CO250 Spring 2020

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June 8, 2020

1 Introduction

1.1 Abstract Optimization Problem

An *abstract optimization problem (P)* is of the following form:

- Given: a set $\mathbf{A} \subseteq \mathbb{R}^n$ and a function $f : \mathbf{A} \to \mathbb{R}$
- Goal: find $x \in \mathbf{A}$ that minimizes/maximizes f
- Bad news: Hard to solve & may not be well defined

We look at 3 special cases of (P) in this course:

1. Linear Programming (LP)
   - $\mathbf{A}$ is simply given by *linear* constrains, and f is a *linear* function
2. Integer Programming (IP)
   - Same as above, but now we want max/min over the *integer* points in $\mathbf{A}$
3. Non-linear Programming (NLP)
   - $\mathbf{A}$ is given by *non-linear* constrains, and f is a *non-linear* function

1.1.1 Example: Water Tech

WaterTech produces 4 products, $P = \{1, 2, 3, 4\}$, from the following resources:

- time on two machines
- skilled and unskilled labour

The following table gives precise requirements:

Product	Machine 1	Machine 2	Skilled Labour	Unskilled Labour	Unit Sale Price
1	11	4	8	7	300
2	7	6	5	8	260
3	6	5	5	7	220
4	5	4	6	4	180

Restrictions:

- WaterTech has 700h on machine 1 and 500h on machine 2 available
- it can purchase 600h of skilled labour at \$8 per hour and at most 650h of unskilled labour at \$6 per hour

Question:
How much of each product should WaterTech produce in order to maximize profit?

1.2 Ingredients of a math model:

- Decision variables: Capture unknown information
- Constraints: Describe which assignments to variables are feasible
- Objective function: A function of the variables that we would like to maximize/minimize

1.3 Variables

WaterTech needs to decide how many units of each product to produce, so introduce some variables:

- x_i for number of labour to purchase
- y_s, y_u for number of hours of skilled/unskilled labour to purchase

1.4 Constrains

What makes an assignment to $\{x_i\} \in P, y_s, y_u$ a feasible assignment?

For example, a production plan described by an assignment may not use more than 700h of time on machine 1

$$11x_1 + 7x_2 + 6x_3 + 5x_4 \leq 700$$

Similarly, we may not use more than 500h of machine 2 time

$$4x_1 + 6x_2 + 5x_3 + 4x_4 \leq 500$$

Producing x_i units of product $i \in P$ must require less than y_s units of skilled labour

$$8x_1 + 5x_2 + 5x_3 + 6x_4 \leq y_s$$

Similar story for unskilled labour:

$$7x_1 + 8x_2 + 7x_3 + 5x_4 \leq y_u$$

Since amount of labour that can be purchased is limited, we also have

$$y_s \leq 600$$

$$y_u \leq 650$$

1.5 Objective Function

Revenue from sales:

$$300x_1 + 260x_2 + 220x_3 + 180x_4$$

Cost of labour:

$$8y_s + 6y_u$$

Objective Function:

$$\text{maximize } 300x_1 + 260x_2 + 220x_3 + 180x_4 - 8y_s - 6y_u$$

The complete model for WaterTech problem is:

$$\begin{array}{rl}
\max & 300x_1 + 260x_2 + 220x_3 + 180x_4 - 8y_s - 6y_u \\
\text{s.t} & 11x_1 + 7x_2 + 6x_3 + 5x_4 \leq 700 \\
 & 4x_1 + 6x_2 + 5x_3 + 4x_4 \leq 500 \\
 & 8x_1 + 5x_2 + 5x_3 + 6x_4 \leq y_s \\
 & 7x_1 + 8x_2 + 7x_3 + 5x_4 \leq y_u \\
 & y_s \leq 600 \\
 & y_u \leq 650 \\
 & x_1, x_2, x_3, x_4, y_u, y_s \geq 0
\end{array}$$

Solution obtained via CPLEX is:

$$\begin{aligned}
x &= (16 + \frac{2}{3}, 50, 0, 33 + \frac{1}{3})^T \\
y_s &= 583 + \frac{1}{3} \\
y_u &= 650 \\
Profit &= 15433 + \frac{1}{3}
\end{aligned}$$

Notice that the solution is fractional, which may or may not be correct depending on the question

1.6 Correctness of Model

First, define some terminologies:

- Word description of problem
  - Similarly, a solution to the word description is an assignment to the unknowns
- Formulation
  - A solution to the formulation is an assignment to all of its variables

A solution feasible if all constrains are satisfied, optimal if no other feasible solution exists

One way to show correctness is to define a mapping between feasible solutions to the word description, and feasible solutions to the model, and vice versa.

