# Game theory - Price of Anarchy

## Reference


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The price of anarchy is independent of the network topology ☆


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Price of anarchy is a measure of degradation of network performance due to selfish behaviour of non-cooperative users.

* here we consider a traffic network.
* Latency is travel time and Total Latency is sum of travel times of all users in the network.

$$\boxed{\text{Price of anarchy} = \max \left\{ \frac{\text{Total Latency (Nash eq.)}}{\text{optimal total Latency}} \right\}}$$

* The worst case / price of anarchy is calculated at a single commodity (between 2 fixed points) instance in a network of parallel links

edges : roads connecting 2 consequtive nodes.

edge congestion : queue length in edges

Latency function ($l(x)$): Latency is the time taken to reach destination. It is a function of edge congestion ($x$).

Commodity : Source - destination pair

links : paths of a commodity.

## The model

$G = (V, E)$ $\quad V =$ nodes , $E =$ edges

Paths $(P)$ = Links

$\mathcal{P}_i = \{P \mid P \text{ joins } s_i \,\&\, t_i\}$

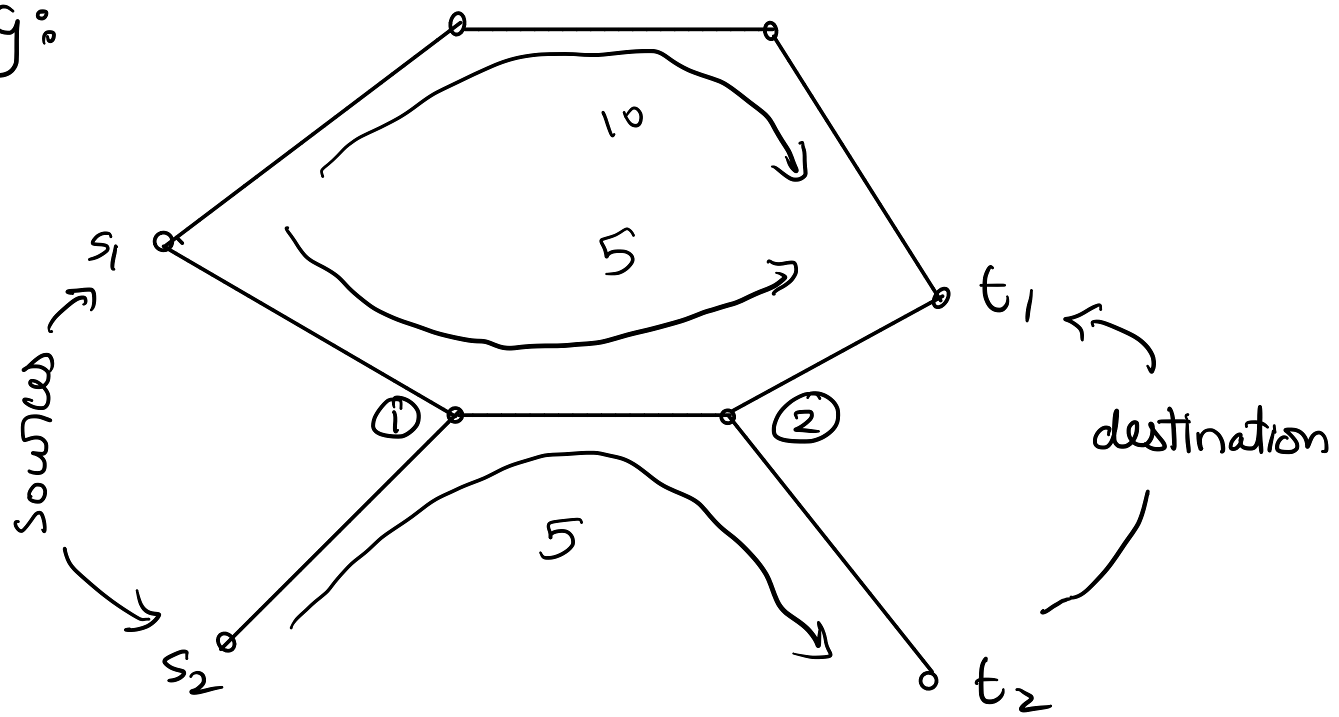
$\mathcal{P} = \bigcup_i \mathcal{P}_i$

* Flow through a path $\left[ f : \mathcal{P} \to \mathbb{R}^+ \right]$

* Edge flow rate $\left[ f_e : E \to \mathbb{R}^+ \right]$

$$\boxed{f_e = \sum_{P : e \in P} f_P}$$

eg:




flow through edge connecting 1 & 2 $= 5 + 5$

$= 10$

* Latency of edge $= l_e(f_e)$

Is a function of flow through the edge.

