{L^\infty} &= \sup_{x \in \mathbb{R}^2} \left| \int_{\mathbb{R}^2} \nabla^\perp K_\varepsilon(x - \Phi_\varepsilon^t(y)) \left(\theta_N^0(y) - \theta^0(y) \right) dy \right| \\
&\le \left\| \nabla^\perp K_\varepsilon \right\|_{L^\infty} \left\| \theta_N^0 - \theta^0 \right\|_{L^1} \le \eta.
\end{aligned}
$$

Since $\left\| \Phi_{\varepsilon, N}^t - \Phi_\varepsilon^t \right\|_{L^\infty}$ is continuous in t, (3.16) yields

$$\max \left\{ \partial_t^+ \left\| \Phi_{\varepsilon, N}^t - \Phi_\varepsilon^t \right\|_{L^\infty}, -\partial_{t-} \left\| \Phi_{\varepsilon, N}^t - \Phi_\varepsilon^t \right\|_{L^\infty}, \right\} \le C_\alpha M \left\| \Phi_{\varepsilon, N}^t - \Phi_\varepsilon^t \right\|_{L^\infty} + \eta$$

for all $t \in [-T_1, T_1]$ and $N \ge N_\eta$. Hence, a Grönwall-type argument shows that

$$\left\| \Phi_{\varepsilon, N}^t - \Phi_\varepsilon^t \right\|_{L^\infty} \le \begin{cases} \frac{\eta}{C_\alpha M} \left(e^{C_\alpha M |t|} - 1 \right) & \text{if } M > 0, \\ \eta |t| & \text{if } M = 0 \end{cases}$$

for all such (t, N), from which the claim follows. $\qquad \square$

Since $\mathcal{L} = \bigcup_{N=1}^\infty \mathcal{L}_N$, Lemma 3.8 shows that $\lim_{N \to \infty} z_{\varepsilon, N}^{t, \lambda} = z_\varepsilon^{t, \lambda}$ in $CC(\mathbb{R}^2)$ for any $\varepsilon > 0$ and $(t, \lambda) \in (-T_1, T_1) \times \mathcal{L}$. Lower semi-continuity of $\ell(\cdot)$ and $\|\cdot\|_{\dot{H}^2}$ on $CC(\mathbb{R}^2)$ (the latter by [10, Corollary B.3]) and (3.15) show that

$$\ell(z_\varepsilon^{t, \lambda}) \left\| z_\varepsilon^{t, \lambda} \right\|_{\dot{H}^2}^2 \le \frac{Q(\Theta)}{1 - C_\alpha (L_\mu(\Theta) + R_\mu(\Theta)) Q(\Theta) |t|}$$

holds for these $(\varepsilon, t, \lambda)$. After taking $\varepsilon \to 0$ and using the same lower semi-continuity again, we obtain the same bound with $z^{t, \lambda}$ in place of $z_\varepsilon^{t, \lambda}$. It follows that

$$Q(\Phi_*^t \Theta) \le \frac{Q(\Theta)}{1 - C_\alpha (L_\mu(\Theta) + R_\mu(\Theta)) Q(\Theta) |t|}$$

for all $t \in (-T_1, T_1)$. Since $L_\mu(\Phi_*^t \Theta)$ and $R_\mu(\Phi_*^t \Theta)$ are bounded on compact subsets of I, an analogous bound holds for all t near any $\tau \in I$ with $Q(\Phi_*^\tau \Theta) < \infty$ (with $(\Phi_*^\tau \Theta, |t - \tau|)$ in place of $(\Theta, |t|)$). This proves Theorem 1.5(ii).

