WELL-POSEDNESS FOR LOW REGULARITY SOLUTIONS TO THE G-SQG EQUATION WITH REGULAR LEVEL SETS

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Abstract. This is the abstract.

1. Introduction

We are concerned with the PDE in two dimensions

$$\partial_t\omega + u\cdot\nabla\omega = 0 \tag{1.1}$$

with

$$u := -\nabla^\perp(-\Delta)^{-1+\alpha}\omega \tag{1.2}$$

and $\alpha \in \left(0, \frac{1}{2}\right)$, where $(x_1, x_2)^\perp := (-x_2, x_1)$ and $\nabla^\perp := (-\partial_{x_2}, \partial_{x_1})$.

Given $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, the velocity field $u(\omega)$ generated by ω as in (1.2) is given as

$$u(\omega; x) := \int_{\mathbb{R}^2} \nabla^\perp K(x-y)\omega(y)\,dy,$$

where kernel $K\colon \mathbb{R}^2 \to (0,\infty]$ is defined as

$$K(x) := \frac{c_\alpha}{2\alpha\,|x|^{2\alpha}}$$

for some $c_\alpha > 0$. Since ω is in $L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$, it can be easily seen that $u(\omega)$ is a well-defined $(1-2\alpha)$-Hölder continuous function $\mathbb{R}^2 \to \mathbb{R}^2$.

More generally, when ω is a finite signed Borel measure on $\mathbb{R}^2$, (1.2) yields

$$u(\omega; x) := \int_{\mathbb{R}^2} \nabla^\perp K(x-y)\,d\omega(y)$$

whenever the integral converges absolutely. (The reason for considering this general case is merely because of convenience in certain aspects of developing the theory, and we will not be concerned with the well-posedness of (1.1)–(1.2) in this general setting.) When ω is an L^1 function, we identify it with the finite signed Borel measure it defines through integration with respect to the Lebesgue measure, so that the interpretations of $u(\omega)$ in both ways are consistent. Note that for any measure-preserving homeomorphism $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$, this correspondence between L^1 functions and finite signed Borel measures identifies the function $\omega\circ\Phi^{-1}$ with the pushforward measure $\Phi_*\omega$.

Our first result is local well-posedness of (1.1)–(1.2) within a class of $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ admitting a decomposition of the form

$$\omega(x) = \int_{\mathcal{L}} \mathbb{1}_{\Omega^\lambda}(x)\, d\theta(\lambda), \tag{1.3}$$

where $\mathcal{L}$ is a measurable space (whose σ-algebra is not explicitly written), θ is a σ-finite signed measure on $\mathcal{L}$, Ω is a set in the product σ-algebra of $\mathbb{R}^2 \times \mathcal{L}$ with finite measure (with respect to the Lebesgue measure and the total variation $|\theta|$ of θ), and $\Omega^\lambda \subseteq \mathbb{R}^2$ is the λ-section of Ω for each $\lambda \in \mathcal{L}$.

A natural choice of (Ω, θ) is that each Ω^λ is a super-level set of ω^+ and ω^-, and θ^+, θ^- are respectively the uniform measures on $(0, \sup \omega]$ and $[\inf \omega, 0)$, so that (1.3) is the standard layer cake representation. In this reason, we call the pair (Ω, θ), or simply Ω, a *generalized layer cake representation* of ω.

Remark. The measurable space $\mathcal{L}$ is implicit from Ω and θ, so we will suppress it in the notation. From now on, $\mathcal{L}$ always denotes the measurable space on which Ω and θ are defined.

It turns out that each Ω^λ being a super-level set of ω is not necessary for the well-posedness theory we develop. Furthermore, this abstract setting allows a simple setup for studying H^2 regularity of level sets, which our second result is mainly about, that encompasses situations where some level sets of ω may consist of multiple (or even infinitely many) disjoint curves. In these reasons, we state and prove our well-posedness result in terms of an arbitrary generalized layer cake representation, rather than the standard layer cake representation.

In addition to merely having a decomposition (1.3), we impose a regularity condition

$$L_\theta(\Omega) \coloneqq \sup_{x \in \mathbb{R}^2} \int_{\mathcal{L}} \frac{d|\theta|(\lambda)}{d(x, \partial\Omega^\lambda)^{2\alpha}} < \infty. \tag{1.4}$$

Note that a standard measure theory argument shows joint measurability of

$$d(x, \partial\Omega^\lambda) = \max\{d(x, \Omega^\lambda), d(x, \mathbb{R}^2 \setminus \Omega^\lambda)\} \tag{1.5}$$

in (x, λ), so $L_\theta(\Omega)$ is always well-defined (with the convention $\frac{1}{0} = \infty$ and $\infty \cdot 0 = 0$). The quantity $L_\theta(\Omega)$ arises in the context of estimating the growth of H^2 norm of a boundary curve $\partial\Omega^\lambda$ (Lemma 3.7), which was the motivation for us to study the well-posedness problem under the condition (1.4).

It turns out that any ω admitting Ω with (1.4) must be 2α-Hölder continuous. In fact, this is the best possible L^∞-type regularity condition (1.4) implies because there exists ω satisfying (1.4) whose modulus of continuity $\rho(\delta)$ is in between constant multiples of $\min\{\delta^{2\alpha}, 1\}$. In the below (Lemma 2.1) we show that (1.4) implies Lipschitz continuity of $u(\omega)$, which does not follow from mere 2α-Hölder continuity in general. This is one of the key components that lead to our well-posedness result.

On the other hand, investigating the distributional derivative of $u(\omega)$ shows that the most general condition on ω *in terms of* ρ that ensures Lipschitz continuity of $u(\omega)$ is

$$\int_0^1 \frac{\min\{\rho(\delta), 1\}}{\delta^{1+2\alpha}}\, d\delta < \infty. \tag{1.6}$$

It can be shown that (1.6) in fact implies (1.4) when (Ω, θ) is the standard layer cake representation. Since (1.6) does not hold for $\rho(\delta) = \min\{\delta^{2\alpha}, 1\}$, we see that (1.4) is strictly more general than any assumption on the modulus of continuity of ω from which Lipschitz continuity of $u(\omega)$ is guaranteed.

Since any ω in the class of functions we consider always generates Lipschitz velocity field, it is natural to consider the following notion of solutions to (1.1)–(1.2).

Definition 1.1. Let $\omega^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. A *Lagrangian solution* to (1.1)–(1.2) on a time interval $I \ni 0$ with the initial data ω^0 is a function $\omega\colon I \to L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ given as $\omega^t := \omega \circ (\Phi^t)^{-1}$, where $\Phi \in C\left(I; C(\mathbb{R}^2;\mathbb{R}^2)\right)$ (with respect to the extended metric $d(F,G) := \|F-G\|_{L^\infty}$) satisfies the initial value problem

$$\begin{aligned} \partial_t \Phi^t &= u(\omega^t) \circ \Phi^t, \\ \Phi^0 &= \mathrm{Id}, \end{aligned} \tag{1.7}$$

where the time derivative is one-sided at any end-point of I, and each Φ^t is a measure-preserving homeomorphism. We call Φ the *flow map* associated to ω.

Remark. It is easy to show that any Lagrangian solution ω is a weak solution to (1.1)–(1.2) in the sense that

$$\frac{d}{dt}\int_{\mathbb{R}^2} \varphi(x)\, \omega^t(x)\, dx = \int_{\mathbb{R}^2} \big(u(\omega^t;x)\cdot \nabla\varphi(x)\big)\, \omega^t(x)\, dx$$

holds for all $\varphi \in C^1(\mathbb{R}^2)$.

Given a measure-preserving homeomorphism $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$, it can be easily seen that $\Phi_*\Omega := \{(\Phi(x),\lambda) \in \mathbb{R}^2 \times \mathcal{L}\colon (x,\lambda) \in \Omega\}$ is a generalized layer cake representation of $\omega \circ \Phi^{-1}$ and

$$\omega(\Phi^{-1}(x)) = \int_{\mathcal{L}} \mathbb{1}_{\Phi(\Omega^\lambda)}(x)\, d\theta(\lambda)$$

for $x \in \mathbb{R}^2$. Also,

$$\begin{aligned} L_\theta(\Phi_*\Omega) &= \sup_{x\in\mathbb{R}^2} \int_{\mathcal{L}} \frac{d|\theta|(\lambda)}{d(x, \partial\Phi(\Omega^\lambda))^{2\alpha}} = \sup_{x\in\mathbb{R}^2} \int_{\mathcal{L}} \frac{d|\theta|(\lambda)}{d(x, \Phi(\partial\Omega^\lambda))^{2\alpha}} \\ &= \sup_{x\in\mathbb{R}^2} \int_{\mathcal{L}} \frac{d|\theta|(\lambda)}{d(\Phi(x), \Phi(\partial\Omega^\lambda))^{2\alpha}} \le \left\|\Phi^{-1}\right\|_{\dot{C}^{0,1}}^{2\alpha} L_\theta(\Omega). \end{aligned} \tag{1.8}$$

Now we state our first main result.

Theorem 1.2. *Let $\omega^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ admit a generalized layer cake representation (Ω, θ) such that $L_\theta(\Omega) < \infty$. Then there is an open interval $I \ni 0$ and a Lagrangian solution $\omega\colon I \to L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ to (1.1)–(1.2) with initial data ω^0 and the associated flow map $\Phi \in C_{\mathrm{loc}}\left(I; C(\mathbb{R}^2;\mathbb{R}^2)\right)$ such that $\sup_{t\in J} L_\theta(\Phi^t_*\Omega) < \infty$ for any compact interval $J \subseteq I$. Let I be the maximal such interval. Then the solution ω is unique and independent of the choice of (Ω,θ), for any compact interval $J \subseteq I$ we have $\sup_{t\in J} \max\left\{\|\Phi^t\|_{\dot{C}^{0,1}}, \|(\Phi^t)^{-1}\|_{\dot{C}^{0,1}}\right\} < \infty$, and for any endpoint T of I we have either $|T| = \infty$ or $\lim_{t\to T} L_\theta(\Phi^t_*\Omega) = \infty$.*

Our second result shows that in the setting of Theorem 1.2, if each $\partial\Omega^\lambda$ is an H^2 curve and certain additional assumptions are satisfied, then these properties must be retained by the solution provided by Theorem 1.2, at least for a short time. To state it precisely, let us give the following definitions. We refer to [3] for notions related to the space of closed plane curves ($\mathrm{CC}(\mathbb{R}^2)$, $\mathrm{PSC}(\mathbb{R}^2)$, $\mathrm{im}(\,\cdot\,)$, $\ell(\,\cdot\,)$, $\|\cdot\|_{\dot{H}^2}$ and $\Delta(\,\cdot\,,\,\cdot\,)$).

Definition 1.3. Let $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Then a generalized layer cake representation (Ω, θ) of ω is said to be *composed of simple closed curves* if for each $\lambda \in \mathcal{L}$, Ω^λ is a bounded open set and $\partial\Omega^\lambda = \mathrm{im}(z^\lambda)$ for some $z^\lambda \in \mathrm{PSC}(\mathbb{R}^2)$. In this case, we define

$$R_\theta(\Omega) := \sup_{\lambda\in\mathcal{L}} \ell(z^\lambda)^{1/2} \int_{\mathcal{L}} \frac{d|\theta|(\lambda')}{\ell(z^{\lambda'})^{1/2}\Delta(z^\lambda, z^{\lambda'})^{2\alpha}} \qquad \text{and} \qquad Q(\Omega) := \sup_{\lambda\in\mathcal{L}} \ell(z^\lambda) \left\|z^\lambda\right\|_{\dot{H}^2}^2.$$

Remark. Note that $\Delta(z^\lambda, z^{\lambda'}) = \inf_{x\in\mathbb{Q}^2} \left(d(x, \partial\Omega^\lambda) + d(x, \partial\Omega^{\lambda'})\right)$ is measurable in λ' since $d(x, \partial\Omega^{\lambda'})$ is jointly measurable in (x, λ'). On the other hand, since $\lambda' \mapsto \partial\Omega^{\lambda'}$ is measurable with respect to the topology on $\wp(\mathbb{R}^2)$ given by the family of pseudometrics $(A, B) \mapsto |(A \triangle B) \cap K|$ for compact $K \subseteq \mathbb{R}^2$, lower semi-continuity of the perimeter functional (in the sense of Caccioppoli) with respect to this topology shows measurability of $\lambda' \mapsto \ell(z^{\lambda'})$; see [1, Proposition 3.38] for lower semi-continuity, and see [1, Proposition 3.62] and [2, Theorem I] for that $\ell(z^{\lambda'})$ equals the perimeter of $\Omega^{\lambda'}$. Hence, $R_\theta(\Omega)$ is well-defined.

For our second result, we will impose in addition to $L_\theta(\Omega) < \infty$ that Ω is composed of simple closed curves and $Q(\Omega), R_\theta(\Omega) < \infty$. The condition $Q(\Omega) < \infty$ ensures a form of *scaling-invariant* uniform H^2 regularity of z^λ's. (Recall that for $a > 0$ and $\gamma \in \mathrm{CC}(\mathbb{R}^2)$, $\|a\gamma\|_{\dot{H}^2}^2 = \frac{1}{a}\|\gamma\|_{\dot{H}^2}^2$ while $\ell(a\gamma) = a\ell(\gamma)$.) The condition $R_\theta(\Omega) < \infty$ on the other hand controls how densely z^λ's of *different scales* can be packed together. More specifically, it prevents too many small z^λ's to be placed near a large z^λ. Interpreting z^λ's as level sets of ω, this in effect rules out too sharp "pinched tops/bottoms". (More elaboration of why this is needed?)

Next we state our second main result.

Theorem 1.4. *Let $\omega^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ admit a generalized layer cake representation (Ω, θ) composed of simple closed curves such that $L_\theta(\Omega), R_\theta(\Omega), Q(\Omega) < \infty$, and let $\omega\colon I \to L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ be the corresponding Lagrangian solution to* (1.1)–(1.2) *from Theorem 1.2.*

(i) Then $\Phi_*^t\Omega$ is composed of simple closed curves for each $t \in I$, and $\sup_{t\in J} R_\theta(\Phi_*^t\Omega) < \infty$ for each compact interval $J \subseteq I$.

(ii) There is an open interval $I' \subseteq I$ containing 0 such that $\sup_{t\in J} Q(\Phi_*^t\Omega) < \infty$ for each compact interval $J \subseteq I'$. Let I' be the maximal such interval. If T is its endpoint that is not an endpoint of I, then $\lim_{t\to T} Q(\Phi_*^t\Omega) = \infty$.

(iii) If $\alpha \in (0, \frac{1}{6}]$, then $I' = I$ and for any endpoint T of I we have either $|T| = \infty$ or $\lim_{t\to T} Q(\Phi_*^t\Omega) = \infty$.

