CO250 Spring 2020

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June 27, 2020

1 Introduction

1.1 Abstract Optimization Problem

An *abstract optimization problem (P)* is of the following form:

- Given: a set $\mathbf{A} \subseteq \mathbb{R}^n$ and a function $f : \mathbf{A} \to \mathbb{R}$
- Goal: find $x \in \mathbf{A}$ that minimizes/maximizes f
- Bad news: Hard to solve & may not be well defined

We look at 3 special cases of (P) in this course:

1. Linear Programming (LP)
    - $\mathbf{A}$ is simply given by *linear* constrains, and f is a *linear* function
2. Integer Programming (IP)
    - Same as above, but now we want max/min over the *integer* points in $\mathbf{A}$
3. Non-linear Programming (NLP)
    - $\mathbf{A}$ is given by *non-linear* constrains, and f is a *non-linear* function

1.1.1 Example: Water Tech

WaterTech produces 4 products, $P = \{1, 2, 3, 4\}$, from the following resources:

- time on two machines
- skilled and unskilled labour

The following table gives precise requirements:

Product	Machine 1	Machine 2	Skilled Labour	Unskilled Labour	Unit Sale Price
1	11	4	8	7	300
2	7	6	5	8	260
3	6	5	5	7	220
4	5	4	6	4	180

Restrictions:

- WaterTech has 700h on machine 1 and 500h on machine 2 available
- it can purchase 600h of skilled labour at \$8 per hour and at most 650h of unskilled labour at \$6 per hour

Question:
How much of each product should WaterTech produce in order to maximize profit?

1.2 Ingredients of a math model:

- Decision variables: Capture unknown information
- Constraints: Describe which assignments to variables are feasible
- Objective function: A function of the variables that we would like to maximize/minimize

1.3 Variables

WaterTech needs to decide how many units of each product to produce, so introduce some variables:

- x_i for number of labour to purchase
- y_s, y_u for number of hours of skilled/unskilled labour to purchase

1.4 Constrains

What makes an assignment to $\{x_i\} \in P, y_s, y_u$ a feasible assignment?

For example, a production plan described by an assignment may not use more than 700h of time on machine 1

$$11x_1 + 7x_2 + 6x_3 + 5x_4 \leq 700$$

Similarly, we may not use more than 500h of machine 2 time

$$4x_1 + 6x_2 + 5x_3 + 4x_4 \leq 500$$

Producing x_i units of product $i \in P$ must require less than y_s units of skilled labour

$$8x_1 + 5x_2 + 5x_3 + 6x_4 \leq y_s$$

Similar story for unskilled labour:

$$7x_1 + 8x_2 + 7x_3 + 5x_4 \leq y_u$$

Since amount of labour that can be purchased is limited, we also have

$$y_s \leq 600$$

$$y_u \leq 650$$

1.5 Objective Function

Revenue from sales:

$$300x_1 + 260x_2 + 220x_3 + 180x_4$$

Cost of labour:

$$8y_s + 6y_u$$

Objective Function:

$$\text{maximize } 300x_1 + 260x_2 + 220x_3 + 180x_4 - 8y_s - 6y_u$$

The complete model for WaterTech problem is:

$$\begin{array}{rl} \max & 300x_1 + 260x_2 + 220x_3 + 180x_4 - 8y_s - 6y_u \\ \text{s.t} & 11x_1 + 7x_2 + 6x_3 + 5x_4 \leq 700 \\ & 4x_1 + 6x_2 + 5x_3 + 4x_4 \leq 500 \\ & 8x_1 + 5x_2 + 5x_3 + 6x_4 \leq y_s \\ & 7x_1 + 8x_2 + 7x_3 + 5x_4 \leq y_u \\ & y_s \leq 600 \\ & y_u \leq 650 \\ & x_1, x_2, x_3, x_4, y_u, y_s \geq 0 \end{array}$$

Solution obtained via CPLEX is:

$$\begin{aligned} x &= (16 + \frac{2}{3}, 50, 0, 33 + \frac{1}{3})^T \\ y_s &= 583 + \frac{1}{3} \\ y_u &= 650 \\ Profit &= 15433 + \frac{1}{3} \end{aligned}$$

Notice that the solution is fractional, which may or may not be correct depending on the question

1.6 Correctness of Model

First, define some terminologies:

- Word description of problem
  - Similarly, a solution to the word description is an assignment to the unknowns
- Formulation
  - A solution to the formulation is an assignment to all of its variables

A solution feasible if all constrains are satisfied, optimal if no other feasible solution exists

One way to show correctness is to define a mapping between feasible solutions to the word description, and feasible solutions to the model, and vice versa.

2 Linear Program Model (LP)

2.1 Linear Functions

Affine Functions

- A function $f : \mathbb{R}^n \to \mathbb{R}$ is affine if $f(x) = \alpha^T x + \beta$ for $\alpha \in \mathbb{R}^n, \beta \in \mathbb{R}$

Linear Functions

- An affine function with $\beta = 0$

2.2 Linear Program

Linear Program

- the optimization problem

  $$max/min\{f(x) : f_i(x) \leq b_i, \forall 1 \leq i \leq m, x \in \mathbb{R}^n\}$$

  is a linear program if f is affine and $g_1, ..., g_m$ is finite number of linear functions

Some notes:

- dividing by variables is not allowed in LP
- can NOT have strict inequalities
- must have FINITE number of constraints

Example:

$$\begin{array}{rll} \max & \frac{-1}{x_1} - x_3 & \\ \text{s.t.} & 2x_1 + x_3 < 3 & \\ & x_1 + \alpha x_2 = 2 & \forall \alpha \in \mathbb{R} \end{array}$$

Going back to the WaterTech problem, the model we created was in fact a linear program!