4. Proof of Theorem 1.5(iii)

Once again, all constants C_α below can change from one inequality to another, but they always only depend on α. Let

$$\Sigma(\Theta) := \inf_{\lambda \in \mathcal{L}} |\Theta^\lambda| > 0 \qquad \text{and} \qquad \Lambda(\Theta) := \int_{\mathcal{L}} \ell(\partial \Theta^\lambda) \, d|\mu|(\lambda) < \infty.$$

We now also assume that $\alpha \in \left(0, \frac{1}{6}\right]$, $|\mu|(\mathcal{L}) < \infty$, and T is an endpoint of the interval I' with $|T| < \infty$ and $\limsup_{t \to T} Q(\Phi_*^t \Theta) < \infty$. We let $S := \sup_{t \in [0,T)} Q(\Phi_*^t \Theta) < \infty$ and will then obtain a contradiction.

Since $\ell(\cdot)$ and $\|\cdot\|_{\dot{H}^2}$ are lower semi-continuous on $\mathrm{CC}(\mathbb{R}^2)$ (see the last paragraph of Section 3), $Q(\Phi_*^t \Theta)$ is lower semi-continuous in t. So if T is not an endpoint of I, we obtain $Q(\Phi_*^T \Theta) < \infty$ and hence a contradiction with Theorems 1.3 and 1.5(ii). Therefore T is an endpoint of I, and the desired contradiction will follow once we show $\limsup_{t \to T} L_\mu(\Phi_*^t \Theta) < \infty$. We can assume that $T > 0$ without loss because the other case is identical.

Lemma 4.1. *For some C_α and all $(t, \lambda) \in [0, T) \times \mathcal{L}$ we have*

$$\ell(z^{t,\lambda}) \leq S \left(\frac{\ell(z^{0,\lambda})}{2} + C_\alpha T \left(\|\theta^0\|_{L^1} + \|\theta^0\|_{L^\infty} \right) \right). \tag{4.1}$$

Proof. Integration by parts and the Schwarz inequality show that for any rectifiable $\gamma \in \mathrm{CC}(\mathbb{R}^2)$ we have

$$\ell(\gamma) \leq \mathrm{diam}(\mathrm{im}(\gamma))^2 \|\gamma\|_{\dot{H}^2}^2 \leq \mathrm{diam}(\mathrm{im}(\gamma)) \, \ell(\gamma) \, \|\gamma\|_{\dot{H}^2}^2.$$

Then it is enough to prove that for all $(t, \lambda) \in [0, T) \times \mathcal{L}$ we have

$$\partial_t^+ \mathrm{diam}(\mathrm{im}(z^{t,\lambda})) \leq C_\alpha \left(\|\theta^0\|_{L^1} + \|\theta^0\|_{L^\infty} \right) \tag{4.2}$$

because then this and $\mathrm{diam}(\mathrm{im}(\gamma)) \leq \frac{\ell(\gamma)}{2}$ yield (4.1).

Let $0 \leq t \leq t + h < T$ and pick any $x, y \in \mathrm{im}(z^{0,\lambda})$ such that $\left| \Phi^{t+h}(x) - \Phi^{t+h}(y) \right| = \mathrm{diam}(\mathrm{im}(z^{t+h,\lambda}))$. Then clearly

$$\mathrm{diam}(\mathrm{im}(z^{t+h,\lambda})) - \mathrm{diam}(\mathrm{im}(z^{t,\lambda})) \leq \left| \Phi^{t+h}(x) - \Phi^{t+h}(y) \right| - \left| \Phi^t(x) - \Phi^t(y) \right|$$

$$\leq \int_t^{t+h} \left(|u(\theta^\tau; \Phi^\tau(x))| + |u(\theta^\tau; \Phi^\tau(y))| \right) d\tau,$$

and for any $\tau \in [0, T)$ we have

$$\|u(\theta^\tau)\|_{L^\infty} \leq C_\alpha \left(\|\theta^\tau\|_{L^1} + \|\theta^\tau\|_{L^\infty} \right) = C_\alpha \left(\|\theta^0\|_{L^1} + \|\theta^0\|_{L^\infty} \right)$$

for some C_α since Φ^τ is measure-preserving. This now indeed yields (4.2). $\square$