Consider some smooth even $\chi\colon \mathbb{R} \to \mathbb{R}$ such that $0 \le \chi \le 1$, $\chi \equiv 1$ on $\mathbb{R} \setminus (-1, 1)$, and $0 \notin \operatorname{supp}\chi$. For each $\varepsilon > 0$, let $K_\varepsilon(x) := \chi\left(\frac{|x|}{\varepsilon}\right) K(x)$. Note that for any $n \in \mathbb{Z}_{\ge 0}$, there is $C_{\alpha,n}$ that only depends on α, n such that the norm of the n-linear form $D^n K_\varepsilon(x)$ is bounded by $\frac{C_{\alpha,n}}{\max\{|x|,\varepsilon\}^{n+2\alpha}}$. For any finite signed Borel measure ω on $\mathbb{R}^2$ we now define the mollified velocity field

$$u_\varepsilon(\omega;x) \coloneqq \int_{\mathbb{R}^2} \nabla^\perp K_\varepsilon(x-y)\,d\omega(y)$$

for $x \in \mathbb{R}^2$. Since $\nabla^\perp K_\varepsilon$ is a smooth function whose all derivatives vanish at infinity, this integral is always well-defined and $u_\varepsilon(\omega)$ is a smooth function such that

$$\left\|D^k(u_\varepsilon(\omega))\right\|_{L^\infty} \le \left\|D^k(\nabla^\perp K_\varepsilon)\right\|_{L^\infty} \|\omega\|_{\mathrm{TV}} \tag{1.9}$$

for all $k \in \mathbb{Z}_{\ge 0}$.

2. Proof of Theorem 1.2

We start with some estimates on the velocity field $u(\omega)$ in terms of $L_\theta(\Omega)$, where (Ω,θ) is a generalized layer cake representation of ω. All constants C_α below can change from one inequality to another, but they always only depend on α.

Lemma 2.1. *There is C_α such that for any $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ with a generalized layer cake representation (Ω,θ), and for any $\varepsilon > 0$ and $x \in \mathbb{R}^2$, we have*

$$|D(u_\varepsilon(\omega))(x)| \le C_\alpha \int_{\mathcal{L}} \frac{d|\theta|(\lambda)}{d(x,\partial\Omega^\lambda)^{2\alpha}}.$$

Therefore,

$$\|u_\varepsilon(\omega)\|_{\dot{C}^{0,1}} \le C_\alpha L_\theta(\Omega) \qquad \textit{and} \qquad \|u(\omega)\|_{\dot{C}^{0,1}} \le C_\alpha L_\theta(\Omega).$$

Proof. For each $\varepsilon > 0$ and $x, h \in \mathbb{R}^2$, oddness of $\nabla^\perp K_\varepsilon$ shows that

$$u_\varepsilon(\omega;x+h) - u_\varepsilon(\omega;x) = \int_{\mathbb{R}^2} \left(\nabla^\perp K_\varepsilon(x+h-y) - \nabla^\perp K_\varepsilon(x-y)\right)(\omega(y)-\omega(x))\,dy.$$

Replacing h by sh with $s \in \mathbb{R}$, and then taking $s \to 0$ yields

$$\begin{aligned} D(u_\varepsilon(\omega))(x)h &= \int_{\mathbb{R}^2} D(\nabla^\perp K_\varepsilon)(x-y)h\,(\omega(y)-\omega(x))\,dy \\ &= \int_{\mathbb{R}^2}\int_{\mathcal{L}} D(\nabla^\perp K_\varepsilon)(x-y)h\,(\mathbb{1}_{\Omega^\lambda}(y) - \mathbb{1}_{\Omega^\lambda}(x))\,d\theta(\lambda)\,dy. \end{aligned}$$

Note that $\mathbb{1}_{\Omega^\lambda}(y) - \mathbb{1}_{\Omega^\lambda}(x) \ne 0$ implies $|x-y| \ge d(x,\partial\Omega^\lambda)$, so

$$|D(u_\varepsilon(\omega))(x)| \le \int_{\mathcal{L}}\int_{|x-y|\ge d(x,\partial\Omega^\lambda)} \frac{C_\alpha}{|x-y|^{2+2\alpha}}\,dy\,d|\theta|(\lambda) \le \int_{\mathcal{L}} \frac{C_\alpha}{d(x,\partial\Omega^\lambda)^{2\alpha}} d|\theta|(\lambda).$$

This proves the first and second claims, and the third follows by taking $\varepsilon \to 0^+$. $\square$

Lemma 2.2. *There is C_α such that for any $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ with generalized layer cake representations (Ω_i,θ_i) and any measure-preserving homeomorphisms $\Phi_i \colon \mathbb{R}^2 \to \mathbb{R}^2$ $(i = 1,2)$,*

$$\|u(\Phi_{1*}\omega) - u(\Phi_{2*}\omega)\|_{L^\infty} \le C_\alpha\left(L_{\theta_1}(\Phi_{1*}\Omega_1) + L_{\theta_2}(\Phi_{2*}\Omega_2)\right)\|\Phi_1 - \Phi_2\|_{L^\infty}.$$

Proof. Let $\mathcal{L}_i$ the measurable space associated to (Ω_i, θ_i) and $\omega_i := \omega \circ \Phi_i^{-1}$ for $i = 1, 2$. Let $d := \|\Phi_1 - \Phi_2\|_{L^\infty}$ and fix any $x \in \mathbb{R}^2$. Then $u(\Phi_{1*}\omega; x) - u(\Phi_{2*}\omega; x)$ is the sum of

$$I_1 := \int_{|x-y|\le 2d} \nabla^\perp K(x-y)(\omega_1(y) - \omega_2(y))\, dy,$$
$$I_2 := \int_{|x-y|>2d} \nabla^\perp K(x-y)(\omega_1(y) - \omega_2(y))\, dy.$$

Estimate for I_1. By oddness of $\nabla^\perp K$, we have

$$I_1 = \int_{|x-y|\le 2d} \nabla^\perp K(x-y)(\omega_1(y) - \omega_1(x))\, dy - \int_{|x-y|\le 2d} \nabla^\perp K(x-y)(\omega_2(y) - \omega_2(x))\, dy.$$

Since

$$I_3 := \int_{|x-y|\le 2d} \frac{|\omega_1(y) - \omega_1(x)|}{|x-y|^{1+2\alpha}}\, dy \le \int_{\mathcal{L}_1} \int_{|x-y|\le 2d} \frac{\left|\mathbb{1}_{\Phi_1(\Omega_1^\lambda)}(y) - \mathbb{1}_{\Phi_1(\Omega_1^\lambda)}(x)\right|}{|x-y|^{1+2\alpha}}\, dy\, d|\theta_1|(\lambda)$$

and (as in the proof of Lemma 2.1) we have $|x-y| \ge d(x, \partial\Phi_1(\Omega_1^\lambda))$ whenever the last integrand is nonzero, we see that

$$I_3 \le \int_{\mathcal{L}_1} \int_{|x-y|\le 2d} \frac{1}{|x-y|\ d(x, \partial\Phi_1(\Omega_1^\lambda))^{2\alpha}}\, dy\, d|\theta_1|(\lambda) \le 4\pi L_{\theta_1}(\Phi_{1*}\Omega_1) d.$$

The same argument for ω_2 in place of ω_1 now yields

$$|I_1| \le C_\alpha \left(L_{\theta_1}(\Phi_{1*}\Omega_1) + L_{\theta_2}(\Phi_{2*}\Omega_2) \right) d. \tag{2.1}$$

Estimate for I_2. For each $R > 2d$ let

$$I_2^R := \int_{2d<|x-y|\le R} \nabla^\perp K(x-y)(\omega_1(y) - \omega_2(y))\, dy,$$

so that $I_2 = \lim_{R\to\infty} I_2^R$. Fix $R > 2d$, and then Φ_i being measure-preserving yields

$$\begin{aligned}
I_2^R &= \int_{2d<|x-y|\le R} \nabla^\perp K(x-y)(\omega_1(y) - \omega_1(x))\, dy \\
&\qquad - \int_{2d<|x-y|\le R} \nabla^\perp K(x-y)(\omega_2(y) - \omega_1(x))\, dy \\
&= \int_{2d<|x-\Phi_1(y)|\le R} \nabla^\perp K(x-\Phi_1(y))(\omega(y) - \omega_1(x))\, dy \\
&\qquad - \int_{2d<|x-\Phi_2(y)|\le R} \nabla^\perp K(x-\Phi_2(y))(\omega(y) - \omega_1(x))\, dy \\
&= \int_{|x-\Phi_1(y)|,|x-\Phi_2(y)|\in(2d,R]} \left[\nabla^\perp K(x-\Phi_1(y)) - \nabla^\perp K(x-\Phi_2(y))\right](\omega(y) - \omega_1(x))\, dy \\
&\qquad + \int_{|x-\Phi_2(y)|\le 2d<|x-\Phi_1(y)|\le R} \nabla^\perp K(x-\Phi_1(y))(\omega(y) - \omega_1(x))\, dy
\end{aligned}$$

$$
\begin{aligned}
&- \int_{|x-\Phi_1(y)|\le 2d<|x-\Phi_2(y)|\le R} \nabla^\perp K(x-\Phi_2(y))(\omega(y)-\omega_1(x))\,dy \\
&+ \int_{2d<|x-\Phi_1(y)|\le R<|x-\Phi_2(y)|} \nabla^\perp K(x-\Phi_1(y))(\omega(y)-\omega_1(x))\,dy \\
&- \int_{2d<|x-\Phi_2(y)|\le R<|x-\Phi_1(y)|} \nabla^\perp K(x-\Phi_2(y))(\omega(y)-\omega_1(x))\,dy.
\end{aligned}
$$

Let us denote the integrals on the right-hand side I_4, I_5, I_6, I_7, I_8 (in the order of appearance).

To estimate I_4, note that for any y in the domain of integration we have

$$
\min_{\eta\in[0,1]} |x-(1-\eta)\Phi_1(y)-\eta\Phi_2(y)| \ge |x-\Phi_1(y)|-d \ge \frac{|x-\Phi_1(y)|}{2},
$$

so the mean value theorem shows that

$$
\left|\nabla^\perp K(x-\Phi_1(y)) - \nabla^\perp K(x-\Phi_2(y))\right| \le \frac{C_\alpha d}{|x-\Phi_1(y)|^{2+2\alpha}}.
$$

The change of variables formula now yields

$$
\begin{aligned}
|I_4| &\le \int_{|x-y|>2d} \frac{C_\alpha d\,|\omega_1(y)-\omega_1(x)|}{|x-y|^{2+2\alpha}}\,dy \\
&\le \int_{\mathcal{L}_1}\int_{|x-y|>2d} \frac{C_\alpha d\left|\mathbb{1}_{\Phi_1(\Omega_1^\lambda)}(y)-\mathbb{1}_{\Phi_1(\Omega_1^\lambda)}(x)\right|}{|x-y|^{2+2\alpha}}\,dy\,d|\theta_1|(\lambda).
\end{aligned}
$$

Again, $|x-y|\ge d(x,\partial\Phi_1(\Omega_1^\lambda))$ holds whenever the last integrand is nonzero, so

$$
|I_4| \le \int_{\mathcal{L}_1}\int_{|x-y|\ge d(x,\partial\Phi_1(\Omega_1^\lambda))} \frac{C_\alpha d}{|x-y|^{2+2\alpha}}\,dy\,d|\theta_1|(\lambda) \le C_\alpha L_{\theta_1}(\Phi_{1*}\Omega_1)d.
$$

For I_5, note that for any y in the domain of integration we have

$$
|x-\Phi_1(y)| \le |x-\Phi_2(y)|+d \le 3d.
$$

By applying again change of variables we obtain

$$
|I_5| \le \int_{|x-y|\le 3d} \frac{C_\alpha\,|\omega_1(y)-\omega_1(x)|}{|x-y|^{1+2\alpha}}\,dy,
$$

so the same argument as in the estimate for I_3 shows $|I_5|\le C_\alpha L_{\theta_1}(\Phi_{1*}\Omega_1)d$. And clearly

$$
|I_6| \le \int_{|x-\Phi_1(y)|\le 2d} \frac{C_\alpha\,|\omega(y)-\omega_1(x)|}{|x-\Phi_1(y)|^{1+2\alpha}}\,dy = \int_{|x-y|\le 2d} \frac{C_\alpha\,|\omega_1(y)-\omega_1(x)|}{|x-y|^{1+2\alpha}}\,dy,
$$

so again $|I_6|\le C_\alpha L_{\theta_1}(\Phi_{1*}\Omega_1)d$.

To estimate I_7, note that for any y in the domain of integration we have

$$
|x-\Phi_1(y)| \ge |x-\Phi_2(y)|-d > R-d,
$$

so the change of variables formula yields

$$|I_7| \le \int_{R-d<|x-y|\le R} \frac{C_\alpha \|\omega\|_{L^\infty}}{|x-y|^{1+2\alpha}}\,dy \le \frac{C_\alpha \|\omega\|_{L^\infty} d}{R^{2\alpha}}$$

because $R > 2d$. In the same way we also obtain $|I_8| \le \frac{C_\alpha \|\omega\|_{L^\infty} d}{R^{2\alpha}}$.

Collecting the above estimates and letting $R \to \infty$, we see that $|I_2| \le C_\alpha L_{\theta_1}(\Phi_{1*}\Omega_1)d$. This and (2.1) now hold uniformly in $x \in \mathbb{R}^2$, finishing the proof. □

Lemma 2.3. *There is C_α such that for any $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $\varepsilon > 0$ we have*

$$\|u(\omega) - u_\varepsilon(\omega)\|_{L^\infty} \le C_\alpha \|\omega\|_{L^\infty} \varepsilon^{1-2\alpha}.$$

Proof. Since $\nabla^\perp K_\varepsilon(x) = \nabla^\perp K(x)$ when $|x| \ge \varepsilon$, for any $x \in \mathbb{R}^2$ we have

$$\begin{aligned} |u(\omega;x) - u_\varepsilon(\omega;x)| &\le \int_{|x-y|\le\varepsilon} \left|\nabla^\perp K(x-y) - \nabla^\perp K_\varepsilon(x-y)\right| \|\omega\|_{L^\infty}\,dy \\ &\le \int_{|x-y|\le\varepsilon} \frac{C_\alpha \|\omega\|_{L^\infty}}{|x-y|^{1+2\alpha}}\,dy = C_\alpha \|\omega\|_{L^\infty} \varepsilon^{1-2\alpha}. \end{aligned}$$

□

Now, fix any initial data $\omega^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ admitting a generalized layer cake representation (Ω, θ) with $L_\theta(\Omega) < \infty$. Fix any $\varepsilon > 0$ and consider the ODE

$$\begin{aligned} \partial_t \Psi_\varepsilon^t &= u_\varepsilon((\mathrm{Id} + \Psi_\varepsilon^t)_*\omega^0) \circ (\mathrm{Id} + \Psi_\varepsilon^t), \\ \Psi_\varepsilon^0 &= 0 \end{aligned} \tag{2.2}$$

with $\Psi_\varepsilon^t \in BC(\mathbb{R}^2;\mathbb{R}^2)$ (the space of bounded continuous functions from $\mathbb{R}^2$ to $\mathbb{R}^2$). That is, $\Phi_\varepsilon^t := \mathrm{Id} + \Psi_\varepsilon^t$ is the flow map generated by the vector field u_ε that is in turn generated by the measure $\Phi_{\varepsilon*}^t\omega^0$. We will next show that (2.2) is globally well-posed in $BC(\mathbb{R}^2;\mathbb{R}^2)$.