2 Linear Program Model (LP)

2.1 Linear Functions

Affine Functions

- A function $f : \mathbb{R}^n \to \mathbb{R}$ is affine if $f(x) = \alpha^T x + \beta$ for $\alpha \in \mathbb{R}^n, \beta \in \mathbb{R}$

Linear Functions

- An affine function with $\beta = 0$

2.2 Linear Program

Linear Program

- the optimization problem

  $$max/min\{f(x) : f_i(x) \leq b_i, \forall 1 \leq i \leq m, x \in \mathbb{R}^n\}$$

  is a linear program if f is affine and $g_1, ..., g_m$ is finite number of linear functions

Some notes:

- dividing by variables is not allowed in LP
- can NOT have strict inequalities
- must have FINITE number of constraints

Example:

$$\begin{array}{rll} \max & \frac{-1}{x_1} - x_3 & \\ \text{s.t.} & 2x_1 + x_3 < 3 & \\ & x_1 + \alpha x_2 = 2 & \forall \alpha \in \mathbb{R} \end{array}$$

Going back to the WaterTech problem, the model we created was in fact a linear program!

2.3 LP Models: Multiperiod Models

A multiperiod model is a problem where:

- time is split into periods
- we have to make a decision in each period
- all decisions influences the final outcome

Example:
KW Oil is a local supplier of heating oil, it needs to decide how much oil to purchase in order to satisfy demand of its customers.

Month	1	2	3	4
Demand(l)	5000	8000	9000	6000
Price($\$/l$)	0.75	0.72	0.92	0.90

Question: When should we purchase how much oil when the goal is to min overall total cost?
Additional Complication: The company has a storage tank that

- has a capacity of 4000 litres of oil
- currently (beginning of month 1) contains 2000 litres of oil

Assumption: Oil is delivered at the beginning of the month, and consumption occurs int he middle of the month

Variables

- Need to decide how many litres of oil to purchase in each month i
  - make variable p_i for $i \in [4]$
- How much oil is stored in the tank at the beginning of month i?
  - make variable t_i for $i \in [4]$

Objective Function

Minimize cost of oil purchased

$$min \qquad 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4$$

Constrains

We need

$$p_i + t_i \geq \text{(demand in month } i\text{)}$$

Balancing equation we get

$$p_i + t_i = \text{(demand in month } i\text{)} + t_{i+1}$$

So we have the following four constrains

$$\begin{aligned} p_1 + 2000 &= 5000 + t_2 \\ p_2 + t2 &= 5000 + t_3 \\ p_3 + t3 &= 5000 + t_4 \\ p_4 + t4 &\geq 6000 \end{aligned}$$

Complete LP for KW Oil

$$
\begin{aligned}
\min \quad & 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4 \\
\text{s.t.} \quad & p_1 + 2000 = 5000 + t_2 \\
& \quad p_2 + t2 = 5000 + t_3 \\
& \quad p_3 + t3 = 5000 + t_4 \\
& \quad p_4 + t4 \geq 6000 \\
& \qquad\quad t_1 = 2000 \\
& \qquad\quad t_i \leq 4000 \qquad (i = 2, 3, 4) \\
& \qquad t_1, p_i \geq 0 \qquad (i = 1, 2, 3, 4)
\end{aligned}
$$

Solving the LP gives the solution:
$p = (3000, 12000, 5000, 6000)^T$
$t = (2000, 0, 4000, 0)^T$

3 Integer Program (IP)

Recall the WaterTech problem

$$\begin{aligned}
\max \quad & 300x_1 + 260x_2 + 220x_3 + 180x_4 - 8y_s - 6y_u \\
\text{s.t} \quad & 11x_1 + 7x_2 + 6x_3 + 5x_4 \leq 700 \\
& 4x_1 + 6x_2 + 5x_3 + 4x_4 \leq 500 \\
& 8x_1 + 5x_2 + 5x_3 + 6x_4 \leq y_s \\
& 7x_1 + 8x_2 + 7x_3 + 5x_4 \leq y_u \\
& y_s \leq 600 \\
& y_u \leq 650 \\
& x_1, x_2, x_3, x_4, y_u, y_s \geq 0
\end{aligned}$$

$$\begin{aligned}
x &= (16 + \frac{2}{3}, 50, 0, 33 + \frac{1}{3})^T \\
y_s &= 583 + \frac{1}{3} \\
y_u &= 650 \\
Profit &= 15433 + \frac{1}{3}
\end{aligned}$$

Fractional solutions are often not desirable! Can we force the solution to be integer?

Integer Program

- an integer program is a linear program with added integrality constraints for some/all the variables
- we call an IP mixed if there are integer and fractional variables, and pure otherwise
- the difference between LPs and IPs is subtle, but LPs are easy to solve, IPs are not!

Integer program is provably difficult to solve!