Next, the isoperimetric inequality and Φ^t being measure-preserving show that

$$M := \sup_{t\in[0,T)} \sup_{\lambda\in\mathcal{L}} \left\|z^{t,\lambda}\right\|_{\dot{H}^2}^2 \le \sup_{t\in[0,T)} \frac{Q(\Phi_*^t\Theta)}{\inf_{\lambda\in\mathcal{L}}\ell(z^{t,\lambda})} \le \sup_{t\in[0,T)} \frac{Q(\Phi_*^t\Theta)}{2\sqrt{\pi}\,\Sigma(\Phi_*^t\Theta)^{1/2}}$$

$$= \sup_{t\in[0,T)} \frac{Q(\Phi_*^t\Theta)}{2\sqrt{\pi}\,\Sigma(\Theta)^{1/2}} < \infty.$$

For any $\eta > 0$ and any $\tilde{\Theta} \subseteq \mathbb{R}^2 \times \mathcal{L}$ from the product σ-algebra define

$$L_\mu^\eta(\tilde{\Theta}) := \sup_{x\in\mathbb{R}^2} \int_{\mathcal{L}} \frac{d|\mu|(\lambda)}{\left(d(x,\partial\tilde{\Theta}^\lambda) + \eta\right)^{2\alpha}}.$$

Then from $\Phi - \mathrm{Id} \in C\left([0,T); BC(\mathbb{R}^2; \mathbb{R}^2)\right)$ (see Proposition 2.8) we see that $L_\mu^\eta(\Phi_*^t\Theta)$ is continuous in $t \in [0,T)$. Since $\lim_{\eta\to 0^+} L_\mu^\eta(\tilde{\Theta}) = L_\mu(\tilde{\Theta})$, it will suffice to obtain the following η-independent upper bound on $L_\mu^\eta(\Phi_*^t\Theta)$.

Lemma 4.2. *There is C_α such that for any $\eta > 0$ and any $t \in [0,T)$ we have*

$$\partial_t^+ L_\mu^\eta(\Phi_*^t\Theta) \le C_\alpha S M^{1+2\alpha} \left(\Lambda(\Theta) + T|\mu|(\mathcal{L})\left(\left\|\theta^0\right\|_{L^1} + \left\|\theta^0\right\|_{L^\infty}\right)\right) L_\mu^\eta(\Phi_*^t\Theta). \qquad (4.3)$$

Proof. Fix $\eta > 0$ and $t \in [0,T)$, and find a decreasing sequence $t_n \to t$ such that

$$\partial_t^+ L_\mu^\eta(\Phi_*^t\Theta) = \lim_{n\to\infty} \frac{L_\mu^\eta(\Phi_*^{t_n}\Theta) - L_\mu^\eta(\Phi_*^t\Theta)}{t_n - t}. \qquad (4.4)$$

Without loss assume that $t_1 \le t + \frac{1}{2C_\alpha L_\mu(\Phi_*^t\Theta)}$ with C_α from Lemma 2.1, so that

$$L_\mu(\Phi_*^\tau\theta) \le 2L_\mu(\Phi_*^t\Theta) \qquad (4.5)$$

holds for all $\tau \in [t, t_1]$ by Proposition 2.8. Fix any $n \in \mathbb{N}$ and pick $x \in \mathbb{R}^2$ such that

$$L_\mu^\eta(\Phi_*^{t_n}\Theta) \le \int_{\mathcal{L}} \frac{d|\mu|(\lambda)}{(d(\Phi^{t_n}(x), \mathrm{im}(z^{t_n,\lambda})) + \eta)^{2\alpha}} + \frac{t_n - t}{n}. \qquad (4.6)$$

Then fix any $\lambda \in \mathcal{L}$ and pick $y \in \mathrm{im}(z^{0,\lambda})$ such that

$$d(\Phi^{t_n}(x), \mathrm{im}(z^{t_n,\lambda})) = \left|\Phi^{t_n}(x) - \Phi^{t_n}(y)\right|.$$