Lemma 2.4. *For each $F \in BC(\mathbb{R}^2;\mathbb{R}^2)$, let*

$$\mathcal{F}(F) := u_\varepsilon((\mathrm{Id} + F)_*\omega^0) \circ (\mathrm{Id} + F).$$

Then $\mathcal{F}\colon BC(\mathbb{R}^2;\mathbb{R}^2) \to BC(\mathbb{R}^2;\mathbb{R}^2)$ is well-defined and Lipschitz continuous.

Proof. Clearly $\mathcal{F}(F) \in C(\mathbb{R}^2;\mathbb{R}^2)$ for any $F \in BC(\mathbb{R}^2;\mathbb{R}^2)$. For any $F_1, F_2 \in BC(\mathbb{R}^2;\mathbb{R}^2)$ and $x \in \mathbb{R}^2$ we see that $(\mathcal{F}(F_1) - \mathcal{F}(F_2))(x)$ equals

$$\begin{aligned} &\int_{\mathbb{R}^2} \nabla^\perp K_\varepsilon(x + F_1(x) - y)\,d(\mathrm{Id} + F_1)_*\omega^0(y) - \int_{\mathbb{R}^2} \nabla^\perp K_\varepsilon(x + F_2(x) - y)\,d(\mathrm{Id} + F_2)_*\omega^0(y) \\ &= \int_{\mathbb{R}^2} \left[\nabla^\perp K_\varepsilon(x - y + F_1(x) - F_1(y)) - \nabla^\perp K_\varepsilon(x - y + F_2(x) - F_2(y))\right] \omega^0(y)\,dy, \end{aligned}$$

so

$$\|\mathcal{F}(F_1) - \mathcal{F}(F_2)\|_{L^\infty} \le 2\left\|D(\nabla^\perp K_\varepsilon)\right\|_{L^\infty} \left\|\omega^0\right\|_{L^1} \|F_1 - F_2\|_{L^\infty}.$$

Since $\mathcal{F}(0) = u_\varepsilon(\omega^0)$ is bounded, both claims follow from this. □

Lemma 2.4 shows that (2.2) is globally well-posed, and we let $\Phi_\varepsilon^t := \mathrm{Id}+\Psi_\varepsilon^t$ and $\omega_\varepsilon^t := \Phi_{\varepsilon *}^t \omega^0$, with the latter being for now only a finite signed Borel measure. We will next show that Φ_ε^t is in fact a measure-preserving homeomorphism, which will mean that ω_ε^t is an $L^1 \cap L^\infty$ function.

Clearly the ODE

$$\partial_t G^t = u_\varepsilon(\omega_\varepsilon^t) \circ (\mathrm{Id} + G^t) \tag{2.3}$$

is globally well-posed in $BC(\mathbb{R}^2;\mathbb{R}^2)$ for any initial data at any initial time. For each $t_0, t_1 \in \mathbb{R}$, let $\Theta_\varepsilon^{t_0,t}$ be the unique solution to (2.3) with initial data $\Theta_\varepsilon^{t_0,t_0} := 0$ at time $t = t_0$, and consider $G^t := \Theta_\varepsilon^{t_0,t_1} + \Theta_\varepsilon^{t_1,t} \circ (\mathrm{Id} + \Theta_\varepsilon^{t_0,t_1})$. Then G^t solves (2.3) and $G^{t_1} = \Theta_\varepsilon^{t_0,t_1}$, so uniqueness of the solution with the initial data $\Theta_\varepsilon^{t_0,t_1}$ at time $t = t_1$ shows that

$$\mathrm{Id} + \Theta_\varepsilon^{t_0,t} = \mathrm{Id} + G^t = (\mathrm{Id} + \Theta_\varepsilon^{t_1,t}) \circ (\mathrm{Id} + \Theta_\varepsilon^{t_0,t_1})$$

for all $t \in \mathbb{R}$. Letting $t := t_0$ shows that $(\mathrm{Id} + \Theta_\varepsilon^{t_1,t_0}) \circ (\mathrm{Id} + \Theta_\varepsilon^{t_0,t_1}) = \mathrm{Id}$, so we conclude that each $\mathrm{Id} + \Theta_\varepsilon^{t_0,t}$ is a homeomorphism. Then so is $\Phi_\varepsilon^t = \mathrm{Id} + \Theta_\varepsilon^{0,t}$.

Letting $BC^1(\mathbb{R}^2;\mathbb{R}^2)$ be the space of bounded C^1 functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ with bounded first derivatives, we see that for any $F \in BC^1(\mathbb{R}^2;\mathbb{R}^2)$ and $x, h \in \mathbb{R}^2$ we have

$$D(u_\varepsilon(\omega_\varepsilon^t) \circ (\mathrm{Id} + F))(x)h = \int_{\mathbb{R}^2} D(\nabla^\perp K_\varepsilon)(x + F(x) - y)(h + DF(x)h)\, d\omega_\varepsilon^t(y).$$

Therefore $F \mapsto u_\varepsilon(\omega_\varepsilon^t) \circ (\mathrm{Id} + F)$ is locally Lipschitz on $BC^1(\mathbb{R}^2;\mathbb{R}^2)$, so (2.3) is locally well-posed there. But since for any $F \in BC^1(\mathbb{R}^2;\mathbb{R}^2)$ we have

$$\left\| D(u_\varepsilon(\omega_\varepsilon^t) \circ (\mathrm{Id} + F)) \right\|_{L^\infty} \le \left\| D(\nabla^\perp K_\varepsilon) \right\|_{L^\infty} \left\| \omega^0 \right\|_{L^1} \left\| \mathrm{Id} + DF \right\|_{L^\infty},$$

a Grönwall-type argument shows that the C^1 norm of any solution to (2.3) can grow no faster than exponentially. Therefore (2.3) is even globally well-posed in $BC^1(\mathbb{R}^2;\mathbb{R}^2)$, and so $\Theta_\varepsilon^{t_0,t} \in BC^1(\mathbb{R}^2;\mathbb{R}^2)$ for all $t \in \mathbb{R}$. This and $\nabla \cdot u_\varepsilon(\omega_\varepsilon^t) \equiv 0$ now show that the map $\mathrm{Id} + \Theta_\varepsilon^{t_0,t}$ is measure-preserving. Then $\omega_\varepsilon^t = \omega^0 \circ (\Phi_\varepsilon^t)^{-1} \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $\Phi_{\varepsilon *}^t \Omega$ is its generalized layer cake representation.

Similarly, with $BC^2(\mathbb{R}^2;\mathbb{R}^2)$ the space of bounded C^2 functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ with bounded first and second derivatives, for each $F \in BC^2(\mathbb{R}^2;\mathbb{R}^2)$ and $x, h_1, h_2 \in \mathbb{R}^2$ we have

$$\begin{aligned} &D^2(u_\varepsilon(\omega_\varepsilon^t) \circ (\mathrm{Id} + F))(x)(h_1, h_2) \\ &\quad = \int_{\mathbb{R}^2} D^2(\nabla^\perp K_\varepsilon)(x + F(x) - y)(h_1 + DF(x)h_1, h_2 + DF(x)h_2)\, \omega_\varepsilon^t(y)\, dy \\ &\qquad\quad + \int_{\mathbb{R}^2} D(\nabla^\perp K_\varepsilon)(x + F(x) - y) \left(D^2F(x)(h_1, h_2) \right) \omega_\varepsilon^t(y)\, dy \end{aligned}$$

and

$$\begin{aligned} \left\| D^2(u_\varepsilon(\omega_\varepsilon^t) \circ (\mathrm{Id} + F)) \right\|_{L^\infty} \le{} & \left\| D^2(\nabla^\perp K_\varepsilon) \right\|_{L^\infty} \left\| \omega^0 \right\|_{L^1} \left\| \mathrm{Id} + DF \right\|_{L^\infty}^2 \\ & + \left\| D(\nabla^\perp K_\varepsilon) \right\|_{L^\infty} \left\| \omega^0 \right\|_{L^1} \left\| D^2F \right\|_{L^\infty}. \end{aligned}$$

Another Grönwall-type argument and the time-exponential bound on the C^1 norms of solutions to (2.3) now shows that (2.3) is globally well-posed in $BC^2(\mathbb{R}^2;\mathbb{R}^2)$, which we will use in Section 3. One can continue and inductively show that (2.3) is globally well-posed in $BC^k(\mathbb{R}^2;\mathbb{R}^2)$ for all $k \in \mathbb{N}$ (then each Φ_ε^t is a diffeomorphism), but we will not need this here.

Next we derive an ε-independent estimate on the growth of $L_\theta(\Phi_{\varepsilon*}^t\Omega)$.

Lemma 2.5. *$\left\|D^k(u_\varepsilon(\omega_\varepsilon^t))\right\|_{L^\infty}$ is continuous in t for all $k \in \mathbb{Z}_{\geq 0}$, and*

$$\begin{aligned}
&\left|\Theta_\varepsilon^{t_0,t_1}(x) - \Theta_\varepsilon^{t_0,t_1}(y) - \Theta_\varepsilon^{t_0,t_2}(x) + \Theta_\varepsilon^{t_0,t_2}(y)\right| \\
&\qquad \leq \left(\exp\left(\left|\int_{t_2}^{t_1} \|u_\varepsilon(\omega_\varepsilon^\tau)\|_{\dot{C}^{0,1}}\, d\tau\right|\right) - 1\right) \left|x + \Theta_\varepsilon^{t_0,t_2}(x) - y - \Theta_\varepsilon^{t_0,t_2}(y)\right| \qquad (2.4)
\end{aligned}$$

holds for all $x, y \in \mathbb{R}^2$ and $t_0, t_1, t_2 \in \mathbb{R}$.

Proof. Change of variables yields

$$\left\|D^k(u_\varepsilon(\omega_\varepsilon^{t_1})) - D^k(u_\varepsilon(\omega_\varepsilon^{t_2}))\right\|_{L^\infty} \leq \left\|D^{k+1}(\nabla^\perp K_\varepsilon)\right\|_{L^\infty} \left\|\omega^0\right\|_{L^1} \left\|\Phi_\varepsilon^{t_1} - \Phi_\varepsilon^{t_2}\right\|_{L^\infty}$$

for any $(k, t_1, t_2) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^2$. This shows the first claim, and in particular that $\|u_\varepsilon(\omega_\varepsilon^t)\|_{\dot{C}^{0,1}}$ is continuous in t.

Next, letting $x' := x + \Theta_\varepsilon^{t_0,t_2}(x)$, we see that

$$\Theta_\varepsilon^{t_0,t_1}(x) = x' + \Theta_\varepsilon^{t_2,t_1}(x') - x = \Theta_\varepsilon^{t_2,t_1}(x') + \Theta_\varepsilon^{t_0,t_2}(x).$$

So with $y' := y + \Theta_\varepsilon^{t_0,t_2}(y)$, the left-hand side of (2.4) is just $|\Theta_\varepsilon^{t_2,t_1}(x') - \Theta_\varepsilon^{t_2,t_1}(y')|$, while the last factor is $|x' - y'|$. The result now follows from the definition of $\Theta_\varepsilon^{t_2,t}$. $\square$

Proposition 2.6. *$L_\theta(\Phi_{\varepsilon*}^t\Omega)$ is continuous in t and*

$$\max\left\{\partial_t^+ L_\theta(\Phi_{\varepsilon*}^t\Omega), -\partial_{t-} L_\theta(\Phi_{\varepsilon*}^t\Omega)\right\} \leq \left\|u_\varepsilon(\omega_\varepsilon^t)\right\|_{\dot{C}^{0,1}} L_\theta(\Phi_{\varepsilon*}^t\Omega).$$

Proof. Fix any $t_1, t_2 \in \mathbb{R}$, $x \in \mathbb{R}^2$, $\lambda \in \mathcal{L}$, and $\eta > 0$. Pick $y \in \partial\Omega^\lambda$ such that

$$\left|\Phi_\varepsilon^{t_1}(x) - \Phi_\varepsilon^{t_1}(y)\right| \leq d(\Phi_\varepsilon^{t_1}(x), \partial\Phi_\varepsilon^{t_1}(\Omega^\lambda)) + \eta.$$

Then Lemma 2.5 and the inequality $\left|\frac{1}{a^{2\alpha}} - \frac{1}{b^{2\alpha}}\right| \leq \frac{|a-b|}{ab^{2\alpha}}$ for $a, b > 0$ show that

$$\begin{aligned}
&\frac{1}{\left(d(\Phi_\varepsilon^{t_1}(x), \partial\Phi_\varepsilon^{t_1}(\Omega^\lambda)) + 2\eta\right)^{2\alpha}} - \frac{1}{\left(d(\Phi_\varepsilon^{t_2}(x), \partial\Phi_\varepsilon^{t_2}(\Omega^\lambda)) + \eta\right)^{2\alpha}} \\
&\qquad \leq \frac{1}{\left(\left|\Phi_\varepsilon^{t_1}(x) - \Phi_\varepsilon^{t_1}(y)\right| + \eta\right)^{2\alpha}} - \frac{1}{\left(\left|\Phi_\varepsilon^{t_2}(x) - \Phi_\varepsilon^{t_2}(y)\right| + \eta\right)^{2\alpha}} \\
&\qquad \leq \frac{|\Phi_\varepsilon^{t_1}(x) - \Phi_\varepsilon^{t_1}(y) - \Phi_\varepsilon^{t_2}(x) + \Phi_\varepsilon^{t_2}(y)|}{\left(\left|\Phi_\varepsilon^{t_1}(x) - \Phi_\varepsilon^{t_1}(y)\right| + \eta\right)\left(\left|\Phi_\varepsilon^{t_2}(x) - \Phi_\varepsilon^{t_2}(y)\right| + \eta\right)^{2\alpha}} \qquad (2.5) \\
&\qquad \leq \frac{\exp\left(\left|\int_{t_2}^{t_1} \|u_\varepsilon(\omega_\varepsilon^\tau)\|_{\dot{C}^{0,1}}\, d\tau\right|\right) - 1}{(d(\Phi_\varepsilon^{t_2}(x), \partial\Phi_\varepsilon^{t_2}(\Omega^\lambda)) + \eta)^{2\alpha}},
\end{aligned}$$

so letting $\eta \to 0^+$, integrating over λ, and then taking supremum over $x \in \mathbb{R}^2$ shows

$$L_\theta(\Phi_{\varepsilon*}^{t_1}\Omega) \leq \exp\left(\left|\int_{t_2}^{t_1} \|u_\varepsilon(\omega_\varepsilon^\tau)\|_{\dot{C}^{0,1}}\, d\tau\right|\right) L_\theta(\Phi_{\varepsilon*}^{t_2}\Omega).$$

Since $t_1, t_2 \in \mathbb{R}$ were arbitrary, both claims follow from this. $\square$

From Lemma 2.1, Proposition 2.6, and a Grönwall-type argument we now obtain the following result.

