Eucledeen Transformations

Translations:

Consider for a minute a vector in $\mathbb{R}^2$ with components (x_1, x_2) we can take it to its new location (x_1', x_2') with a displacement (h, k) thus:

$$x_1' = x_1 + h \qquad x_2' = x_2 + k \qquad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_1 + h \\ x_2 + k \end{bmatrix}$$

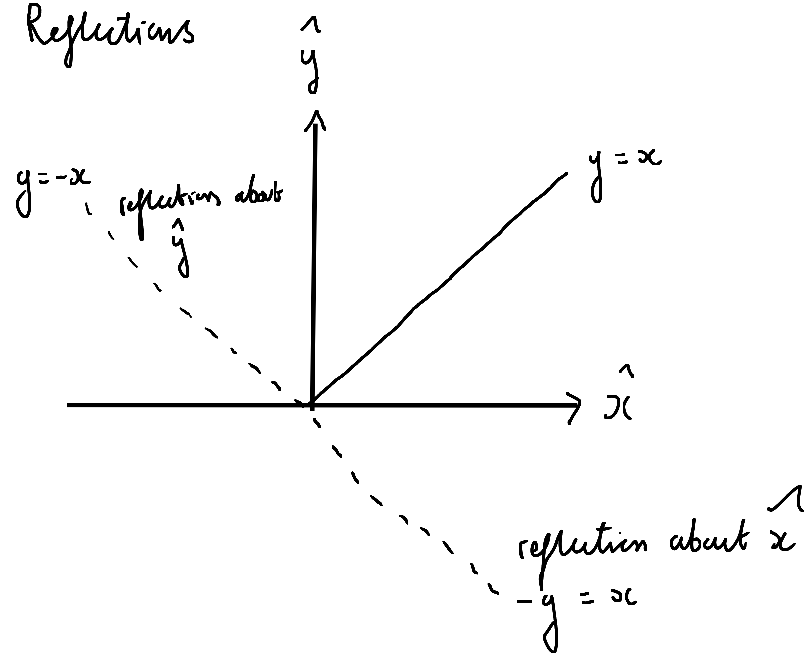
Lets transform into the form $A\vec{x} = \vec{b}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \begin{matrix} 1 \cdot x_1 + 0 \cdot x_2 \\ 0 \cdot x_1 + 1 \cdot x_2 \end{matrix}$$

In order to add (h, k) to (x_1, x_2) we need a new dimension, introducing homogeneous coordinates:

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot x_1 + \cancel{0 \cdot x_2} + 1 \cdot h \\ \cancel{0 \cdot x_1} + 1 \cdot x_2 + 1 \cdot k \\ \cancel{0 \cdot x_1} + \cancel{0 \cdot x_2} + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} x_1 + h \\ x_2 + k \\ 1 \end{bmatrix}$$

Reflections



Reflection about $\hat{x}$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \cdot x + 0 \cdot y \\ 0 \cdot x - 1y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

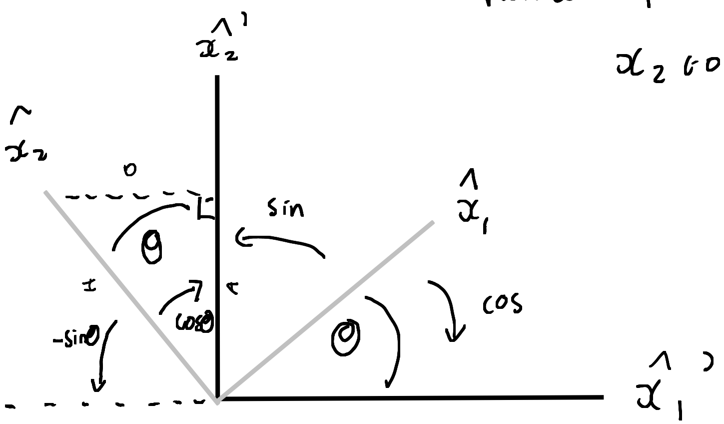
Reflection about $\hat{y}$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \cdot x + 0 \cdot y \\ 0 \cdot x + 1y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

Rotations

Rotate x_1 to x_1' by θ degrees

x_2 to x_2'



$$x_1' = x_1 \cos\theta - x_2 \sin\theta$$

$$x_2' = x_1 \sin\theta + x_2 \cos\theta$$

Clockwise

Now in the form $Ax = y$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

Anticlockwise

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Affine transformations

Are ones which preserves angles.

Scaling

To do this we need a factor for each dimension so for $\mathbb{R}^2$

lets have p & q for axis $\hat{x}_1$ & $\hat{x}_2$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \begin{matrix} px_1 + 0x_2 \\ 0x_1 + qx_2 \end{matrix}$$

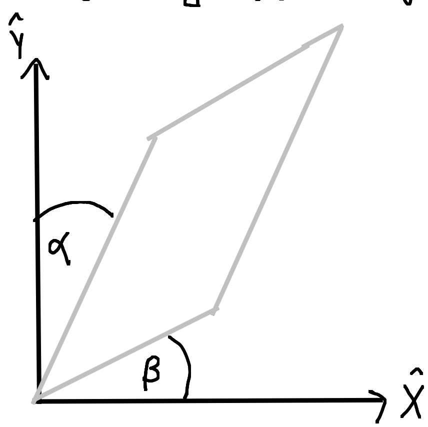
If $p=q$ then the scaling factor is simply a scaler which can be factored out.

$$\begin{bmatrix} x_1' \\ x_2 \end{bmatrix} = \mu \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{where } \mu = p = q$$

Skew

$$\begin{bmatrix} 1 & S_x \\ S_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$S_x = \tan\alpha$
$S_y = \tan\beta$



$$1 \cdot x + S_x \cdot y = x'$$
$$S_y \cdot x + 1 \cdot y = y'$$