2.3 LP Models: Multiperiod Models

A multiperiod model is a problem where:

- time is split into periods
- we have to make a decision in each period
- all decisions influences the final outcome

Example:
KW Oil is a local supplier of heating oil, it needs to decide how much oil to purchase in order to satisfy demand of its customers.

Month	1	2	3	4
Demand(l)	5000	8000	9000	6000
Price($\$/l$)	0.75	0.72	0.92	0.90

Question: When should we purchase how much oil when the goal is to min overall total cost?
Additional Complication: The company has a storage tank that

- has a capacity of 4000 litres of oil
- currently (beginning of month 1) contains 2000 litres of oil

Assumption: Oil is delivered at the beginning of the month, and consumption occurs int he middle of the month

Variables

- Need to decide how many litres of oil to purchase in each month i
  - make variable p_i for $i \in [4]$
- How much oil is stored in the tank at the beginning of month i?
  - make variable t_i for $i \in [4]$

Objective Function

Minimize cost of oil purchased

$$min \qquad 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4$$

Constrains

We need

$$p_i + t_i \geq \text{(demand in month } i\text{)}$$

Balancing equation we get

$$p_i + t_i = \text{(demand in month } i\text{)} + t_{i+1}$$

So we have the following four constrains

$$\begin{aligned} p_1 + 2000 &= 5000 + t_2 \\ p_2 + t2 &= 5000 + t_3 \\ p_3 + t3 &= 5000 + t_4 \\ p_4 + t4 &\geq 6000 \end{aligned}$$

Complete LP for KW Oil

$$
\begin{aligned}
\min \quad & 0.75p_1 + 0.72p_2 + 0.92p_3 + 0.90p_4 \\
\text{s.t.} \quad & p_1 + 2000 = 5000 + t_2 \\
& p_2 + t2 = 5000 + t_3 \\
& p_3 + t3 = 5000 + t_4 \\
& p_4 + t4 \geq 6000 \\
& t_1 = 2000 \\
& t_i \leq 4000 \qquad (i = 2, 3, 4) \\
& t_1, p_i \geq 0 \qquad (i = 1, 2, 3, 4)
\end{aligned}
$$

Solving the LP gives the solution:
$p = (3000, 12000, 5000, 6000)^T$
$t = (2000, 0, 4000, 0)^T$

3 Integer Program (IP)

Recall the WaterTech problem

$$
\begin{array}{rl}
\max & 300x_1 + 260x_2 + 220x_3 + 180x_4 - 8y_s - 6y_u \\
\text{s.t} & 11x_1 + 7x_2 + 6x_3 + 5x_4 \leq 700 \\
& 4x_1 + 6x_2 + 5x_3 + 4x_4 \leq 500 \\
& 8x_1 + 5x_2 + 5x_3 + 6x_4 \leq y_s \\
& 7x_1 + 8x_2 + 7x_3 + 5x_4 \leq y_u \\
& y_s \leq 600 \\
& y_u \leq 650 \\
& x_1, x_2, x_3, x_4, y_u, y_s \geq 0
\end{array}
$$

$$
\begin{aligned}
x &= (16 + \frac{2}{3}, 50, 0, 33 + \frac{1}{3})^T \\
y_s &= 583 + \frac{1}{3} \\
y_u &= 650 \\
Profit &= 15433 + \frac{1}{3}
\end{aligned}
$$

Fractional solutions are often not desirable! Can we force the solution to be integer?

Integer Program

- an integer program is a linear program with added integrality constraints for some/all the variables
- we call an IP mixed if there are integer and fractional variables, and pure otherwise
- the difference between LPs and IPs is subtle, but LPs are easy to solve, IPs are not!

Integer program is provably difficult to solve!

- An algorithm is efficient if its running can be bounded by a polynomial of the input size of the instance
- LPs can be solved efficiently
- IPs are very unlikely to have efficient algorithms!

3.1 IP Models: Knapsack

Example:
KitchTech Shipping is a company wishes to ship crates from Toronto to Kitchener. Each crate type has a weight and value, and the total weight of crates shipped must not exceed 10,000 lbs.
Goal: Maximize the total value of shipped goods.

Type	1	2	3	4	5	6
weight (lbs)	30	20	30	90	30	70
value ($)	60	70	40	70	20	90

Variables:
One variable x_i for the number of crates of type i to pack.

Constraints:
The total weight of crates picked must not exceed 10000 lbs.

$$30x_1 + 20x_2 + 30x_3 + 90x_4 + 30x_5 + 70x_6 \leq 10000$$

Objective function:
Maximize the total value

$$max \qquad 60x_1 + 70x_2 + 40x_3 + 70x_4 + 20x_5 + 90x_6$$

Complete IP model for KitchTech Shipping:

$$\begin{array}{rl} \max & 60x_1 + 70x_2 + 40x_3 + 70x_4 + 20x_5 + 90x_6 \\ \text{s.t.} & 30x_1 + 20x_2 + 30x_3 + 90x_4 + 30x_5 + 70x_6 \leq 10000 \\ & x_i \geq 0 \qquad (i \in [6]) \\ & x_i \text{ integer } (i \in [6]) \end{array}$$

Let's make this shit more complicated with more rules...
Suppose that:

1. we must not send more than 10 crates of the same type

2. we can only send crates of type 3, if we send at least 1 crate of type 4

Note that we can send at most 10 crates of type 3 by the previous constraints!
By adding the following constraint, the added requirements is fulfilled:

$$x_3 \leq 10x_4$$

proving correctness of the added constraint:

- $x_4 \geq 1 \rightarrow$ new constraint is redundant
- $x_4 = 0 \rightarrow$ new constraint becomes $x_3 \leq 0$

Suppose we add another rule where we must:

1. take a total of at least 4 crates of type 1 or 2, or
2. take at least 4 crates of type 5 or 6

strategy:
Create a new variable y such that:

- $y = 1 \rightarrow x_1 + x_2 \geq 4$
- $y = 0 \rightarrow x_5 + x_6 \geq 4$
- and force y to take on value 0 or 1

So we add the following constraints:

- $x_1 + x_2 \geq 4y$
- $x_5 + x_6 \geq 4(1 - y)$
- $0 \leq y \leq 1$
- y integer

The variable y we added is called a binary variable. These are very useful for modelling logical constraints of the form:

- Condition (A or B) and C $\rightarrow$ D

So the finalized model would be:

$$
\begin{array}{rl}
\max & 60x_1 + 70x_2 + 40x_3 + 70x_4 + 20x_5 + 90x_6 \\
\text{s.t.} & 30x_1 + 20x_2 + 30x_3 + 90x_4 + 30x_5 + 70x_6 \leq 10000 \\
& x_3 \leq 10x_4 \\
& x_1 + x_2 \geq 4y \\
& x_5 + x_6 \geq 4(1 - y) \\
& x_i \geq 0 \qquad (i \in [6]) \\
& 0 \leq y \leq 1 \\
& \text{y integer} \\
& x_i \text{ integer } (i \in [6])
\end{array}
$$

3.2 IP Models: Scheduling

Example:
The neighborhood coffee shop is open on workdays. The daily demand for workers is given in the table. Each worker works for 4 consecutive days and has one day off.
Goal: Hire the smallest number of workers so that the demand can be met

Mon	Tues	Wed	Thurs	Fri
3	5	9	2	7

Variables:
Introduce variable x_d for every $d \in \{M, T, W, Th, F\}$ counting the number of people to hire with starting day d

Objective function:
Minimize the total number of people hired:

$$min \qquad x_M + x_T + x_W + x_{Th} + x_F$$

Constraints:
We need to ensure that enough people work on each of the days.
Question: Given a solution, how many people work on Monday?
Answer: All but those that start on Tuesday, i.e.

$$x_M + x_W + x_{Th} + x_F$$

And it must be greater than or equal to the number of workers required
So the complete LP is:

$$\begin{array}{rl} \text{min} & x_M + x_T + x_W + x_{Th} + x_F \\ \text{s.t.} & x_M + x_W + x_{Th} + x_F \geq 3 \\ & x_M + x_T + x_{Th} + x_F \geq 5 \\ & x_M + x_T + x_W + x_F \geq 9 \\ & x_M + x_T + x_W + x_{Th} \geq 2 \\ & x_T + x_W + x_{Th} + x_F \geq 7 \\ & x \geq 0, x \text{ integer} \end{array}$$

4 Optimization on graphs

4.1 Graph Theory 101:

A graph G consist of:

- vertices $u, w, ... \in V$ (circles)
- edges $uw, wz, ... \in E$ (lines connecting circles)

Two vertices u and v are adjacent if $uv \in E$.
Vertices u and v are the endpoints of edge $uv \in E$
An edge $e \in E$ is incident to $u \in V$ if u is an endpoint of e.

An $s, t - path$ in $G = (V, E)$ is a sequence

$$v_1v_2, v_2v_3, v_3v_4, ..., v_{k-2}v_{k-1}, v_{k-1}v_k$$

Where

- $v_i \in V$ and $v_iv_{i+1} \in E$ for all i, and
- $v_1 = s, v_k = t$ and $\underbrace{v_i \neq v_j \text{ for all } i \neq j}_{\text{Without this, it is called an s,t-walk}}$

The length of a path $P = v_1v_2, v_2v_3, v_3v_4, ..., v_{k-1}v_k$ is the sum of the lengths of the edges on P

$$c(P) = \sum(c_e : e \in P)$$

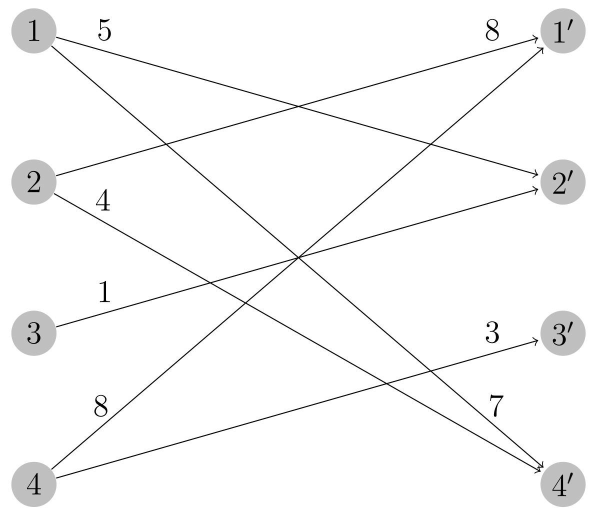
4.2 IP Models: Matchings

Example:
WaterTech has a collection of important jobs that it needs to handle urgently.
It also has 4 employees that need to handle these jobs.
Employees have different skill-sets and may take different amount of times to execute a job, note that some workers are not able to handle certain jobs!

Goal: Assign each worker to exactly one task so that the total execution time is smallest!
We will rephrase this in the language of graphs.