Corollary 2.7. *With C_α from Lemma 2.1, for all $t \in \mathbb{R}$ we have*

$$\max\left\{\partial_t^+ L_\theta(\Phi^t_{\varepsilon *}\Omega), -\partial_{t-} L_\theta(\Phi^t_{\varepsilon *}\Omega)\right\} \le C_\alpha L_\theta(\Phi^t_{\varepsilon *}\Omega)^2.$$

In particular, for all $t \in (-\frac{1}{C_\alpha L_\theta(\Omega)}, \frac{1}{C_\alpha L_\theta(\Omega)})$ we have

$$L_\theta(\Phi^t_{\varepsilon *}\Omega) \le \frac{L_\theta(\Omega)}{1 - C_\alpha L_\theta(\Omega)\,|t|}.$$

Proposition 2.8. *Let $T_0 := \frac{1}{2C_\alpha L_\theta(\Omega)}$, with C_α from Lemma 2.1. There is $\Psi := \lim_{\varepsilon\to 0}\Psi_\varepsilon \in C\left([-T_0,T_0]; BC(\mathbb{R}^2;\mathbb{R}^2)\right)$, and $\Phi^t := \mathrm{Id} + \Psi^t$ is a measure-preserving homeomorphism for each $t \in [-T_0,T_0]$ that solves (1.7). Moreover, for each $t \in [-T_0,T_0]$ we have*

$$L_\theta(\Phi^t_*\Omega) \le \sup_{\varepsilon>0} L_\theta(\Phi^t_{\varepsilon *}\Omega) \le 2L_\theta(\Omega)$$

and

$$\max\left\{\left\|\Phi^t\right\|_{\dot C^{0,1}}, \left\|(\Phi^t)^{-1}\right\|_{\dot C^{0,1}}\right\} \le \sup_{\varepsilon>0}\max\left\{\left\|\Phi^t_\varepsilon\right\|_{\dot C^{0,1}}, \left\|(\Phi^t_\varepsilon)^{-1}\right\|_{\dot C^{0,1}}\right\} \le e^{2C_\alpha L_\theta(\Omega)|t|}.$$

Proof. Corollary 2.7 shows that

$$M := \sup_{\varepsilon>0}\sup_{t\in[-T_0,T_0]} L_\theta(\Phi^t_{\varepsilon *}\Omega) \in [L_\theta(\Omega), 2L_\theta(\Omega)].$$

We may assume that $L_\theta(\Omega) > 0$ because otherwise $\omega^0 \equiv 0$ and the result follows trivially. Fix any $t_0 \in [-T_0,T_0]$ and pick any $t \in [-T_0,T_0]$, $\varepsilon > 0$, and $\varepsilon' \in (0,\varepsilon)$. Then Lemmas 2.1, 2.2, and 2.3 show that

$$\begin{aligned}
&\left\|u_\varepsilon(\omega^t_\varepsilon)\circ(\mathrm{Id}+\Theta^{t_0,t}_\varepsilon) - u_{\varepsilon'}(\omega^t_{\varepsilon'})\circ(\mathrm{Id}+\Theta^{t_0,t}_{\varepsilon'})\right\|_{L^\infty}\\
&\qquad\le \left\|u_\varepsilon(\omega^t_\varepsilon)\right\|_{\dot C^{0,1}}\left\|\Theta^{t_0,t}_\varepsilon - \Theta^{t_0,t}_{\varepsilon'}\right\|_{L^\infty} + \left\|u_\varepsilon(\omega^t_\varepsilon) - u(\omega^t_\varepsilon)\right\|_{L^\infty}\\
&\qquad\qquad + \left\|u(\omega^t_\varepsilon) - u(\omega^t_{\varepsilon'})\right\|_{L^\infty} + \left\|u(\omega^t_{\varepsilon'}) - u_{\varepsilon'}(\omega^t_{\varepsilon'})\right\|_{L^\infty}\\
&\qquad\le C_\alpha M\left\|\Theta^{t_0,t}_\varepsilon - \Theta^{t_0,t}_{\varepsilon'}\right\|_{L^\infty} + C_\alpha M\left\|\Phi^t_\varepsilon - \Phi^t_{\varepsilon'}\right\|_{L^\infty} + C_\alpha\left\|\omega^0\right\|_{L^\infty}\varepsilon^{1-2\alpha}
\end{aligned}\tag{2.6}$$

where C_α (which we now fix for the rest of the proof) is two times the maximum of all the C_α's appearing in those lemmas. Integrating (2.6) between any $t_1, t_2 \in [-T_0,T_0]$ yields

$$\begin{aligned}
&\left\|\Theta^{t_0,t_1}_\varepsilon - \Theta^{t_0,t_1}_{\varepsilon'} - \Theta^{t_0,t_2}_\varepsilon + \Theta^{t_0,t_2}_{\varepsilon'}\right\|_{L^\infty}\\
&\qquad\le C_\alpha M\left|\int_{t_2}^{t_1}\left\|\Theta^{t_0,\tau}_\varepsilon - \Theta^{t_0,\tau}_{\varepsilon'}\right\|_{L^\infty}d\tau\right| + C_\alpha M\left|\int_{t_2}^{t_1}\|\Phi^\tau_\varepsilon - \Phi^\tau_{\varepsilon'}\|_{L^\infty}d\tau\right|\\
&\qquad\qquad + C_\alpha\left\|\omega^0\right\|_{L^\infty}|t_1-t_2|\,\varepsilon^{1-2\alpha}.
\end{aligned}\tag{2.7}$$

In particular, taking $t_0 = 0$, dividing by $|t_1 - t_2|$, and letting $t_1 \to t_2^\pm$ shows that

$$\max\left\{\partial_t^+\left\|\Psi^t_\varepsilon - \Psi^t_{\varepsilon'}\right\|_{L^\infty}, -\partial_{t-}\left\|\Psi^t_\varepsilon - \Psi^t_{\varepsilon'}\right\|_{L^\infty}\right\} \le 2C_\alpha M\left\|\Psi^t_\varepsilon - \Psi^t_{\varepsilon'}\right\|_{L^\infty} + C_\alpha\left\|\omega^0\right\|_{L^\infty}\varepsilon^{1-2\alpha}$$

for each $t \in [-T_0, T_0]$, and then a Grönwall-type argument yields

$$\left\|\Phi^t_\varepsilon - \Phi^t_{\varepsilon'}\right\|_{L^\infty} = \left\|\Psi^t_\varepsilon - \Psi^t_{\varepsilon'}\right\|_{L^\infty} \le \frac{\|\omega^0\|_{L^\infty}}{2M}(e^{2C_\alpha M|t|} - 1)\varepsilon^{1-2\alpha}.$$

Applying this inequality to (2.7), dividing by $|t_1 - t_2|$ and then sending $t_1 \to t_2^\pm$ shows that

$$\begin{aligned}\max\left\{\partial_t^+ \left\|\Theta^{t_0,t}_\varepsilon - \Theta^{t_0,t}_{\varepsilon'}\right\|_{L^\infty}, -\partial_{t-} \left\|\Theta^{t_0,t}_\varepsilon - \Theta^{t_0,t}_{\varepsilon'}\right\|_{L^\infty}\right\}& \\ \le C_\alpha M \left\|\Theta^{t_0,t}_\varepsilon - \Theta^{t_0,t}_{\varepsilon'}\right\|_{L^\infty} + C_\alpha \left\|\omega^0\right\|_{L^\infty} e^{2C_\alpha M|t|}\varepsilon^{1-2\alpha}&\end{aligned}$$

for all $t \in [-T_0, T_0]$, so a Grönwall-type argument yields

$$\left\|\Theta^{t_0,t}_\varepsilon - \Theta^{t_0,t}_{\varepsilon'}\right\|_{L^\infty} \le \frac{\|\omega^0\|_{L^\infty} e^{2C_\alpha M T_0}\varepsilon^{1-2\alpha}}{M}\left(e^{C_\alpha M|t-t_0|} - 1\right).$$

Therefore, $\Theta^{t_0,\cdot}_\varepsilon$ converges uniformly to some $\Theta^{t_0,\cdot} : [-T_0, T_0] \to BC(\mathbb{R}^2; \mathbb{R}^2)$ as $\varepsilon \to 0$.

Let $\Psi^t := \Theta^{0,t}$. Since $(\mathrm{Id} + \Theta^{t_0,t_1}_\varepsilon) \circ (\mathrm{Id} + \Theta^{t_1,t_0}_\varepsilon) = \mathrm{Id}$ for all $t_0, t_1 \in [-T_0, T_0]$ and $\varepsilon > 0$, sending $\varepsilon \to 0$ shows that $(\mathrm{Id} + \Theta^{t_0,t_1}) \circ (\mathrm{Id} + \Theta^{t_1,t_0}) = \mathrm{Id}$. In particular, $\Phi^t := \mathrm{Id} + \Psi^t$ is a homeomorphism whose inverse is $\mathrm{Id} + \Theta^{t,0}$. Also, Lemma 2.5 and the definition of C_α show that $\|\mathrm{Id} + \Theta^{t_0,t}_\varepsilon\|_{\dot{C}^{0,1}} \le e^{C_\alpha M|t-t_0|}$ for all $t_0, t \in [-T_0, T_0]$ and $\varepsilon > 0$, thus $\max\{\|\Phi^t\|_{\dot{C}^{0,1}}, \|(\Phi^t)^{-1}\|_{\dot{C}^{0,1}}\} \le e^{C_\alpha M|t|}$ holds for all $t \in [-T_0, T_0]$. By Fatou's lemma we also have $L_\theta(\Phi^t_*\Omega) \le \liminf_{\varepsilon\to 0} L_\theta(\Phi^t_{\varepsilon *}\Omega) \le M$ for each $t \in [-T_0, T_0]$. And since each Φ^t_ε is measure-preserving, their uniform limit Φ^t is also such because for any open set $U \subseteq \mathbb{R}^2$ we have that $\mathbb{1}_U \circ \Phi^t_\varepsilon \to \mathbb{1}_U \circ \Phi^t$ pointwise as $\varepsilon \to 0$.

It remains to show that Φ^t satisfies (1.7), that is, with $\omega^t := \omega^0 \circ (\Phi^t)^{-1}$ we have

$$\Phi^t = \mathrm{Id} + \int_0^t u(\omega^\tau) \circ \Phi^\tau \, d\tau \tag{2.8}$$

for each $t \in [-T_0, T_0]$. Taking $t_0 = 0$ and letting $\varepsilon' \to 0^+$ in (2.6) yields

$$\left\|u_\varepsilon(\omega^t_\varepsilon) \circ \Phi^t_\varepsilon - u(\omega^t) \circ \Phi^t\right\|_{L^\infty} \le 2C_\alpha M \left\|\Phi^t_\varepsilon - \Phi^t\right\|_{L^\infty} + C_\alpha \left\|\omega^0\right\|_{L^\infty} \varepsilon^{1-2\alpha},$$

which shows that the right-hand side of

$$\Phi^t_\varepsilon = \mathrm{Id} + \int_0^t u(\omega^\tau_\varepsilon) \circ \Phi^\tau_\varepsilon \, d\tau$$

converges uniformly to the right-hand side of (2.8) as $\varepsilon \to 0$. This now proves (1.7). □

Proposition 2.9. *Let (Ω_1, θ_1), (Ω_2, θ_2) be generalized layer cake representations of ω^0 and $\Phi_1, \Phi_2 \in C(I; C(\mathbb{R}^2; \mathbb{R}^2))$ be solutions to (1.7) on a compact interval $I \ni 0$ that are both measure-preserving homeomorphisms and $\sup_{t\in I} \max_{i\in\{1,2\}} L_{\theta_i}(\Phi^t_{i*}\Omega_i) < \infty$. Then $\Phi_1 = \Phi_2$.*

Proof. Let

$$M := \sup_{t\in I} \max_{i=1,2} L_{\theta_i}(\Phi^t_{i*}\Omega_i).$$

Then Lemmas 2.1 and 2.2 show that

$$\begin{aligned}\left\|u(\Phi^t_{1*}\omega^0) \circ \Phi^t_1 - u(\Phi^t_{2*}\omega^0) \circ \Phi^t_2\right\|_{L^\infty} &\le \left\|u(\Phi^t_{1*}\omega^0)\right\|_{\dot{C}^{0,1}} \left\|\Phi^t_1 - \Phi^t_2\right\|_{L^\infty} + \left\|u(\Phi^t_{1*}\omega^0) - u(\Phi^t_{2*}\omega^0)\right\|_{L^\infty} \\ &\le C_\alpha M \left\|\Phi^t_1 - \Phi^t_2\right\|_{L^\infty}\end{aligned}$$

with some C_α, which together with continuity of $\|\Phi_1^t - \Phi_2^t\|_{L^\infty}$ in t yields

$$\max\left\{\partial_t^+ \left\|\Phi_1^t - \Phi_2^t\right\|_{L^\infty}, -\partial_{t-} \left\|\Phi_1^t - \Phi_2^t\right\|_{L^\infty}\right\} \le C_\alpha M \left\|\Phi_1^t - \Phi_2^t\right\|_{L^\infty}.$$

A Grönwall-type argument finishes the proof. □

Combining Propositions 2.8 and 2.9 with (1.8), the latter showing that the time spans of maximal solutions for any two generalized layer cake representations of ω^0 must coincide (recall that $\sup_{t\in J} \|(\Phi^t)^{-1}\|_{\dot{C}^{0,1}} < \infty$ for any compact interval $J \subseteq I$), now yields Theorem 1.2.

3. Proof of Theorem 1.4

Again, all constants C_α below can change from one inequality to another, but they always only depend on α. Fix any ω^0 satisfying the hypotheses and for each $\lambda \in \mathcal{L}$, let $z^{0,\lambda} \in \mathrm{PSC}(\mathbb{R}^2)$ be such that $\partial\Omega^\lambda = \mathrm{im}(z^{0,\lambda})$.