Since $L_\mu^\eta(\Phi_*^{t_n}\Theta) \le L_\mu(\Phi_*^{t_n}\Theta) < \infty$, for $|\mu|$-almost every λ we have $|\Phi^{t_n}(x) - \Phi^{t_n}(y)| > 0$ and so $x \ne y$. For any such λ, an argument similar to (2.7) yields the first inequality in

$$\frac{1}{(d(\Phi^{t_n}(x), \mathrm{im}(z^{t_n,\lambda})) + \eta)^{2\alpha}} - \frac{1}{(d(\Phi^t(x), \mathrm{im}(z^{t,\lambda})) + \eta)^{2\alpha}}$$

$$\le \frac{|\Phi^t(x) - \Phi^t(y)| - |\Phi^{t_n}(x) - \Phi^{t_n}(y)|}{(|\Phi^{t_n}(x) - \Phi^{t_n}(y)| + \eta)(|\Phi^t(x) - \Phi^t(y)| + \eta)^{2\alpha}}$$

$$\le \frac{\Phi^t(x) - \Phi^t(y)}{|\Phi^t(x) - \Phi^t(y)|} \cdot \frac{(\Phi^t(x) - \Phi^{t_n}(x)) - (\Phi^t(y) - \Phi^{t_n}(y))}{(|\Phi^{t_n}(x) - \Phi^{t_n}(y)| + \eta)(|\Phi^t(x) - \Phi^t(y)| + \eta)^{2\alpha}}.$$

We now write the right-hand side as the sum of the following terms which we estimate separately:

$$P_1^n := \frac{\Phi^{t_n}(x) - \Phi^{t_n}(y)}{|\Phi^{t_n}(x) - \Phi^{t_n}(y)|} \cdot \frac{(t_n - t)\left(u(\theta^{t_n}; \Phi^{t_n}(y)) - u(\theta^{t_n}; \Phi^{t_n}(x))\right)}{\left(|\Phi^{t_n}(x) - \Phi^{t_n}(y)| + \eta\right)\left(|\Phi^t(x) - \Phi^t(y)| + \eta\right)^{2\alpha}},$$

$$P_2^n := \left[\frac{\Phi^t(x) - \Phi^t(y)}{|\Phi^t(x) - \Phi^t(y)|} - \frac{\Phi^{t_n}(x) - \Phi^{t_n}(y)}{|\Phi^{t_n}(x) - \Phi^{t_n}(y)|}\right] \cdot \frac{(t_n - t)\left(u(\theta^{t_n}; \Phi^{t_n}(y)) - u(\theta^{t_n}; \Phi^{t_n}(x))\right)}{\left(|\Phi^{t_n}(x) - \Phi^{t_n}(y)| + \eta\right)\left(|\Phi^t(x) - \Phi^t(y)| + \eta\right)^{2\alpha}},$$

$$P_3^n := \frac{\Phi^t(x) - \Phi^t(y)}{|\Phi^t(x) - \Phi^t(y)|} \cdot \frac{\int_t^{t_n} \left[u(\theta^\tau; \Phi^\tau(y)) - u(\theta^{t_n}; \Phi^{t_n}(y)) - u(\theta^\tau; \Phi^\tau(x)) + u(\theta^{t_n}; \Phi^{t_n}(x))\right] d\tau}{\left(|\Phi^{t_n}(x) - \Phi^{t_n}(y)| + \eta\right)\left(|\Phi^t(x) - \Phi^t(y)| + \eta\right)^{2\alpha}}.$$