Employees	Jobs			
	1’	2’	3’	4’
1	-	5	-	7
2	8	-	-	4
3	-	1	-	-
4	8	-	3	-

Create a graph with one vertex for each employee and job.
Add an edge ij with cost c_{ij} for $i \in E$ and $j \in J$ if employee i can handle job j in time c_{ij}



Matching

- A collection $M \subseteq E$ is matching if no two edges $ij, i'j' \in M (ij \neq i'j')$ share an endpoint.
- i.e. $\{ij\} \cap \{i'j'\} = 0$

For example:

- $M = \{14', 21', 32', 43'\}$ is a matching
- $M = \{14', 32', 41', 43'\}$ is NOT a matching

The cost of a matching M is the sum of costs of its edges:

$$c(M) = \sum (c_e : e \in M)$$

A matching M is perfect if every vertex v in the graph is incident to an edge in M
Note: a perfect matching correspond to feasible assignments of workers to jobs!

More notations:
Use $\delta(v)$ to denote the set of edges incident to v, i.e.:

$$\delta(v) = \{e \in E : e = vu \text{ for some } u \in V\}$$

This definition is improved later!
For example:

- $\delta(2) = \{21', 24'\}$
- $\delta(3') = \{43'\}$

So another definition of a perfect matching is:

- Given $G = (V, E), M \subseteq E$ is a perfect matching iff $M \cap \delta(v)$ contains a single edge for all $v \in V$

So the IP will have a binary variable x_e for every edge $e \in E$, the idea is:

- $x_e = 1 \longleftrightarrow e \in M$

So the constraints for perfect matching is:
For all $v \in V$, we need

$$\sum(x_e : e \in \delta(v)) = 1$$

The objective function would be:

$$\sum(c_e x_e : e \in E)$$

Complete IP for any perfect matching problem:

$$\begin{array}{rl} \min & \sum(c_e x_e : e \in E) \\ \text{s.t} & \sum(x_e : e \in \delta(v)) = 1 (v \in V) \\ & x \geq 0, \text{ x is integer} \end{array}$$

For the example question, we have:

$$\begin{array}{rl} \min & (5, 1, 3, 4)x) \\ \text{s.t} & \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 12 & 13 & 14 & 23 \end{array} \\ \left(\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right) \end{array} x = \mathbf{1} \\ & x \geq 0, \text{ x is integer} \end{array}$$

4.3 Shortest Path Problem

Input:

- Graph $G = (V, E)$
- Non-negative edge lengts c_e for all $e \in E$
- Vertices $s, t \in V$

Goal: Compute an s, t-path of the smallest total length

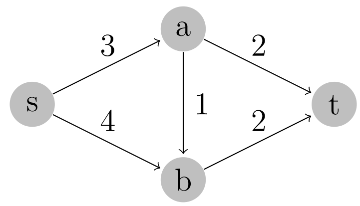
The shortest path problem is:

- Given: Graph $G = (V, E)$, lengths $c_e \geq 0$ for all $e \in E$, and $s, t \in V$, compute an s, t-path of smallest total length

Useful Observation:

- Let $C \in E$ be a set of edges whose removal disconnects s and t

For example:



Let $C = \{sb, ab, at\}$, notice that removing all edges in C eliminates all paths from s to t
Therefore, every s, t-path must have at least one edge in C

A more precise definition of notation δ:
For $S \in V$, we let $\delta(S)$ be the set of edges with exactly one endpoint in S

$$\delta(S) = \{uv \in E : u \in S, v \notin S\}$$

Examples:

1. $S = \{s\} \to \{sa, ab\}$
2. $S = \{s, a\} \to \{ab, at, sb\}$
3. $S = \{a, b\} \to \{sa, sb, at, bt\}$

Definition of s, t-cut:

- $\delta(S)$ is an s, t-cut if $s \in S$ and $t \notin S$

Using the 3 $\delta(S)$ examples above, 1 and 2 are s, t-cuts, 3 is not

Remark:

1. if P is an s, t-path and $\delta(S)$ is an s, t-cut, then P must have an edge from $\delta(S)$
2. if $S \subseteq E$ contains at least one edge from every s, t-cut, then S contains an s, t-path

Prove #2 by contradiction:

- suppose S has an edge from every s,t-cut, but S has no s,t-path
- Let R be the set of vertices reachable from s in S: $R = \{u \in V : S \text{ has an } s,u \text{ path}\}$
- $\delta(R)$ is an s,t-cut since $s \in R$ and $t \notin R$
- Note: there cannot be an edge $uv \in S$ with $u \in R$ and $v \notin R$. Otherwise v should have been in R!
- So $\delta(R) \cap S = \emptyset$
- Contradiction!

Generic IP for Shortest Path problem:
Variables:
We have one binary variable x_e for each edge $e \in E$. We want:

$$x_e = \begin{cases} 1 & : e \in P \\ 0 & : \text{otherwise} \end{cases}$$

Constraints:
We have one constraint for each s,t-cut $\delta(U)$, forcing P to have an edge from $\delta(S)$

$$\sum x_e : e \in \delta(U)) \geq 1 \qquad \text{for all } s,t\text{-cuts } \delta(U) \tag{1}$$

Objective Function:

$$\sum(c_e x_e : e \in E)$$

Complete Model:

$$\begin{array}{rl} \min & \sum(c_e x_e : e \in E) \\ \text{s.t.} & \sum(x_e : e \in \delta(U)) \geq 1 (U \subseteq V, s \in U, t \notin U) \\ & x_e \geq 0,\ x_e \text{ integer} \end{array}$$

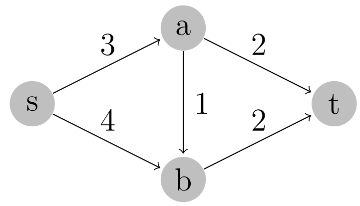
Suppose $c_e > 0$ for all $e \in E$, then in an optimal solution, $x_e \leq 1$ for all $e \in E$

- Suppose $x_e \geq 1$
- Then let $x_e = 1$. This is cheaper and maintains feasibility!