Let $\omega\colon I \to L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ be the Lagrangian solution to (1.1)–(1.2) from Theorem 1.2, with initial data ω^0 and flow map $\Phi \in C\left(I; C(\mathbb{R}^2;\mathbb{R}^2)\right)$, and consider $T_0 := \frac{1}{2C_\alpha L_\theta(\Omega)}$ as in Proposition 2.8 (note that $[-T_0, T_0] \subseteq I$ because I is maximal). Then since Φ^t is a homeomorphism for each $t \in I$, it follows that $\Phi^t_*\Omega$ is composed of simple closed curves (this proves the first claim in Theorem 1.4(i)). We denote these $z^{t,\lambda} := \Phi^t \circ z^{0,\lambda} \in \mathrm{PSC}(\mathbb{R}^2)$, where $\Phi^t \circ z^{0,\lambda} \in \mathrm{CC}(\mathbb{R}^2)$ is the curve whose representative is $\Phi^t \circ \tilde{z}^{0,\lambda}$ whenever $\tilde{z}^{0,\lambda}$ is a representative of $z^{0,\lambda}$ (since $\left\{z^{t,\lambda}\right\}_{t\in I}$ is clearly a connected subset of $\mathrm{CC}(\mathbb{R}^2)$, [3, Lemma B.4] shows that each $z^{t,\lambda}$ is positively oriented).

Fix any $\varepsilon > 0$ and recall that $\Phi_\varepsilon^t := \mathrm{Id} + \Psi_\varepsilon^t$, where Ψ_ε^t is the solution to (2.2). For each $(t,\lambda) \in \mathbb{R}\times\mathcal{L}$ let $z_\varepsilon^{t,\lambda} := \Phi_\varepsilon^t \circ z^{0,\lambda}$ and $\omega_\varepsilon^t := \omega^0 \circ (\Phi_\varepsilon^t)^{-1}$, then fix any arclength parametrization of $z_\varepsilon^{t,\lambda}$ (we denote it again $z_\varepsilon^{t,\lambda}(\cdot)$) and for $s \in [0, \ell(z_\varepsilon^{t,\lambda})]$ define

- $\ell_\varepsilon^{t,\lambda} := \ell(z_\varepsilon^{t,\lambda})$,
- $\mathbf{T}_\varepsilon^{t,\lambda}(s) := \partial_s z_\varepsilon^{t,\lambda}(s)$,
- $\mathbf{N}_\varepsilon^{t,\lambda}(s) := \mathbf{T}^{t,\lambda}(s)^\perp$,
- $\kappa_\varepsilon^{t,\lambda}(s) := \partial_s^2 z_\varepsilon^{t,\lambda}(s) \cdot \mathbf{N}^{t,\lambda}(s)$,
- $\Delta_\varepsilon^{t,\lambda,\lambda'} := \Delta(z_\varepsilon^{t,\lambda}, z_\varepsilon^{t,\lambda'})$,
- $u_\varepsilon^{t,\lambda}(s) := u_\varepsilon(\omega_\varepsilon^t; z_\varepsilon^{t,\lambda}(s))$.

Proposition 2.8 shows that $\lim_{\varepsilon\to 0} z_\varepsilon^{t,\lambda} = z^{t,\lambda}$ in $\mathrm{CC}(\mathbb{R}^2)$, and as noted in [3, Section 4],

$$\partial_s^2 z_\varepsilon^{t,\lambda}(s) = \partial_s \mathbf{T}_\varepsilon^{t,\lambda}(s) = \kappa_\varepsilon^{t,\lambda}(s)\mathbf{N}_\varepsilon^{t,\lambda}(s) \qquad \text{and} \qquad \partial_s \mathbf{N}_\varepsilon^{t,\lambda}(s) = -\kappa_\varepsilon^{t,\lambda}(s)\mathbf{T}_\varepsilon^{t,\lambda}(s)$$

holds as well. Then the argument in [3, Lemma 4.1] also applies here, and we obtain

$$\begin{aligned}\partial_t \left\|z_\varepsilon^{t,\lambda}\right\|_{\dot{H}^2}^2 = -3\int_{\ell_\varepsilon^{t,\lambda}\mathbb{T}} &\kappa_\varepsilon^{t,\lambda}(s)^2 \left(\partial_s u_\varepsilon^{t,\lambda}(s)\cdot \mathbf{T}_\varepsilon^{t,\lambda}(s)\right) ds \\ &+ 2\int_{\ell_\varepsilon^{t,\lambda}\mathbb{T}} \kappa_\varepsilon^{t,\lambda}(s)\left(\partial_s^2 u_\varepsilon^{t,\lambda}(s)\cdot \mathbf{N}_\varepsilon^{t,\lambda}(s)\right) ds.\end{aligned} \tag{3.1}$$

In the proof of this we fix some constant-speed parametrization $\tilde{z}_\varepsilon^{t,\lambda}\colon \mathbb{T} \to \mathbb{R}^2$ of $z_\varepsilon^{t,\lambda}$, and for each $h \in \mathbb{R}$ let $\tilde{z}_\varepsilon^{t+h,\lambda} := \Phi_\varepsilon^{t+h} \circ (\Phi_\varepsilon^t)^{-1} \circ \tilde{z}_\varepsilon^{t,\lambda}$. Since (2.3) is globally well-posed in $BC^2(\mathbb{R}^2;\mathbb{R}^2)$

(see the paragraph before Lemma 2.5), it easily follows that $\tilde{z}_\varepsilon^{t,\lambda} \in H^2(\mathbb{T};\mathbb{R}^2)$ and

$$\tilde{z}_\varepsilon^{t+h,\lambda} = \tilde{z}_\varepsilon^{t,\lambda} + \int_t^{t+h} u_\varepsilon(\omega_\varepsilon^\tau) \circ \tilde{z}_\varepsilon^{\tau,\lambda}\, d\tau \tag{3.2}$$

holds for all $h \in \mathbb{R}$ (in $H^2(\mathbb{T};\mathbb{R}^2)$).

Remark. Below we will also consider the above setup with initial data being (ω^{t_0}, z^{t_0}) for some $t_0 \in I$ instead of (ω^0, z^0). Since we do not yet know whether the $z^{t_0,\lambda}$ are H^2 curves when $t_0 \neq 0$, we cannot define $\kappa_\varepsilon^{t,\lambda}$ for these initial data and also cannot yet claim (3.2) to hold in $H^2(\mathbb{T};\mathbb{R}^2)$. However, since Φ^{t_0} is Lipschitz by Proposition 2.8, so are the $z^{t_0,\lambda}$ and then (3.2) holds in $C^{0,1}(\mathbb{T};\mathbb{R}^2)$. We will use this in the following results, up to Lemma 3.4.

Lemma 3.1. *With C_α from Lemma 2.1, for any $(t,\lambda) \in [-T_0, T_0] \times \mathcal{L}$ and $\varepsilon > 0$ we have*

$$\left|\partial_t \ell_\varepsilon^{t,\lambda}\right| \le 2C_\alpha L_\theta(\Omega) \ell_\varepsilon^{t,\lambda} \tag{3.3}$$

Proof. With $\tilde{z}_\varepsilon^{t,\lambda}$ as above, we have $\left|\partial_\xi \tilde{z}_\varepsilon^{t,\lambda}(\xi)\right| = \ell_\varepsilon^{t,\lambda} > 0$ for any $(t,\lambda) \in \mathbb{R} \times \mathcal{L}$. Then for any $(h,\xi) \in \mathbb{R} \times \mathbb{T}$ we get

$$\begin{aligned}
&\left|\partial_\xi \tilde{z}_\varepsilon^{t+h,\lambda}(\xi)\right| - \left|\partial_\xi \tilde{z}_\varepsilon^{t,\lambda}(\xi)\right| \\
&\quad = \frac{2\int_t^{t+h} \frac{d}{d\xi} u_\varepsilon(\omega_\varepsilon^\tau; \tilde{z}_\varepsilon^{\tau,\lambda}(\xi)) \cdot \partial_\xi \tilde{z}_\varepsilon^{t,\lambda}(\xi)\, d\tau}{\left|\partial_\xi \tilde{z}_\varepsilon^{t+h,\lambda}(\xi)\right| + \left|\partial_\xi \tilde{z}_\varepsilon^{t,\lambda}(\xi)\right|} + \frac{\left|\int_t^{t+h} \frac{d}{d\xi} u_\varepsilon(\omega_\varepsilon^\tau; \tilde{z}_\varepsilon^{\tau,\lambda}(\xi))\, d\tau\right|^2}{\left|\partial_\xi \tilde{z}_\varepsilon^{t+h,\lambda}(\xi)\right| + \left|\partial_\xi \tilde{z}_\varepsilon^{t,\lambda}(\xi)\right|}.
\end{aligned}$$

Since $\left\|D(u_\varepsilon(\omega_\varepsilon^\tau))\right\|_{L^\infty}$ is continuous in τ by Lemma 2.5, integrating in ξ, dividing by h, and then letting $h \to 0$ shows that

$$\partial_t \ell_\varepsilon^{t,\lambda} = \int_{\mathbb{T}} \frac{d}{d\xi} u_\varepsilon(\omega_\varepsilon^t; \tilde{z}_\varepsilon^{t,\lambda}(\xi)) \cdot \frac{\partial_\xi \tilde{z}_\varepsilon^{t,\lambda}(\xi)}{\left|\partial_\xi \tilde{z}_\varepsilon^{t,\lambda}(\xi)\right|}\, d\xi = \int_{\ell_\varepsilon^{t,\lambda}\mathbb{T}} \partial_s u_\varepsilon^{t,\lambda}(s) \cdot \mathbf{T}_\varepsilon^{t,\lambda}(s)\, ds.$$

The result now follows from Lemma 2.1 and Corollary 2.7. ☐

Lemma 3.2. *With C_α from Lemma 2.1, for any $(t,\lambda) \in I \times \mathcal{L}$ and all small enough $h \in \mathbb{R}$ we have*

$$e^{-3C_\alpha L_\theta(\Phi_*^t \Omega)|h|} \ell(z^{t,\lambda}) \le \ell(z^{t+h,\lambda}) \le e^{3C_\alpha L_\theta(\Phi_*^t \Omega)|h|} \ell(z^{t,\lambda}).$$

Proof. Since $\ell\colon \mathrm{CC}(\mathbb{R}^2) \to [0,\infty]$ is lower semi-continuous (by definition) and $\lim_{\varepsilon\to 0} z_\varepsilon^{t,\lambda} = z^{t,\lambda}$ in $\mathrm{CC}(\mathbb{R}^2)$, a Grönwall-type argument applied to (3.3) shows that for $|t| \le T_0$ we have

$$\ell(z^{t,\lambda}) \le e^{2C_\alpha L_\theta(\Omega)|t|} \ell(z^{0,\lambda}). \tag{3.4}$$

Next fix any $t_0 \in I$ with $|t_0| \le \frac{T_0}{2}$. Repeating the proof of (3.4) with $z^{t_0,\lambda}$ in place of $z^{0,\lambda}$ (recall the remark after (3.2)) shows that

$$\ell(z^{t,\lambda}) \le e^{2C_\alpha L_\theta(\Phi_*^{t_0}\Omega)|t-t_0|} \ell(z^{t_0,\lambda})$$

whenever $|t - t_0| \le \frac{1}{2C_\alpha L_\theta(\Phi_*^{t_0}\Omega)}$. Since $L_\theta(\Phi_*^{t_0}\Omega) \le \frac{4}{3} L_\theta(\Omega)$ by Corollary 2.7, we see that

$$\ell(z^{0,\lambda}) \le e^{3C_\alpha L_\theta(\Omega)|t_0|} \ell(z^{t_0,\lambda}),$$

and so

$$e^{-3C_\alpha L_\theta(\Omega)|t|}\ell(z^{0,\lambda}) \le \ell(z^{t,\lambda}) \le e^{3C_\alpha L_\theta(\Omega)|t|}\ell(z^{0,\lambda})$$

holds whenever $|t| \le \frac{T_0}{2}$. Applying this with (ω^t, t, h) in place of $(\omega^0, 0, t)$, for any $t \in I$, now shows the claim. □

Proposition 3.3. *$R_\theta(\Phi_*^t\Omega)$ is continuous in t. In particular, for any compact interval $J \subseteq I$ we have*

$$\sup_{t\in J} R_\theta(\Phi_*^t\Omega) < \infty.$$

Proof. Take any compact interval $J \subseteq I$ containing 0. Lemmas 2.1, 2.5 and Corollary 2.7 show that

$$\left|\Phi_\varepsilon^{t_1}(x) - \Phi_\varepsilon^{t_1}(y) - \Phi_\varepsilon^{t_2}(x) + \Phi_\varepsilon^{t_2}(y)\right| \le (e^{2C_\alpha L_\theta(\Omega)|t_1-t_2|} - 1)\left|\Phi_\varepsilon^{t_1}(x) - \Phi_\varepsilon^{t_1}(y)\right| \tag{3.5}$$

for all $\varepsilon > 0$, $t_1, t_2 \in [-T_0, T_0]$ and $x, y \in \mathbb{R}^2$. Hence, taking $\varepsilon \to 0$ shows that (3.5) continues to hold with Φ in place of Φ_ε. Applying the same argument with $\Phi_*^t\Omega$ in place of Ω (recall the remark after (3.2)), shows that each $t \in J$ has a neighborhood such that

$$\left|\Phi^{t_1}(x) - \Phi^{t_1}(y) - \Phi^{t_2}(x) + \Phi^{t_2}(y)\right| \le (e^{2C_\alpha M|t_1-t_2|} - 1)\left|\Phi^{t_1}(x) - \Phi^{t_1}(y)\right| \tag{3.6}$$

holds for any $t_1, t_2 \in J$ in that neighborhood, where

$$M \coloneqq \sup_{t\in J} L_\theta(\Phi_*^t\Omega) < \infty.$$

Since J is an interval, it follows that (3.6) in fact holds for all $t_1, t_2 \in J$.