* latency in path

$$l_p = \sum_{e:\, e \in p} l_e$$

Latency in a path is a function of network flow. $(l_p(f))$

* In the model we assume users' utility to be negative of latency

* Cost function : function of flow in G $(f)$

$$C(f) = \sum_{e \in E} l_e(f_e) f_e$$

sum of latencies of all users.

$$= \sum_{p \in \mathcal{P}} l_p(f_p)\, f_p$$

* We try to minimize the cost function.

* Instance : $(G, r, l)$

An instance can have different network flow.

$$r_i = \sum_{p \in \mathcal{P}_i} f_p$$

(Total flow from a source to a destination)

## * Nash Equillibrium in Networks

A flow '$f$' is in nash equillibrium if

$\forall\ i\ ,\ P_1, P_2 \in \mathcal{P}_i\ \&\ \delta \in (0, f_{P_1}]$

$$l_{P_1}(f) \leq l_{P_2}(\tilde{f})$$

where

$$\tilde{f} = \begin{cases} f_{P_1} - \delta & P = P_1 \\ f_{P_2} + \delta & P = P_2 \\ f_P & \text{else} \end{cases}$$

* In our model we assume each Person contributes a small part in the network flow

as $\delta \to 0$

$l_{P_1}(f) \leq l_{P_2}(f)$ (Proposition 2.2 from reference)

Similarly

$$l_{P_2}(f) \leq l_{P_1}(f)$$

$\therefore \quad l_{P_1}(f) = l_{P_2}(f)$

for a particular source - destination pair latencies of different paths are equal.

$$* \quad C(f) = \sum_i L_i(f)\, r_i$$

$$L_i(f) = l_p(f) \quad \forall\, p \in \mathcal{P}_i$$

(Proposition 2.3 from refferance)

Marginal Cost function

$$l^*(x) = \frac{d}{dx}(x\, l)$$

It gives you rate of change in network cost due to a small increment in the flow in the path or edge.

Standard latency function : $l(f)$ $\underset{\cdot\cdot}{S}$

$l^*(f)$ is increasing (or) $f\, l(f)$ is convex

$\Rightarrow$ This is a fair assumption to make. Idividuals contribution to network latency is more if there are more vechicles.

* We only consider classes of standard latency function ($\mathcal{L}$) in this paper.

⊛ Proposition 2.7 from refference

Nash equillibrium flow 'f' for $(G, r, \ell^*)$ corresponds to optimal cost in $(G, r, \ell)$

Proof: For a nash equillibrium in $(G, r, \ell^*)$ each player tries to take a path with less $\ell^*$.

By taking a path with less $\ell^*$

$$\left\{ \text{or } \frac{d}{df}(c) = \frac{d}{df}\left(f\,\ell(f)\right) \right\}$$ they

are reducing their contribution to network latency.

⊛ A good way to force Network users to take network optimal route is to somehow make their utilities dependent on $\ell^*$ (Marginal Cost function).

eg: You could tax everyone $\ell^*$ for using the road.

## * Optimal flow

minimize $\displaystyle\sum_{e \in E} l_e(f_e)\, f_e$

Subject to

$$\sum_{p \in \mathcal{P}_i} f_p = r_i$$

$$f_e = \sum_{e \in p} f_p$$

$$f_p \geqslant 0 \quad \forall p \in \mathcal{P}$$

This is a convex optimisation problem, the objective function is convex.

$\therefore$ It will only have 1 local optimal value = global optimal.

## * Nash equillibrium as a convex optimisation problem

$$l^* = \frac{d}{dx}(x l) \longrightarrow \int l = \int_0^{f} l^*(x)\,dx$$

$$\text{minimize}\left[\sum_{e\in E}\int_0^{f_e} l_e^*(x)\,dx\right] \equiv \text{minimize}\left[\sum_{e\in E} l_e f_e\right]$$

(Proposition 2.7) $\Vert$

$$\text{Nash eq: in } (G, \pi, l^*) \equiv \text{optimal in } (G, \pi, l)$$

$\therefore$ To get nash equillibrium for any instance $(G, \pi, l)$ we have to

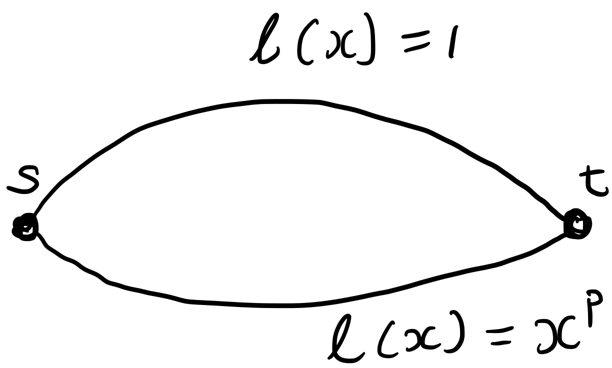
$$\text{minimize}\left[\sum_{e\in E}\int_0^{f_e} l_e(x)\,dx\right]$$

Since this is a convex expression, it becomes a convex optimisation problem.