For binary solution x, define

$$S_x = \{e \in E : x_e = 1\}$$

Note: If x is a feasible for an IP, then S_x has at least one edge from every s,t-cut and S_x has a s,t-path, but S_x may contain more than just an s,t-path! Consider this diagram again:



Let $x_e = 1$ for $e \in \{sa, ab, at\}$, and $x_e = 0$ otherwise.
So $S_x = \{sa, ab, at\}$
x cannot be optimal for the IP because we can reduce S_x and get a better solution!
i.e. let $x_a b = 0$ and the solution is more efficient

So if x is an optimal solution for the IP and $c_e \geq 0$ for all $e \in E$ then S_x contains the edges of a shortest s,t-path

5 Nonlinear Programs (NLP)

A nonlinear program (NLP) is of the form:

$$\begin{array}{rl} \min & f(x) \\ \text{s.t.} & g_1(x) \leq 0 \\ & g_2(x) \leq 0 \\ & \dots \\ & g_m(x) \leq 0 \end{array}$$

Where:

- $x \in \mathbb{R}$
- $f : \mathbb{R}^n \to \mathbb{R}$
- $g_i : \mathbb{R}^n \to \mathbb{R}$

Note: Linear programs (LPs) are NLPs!

5.1 NLP Models: Finding Close Points in an LP

Problem:
We are given an LP (P), and an infeasible point $\bar{x}$
Goal:
Find a point $x \in P$ that is as close as possible to $\bar{x}$

i.e. find a point $x \in P$ that minimizes the Euclidean distance to $\bar{x}$

$$||x - \bar{x}||_2 = \sqrt{\sum_{i=1}^{n} (x_i - \bar{x}_i)^2}$$

Remark: $||p||_2$ is called the L^2-norm of p

Generic model:

$$\begin{array}{ll|ll} \text{Given LP:} & & \text{Solve:} & \\ \min & c^T x & \min & ||x - \bar{x}||_2 \\ \text{s.t.} & x \in P & \text{s.t.} & x \in P \text{ where } P = \{x : Ax \leq b\} \end{array}$$

5.2 NLP Models: Binary IPs

Suppose we are given a binary IP (i.e. an IP with all variables are binary)
Recall that Binary IPs are generally hard to solve!
We need to rewrite binary IPs as NLPs

Generic model:

Given IP:	Rewrite:
$\begin{array}{lll} \max & c^T x & \\ \text{s.t.} & Ax \leq b & \\ & x \geq 0 & \\ & x_j \in \{0,1\} & (j \in \{1, /dots, n\}) \end{array}$	$\begin{array}{llll} \max & c^T x & & \\ \text{s.t.} & Ax \leq b & & \\ & x \geq 0 & & \\ & x_j(1-x_j) = 0 & (j \in [n]) & (*) \end{array}$

Correctness:
For $j \in [n]$, $(*)$ holds iff $x_j = 0$ or $x_j = 1$

Question:
Can you change the NLP to express teh fact that x_j is any non-negative integer instead of binary?

$$\begin{array}{llll} \max & c^T x & & \\ \text{s.t.} & Ax \leq b & & \\ & x \geq 0 & & \\ & sin(\pi x_j) = 0 & (j \in [n]) & (**) \end{array}$$

Correctness:
Note that $sin(\pi x_j) = 0$ only if x_j is an integer. Combining with non negative constraint it limits x_j to be any non-negative integer.

5.3 Fermat's last Theorem

Conjecture:
There are no integers $x, y, z \geq 1$ and $n \geq 3$ such that

$$x^n + y^n = z^n$$

The proof is 150 pages long!

NLP for Fermat's Last Theorem

$$\begin{array}{lll} \min & ({x_1}^{x_4} + {x_2}^{x_4} - {x_3}^{x_4})^2 + (sin\ \pi x_1)^2 + (sin\ \pi x_2)^2 + (sin\ \pi x_3)^2 + (sin\ \pi x_4)^2 & \\ \text{s.t.} & x_i \geq 1 \qquad (i = 1 \dots 3) & \\ & x_4 \geq 3 & \end{array}$$

- This NLP is trivially feasible, and the value of any feasible solution is non-negative as its objective is the sum of squares, which means the min possible value is 0
- In fact, the value of a solution (x_1, x_2, x_3, x_4) is 0 iff
  - ${x_1}^{x_4} + {x_2}^{x_4} = {x_3}^{x_4}$, and
  - $(sin\ \pi x_i)^2 = 0$, for all $(i = 1 \ldots 3)$

Notes:

- Fermat's Last Theorem is true iff the NLP has optimal value greater than 0
- It is well known that there is an infinite sequence of feasible solutions whose objective value converges to 0!
- To prove Fermat's Last Theorem we must show that the value 0 can not be attained!

6 Linear Programs: Possible Outcomes

What does solving an optimization problem mean?
$\boxed{\text{LP/IP/NLP}} \rightarrow \boxed{\text{algorithm (software)}} \rightarrow \boxed{\text{solution}}$

For example:

$$\begin{array}{rl} \max & 2x_1 - 3x_2 \\ \text{s.t.} & x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

Trivially, the solution for this problem is:

$\begin{array}{l} x_1 = 1 \\ x_2 = 0 \end{array}$ But sometimes, the answer is not so straightforward!