Fix $t_1, t_2 \in J$, $\lambda, \lambda' \in \mathcal{L}$, and $\eta > 0$. Then there are $x \in \operatorname{im}(z^{0,\lambda})$ and $y \in \operatorname{im}(z^{0,\lambda'})$ such that $\Delta(z^{t_1,\lambda}, z^{t_1,\lambda'}) = |\Phi^{t_1}(x) - \Phi^{t_1}(y)|$. A similar argument as in (2.5) now shows that

$$\begin{aligned}
&\frac{1}{\left(\Delta(z^{t_1,\lambda}, z^{t_1,\lambda'}) + \eta\right)^{2\alpha}} - \frac{1}{\left(\Delta(z^{t_2,\lambda}, z^{t_2,\lambda'}) + \eta\right)^{2\alpha}} \\
&\qquad \le \frac{|\Phi^{t_1}(x) - \Phi^{t_1}(y) - \Phi^{t_2}(x) + \Phi^{t_2}(y)|}{\left(|\Phi^{t_1}(x) - \Phi^{t_1}(y)| + \eta\right)\left(|\Phi^{t_2}(x) - \Phi^{t_2}(y)| + \eta\right)^{2\alpha}} \\
&\qquad \le \frac{e^{2C_\alpha M|t_1-t_2|} - 1}{(\Delta(z^{t_2,\lambda}, z^{t_2,\lambda'}) + \eta)^{2\alpha}},
\end{aligned}$$

thus

$$\frac{1}{\left(\Delta(z^{t_1,\lambda}, z^{t_1,\lambda'}) + \eta\right)^{2\alpha}} \le \frac{e^{2C_\alpha M|t_1-t_2|}}{\Delta(z^{t_2,\lambda}, z^{t_2,\lambda'})^{2\alpha}}. \tag{3.7}$$

Lemma 3.2 yields

$$e^{-3C_\alpha M|t_1-t_2|}\ell(z^{t_2,\lambda}) \le \ell(z^{t_1,\lambda}) \le e^{3C_\alpha M|t_1-t_2|}\ell(z^{t_2,\lambda})$$

for any $t_1, t_2 \in J$ because J is an interval, and so we get

$$\frac{\ell(z^{t_1,\lambda})^{1/2}}{\ell(z^{t_1,\lambda'})^{1/2}\left(\Delta(z^{t_1,\lambda}, z^{t_1,\lambda'}) + \eta\right)^{2\alpha}} \le \frac{e^{5C_\alpha M|t_1-t_2|}\ell(z^{t_2,\lambda})^{1/2}}{\ell(z^{t_2,\lambda'})^{1/2}\Delta(z^{t_2,\lambda}, z^{t_2,\lambda'})^{2\alpha}}. \tag{3.8}$$

Letting $\eta \to 0^+$, integrating in λ', and then taking the supremum over $\lambda \in \mathcal{L}$ shows that

$$R_\theta(\Phi_*^{t_1}\Omega) \le e^{5C_\alpha M|t_1-t_2|} R_\theta(\Phi_*^{t_2}\Omega). \tag{3.9}$$

Since $t_1, t_2 \in J$ were arbitrary, the claim follows. □

This proves the second claim in Theorem 1.4(i). Our proof of Theorem 1.4(ii) will use the following extension to $R_\theta(\Phi_{\varepsilon *}^t\Omega)$ of the bound on $R_\theta(\Phi_*^t\Omega)$ from the last proof.

Lemma 3.4. *$R_\theta(\Phi_{\varepsilon *}^t\Omega)$ is continuous in t for each $\varepsilon > 0$, and*

$$R_\theta(\Phi_{\varepsilon *}^t\Omega) \le e^{4C_\alpha L_\theta(\Omega)|t|} R_\theta(\Omega) \le 8R_\theta(\Omega) \tag{3.10}$$

holds for any $t \in [-T_0, T_0]$, with C_α from Lemma 2.1.

Proof. We have (3.3) for $t \in [-T_0, T_0]$, so a Grönwall-type argument shows that

$$e^{-2C_\alpha L_\theta(\Omega)|t_1-t_2|}\ell_\varepsilon^{t_2,\lambda} \le \ell_\varepsilon^{t_1,\lambda} \le e^{2C_\alpha L_\theta(\Omega)|t_1-t_2|}\ell_\varepsilon^{t_2,\lambda}$$

for any $t_1, t_2 \in [-T_0, T_0]$ and $\lambda \in \mathcal{L}$. Since (3.5) holds, a similar argument as in (3.7) shows

$$\frac{1}{\left(\Delta_\varepsilon^{t_1,\lambda,\lambda'} + \eta\right)^{2\alpha}} \le \frac{e^{2C_\alpha L_\theta(\Omega)|t_1-t_2|}}{(\Delta_\varepsilon^{t_2,\lambda,\lambda'})^{2\alpha}}$$

for any $t_1, t_2 \in [-T_0, T_0]$, $\lambda, \lambda' \in \mathcal{L}$ and $\eta > 0$. Hence we see that (3.8), and thus also (3.9), continue to hold with $(z_\varepsilon, 4C_\alpha L_\theta(\Omega), \Phi_\varepsilon)$ in place of $(z, 5C_\alpha M, \Phi)$. This now shows both claims (the second inequality in (3.10) follows by the definition of T_0 and $e^2 \le 8$). □

We now turn to Theorem 1.4(ii), which we prove by using Lemma 3.1 and by estimating the right-hand side of (3.1) in lemmas below.

Lemma 3.5. *There is C_α such that for any $\beta \in (0,1]$, any $C^{1,\beta}$ closed curve $\gamma\colon \ell\mathbb{T} \to \mathbb{R}^2$ parametrized by arclength, and any $x \in \mathbb{R}^2$ we have*

$$\int_{\ell\mathbb{T}} \frac{ds}{|x-\gamma(s)|^{1+2\alpha}} \le C_\alpha \frac{\ell\,\|\gamma\|_{\dot C^{1,\beta}}^{1/\beta}}{d(x, \operatorname{im}(\gamma))^{2\alpha}}.$$

Proof. Let $d := \frac14 \|\gamma\|_{\dot C^{1,\beta}}^{-1/\beta}$ and $\Delta := d(x, \operatorname{im}(\gamma))$. Then [3, Lemma A.2] shows that

$$\begin{aligned}\int_{\ell\mathbb{T}} \frac{ds}{|x-\gamma(s)|^{1+2\alpha}} &\le \frac{\ell}{4d}\left(\int_{|s|\le\Delta} \frac{ds}{\Delta^{1+2\alpha}} + \int_{\Delta\le|s|\le 2d} \frac{ds}{|s/2|^{1+2\alpha}}\right) + \frac{1}{\Delta^{2\alpha}}\int_{\ell\mathbb{T}} \frac{ds}{d} \\ &\le \frac{\ell}{2d\Delta^{2\alpha}} + \frac{\ell}{2^{1-2\alpha}\alpha d\Delta^{2\alpha}} + \frac{\ell}{d\Delta^{2\alpha}} = C_\alpha \frac{\ell\,\|\gamma\|_{\dot C^{1,\beta}}^{1/\beta}}{\Delta^{2\alpha}}.\end{aligned}$$

□

Lemma 3.6. *For any $\omega \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ with a generalized layer cake representation (Ω, θ) such that $L_\theta(\Omega) < \infty$, and for any $\varepsilon > 0$ and $x, h_1, h_2 \in \mathbb{R}^2$, we have*

$$D^2(u_\varepsilon(\omega))(x)(h_1,h_2) = \int_{\mathcal{L}}\int_{\mathbb{R}^2} D^2(\nabla^\perp K_\varepsilon)(x-y)(h_1,h_2)\left(\mathbb{1}_{\Omega^\lambda}(y) - \mathbb{1}_{\Omega^\lambda}(x)\right) dy\, d\theta(\lambda)$$

$$= \int_{\mathcal{L}} \int_{\Omega^\lambda} D^2(\nabla^\perp K_\varepsilon)(x-y)(h_1,h_2)\,dy\,d\theta(\lambda).$$

Moreover, there is C_α such that

$$\int_{\mathcal{L}} \int_{\mathbb{R}^2} \left|D^2(\nabla^\perp K_\varepsilon)(x-y)\right| \left|\mathbb{1}_{\Omega^\lambda}(y) - \mathbb{1}_{\Omega^\lambda}(x)\right| d|\theta|(\lambda)\,dy \le \frac{C_\alpha}{\varepsilon} \int_{\mathcal{L}} \frac{d|\theta|(\lambda)}{d(x,\partial\Omega^\lambda)^{2\alpha}}.$$

Proof. Oddness of $D^2(\nabla^\perp K_\varepsilon)$ shows that

$$\begin{aligned} D^2(u_\varepsilon(\omega))(x)(h_1,h_2) &= \int_{\mathbb{R}^2} D^2(\nabla^\perp K_\varepsilon)(x-y)(h_1,h_2)(\omega(y)-\omega(x))\,dy \\ &= \int_{\mathbb{R}^2} \int_{\mathcal{L}} D^2(\nabla^\perp K_\varepsilon)(x-y)(h_1,h_2)\left(\mathbb{1}_{\Omega^\lambda}(y) - \mathbb{1}_{\Omega^\lambda}(x)\right) d\theta(\lambda)\,dy. \end{aligned}$$

Then proceeding as in Lemma 2.1 and using $\left|D^2(\nabla^\perp K_\varepsilon)(x-y)\right| \le \frac{C_\alpha}{\varepsilon|x-y|^{2+2\alpha}}$ in place of $\left|D(\nabla^\perp K_\varepsilon)(x-y)\right| \le \frac{C_\alpha}{|x-y|^{2+2\alpha}}$ proves the second claim. Fubini's theorem now yields the first equality of the first claim, and the second one follows by oddness of $D^2(\nabla^\perp K_\varepsilon)$. $\square$

Lemma 3.7. *There is C_α such that for each $(t,\lambda) \in \mathbb{R}\times\mathcal{L}$ and $\varepsilon > 0$ we have*

$$\left|\partial_t \left\|z_\varepsilon^{t,\lambda}\right\|_{\dot H^2}^2\right| \le C_\alpha (L_\theta(\Phi_{\varepsilon*}^t\Omega) + R_\theta(\Phi_{\varepsilon*}^t\Omega))\, Q(\Phi_{\varepsilon*}^t\Omega) \left\|z_\varepsilon^{t,\lambda}\right\|_{\dot H^2}^2 \tag{3.11}$$

Proof. With $z_\varepsilon^{t,\lambda}(\cdot)$ being the previously fixed arclength parametrization of $z_\varepsilon^{t,\lambda}$, we have

$$\partial_s^2 u_\varepsilon^{t,\lambda}(s) = D^2(u_\varepsilon(\omega_\varepsilon^t))(z_\varepsilon^{t,\lambda}(s))(\mathbf{T}_\varepsilon^{t,\lambda}(s), \mathbf{T}_\varepsilon^{t,\lambda}(s)) + \kappa_\varepsilon^{t,\lambda}(s) D(u_\varepsilon(\omega_\varepsilon^t))(z_\varepsilon^{t,\lambda}(s))(\mathbf{N}_\varepsilon^{t,\lambda}(s)).$$

Hence (3.1) yields

$$\begin{aligned} \partial_t \left\|z_\varepsilon^{t,\lambda}\right\|_{\dot H^2}^2 = &\int_{\ell_\varepsilon^{t,\lambda}\mathbb{T}} \kappa_\varepsilon^{t,\lambda}(s)^2 \left[2D(u_\varepsilon(\omega_\varepsilon^t))(z_\varepsilon^{t,\lambda}(s))(\mathbf{N}_\varepsilon^{t,\lambda}(s)) \cdot \mathbf{N}_\varepsilon^{t,\lambda}(s) - 3\,\partial_s u_\varepsilon^{t,\lambda}(s)\cdot \mathbf{T}_\varepsilon^{t,\lambda}(s)\right] ds \\ &+ 2\int_{\ell_\varepsilon^{t,\lambda}\mathbb{T}} \kappa_\varepsilon^{t,\lambda}(s) \left[D^2(u_\varepsilon(\omega_\varepsilon^t))(z_\varepsilon^{t,\lambda}(s))(\mathbf{T}_\varepsilon^{t,\lambda}(s), \mathbf{T}_\varepsilon^{t,\lambda}(s)) \cdot \mathbf{N}_\varepsilon^{t,\lambda}(s)\right] ds. \end{aligned} \tag{3.12}$$

Lemma 2.1 shows that the absolute value of the first integral is bounded by

$$5\left\|u_\varepsilon(\omega_\varepsilon^t)\right\|_{\dot C^{0,1}} \left\|z_\varepsilon^{t,\lambda}\right\|_{\dot H^2}^2 \le C_\alpha L_\theta(\Phi_{\varepsilon*}^t\Omega)\left\|z_\varepsilon^{t,\lambda}\right\|_{\dot H^2}^2 \le C_\alpha L_\theta(\Phi_{\varepsilon*}^t\Omega) Q(\Phi_{\varepsilon*}^t\Omega)\left\|z_\varepsilon^{t,\lambda}\right\|_{\dot H^2}^2,$$

where the second inequality holds by $Q(\Phi_{\varepsilon*}^t\Omega) \ge 4$, which in turn follows from [3, Lemma A.1]. Hence, it remains to estimate the second integral, which we denote G_1. We will suppress t from the notation for the sake of simplicity because it will be fixed in the arguments below.

Since $L_\theta(\Phi_{\varepsilon*}\Omega) < \infty$, Lemma 3.6, Fubini's theorem, and Green's theorem show that

$$\begin{aligned} G_1 &= \int_{\ell_\varepsilon^\lambda\mathbb{T}} \int_{\mathcal{L}} \int_{\Phi_\varepsilon(\Omega^{\lambda'})} \kappa_\varepsilon^\lambda(s)\mathbf{N}_\varepsilon^\lambda(s)\cdot D^2(\nabla^\perp K_\varepsilon)(z_\varepsilon^\lambda(s)-y)(\mathbf{T}_\varepsilon^\lambda(s), \mathbf{T}_\varepsilon^\lambda(s))\,dy\,d\theta(\lambda')\,ds \\ &= \int_{\mathcal{L}} \int_{\ell_\varepsilon^\lambda\mathbb{T}} \int_{\Phi_\varepsilon(\Omega^{\lambda'})} \kappa_\varepsilon^\lambda(s)\mathbf{N}_\varepsilon^\lambda(s)\cdot D^2(\nabla^\perp K_\varepsilon)(z_\varepsilon^\lambda(s)-y)(\mathbf{T}_\varepsilon^\lambda(s), \mathbf{T}_\varepsilon^\lambda(s))\,dy\,ds\,d\theta(\lambda') \end{aligned}$$

$$= -\int_{\mathcal{L}} \int_{\ell_\varepsilon^\lambda \mathbb{T} \times \ell_\varepsilon^{\lambda'} \mathbb{T}} \kappa_\varepsilon^\lambda(s) D^2 K_\varepsilon(z_\varepsilon^\lambda(s) - z_\varepsilon^{\lambda'}(s'))(\mathbf{T}_\varepsilon^\lambda(s), \mathbf{T}_\varepsilon^\lambda(s))(\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{N}_\varepsilon^\lambda(s))\, ds\, ds' d\theta(\lambda').$$

Note that the last integrand is jointly measurable in (s, s') but not necessarily in (s, s', λ') because the parametrizations $z_\varepsilon^{\lambda'}(\cdot)$ are chosen independently from each other. From

$$\mathbf{T}_\varepsilon^\lambda(s) = (\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{T}_\varepsilon^\lambda(s))\mathbf{T}_\varepsilon^{\lambda'}(s') + (\mathbf{N}_\varepsilon^{\lambda'}(s') \cdot \mathbf{T}_\varepsilon^\lambda(s))\mathbf{N}_\varepsilon^{\lambda'}(s')$$

and $\mathbf{N}_\varepsilon^{\lambda'}(s') \cdot \mathbf{T}_\varepsilon^\lambda(s) = -\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{N}_\varepsilon^\lambda(s)$ it now follows that $|G_1|$ is bounded by the sum of

$$G_2 := \overline{\int_{\mathcal{L}}} \Bigg| \int_{\ell_\varepsilon^\lambda \mathbb{T} \times \ell_\varepsilon^{\lambda'} \mathbb{T}} \kappa_\varepsilon^\lambda(s) D^2 K_\varepsilon(z_\varepsilon^\lambda(s) - z_\varepsilon^{\lambda'}(s'))(\mathbf{T}_\varepsilon^\lambda(s), \mathbf{T}_\varepsilon^{\lambda'}(s'))$$
$$(\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{N}_\varepsilon^\lambda(s))(\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{T}_\varepsilon^\lambda(s))\, ds\, ds' \Bigg|\, d|\theta|(\lambda'),$$

$$G_3 := \overline{\int_{\mathcal{L}}} \Bigg| \int_{\ell_\varepsilon^\lambda \mathbb{T} \times \ell_\varepsilon^{\lambda'} \mathbb{T}} \kappa_\varepsilon^\lambda(s) D^2 K_\varepsilon(z_\varepsilon^\lambda(s) - z_\varepsilon^{\lambda'}(s'))(\mathbf{T}_\varepsilon^\lambda(s), \mathbf{N}_\varepsilon^{\lambda'}(s'))(\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{N}_\varepsilon^\lambda(s))^2\, ds\, ds' \Bigg|\, d|\theta|(\lambda'),$$

which we estimate separately next. Here $\overline{\int_{\mathcal{L}}} f(\lambda')\, d|\theta|(\lambda')$ for $f\colon \mathcal{L} \to [0, \infty]$ is the upper Lebesgue integral

$$\overline{\int_{\mathcal{L}}} f(\lambda')\, d|\theta|(\lambda') := \inf_g \int_{\mathcal{L}} g(\lambda')\, d|\theta|(\lambda'),$$

where the inf ranges over all measurable $g \geq f$.

