Quaternions

Given a vector $\vec{v}$ in $\mathbb{R}^3$ & a scalar s we can define a quaternion q:

$$q = s + v_1 i + v_2 j + v_3 k \quad \text{where } ijk = i^2 = j^2 = k^2 = -1$$

$$= (s, \vec{v}) = (s, v_1, v_2, v_3)$$

Addition

$$q_1 + q_2 = (a, b, c, d) + (e, f, g, h) = (a+e, b+f, c+g, d+h)$$

$$= (a + bi + cj + dk) + (e + fi + gj + hk) = (a+e) + (b+f)i + (c+g)j + (d+h)k$$

Multiplication

$$q_1 \odot q_2 = (a + bi + cj + dk)(e + fi + gj + hk)$$

$$ae + afi + agj + ahk + bei + bfi^2 + bgij + bhik$$

$$+ cej + cfji + cgj^2 + chjk + dek + dfki + dgkj + dhk^2$$

$*^0$

	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

ij? $\quad ijk = -1 \quad ijk^2 = -k$

$ij = k \quad *^1$

jk? $\quad ijk = -1 \quad i^2jk = -i$

$jk = i \quad *^2$

ji? $\quad *^2 \; j^2k = ji$

$-k = ji \quad *^3$

kj? $\quad *^1 \; ij^2 = kj$

$kj = -i$

ik? $\quad *^1 \; i^2j = ik$

$ik = -j$

$-ki = ji^2$

ki? $\quad *^3 \quad ki = j$

$$
\ast^{\circ}\quad q_1 \odot q_2 = ae + afi + agj + ahk
$$
$$
+ bei - bf + bgk - bhj
$$
$$
+ cej - cfk - cg + chi
$$
$$
+ dek + dfj - dgi - dh
$$

	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	-1	-1

$$
q_1 \odot q_2 = (ae - bf - cg - dh, \quad \text{collect scalers}
$$
$$
be + af - dg + ch, \quad \text{collect } i\text{'s}
$$
$$
ce + df + ag - bh, \quad \text{collect } j\text{'s}
$$
$$
de - cf + bg + ah) \quad \text{collet } k\text{'s}
$$

$$
q_1 \odot q_2 = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix}
$$

$$
1 = (1,0,0,0) = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix} = \mathbb{1} \qquad i = (0,1,0,0) = \begin{bmatrix} 0&-1&0&0 \\ 1&0&0&0 \\ 0&0&0&-1 \\ 0&0&1&0 \end{bmatrix}
$$

$$
j = (0,0,1,0) = \begin{bmatrix} 0&0&-1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&-1&0&0 \end{bmatrix} \qquad k = (0,0,0,1) = \begin{bmatrix} 0&0&0&-1 \\ 0&0&-1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{bmatrix}
$$