6.1 3 types of outcomes

Feasibility

- a assignment of values to each of the variables is a feasible solution if all the constraints are satisfied
- an optimization problem is feasible if it has at least one feasible solution
- It is infeasible otherwise

Note: a feasible optimization problem can have infeasible solutions!

Optimal

- For a maximization problem, an optimal solution is a feasible solution that max the objective function.
- For a minimization problem, an optimal solution is a feasible solution that min the objective function.

Notes:

- an optimization problem can have several optimal solutions!
- an infeasible problem can NOT have optimal solutions
- not every feasible problem have an optimal solution

Examples:

$\boxed{\begin{array}{rl} \max & x_1 \\ \text{s.t} & x1 \geq 2 \\ & x_1 \leq 1 \end{array}}$ Infeasible problem, so no optimal solution

$\boxed{\begin{array}{rl} \max & x_1 \\ \text{s.t} & x1 \geq 1 \end{array}}$ Feasible $x_1 = 1$, but still no optimal solution
Also called unbounded problem

Unboundedness:

- A maximization problem is unbounded if for every value M, there exists a feasible solution with objective value greater than M
- A minimization problem is unbounded if for every value M, there exists a feasible solution with objective value smaller than M

Note:
There exist problems that are NOT infeasible, NO optimal solution, and NOT unbounded

For example:

$$\begin{array}{rl} \max & x \\ \text{s.t.} & x < 1 \end{array}$$

- It is feasible as $x = 0$ is a feasible solution
- It is not unbounded since 1 is an upper bound
- It does not have an optimal solution **(requires proof!)**

Proof:
Suppose for a contradiction x is an optimal solution. Let:

$$x' := \frac{x+1}{2}$$

Then $x' < 1$ is feasible. Moreover, $x' > x$, meaning x is not optimal.
Contradiction! Thus, x is not optimal.

Another example:

$$\begin{array}{rl} \min & \frac{1}{x} \\ \text{s.t.} & x \geq 1 \end{array}$$

- It is feasible as $x = 1$ is a feasible solution
- It is not unbounded since 0 is a lower bound
- It does not have an optimal solution **(Proof below)**

Proof:
Suppose for a contradiction x is an optimal solution. Let:

$$x' := x + 1$$

Then $x' > x \geq 1$ is feasible. Moreover, $\frac{1}{x'} < \frac{1}{x}$ since $x' \geq x$, meaning x is not optimal.
Contradiction! Thus, x is not optimal.

6.2 Fundamental Theorem of Linear Programming

For any linear program, exactly one of the following holds:

- it has an optimal solution
- it is infeasible
- it is unbounded

Prove later in the course.

6.3 Solving a LP

What is an algorithm that solves LP?

- if the LP has an optimal solution
  - return an optimal solution $\bar{x}$ + proof that $\bar{x}$ is optimal
- if the LP is infeasible
  - return a proof the LP is infeasible
- if the LP is unbounded
  - return a proof the LP is unbounded

7 Certificates

7.1 Proving Infeasibility

Consider the following problem:

$$\begin{array}{rl} \max & (3,4,-1,2)^T x \\ \text{s.t.} & \begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \\ & x \geq 0 \end{array}$$

We cannot try all possible assignments of values to x_1, x_2, x_3, and x_4

Claim:
There is no solution to (1), (2) and $x \geq 0$ where:

$$\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

Proof:
Construct a new equation with $-1 \times (1) + 2 \times (2)$

$$(1\ 0\ 2\ 1)x = -2 \qquad (*)$$

Suppose for a contradiction there exists $\bar{x} \geq 0$ satisfying (1) and (2). Then $\bar{x}$ satisfies (*):

$$\underbrace{(1\ 0\ 2\ 1)\bar{x}}_{\geq 0} = \underbrace{-2}_{<0}$$

But it doesn't, so we have a contradiction. Therefore the program is infeasible.

Same proof but using matrix:
Suppose there exists a contracdition $\bar{x} \geq 0$ and
$\begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$
Construct a new equation:

$$\begin{pmatrix} -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & -2 & -6 & 7 \\ 2 & -1 & -2 & 4 \end{pmatrix} x = \begin{pmatrix} -1 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$(1\ 0\ 2\ 1)x = -2 \qquad (*)$$

Since $\bar{x}$ satisfies the equations, it satisfies (*):

$$\underbrace{(1\ 0\ 2\ 1)}_{\geq 0^T} \underbrace{\bar{x}}_{\geq 0} = \underbrace{-2}_{<0}$$

Contradiction! So the original must be infeasible.

Proposition:
There is no solution to $Ax = b, x \geq 0$ if there exists y where:

$$y^T A \geq 0^T \text{ and } y^T b < 0$$

Proof:
Suppose for a contradiction there is a solution $\bar{x} \geq 0$
Then $\bar{x}$ satisfies the equation $Ax = b$ and should also satisfy:

$$\underbrace{y^T A}_{\geq 0^T} \underbrace{\bar{x}}_{\geq 0} = \underbrace{y^T b}_{<0}$$

But based on the positivity and negativity of the terms, $\bar{x}$ does not satisfy the equation above.
So, by contradiction, $Ax = b, x \geq 0$ has no solution. $\square$

Farkas' Lemma:
If there is no solution to $Ax = b, x \geq 0$, then there exist y where

$$y^T A \geq 0^T \text{ and } y^T b < 0$$

In other words, the ONLY reason why there is no solution, is because such y^T exist!