- An algorithm is efficient if its running can be bounded by a polynomial of the input size of the instance
- LPs can be solved efficiently
- IPs are very unlikely to have efficient algorithms!

3.1 IP Models: Knapsack

Example:
KitchTech Shipping is a company wishes to ship crates from Toronto to Kitchener. Each crate type has a weight and value, and the total weight of crates shipped must not exceed 10,000 lbs.
Goal: Maximize the total value of shipped goods.

Type	1	2	3	4	5	6
weight (lbs)	30	20	30	90	30	70
value ($)	60	70	40	70	20	90

Variables:
One variable x_i for the number of crates of type i to pack.

Constraints:
The total weight of crates picked must not exceed 10000 lbs.

$$30x_1 + 20x_2 + 30x_3 + 90x_4 + 30x_5 + 70x_6 \leq 10000$$

Objective function:
Maximize the total value

$$max \qquad 60x_1 + 70x_2 + 40x_3 + 70x_4 + 20x_5 + 90x_6$$

Complete IP model for KitchTech Shipping:

$$\begin{array}{rl} \max & 60x_1 + 70x_2 + 40x_3 + 70x_4 + 20x_5 + 90x_6 \\ \text{s.t.} & 30x_1 + 20x_2 + 30x_3 + 90x_4 + 30x_5 + 70x_6 \leq 10000 \\ & x_i \geq 0 \qquad (i \in [6]) \\ & x_i \text{ integer } (i \in [6]) \end{array}$$

Let's make this shit more complicated with more rules...
Suppose that:

1. we must not send more than 10 crates of the same type
2. we can only send crates of type 3, if we send at least 1 crate of type 4

Note that we can send at most 10 crates of type 3 by the previous constraints!
By adding the following constraint, the added requirements is fulfilled:

$$x_3 \leq 10x_4$$

proving correctness of the added constraint:

- $x_4 \geq 1 \to$ new constraint is redundant
- $x_4 = 0 \to$ new constraint becomes $x_3 \leq 0$

Suppose we add another rule where we must:

1. take a total of at least 4 crates of type 1 or 2, or
2. take at least 4 crates of type 5 or 6

strategy:
Create a new variable y such that:

- $y = 1 \rightarrow x_1 + x_2 \geq 4$
- $y = 0 \rightarrow x_5 + x_6 \geq 4$
- and force y to take on value 0 or 1

So we add the following constraints:

- $x_1 + x_2 \geq 4y$
- $x_5 + x_6 \geq 4(1 - y)$
- $0 \leq y \leq 1$
- y integer

The variable y we added is called a binary variable. These are very useful for modelling logical constraints of the form:

- Condition (A or B) and C $\rightarrow$ D

So the finalized model would be:

$$\begin{array}{rl} \max & 60x_1 + 70x_2 + 40x_3 + 70x_4 + 20x_5 + 90x_6 \\ \text{s.t.} & 30x_1 + 20x_2 + 30x_3 + 90x_4 + 30x_5 + 70x_6 \leq 10000 \\ & x_3 \leq 10x_4 \\ & x_1 + x_2 \geq 4y \\ & x_5 + x_6 \geq 4(1 - y) \\ & x_i \geq 0 \qquad (i \in [6]) \\ & 0 \leq y \leq 1 \\ & \text{y integer} \\ & x_i \text{ integer } (i \in [6]) \end{array}$$

3.2 IP Models: Scheduling

Example:
The neighborhood coffee shop is open on workdays. The daily demand for workers is given in the table. Each worker works for 4 consecutive days and has one day off.
Goal: Hire the smallest number of workers so that the demand can be met

Mon	Tues	Wed	Thurs	Fri
3	5	9	2	7

Variables:
Introduce variable x_d for every $d \in \{M, T, W, Th, F\}$ counting the number of people to hire with starting day d

Objective function:
Minimize the total number of people hired:

$$min \qquad x_M + x_T + x_W + x_{Th} + x_F$$

Constraints:
We need to ensure that enough people work on each of the days.
Question: Given a solution, how many people work on Monday?
Answer: All but those that start on Tuesday, i.e.

$$x_M + x_W + x_{Th} + x_F$$

And it must be greater than or equal to the number of workers required
So the complete LP is:

$$\begin{array}{rl}
\text{min} & x_M + x_T + x_W + x_{Th} + x_F \\
\text{s.t.} & x_M + x_W + x_{Th} + x_F \geq 3 \\
& x_M + x_T + x_{Th} + x_F \geq 5 \\
& x_M + x_T + x_W + x_F \geq 9 \\
& x_M + x_T + x_W + x_{Th} \geq 2 \\
& x_T + x_W + x_{Th} + x_F \geq 7 \\
& x \geq 0, x \text{ integer}
\end{array}$$