Estimate for G_2. Since

$$\frac{\partial}{\partial s'} \Big(DK_\varepsilon(z_\varepsilon^\lambda(s) - z_\varepsilon^{\lambda'}(s'))(\mathbf{T}_\varepsilon^\lambda(s))(\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{N}_\varepsilon^\lambda(s))(\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{T}_\varepsilon^\lambda(s)) \Big)$$
$$= -D^2 K_\varepsilon(z_\varepsilon^\lambda(s) - z_\varepsilon^{\lambda'}(s'))(\mathbf{T}_\varepsilon^\lambda(s), \mathbf{T}_\varepsilon^{\lambda'}(s'))(\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{N}_\varepsilon^\lambda(s))(\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{T}_\varepsilon^\lambda(s))$$
$$+ \kappa_\varepsilon^{\lambda'}(s') DK_\varepsilon(z_\varepsilon^\lambda(s) - z_\varepsilon^{\lambda'}(s')) \Big((\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{T}_\varepsilon^\lambda(s))^2 - (\mathbf{T}_\varepsilon^{\lambda'}(s') \cdot \mathbf{N}_\varepsilon^\lambda(s))^2 \Big),$$

we see that

$$G_2 \leq C_\alpha \overline{\int_{\mathcal{L}}} \int_{\ell_\varepsilon^\lambda \mathbb{T} \times \ell_\varepsilon^{\lambda'} \mathbb{T}} \frac{\left|\kappa_\varepsilon^\lambda(s) \kappa_\varepsilon^{\lambda'}(s')\right|}{\left|z_\varepsilon^\lambda(s) - z_\varepsilon^{\lambda'}(s')\right|^{1+2\alpha}}\, ds\, ds' d|\theta|(\lambda').$$

By the Schwarz inequality, the inner integral of the right-hand side is bounded by

$$\left(\int_{\ell_\varepsilon^\lambda \mathbb{T}} \kappa_\varepsilon^\lambda(s)^2 \int_{\ell_\varepsilon^{\lambda'} \mathbb{T}} \frac{ds'}{\left|z_\varepsilon^\lambda(s) - z_\varepsilon^{\lambda'}(s')\right|^{1+2\alpha}}\, ds \right)^{1/2} \left(\int_{\ell_\varepsilon^{\lambda'} \mathbb{T}} \kappa_\varepsilon^{\lambda'}(s')^2 \int_{\ell_\varepsilon^\lambda \mathbb{T}} \frac{ds}{\left|z_\varepsilon^\lambda(s) - z_\varepsilon^{\lambda'}(s')\right|^{1+2\alpha}}\, ds' \right)^{1/2},$$

and Lemma 3.5 with $\beta = \frac{1}{2}$ shows that this is bounded by

$$C_\alpha \left(\left\|z_\varepsilon^\lambda\right\|_{\dot{H}^2}^2 \frac{\ell_\varepsilon^{\lambda'} \left\|z_\varepsilon^{\lambda'}\right\|_{\dot{H}^2}^2}{(\Delta_\varepsilon^{\lambda,\lambda'})^{2\alpha}} \right)^{1/2} \left(\left\|z_\varepsilon^{\lambda'}\right\|_{\dot{H}^2}^2 \frac{\ell_\varepsilon^\lambda \left\|z_\varepsilon^\lambda\right\|_{\dot{H}^2}^2}{(\Delta_\varepsilon^{\lambda,\lambda'})^{2\alpha}} \right)^{1/2} = C_\alpha \left\|z_\varepsilon^\lambda\right\|_{\dot{H}^2}^2 \frac{\ell_\varepsilon^{\lambda'} \left\|z_\varepsilon^{\lambda'}\right\|_{\dot{H}^2}^2 (\ell_\varepsilon^\lambda)^{1/2}}{(\ell_\varepsilon^{\lambda'})^{1/2} (\Delta_\varepsilon^{\lambda,\lambda'})^{2\alpha}}.$$

Therefore

$$|G_2| \leq C_\alpha R_\theta(\Phi_{\varepsilon *}\Omega) Q(\Phi_{\varepsilon *}\Omega) \left\|z_\varepsilon^\lambda\right\|_{\dot{H}^2}^2.$$

Estimate for G_3. We can assume $R_\theta(\Phi_{\varepsilon *}\Omega) < \infty$, in which case $\Delta_\varepsilon^{\lambda,\lambda'} > 0$ for $|\theta|$-almost all λ'. Thus we can apply [3, Lemma A.4] to conclude that

$$G_3 \le C_\alpha \overline{\int_{\mathcal{L}}} \int_{\ell_\varepsilon^\lambda \mathbb{T} \times \ell_\varepsilon^{\lambda'} \mathbb{T}} \frac{\left|\kappa_\varepsilon^\lambda(s)\right| \left(\mathcal{M}\kappa_\varepsilon^\lambda(s) + \mathcal{M}\kappa_\varepsilon^{\lambda'}(s')\right)}{\left|z_\varepsilon^\lambda(s) - z_\varepsilon^{\lambda'}(s')\right|^{1+2\alpha}} \, ds\, ds' d|\theta|(\lambda'), \tag{3.13}$$

where $\mathcal{M}$ is the maximal operator given by

$$\mathcal{M}f(s) \coloneqq \max\left\{ \sup_{h\in\left(0,\frac{\ell}{2}\right]} \frac{1}{h}\int_s^{s+h} |f(s')|\, ds', \sup_{h\in\left(0,\frac{\ell}{2}\right]} \frac{1}{h}\int_{s-h}^{s} |f(s')|\, ds' \right\}$$

for $\ell \in (0,\infty)$ and $f \in L^1(\ell\mathbb{T})$. Let us split the integrand of (3.13) into the sum of terms with numerators $\left|\kappa_\varepsilon^\lambda(s)\right| \mathcal{M}\kappa_\varepsilon^\lambda(s)$ and $\left|\kappa_\varepsilon^\lambda(s)\right| \mathcal{M}\kappa_\varepsilon^{\lambda'}(s')$. Then the maximal inequality

$$\left\|\mathcal{M}f\right\|_{L^2} \le C\left\|f\right\|_{L^2},$$

which holds for all $\ell \in (0,\infty)$ and $f \in L^2(\ell\mathbb{T})$ with some universal constant C, shows that the same argument as in the estimate for G_2 bounds the integral of the second term by $C_\alpha R_\theta(\Phi_{\varepsilon *}\Omega) Q(\Phi_{\varepsilon *}\Omega) \left\|z_\varepsilon^\lambda\right\|_{\dot{H}^2}^2$. As for the first term, Lemma 3.5 with $\beta = \frac{1}{2}$, Schwarz inequality, and the maximal inequality show that

$$\begin{aligned}
&\overline{\int_{\mathcal{L}}} \int_{\ell_\varepsilon^\lambda \mathbb{T} \times \ell_\varepsilon^{\lambda'} \mathbb{T}} \frac{\left|\kappa_\varepsilon^\lambda(s)\right| \mathcal{M}\kappa_\varepsilon^\lambda(s)}{\left|z_\varepsilon^\lambda(s) - z_\varepsilon^{\lambda'}(s')\right|^{1+2\alpha}} \, ds\, ds' d|\theta|(\lambda') \\
&\quad \le \overline{\int_{\mathcal{L}}} \int_{\ell_\varepsilon^\lambda \mathbb{T}} \left|\kappa_\varepsilon^\lambda(s)\right| \mathcal{M}\kappa_\varepsilon^\lambda(s) \frac{C_\alpha \ell_\varepsilon^{\lambda'} \left\|z_\varepsilon^{\lambda'}\right\|_{\dot{H}^2}^2}{d(z_\varepsilon^\lambda(s), \mathrm{im}(z_\varepsilon^{\lambda'}))^{2\alpha}} \, ds\, d|\theta|(\lambda') \\
&\quad \le C_\alpha Q(\Phi_{\varepsilon *}\Omega) \int_{\mathcal{L}} \int_{\ell_\varepsilon^\lambda \mathbb{T}} \frac{\left|\kappa_\varepsilon^\lambda(s)\right| \mathcal{M}\kappa_\varepsilon^\lambda(s)}{d(z_\varepsilon^\lambda(s), \mathrm{im}(z_\varepsilon^{\lambda'}))^{2\alpha}} \, ds\, d|\theta|(\lambda') \\
&\quad \le C_\alpha L_\theta(\Phi_{\varepsilon *}\Omega) Q(\Phi_{\varepsilon *}\Omega) \int_{\ell_\varepsilon^\lambda \mathbb{T}} \left|\kappa_\varepsilon^\lambda(s)\right| \mathcal{M}\kappa_\varepsilon^\lambda(s)\, ds \\
&\quad \le C_\alpha L_\theta(\Phi_{\varepsilon *}\Omega) Q(\Phi_{\varepsilon *}\Omega) \left\|z_\varepsilon^\lambda\right\|_{\dot{H}^2}^2.
\end{aligned}$$

Here, the regular (rather than upper) integral can be taken after the second inequality because the integrand is jointly measurable in (s,λ') (recall from (1.5) that $d(x, \mathrm{im}(z_\varepsilon^{\lambda'})) = d(x, \Phi_\varepsilon(\partial\Omega^{\lambda'}))$ is jointly measurable in (x,λ')), which was used in the following step when applying Fubini's theorem. Aggregating the estimates for G_2 and G_3 now yields the desired conclusion. ∎

Lemmas 3.1 and 3.7 suggest existence of an ε-independent estimate

$$\max\left\{\partial_t^+ Q(\Phi_{\varepsilon *}^t\Omega), -\partial_{t-} Q(\Phi_{\varepsilon *}^t\Omega)\right\} \le C_\alpha (L_\theta(\Phi_{\varepsilon *}^t\Omega) + R_\theta(\Phi_{\varepsilon *}^t\Omega)) Q(\Phi_{\varepsilon *}^t\Omega)^2, \tag{3.14}$$

so that a Grönwall-type argument can be used to show that $Q(\Phi_*^t\Omega)$ is finite for all t near 0. However, to apply such an argument, we first need to prove that $Q(\Phi_{\varepsilon *}^t\Omega)$ is bounded on some (ε-independent) time interval $(-t_0, t_0)$, even if the bound depends on ε, or that

$Q(\Phi^t_{\varepsilon*}\Omega)$ is upper semi-continuous for each $\varepsilon > 0$. For instance, (3.11) would become useless if $Q(\Phi^t_{\varepsilon*}\Omega) = \infty$ for all small $\varepsilon, |t| > 0$, and none of the above a priori excludes this.

It turns out that this problem can be overcome by estimating the second integral in (3.12) via the Schwarz inequality, provided $\sup_{\lambda\in\mathcal{L}} \ell(z^{0,\lambda}) < \infty$ (using also Lemma 3.1). Moreover, we will now show that one can similarly deal with general ω^0 considered in this section via a sequence of approximations satisfying this property. Take an increasing sequence $\{\mathcal{L}'_N\}_{N=1}^\infty$ of measurable subsets of $\mathcal{L}$ such that $|\theta|(\mathcal{L}'_N) < \infty$ for each $N \in \mathbb{N}$ and $\mathcal{L} = \bigcup_{N=1}^\infty \mathcal{L}'_N$. For ω^0 as above and any $N \in \mathbb{N}$, let

$$\mathcal{L}_N := \left\{\lambda \in \mathcal{L}'_N \colon \ell(z^{0,\lambda}) \le N\right\}, \quad \Omega_N := \Omega \cap (\mathbb{R}^2 \times \mathcal{L}_N), \quad \omega^0_N(x) := \int_{\mathcal{L}_N} \mathbb{1}_{\Omega^\lambda}(x)\, d\theta(\lambda).$$

Then $\omega^0_N \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ because $\|\omega^0_N\|_{L^\infty} \le |\theta|(\mathcal{L}'_N)$ and $\|\omega^0_N\|_{L^1} \le \frac{N^2}{4\pi}|\theta|(\mathcal{L}'_N)$ (the latter by the isoperimetric inequality), and clearly also $L_\theta(\Omega_N) \le L_\theta(\Omega)$. Let $\Phi_{\varepsilon,N} \in C_{\mathrm{loc}}(\mathbb{R}; C(\mathbb{R}^2;\mathbb{R}^2))$ be the corresponding ε-mollified flow map, that is, the identity map plus the solution to (2.2) with ω^0_N in place of ω^0. Let $z^{t,\lambda}_{\varepsilon,N} := \Phi^t_{\varepsilon,N} \circ z^{0,\lambda}$ and $\omega^t_{\varepsilon,N} := \omega^0_N \circ (\Phi^t_{\varepsilon,N})^{-1}$.