7.2 Proving Optimality

Consider the following problem:

$$\begin{array}{ll} \max & z(x) := (-1 \;\; -4 \; 0 \; 0)x + 4 \\ \text{s.t.} & \begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & 6 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ & x \geq 0 \end{array}$$

The optimal solution is:

$$\begin{aligned} \bar{x}_1 &= 0 \\ \bar{x}_2 &= 0 \\ \bar{x}_3 &= 4 \\ \bar{x}_4 &= 5 \end{aligned}$$

How can we show that it is an optimal solution?
If we show:

- $\bar{x}$ is a feasible solution that gives the value 4 for the objective function
- 4 is an upper bound

Then we can conclude that $\bar{x}$ is an optimal solution
Note: this does not show that $\bar{x}$ is the ONLY optimal solution!

Proof:
Let x' be an arbitrary feasible solution. Then

$$z(x') = \underbrace{(-1 \;\; -4 \; 0 \; 0)x'}_{\leq 0 \text{ since matrix non-positive}} \quad +4 \leq 4$$

So 4 is an upper bound, and thus x' is an optimal solution. $\square$

7.3 Proving Unboundedness

Consider the following problem:

$$\begin{array}{ll} \max & z(x) := (-1 \; 0 \; 0 \; 1)x \\ \text{s.t.} & \begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x \geq 0 \end{array}$$

How can we prove that this problem is unbounded?

- construct a family of feasible solutions xt for all $t \geq 0$ and show that as t goes to infinity, the value of the objective function goes to infinity.
- this means that we can have feasible solution of arbitrary high value, aka we have an unbounded program

Consider the family:

$$x(t) := \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

Claim 1:
$x(t)$ is feasible for all $t \geq 0$
Proof:

Let $A = \begin{pmatrix} -1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\bar{x} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}$, and $r = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$

So we are trying to show:

$$x(t) = \bar{x} + tr \geq 0 \text{ for all } t \geq 0 \text{ as } \bar{x}, r \geq 0$$

$$Ax(t) = A[\bar{x} + tr] = \underbrace{A\bar{x}}_{b} + \underbrace{tAr}_{0} = b$$

So $x(t)$ is feasible for all $t \geq 0$, as desired. $\square$

Claim 2:
As $t \to \infty$, $z \to \infty$
Proof:
Let $c = (-1\ 0\ 0\ 1)$

$$z = c^T x(t) = c^T[\bar{x} + tr] = c^T\bar{x} + t\underbrace{c^T r}_{=1>0} = c^T\bar{x} + t$$

So, as $t \to \infty$, $z \to \infty$, as desired. $\square$

Generalization:
It is provable in a similar way that the linear program:

$$max\{c^T x : Ax = b, x \geq 0\}$$

is unbounded if we can find $\bar{x}$ and r such that

$$\bar{x} \geq 0, r \geq 0, A\bar{x} = b, Ar = 0, \text{ and } c^T r > 0$$

8 Standard Equality Forms

A LP is in Standard Equality Form (SEF) if

- it is a maximization problem, and
- for every variable x_j we have the constraint $x_j \geq 0$, and
- all other constraints are equality constraints

For example:

$$\begin{array}{rl} \max & x_1 + x_2 + 17 \\ \text{s.t.} & x_1 - x_2 = 0 \\ & x_1 \geq 0 \end{array}$$

is an LP that is not in SEF because there is no constraint $x_2 \geq 0$
We say x_2 is free

Note:

- $x_2 \geq 0$ is implied by the problem
- x_2 is still free since $x_2 \geq 0$ is not given explicitly

Motivation:
We will develop an algorithm called the Simplex that can solve any LP if it is in SEF
For any LP that is not in SEF, we can:

- Find an "equivalent" LP in SEF
- Solve the "equivalent" LP using Simplex
- use the solution of "equivalent" LP to get the solution of the original LP

"Equivalent LPs"

- Linear programs (P) and (Q) are equivalent if:
  - (P) infeasible $\longleftrightarrow$ (Q) infeasible
  - (P) unbounded $\longleftrightarrow$ (Q) unbounded
  - can construct optimal solution of (P) from optimal solution of (Q)
  - can construct optimal solution of (Q) from optimal solution of (P)

Theorem:
Every LP has an equivalent LP in SEF
Examples/special cases:

- dealing with minimization
  - $minf(x)$ is the same as $max - f(x)$
- replacing inequalities by equalities
  - The following two constraints are the same:
    * $\alpha \leq \beta$
    * $\alpha + s = \beta$, where $s \geq 0$
  - using similar technique, we can convert any inequality to an equality constraint
- free variables
  - any number is the difference between two non-negative numbers
  - so we can set a free variable $x = a - b$ where $a, b \geq 0$
  - and then rewrite the objective function and constraints by substitution and simplification

So, to solve any LP, it suffices to know how to solve LPs in SEF.

9 Simplex - First Attempt

A naive strategy for solving an LP:

- find a feasible solution x
- if x is optimal, stop
- if LP is unbounded, stop
- find a "better" feasible solution (repeat)

but some problems arise:

- how do we find a feasible solution?
- how do we find a "better" solution?
- will this algorithm ever terminate?

Example 1:

$$\begin{array}{rl} \max & (4,3,0,0)x + 7 \\ \text{s.t} & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

It is obvious that we have a feasible solution:

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 2 \\ x_4 &= 1 \end{aligned}$$

The objective function is $z = 4x_1 + 3x_2 + 7$, the feasible solution has objective value 7
Can we find a feasible solution with value larger than 7? Yes

The idea is to make as little change to the current feasible solution as possible
So we change only 1 variable at a time
We can either increase x_1 or x_2 to affect the objective value
Let's arbitrarily decide to increase x_1 and leave x_2 unchanged, i.e.