## * Proposition 2.4 (from reference)

If two flows $f$ & $\tilde{f}$ both correspondes to nash equilibrium then $C(f) = C(\tilde{f})$

## * Pigou's example




Suppose total flow rate is $\underline{\underline{1}}$

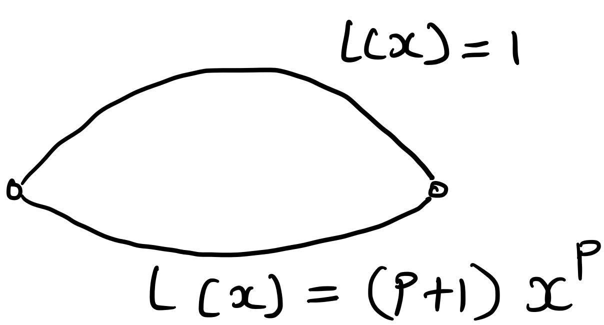
$$C = x \times x^p + (1-x) \times 1$$

$$\boxed{C = x^{p+1} - x + 1}$$

$$\frac{dC}{dx} = (p+1)x^p - 1 = 0$$

$$\boxed{\text{for minimizing } C} \Rightarrow x_c = \left(\frac{1}{p+1}\right)^{1/p}$$

↓ using (proposition 2.7)




at nash equilibrium

$$(p+1)x^p = 1$$

$$x_c = \left(\frac{1}{1+p}\right)^{1/p}$$

$$C = (1+p)^{-(p+1)/p} - (1+p)^{-1/p} + 1$$

(optimal cost function)

$$C = 1 - p\cdot(1+p)^{-(1+p)/p}$$

$C$ (at Nash eq.) $= 1^p \times 1 = 1$

(at nash eq: everyone goes through lower path.)

$$\text{Price of anarchy} = \left[1 - p\cdot(1+p)^{-(1+p)/p}\right]^{-1}$$

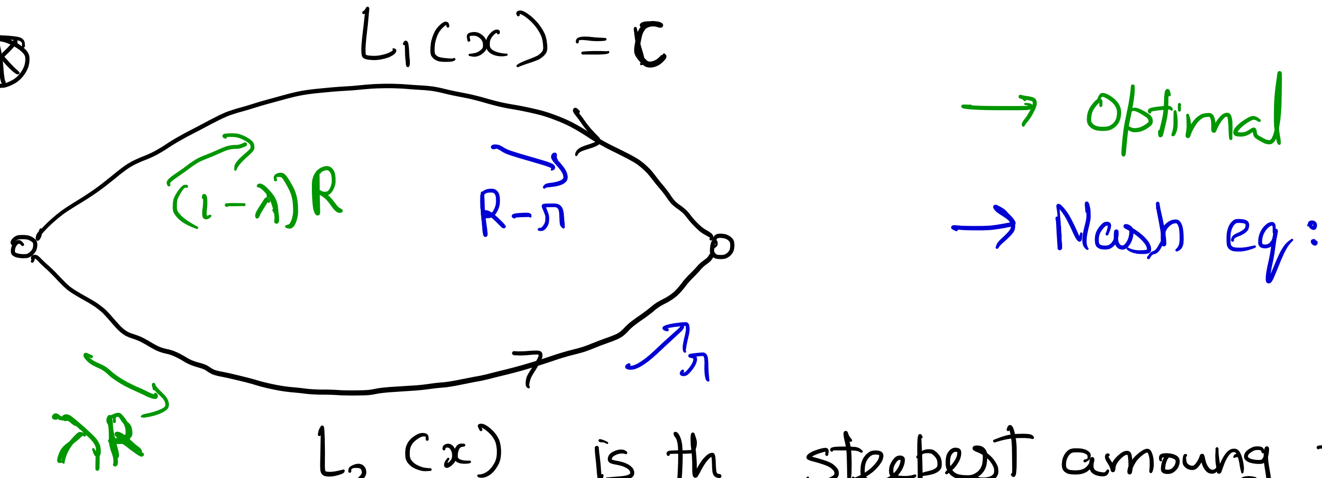
* as $p \to \infty$ price of anarchy $\to \infty$

$\therefore$ having steeper arbitrary latency functions is bad. (according to reference material)

But how do we charecterise steepness?