Then (3.12), the Schwarz inequality, $\ell(\gamma)\,\|\gamma\|^2_{\dot H^2} \ge 4$ for any $\gamma \in \mathrm{CC}(\mathbb{R}^2)$ (see [3, Lemma A.1]), Lemma 3.1, and (1.9) show that

$$\begin{aligned}\left|\partial_t \left\|z^{t,\lambda}_{\varepsilon,N}\right\|^2_{\dot H^2}\right| &\le 5\left\|u_\varepsilon(\omega^t_{\varepsilon,N})\right\|_{\dot C^{0,1}} \left\|z^{t,\lambda}_{\varepsilon,N}\right\|^2_{\dot H^2} + 2\left\|D^2(u_\varepsilon(\omega^t_{\varepsilon,N}))\right\|_{L^\infty} \ell(z^{t,\lambda}_{\varepsilon,N})^{1/2} \left\|z^{t,\lambda}_{\varepsilon,N}\right\|_{\dot H^2} \\ &\le \left(5\left\|D(\nabla^\perp K_\varepsilon)\right\|_{L^\infty} + 2C_\alpha L_\theta(\Omega) N \left\|D^2(\nabla^\perp K_\varepsilon)\right\|_{L^\infty}\right)\left\|\omega^0_N\right\|_{L^1} \left\|z^{t,\lambda}_{\varepsilon,N}\right\|^2_{\dot H^2}\end{aligned}$$

for $(t,\lambda) \in [-T_0, T_0] \times \mathcal{L}_N$. This implies

$$\left\|z^{t+h,\lambda}_{\varepsilon,N}\right\|^2_{\dot H^2} \le e^{C|h|}\left\|z^{t,\lambda}_{\varepsilon,N}\right\|^2_{\dot H^2}$$

whenever $(t, t+h, \lambda) \in [-T_0, T_0]^2 \times \mathcal{L}_N$, with some C depending on $\alpha, \varepsilon, N, \Omega, \theta$. This and Lemma 3.1 now yield

$$Q(\Phi^{t+h}_{\varepsilon,N*}\Omega_N) \le e^{C|h|} Q(\Phi^t_{\varepsilon,N*}\Omega_N)$$

for the same (t,h,λ), with a new C depending again only on $\alpha, \varepsilon, N, \Omega, \theta$, from which we conclude the $Q(\Phi^t_{\varepsilon,N*}\Omega_N)$ is upper semi-continuous in $t \in [-T_0, T_0]$ for all N, ε (this then clearly extends to all $t \in \mathbb{R}$).

Lemmas 3.1, 3.4, 3.7, and Corollary 2.7 with $(\omega_{\varepsilon,N}, \Omega_N, z_{\varepsilon,N}, \Phi_{\varepsilon,N})$ in place of $(\omega_\varepsilon, \Omega, z_\varepsilon, \Phi_\varepsilon)$, and inequalities $L_\theta(\Omega_N) \le L_\theta(\Omega)$, $R_\theta(\Omega_N) \le R_\theta(\Omega)$, and $Q(\Phi^t_{\varepsilon,N*}\Omega_N) \ge 4$, show that

$$\begin{aligned}\ell(z^{t+h,\lambda}_{\varepsilon,N})\left\|z^{t+h,\lambda}_{\varepsilon,N}\right\|^2_{\dot H^2} - Q(\Phi^t_{\varepsilon,N*}\Omega_N) &\le \ell(z^{t+h,\lambda}_{\varepsilon,N})\left\|z^{t+h,\lambda}_{\varepsilon,N}\right\|^2_{\dot H^2} - \ell(z^{t,\lambda}_{\varepsilon,N})\left\|z^{t,\lambda}_{\varepsilon,N}\right\|^2_{\dot H^2} \\ &\le C_\alpha \left|\int_t^{t+h} (L_\theta(\Omega) + R_\theta(\Omega)) Q(\Phi^\tau_{\varepsilon,N*}\Omega_N)^2\, d\tau\right|\end{aligned}$$

holds whenever $(t, t+h, \lambda) \in [-T_0, T_0]^2 \times \mathcal{L}_N$ (with C_α depending only on α, as always). Using upper semi-continuity of $Q(\Phi^t_{\varepsilon,N*}\Omega_N)$ in t, taking the supremum over $\lambda \in \mathcal{L}_N$, dividing by $|h|$, and letting $h \to 0$ now yields

$$\max\left\{\partial_t^+ Q(\Phi^t_{\varepsilon,N*}\Omega_N), -\partial_{t-} Q(\Phi^t_{\varepsilon,N*}\Omega_N)\right\} \le C_\alpha(L_\theta(\Omega)+R_\theta(\Omega))Q(\Phi^t_{\varepsilon,N*}\Omega_N)^2,$$

so via a Grönwall-type argument we conclude that

$$Q(\Phi^t_{\varepsilon,N*}\Omega_N) \le \frac{Q(\Omega_N)}{1-C_\alpha(L_\theta(\Omega)+R_\theta(\Omega))Q(\Omega_N)\,|t|} \le \frac{Q(\Omega)}{1-C_\alpha(L_\theta(\Omega)+R_\theta(\Omega))Q(\Omega)\,|t|} \tag{3.15}$$

whenever $|t| < T_1 \coloneqq \frac{1}{C_\alpha(L_\theta(\Omega)+R_\theta(\Omega))Q(\Omega)}$. We let this C_α be no smaller than the one from Lemma 2.1 (which was used to define T_0), so that $T_1 \in (0, T_0]$.

The following result will allow us to turn (3.15) into a bound on $Q(\Phi^t_*\Omega)$.

Lemma 3.8. *With T_1 as above, for each $\varepsilon > 0$ we have*

$$\lim_{N\to\infty} \sup_{t\in[-T_1,T_1]} \left\|\Phi^t_{\varepsilon,N} - \Phi^t_\varepsilon\right\|_{L^\infty} = 0.$$

Proof. Since $L_\theta(\Omega_N) \le L_\theta(\Omega)$ for all N, the last claim in Proposition 2.8 and (1.8) show that

$$M \coloneqq \sup_{(t,N)\in[-T_1,T_1]\times\mathbb{N}} \max\left\{L_\theta(\Phi^t_{\varepsilon,N*}\omega^0_N), L_\theta(\Phi^t_{\varepsilon*}\omega^0_N)\right\} \le e^{2\alpha}L_\theta(\Omega_N) \le 3L_\theta(\Omega).$$

Then Lemmas 2.1 and 2.2 show for any $t \in [-T_1, T_1]$ that

$$\begin{aligned}
&\left\|u_\varepsilon(\Phi^t_{\varepsilon,N*}\omega^0_N)\circ\Phi^t_{\varepsilon,N} - u_\varepsilon(\Phi^t_{\varepsilon*}\omega^0)\circ\Phi^t_\varepsilon\right\|_{L^\infty} \\
&\quad\le \left\|u_\varepsilon(\Phi^t_{\varepsilon,N*}\omega^0_N)\right\|_{\dot C^{0,1}}\left\|\Phi^t_{\varepsilon,N}-\Phi^t_\varepsilon\right\|_{L^\infty} + \left\|u_\varepsilon(\Phi^t_{\varepsilon,N*}\omega^0_N) - u_\varepsilon(\Phi^t_{\varepsilon*}\omega^0_N)\right\|_{L^\infty} \\
&\qquad\qquad + \left\|u_\varepsilon(\Phi^t_{\varepsilon*}\omega^0_N) - u_\varepsilon(\Phi^t_{\varepsilon*}\omega^0)\right\|_{L^\infty} \\
&\quad\le C_\alpha M\left\|\Phi^t_{\varepsilon,N}-\Phi^t_\varepsilon\right\|_{L^\infty} + \left\|u_\varepsilon(\Phi^t_{\varepsilon*}\omega^0_N) - u_\varepsilon(\Phi^t_{\varepsilon*}\omega^0)\right\|_{L^\infty}
\end{aligned} \tag{3.16}$$

for some C_α. We see that $\omega^0_N \to \omega^0$ in L^1 because $\Omega \subseteq \mathbb{R}^2 \times \mathcal{L}$ has finite measure (with the Lebesgue measure on $\mathbb{R}^2$ and $|\theta|$ on $\mathcal{L}$). So for any $\eta > 0$ there is t-independent $N_\eta \in \mathbb{N}$ such that for all $N \ge N_\eta$ we have

$$\begin{aligned}
\left\|u_\varepsilon(\Phi^t_{\varepsilon*}\omega^0_N) - u_\varepsilon(\Phi^t_{\varepsilon*}\omega^0)\right\|_{L^\infty} &= \sup_{x\in\mathbb{R}^2}\left|\int_{\mathbb{R}^2}\nabla^\perp K_\varepsilon(x-\Phi^t_\varepsilon(y))\left(\omega^0_N(y)-\omega^0(y)\right)dy\right| \\
&\le \left\|\nabla^\perp K_\varepsilon\right\|_{L^\infty}\left\|\omega^0_N-\omega^0\right\|_{L^1} \le \eta.
\end{aligned}$$

Since $\left\|\Phi^t_{\varepsilon,N}-\Phi^t_\varepsilon\right\|_{L^\infty}$ is continuous in t, (3.16) yields

$$\max\left\{\partial_t^+\left\|\Phi^t_{\varepsilon,N}-\Phi^t_\varepsilon\right\|_{L^\infty}, -\partial_{t-}\left\|\Phi^t_{\varepsilon,N}-\Phi^t_\varepsilon\right\|_{L^\infty},\right\} \le C_\alpha M\left\|\Phi^t_{\varepsilon,N}-\Phi^t_\varepsilon\right\|_{L^\infty} + \eta$$

for all $t \in [-T_1, T_1]$ and $N \ge N_\eta$. Hence, a Grönwall-type argument shows that

$$\left\|\Phi^t_{\varepsilon,N}-\Phi^t_\varepsilon\right\|_{L^\infty} \le \begin{cases} \frac{\eta}{C_\alpha M}\left(e^{C_\alpha M|t|}-1\right) & \text{if } M > 0, \\ \eta\,|t| & \text{if } M = 0 \end{cases}$$

for all such (t, N), from which the claim follows. ∎

Since $\mathcal{L} = \bigcup_{N=1}^{\infty} \mathcal{L}_N$, Lemma 3.8 shows that $\lim_{N\to\infty} z_{\varepsilon,N}^{t,\lambda} = z_{\varepsilon}^{t,\lambda}$ in $\mathrm{CC}(\mathbb{R}^2)$ for any $\varepsilon > 0$ and $(t,\lambda) \in (-T_1, T_1) \times \mathcal{L}$. Lower semi-continuity of $\ell(\cdot)$ and $\|\cdot\|_{\dot{H}^2}$ on $\mathrm{CC}(\mathbb{R}^2)$ (the latter by [3, Corollary B.3]) shows that

$$\ell(z_{\varepsilon}^{t,\lambda}) \left\| z_{\varepsilon}^{t,\lambda} \right\|_{\dot{H}^2}^2 \le \frac{Q(\Omega)}{1 - C_\alpha(L_\theta(\Omega) + R_\theta(\Omega))Q(\Omega)\,|t|}$$

holds for these $(\varepsilon, t, \lambda)$. After taking $\varepsilon \to 0$ and using the same lower semi-continuity again, we obtain the same bound with $z^{t,\lambda}$ in place of $z_{\varepsilon}^{t,\lambda}$. It follows that

$$Q(\Phi_*^t\Omega) \le \frac{Q(\Omega)}{1 - C_\alpha(L_\theta(\Omega) + R_\theta(\Omega))Q(\Omega)\,|t|}$$

for all $t \in (-T_1, T_1)$. Since $L_\theta(\Phi_*^t\Omega)$ and $R_\theta(\Phi_*^t\Omega)$ are bounded on compact subsets of I, an analogous bound holds for all t near any $\tau \in I$ with $Q(\Phi_*^\tau\Omega) < \infty$ (with $(\Phi_*^\tau\Omega, |t-\tau|)$ in place of $(\Omega, |t|)$). This proves Theorem 1.4(ii).

Appendix A. Regularity of Functions with $L_\theta(\Omega) < \infty$

Is it ever possible to have a generalized layer cake representation that is better than the usual super-level sets?

Proposition A.1. *For any $\omega\colon \mathbb{R}^2 \to \mathbb{R}$ and a generalized layer cake representation (Ω, θ) of ω,*

$$\|\omega\|_{\dot{C}^{0,2\alpha}} \le L_\theta(\Omega)$$

holds.

Proof. Let $x, y \in \mathbb{R}^2$, $x \neq y$ be given. Then

$$|\omega(x) - \omega(y)| \le \int_{\mathcal{L}} |\mathbb{1}_{\Omega^\lambda}(x) - \mathbb{1}_{\Omega^\lambda}(y)|\; d|\theta|(\lambda)$$

holds. Since the integrand in the right-hand side is nonzero only if $|x - y| \ge d(x, \partial\Omega^\lambda)$, we obtain

$$|\omega(x) - \omega(y)| \le \int_{\mathcal{L}} \frac{|x-y|^{2\alpha}}{d(x, \partial\Omega^\lambda)^{2\alpha}}\, d|\theta|(\lambda) \le L_\theta(\Omega)\,|x-y|^{2\alpha},$$

thus $\|\omega\|_{\dot{C}^{0,2\alpha}} \le L_\theta(\Omega)$ follows. □

Proposition A.2. *Let $\omega\colon \mathbb{R}^2 \to \mathbb{R}$ be a uniformly continuous bounded function whose modulus of continuity ρ satisfies $\int_0^1 \frac{\min\{\rho(\delta),1\}}{\delta^{1+2\alpha}}\, d\delta < \infty$. Let $\mathcal{L} \coloneqq [\inf\omega, \sup\omega]$, θ the signed measure on $\mathcal{L}$ given as*

$$\theta(A) \coloneqq \int_A \mathrm{sgn}(\lambda)\, d\lambda,$$

and

$$\Omega \coloneqq \left\{(x,\lambda) \in \mathbb{R}^2 \times (0, \sup\omega] : \omega(x) > \lambda\right\} \cup \left\{(x,\lambda) \in \mathbb{R}^2 \times [\inf\omega, 0) : \omega(x) < \lambda\right\}.$$

Then (Ω, θ) is a generalized layer cake representation of ω satisfying $L_\theta(\Omega) < \infty$.

Proof. TBD. □

Proposition A.3. *There exist* $\omega\colon \mathbb{R}^2 \to \mathbb{R}$ *and a generalized layer cake representation* (Ω, θ) *of* ω *such that* $L_\theta(\Omega) < \infty$ *and*

$$\min\left\{\delta^{2\alpha}, 1\right\} \le \rho(\delta) \le 2\min\left\{\delta^{2\alpha}, 1\right\}$$

hold for all $\delta \ge 0$ *where* ρ *is the modulus of continuity of* ω*.*

Proof. TBD. □

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