Estimate for P_1^n. Since $|\mu|(\mathcal{L}) < \infty$ and $\nabla^\perp K(\Phi^{t_n}(y) - \cdot) - \nabla^\perp K(\Phi^{t_n}(x) - \cdot)$ is integrable, Fubini's theorem and Green's theorem show that

$$u(\theta^{t_n}; \Phi^{t_n}(y)) - u(\theta^{t_n}; \Phi^{t_n}(x))$$

$$= \int_{\mathbb{R}^2} \left(\nabla^\perp K(\Phi^{t_n}(y) - y') - \nabla^\perp K(\Phi^{t_n}(x) - y')\right) \theta^{t_n}(y')\, dy'$$

$$= \int_{\mathcal{L}} \int_{\Phi^{t_n}(\Theta^{\lambda'})} \left(\nabla^\perp K(\Phi^{t_n}(y) - y') - \nabla^\perp K(\Phi^{t_n}(x) - y')\right) dy'\, d\mu(\lambda')$$

$$= \int_{\mathcal{L}} \int_{\ell(z^{t_n,\lambda'})\mathbb{T}} \left(K(\Phi^{t_n}(y) - z^{t_n,\lambda'}(s)) - K(\Phi^{t_n}(x) - z^{t_n,\lambda'}(s))\right) \partial_s z^{t_n,\lambda'}(s)\, ds\, d\mu(\lambda')$$

where $z^{t_n,\lambda'}(\cdot)$ is an arbitrary arclength parametrization of $z^{t_n,\lambda'}$. Moreover, the definition of y shows that $\Phi^{t_n}(x) - \Phi^{t_n}(y)$ is normal to $z^{t_n,\lambda}$ at $\Phi^{t_n}(y)$, and from $R_\mu(\Phi_*^{t_n}\Theta) < \infty$ we know that $\text{im}(z^{t_n,\lambda}) \cap \text{im}(z^{t_n,\lambda'}) = \emptyset$ for $|\mu|$-almost every λ'. Then for each such λ', [10, Lemma 2.7] (with $\beta := \frac{1}{2}$ and $(\gamma_1, \gamma_2) := (z^{t_n,\lambda}, z^{t_n,\lambda'})$) and Lemma 4.1 show that

$$\left|\left(u(\theta^{t_n}; \Phi^{t_n}(y)) - u(\theta^{t_n}; \Phi^{t_n}(x))\right) \cdot \frac{\Phi^{t_n}(x) - \Phi^{t_n}(y)}{|\Phi^{t_n}(x) - \Phi^{t_n}(y)|}\right|$$

$$\leq C_\alpha M^{1+2\alpha} \left|\Phi^{t_n}(y) - \Phi^{t_n}(x)\right| \int_{\mathcal{L}} \ell(z^{t_n,\lambda'})\, d|\mu|(\lambda')$$

$$\leq C_\alpha S M^{1+2\alpha} \left(\Lambda(\Theta) + T|\mu|(\mathcal{L})\left(\|\theta^0\|_{L^1} + \|\theta^0\|_{L^\infty}\right)\right) \left|\Phi^{t_n}(y) - \Phi^{t_n}(x)\right|$$

for some constant C_α. This then yields

$$|P_1^n| \leq \frac{C_\alpha S M^{1+2\alpha} \left(\Lambda(\Theta) + T|\mu|(\mathcal{L})\left(\|\theta^0\|_{L^1} + \|\theta^0\|_{L^\infty}\right)\right)}{\left(d(\Phi^t(x), \text{im}(z^{t,\lambda})) + \eta\right)^{2\alpha}} (t_n - t). \tag{4.7}$$