$\hookrightarrow \frac{d}{dx}(\ln(\ell(x)))$

⊛




$L_2(x)$ is th steepest amoung the class of functions ($\mathcal{L}$)

for Nash eq:

$$l_2(r) = C$$

total traffic rate $= R$

$$C(\text{Nash eq:}) = C \times r + (R-r) \times C = CR \,//$$

<u>for optimal cost</u>

Suppose for some $\lambda \in (0,1)$

$$l_2^*(\lambda R) = C$$

$$C(\text{optimal}) = \lambda R \, l_2(\lambda R) + (1-\lambda) R C$$

$$\mu = \frac{L_2(\lambda R)}{L_2(r)}$$

$$\text{Price of anarchy} = \left[\mu\lambda + \overline{1-\lambda}\right]^{-1}$$

* you can have $R = r$

$$L_2^*(\lambda r) = l_2(r) \quad (= c)$$

$$\mu = \frac{l_2(\lambda r)}{l_2(r)}$$

$$\frac{C(\text{Nash})}{C(\text{optimal})} = [\mu\lambda + 1 - \lambda]^{-1}$$

* Anarchy value for a latency function ($\alpha(\ell)$)

$$\alpha(\ell) = \sup_{r>0} [\mu\lambda + 1 - \lambda]^{-1}$$

where $l^*(\lambda r) = l(r)$

& $\mu = \dfrac{l(\lambda r)}{l(r)}$

* Anarchy value for a standard Latency class ($\alpha(\mathcal{L})$)

$$\alpha(\mathcal{L}) = \sup_{\ell \in \mathcal{L}} \alpha(\ell)$$

* Lemma 1 (3.5)

$$\boxed{C(f^*) \geq \sum_e l_e(\lambda_e f_e)\lambda_e f_e + (f_e^* - \lambda_e f_e)\, l_e(f_e)}$$

$$\text{s.t.} \quad l_e^*(\lambda_e f_e) = l_e(f_e)$$

Proof:

$$l_e(f_e^*)f_e^* = \lambda_e f_e\, l_e(\lambda_e f_e) + \int_{\lambda_e f_e}^{f_e^*} l_e^*(x)\,dx$$

$$\geq \lambda_e f_e\, l_e(\lambda_e f_e) + (f_e^* - \lambda_e f_e)\, l_e^*(\lambda_e f_e)$$

$$= \lambda_e f_e\, l_e(\lambda_e f_e) + (f_e^* - \lambda_e f_e)\, l_e(f_e)$$

* Lemma 2 (3.6)

$$\mu\lambda + 1 - \lambda \geq \frac{1}{\alpha(\mathcal{L})}$$

{ definition of anarchy value

* Lemma 3 (3.7) : If '$f$' corresponds to Nash equillibrium then.

$$\boxed{\sum_e l_e(f_e)f_e \leq \sum_e l_e(f_e)f_e^*}$$

Proof:

$$\sum_{i=1}^{k} L_i(f)\, r_i = C(f) = \sum_e l_e(f_e) f_e$$

(Proposition 2.3 & 2.2) $\quad l_p(f) \geq L_i(f)$

($\forall$ $s_i$-$t_i$ path $p$)

$$\sum_e l_e(f_e) f_e^* = \sum_i \sum_{p \in \mathcal{P}_i} l_p(f) f_p^*$$

$$= \sum_i l_p(f) \sum_{p \in P_i} f_p^*$$

$$= \sum_i \underset{p \in \mathcal{P}_i}{l_p(f)}\, r_i \geq \sum_i L_i(f)\, r_i$$

$$\therefore \quad \sum_e l_e(f_e) f_e^* = \sum_e l_e(f_e) f_e$$

* With these Lemma we try to Prove our initial assumtion that $\alpha(\mathcal{L})$ is the upper bound for $\dfrac{C\ (\text{Nash})}{C\ (\text{optimal})}$

Let $f$ correspond to Nash equillibrium.

and $\quad l_e^*(\lambda_e f_e) = l_e(f_e)$

(Lemma-1)

$$C(f^*) \geq \sum_e l_e(\lambda_e f_e)\lambda_e f_e + \left(f_e^* - \lambda_e f_e\right) l_e(f_e)$$

$$\geq \sum_e \left[\mu_e \lambda_e + 1 - \lambda_e\right] l_e(f_e) f_e$$

$$+ \underbrace{\sum_e \left[f_e^* - f_e\right] l_e(f_e) f_e}_{\text{This term is always +ve, Lemma-3}}$$

$$C(f^*) \geq \sum_e (\mu_e \lambda_e + 1 - \lambda_e)\, l_e(f_e) f_e$$

(Lemma-2)

$$\geq \left[\sum_e l_e(f_e) f_e\right] \Big/ \alpha(\mathcal{L})$$

$$c(f^*) \geqslant \frac{c(f)}{\alpha(\mathcal{L})}$$

$$\alpha(\mathcal{L}) \geqslant \frac{c(f)}{c(f^*)}$$

* Here we do not assume $f^*$ is optimal