- Increase x_1 as much as possible, and keep x_2 unchanged
  - $x_1 = t$ for some $t \geq 0$ as large as possible
  - $x_1 = 0$

Since we have

$$\begin{aligned}
x_1 &= t \\
x_2 &= 0 \\
x_3 &= ? \\
x_4 &= ?
\end{aligned}$$

The objective function becomes

$$z = 4t + 7$$

So to have z as large as possible, we need to choose t as large as possible
Note:

- we need to satisfy all equality constraints, and
- the non-negativity constraints

Since we have 2 equations and 2 variables of freedom, we can attempt to get a formula that obtains the required value of x_3 and x_4

$$\begin{aligned}
\begin{pmatrix} 2 \\ 1 \end{pmatrix} &= \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x \\
&= x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= t \begin{pmatrix} 3 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_4 \end{pmatrix} \\
&= t \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}
\end{aligned}$$

So we have:

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - t \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Note that this constraint hold for any choice of t

Check non-negativity constraints:

- $x_1 = t \geq 0$
- $x_2 = 0$
- $x_3 = 2 - 3t \geq 0 \rightarrow t \leq \frac{2}{3}$
- $x_3 = 1 - t \geq 0 \rightarrow t \leq 1$

So the largest possible t is $min\{\frac{2}{3}, 1\}$, and we have $t = \frac{2}{3}$
The new solution is:

$$x = (t, 0, 2 - 3t, 1 - t)^T = (\frac{2}{3}, 0, 0, \frac{1}{3})^T$$

The new solution is better, but still not optimal! Can we use the same trick to get a better solution?

- performing the same trick on x_2 show that we can not increase it

So what made the trick to work the first time?

Canonical form

In our example:

$$\begin{aligned} \text{max} \quad & \text{(4,3,0,0)x + 7} \\ \text{s.t} \quad & \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

- Column 3 and 4 in constraint matrix corresponds to an identity matrix
- The corresponding entry has 0 values in the objective function
- The right hand side matrix in the constraint is non-negative

Revised strategy:

- find a feasible solution, x
- rewrite LP so that it is in "canonical" form
- if x is optimal, stop
- if x is unbounded, stop
- find a better feasible solution (repeat)

So we still need to answer the following questions:

- define "canonical" form formally, which require us to define basis and basic solutions
- prove that we can always rewrite LPs in canonical form

10 Basis

Consider:

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \text{A} = & \begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix} \end{array}$$

Notation:
Let B be a subset of column indices.
Then A_B is a column sub-matrix of A indexed by set B

For example:

- $B = \{1, 2, 3\}$
  - $A_B = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- $B = \{1, 3, 4\}$
  - $A_B = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Suppose $B = \{5\}$, then A_B can also be written as A_5

Basis

- Let B be a subset of column indices.
- B is a basis if
  1. A_B is a square matrix
  2. A_B is non-singular (columns are independents)

Examples:

- $B = \{1, 2, 3\}$, B is a Basis
- $B = \{1, 5\}$, B is not a Basis since its not square
- $B = \{2, 3, 4\}$, B is not a Basis since the columns are not independent

Does every matrix have a basis? NO!

Theorem:

- Max # of independent columns = Max # of independent rows

Remark:

- Let A be a matrix with independent rows.
- Then B is a basis iff B is a maximal set of independent columns of A

10.1 Basic Solutions:

$$\underbrace{\begin{pmatrix} 1 & 2 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{b}$$

Let B be a basis for A

- if $j \in B$, then x_j is a basic variable
- if $j \notin B$, then x_j is a non-basic variable

Example:
Basis $B = \{1, 2, 4\}$. Then:

- x_1, x_2, x_4 are basic variables, and
- x_3, x_5 are the non-basic variables

x is a basic solution for basis B if:

- $A_x = b$, and
- $x_j = 0$ whenever $j \notin B$

For example:

- $x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ is a basic solution for $B = \{1, 2, 3\}$
  1. $Ax = b$, checked
  2. $x_4 = x_5 = 0$, checked

- $x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ is a basic solution for $B = \{1, 3, 4\}$

    1. $Ax = b$, checked
    2. $x_2 = x_5 = 0$, checked

Given:

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 2 \\ 2 \end{pmatrix}}_{b}$$

Find a basic solution for $B = \{1, 4\}$

$$\begin{aligned}
\begin{pmatrix} 2 \\ 2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix} x \\
&= x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \underbrace{x_2}_{=0} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \underbrace{x_3}_{=0} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \end{pmatrix}
\end{aligned}$$

which means:

$$\begin{aligned}
\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} &= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} 4 \\ 2 \end{pmatrix}
\end{aligned}$$

Thus, the basic solution is $x = \{4, 0, 0, 2\}^T$
Notice that we had no choice of x once B is given!

Theorem:

- Consider $Ax = b$ and a basis B of A
- Then there exists a unique basic solution x for B

Convention: Columns of A_B and elements of x_B are ordered by B

Proposition:
Consider $Ax = b$ and a basis B of A
Then there exists a unique basic solution x for B

Proof:

$$\begin{aligned}
b &= Ax \\
&= \sum_j A_j x_j \\
&= \sum_{j \in B} A_j x_j + \sum_{j \notin B} \underbrace{A_j x_j}_{=0} \\
&= \sum_{j \in B} A_j x_j \\
&= A_B x_B
\end{aligned}$$

Since B is a basis, it implies A_B is non-singular, which means A_B^{-1} exists
Hence, $x_B = A_B^{-1} b$

10.2 Vector Basic

Consider $Ax = b$ with independent rows
Vector x is a basic solution if it is a basic solution for some basis B