Estimate for P_2^n. By the same argument as in (3.6) we have

$$\left|\Phi^\tau(x) - \Phi^\tau(y) - \Phi^{t_n}(x) + \Phi^{t_n}(y)\right| \leq \left(e^{2C_\alpha L_\mu(\Phi_*^t\Theta)(t_n - \tau)} - 1\right) \left|\Phi^{t_n}(x) - \Phi^{t_n}(y)\right|$$

for any $\tau \in [t, t_n]$. Therefore

$$\left|\frac{\Phi^t(x) - \Phi^t(y)}{|\Phi^t(x) - \Phi^t(y)|} - \frac{\Phi^{t_n}(x) - \Phi^{t_n}(y)}{|\Phi^{t_n}(x) - \Phi^{t_n}(y)|}\right|$$

$$\leq \frac{|\Phi^t(x) - \Phi^t(y) - \Phi^{t_n}(x) + \Phi^{t_n}(y)|}{|\Phi^{t_n}(x) - \Phi^{t_n}(y)|} + \frac{||\Phi^{t_n}(x) - \Phi^{t_n}(y)| - |\Phi^t(x) - \Phi^t(y)||}{|\Phi^{t_n}(x) - \Phi^{t_n}(y)|}$$

$$\leq 2\left(e^{2C_\alpha L_\mu(\Phi_*^t\Theta)(t_n - \tau)} - 1\right),$$

which together with Lemma 2.1 and (4.5) yields

$$|P_2^n| \leq \frac{C_\alpha\left(e^{2C_\alpha L_\mu(\Phi_*^t\Theta)(t_n-t)} - 1\right)L_\mu(\Phi_*^t\Theta)}{(d(\Phi^t(x), \text{im}(z^{t,\lambda})) + \eta)^{2\alpha}}(t_n - t). \tag{4.8}$$

Estimate for P_3^n. By Lemma 2.1, Lemma 2.2, and (4.5) we have

$$\int_t^{t_n}\left|u(\theta^\tau; \Phi^\tau(y)) - u(\theta^{t_n}; \Phi^{t_n}(y)) - u(\theta^\tau; \Phi^\tau(x)) + u(\theta^{t_n}; \Phi^{t_n}(x))\right|d\tau$$

$$\leq 2\int_t^{t_n}\left(\left\|u(\Phi_*^\tau\Theta) - u(\Phi_*^{t_n}\Theta)\right\|_{L^\infty} + \left\|u(\theta^\tau)\right\|_{\dot{C}^{0,1}}\left\|\Phi^\tau - \Phi^{t_n}\right\|_{L^\infty}\right)d\tau$$

$$\leq C_\alpha L_\mu(\Phi_*^t\Theta)\int_t^{t_n}\left\|\Phi^\tau - \Phi^{t_n}\right\|_{L^\infty}d\tau,$$

so

$$|P_3^n| \leq \frac{C_\alpha L_\mu(\Phi_*^t\Theta)\int_t^{t_n}\left\|\Phi^\tau - \Phi^{t_n}\right\|_{L^\infty}d\tau}{\eta\left(d(\Phi^t(x), \text{im}(z^{t,\lambda})) + \eta\right)^{2\alpha}}. \tag{4.9}$$

Estimates (4.8) and (4.9) show that $\lim_{n\to\infty}\frac{|P_j^n|}{t_n - t} = 0$ uniformly in $\lambda \in \mathcal{L}$ for $j = 2, 3$, and then (4.4), (4.6), and (4.7) yield (4.3). □

Lemma 4.2 and a Grönwall-type argument now show

$$L_\mu^\eta(\Phi_*^t\Theta) \leq e^{M_1 T}L_\mu^\eta(\Theta) \leq e^{M_1 T}L_\mu(\Theta)$$

for any $t \in [0, T)$, where

$$M_1 \coloneqq C_\alpha S M^{1+2\alpha}\left(\Lambda(\Theta) + T|\mu|(\mathcal{L})\left(\left\|\theta^0\right\|_{L^1} + \left\|\theta^0\right\|_{L^\infty}\right)\right)$$

and C_α is from Lemma 4.2. Letting $\eta \to 0$ then yields

$$\limsup_{t\to T} L_\mu(\Phi_*^t\Theta) \leq e^{M_1 T}L_\mu(\Theta) < \infty,$$

which is the desired contradiction and finishes the proof of Theorem 1.5(iii).

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