Classical information

$S_i \in \{0, 1\}$ state i has value of either 0 or 1

Quantum Information is differentiated by allowing a superposition of classical states allowing for a mixture. classical states are represented as vector using BRA-KET notation, also known as dirac notation

$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ for the classical states

Definition of BRA-KET vectors

* indicates a conjugate

$$\langle A| = (A_1^* \ A_2^* \dots A_n^*) \qquad |B\rangle = \begin{pmatrix}B_1\\B_2\\ \vdots \\B_n\end{pmatrix}$$

Inner product:

$$\langle A|B\rangle = A_1^* B_1 + A_2^* B_2 + \dots + A_n^* B_n$$

Examples

$$\langle 0|0\rangle = [1 0]\cdot\begin{bmatrix}1\\0\end{bmatrix} = 1+0 = 1$$

$$\langle 0|1\rangle = [1 0]\cdot\begin{bmatrix}0\\1\end{bmatrix} = 0+0 = 0$$

$$\langle A|A\rangle = \langle B|B\rangle = 1$$

$$\langle A|B\rangle = \langle B|A\rangle = 0$$

Outer products of the form $|\phi\rangle\langle\rho|$ are operators on state $|\psi\rangle$

Example $|\phi\rangle = |\rho\rangle = |0\rangle$

$$|0\rangle\langle 0| = \underset{(2,1)}{\begin{bmatrix}1\\0\end{bmatrix}} \underset{(1,2)}{[1 0]} = \underset{(2,2)}{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}}$$

Example $|\phi\rangle = |\rho\rangle = |1\rangle$

$$|1\rangle\langle 1| = \underset{(2,1)}{\begin{bmatrix}0\\1\end{bmatrix}} \underset{(1,2)}{[0 1]} = \underset{(2,2)}{\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}}$$

Example $|\phi\rangle = |0\rangle$, $|\rho\rangle = |1\rangle$

$$|0\rangle\langle 1| = \begin{bmatrix}1\\0\end{bmatrix}[0 1] = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$$

Example $|\phi\rangle = |1\rangle$, $|\rho\rangle = |0\rangle$

$$|1\rangle\langle 0| = \begin{bmatrix}0\\1\end{bmatrix}[1 0] = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$$

$$\alpha \overset{\text{row}}{|0\rangle}\overset{\text{col}}{\langle 0|} + \beta \overset{\text{row}}{|0\rangle}\overset{\text{col}}{\langle 1|} + \gamma \overset{\text{row}}{|1\rangle}\overset{\text{col}}{\langle 0|} + \varepsilon \overset{\text{row}}{|1\rangle}\overset{\text{col}}{\langle 1|} = A$$

$$A = \begin{bmatrix}\alpha & \beta\\ \gamma & \varepsilon\end{bmatrix}$$

Simple gates

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Identity does nothing to state $|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Not gate given (pauli-X)

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $X|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$

Or more generally

$X\psi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha|0\rangle + \beta|1\rangle$

$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $Z\psi = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha|0\rangle - \beta|1\rangle$

$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $H\psi = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{\sqrt{2}}\left(\alpha + \beta|0\rangle + \alpha - \beta|1\rangle\right)$

$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ $S\psi = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ i\beta \end{bmatrix} = \alpha|0\rangle + i\beta|1\rangle$

BRA-KET equivalent notation:

$|0\rangle\langle 0| = \underset{(2,1)}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \underset{(1,2)}{[1\ 0]} = \underset{(2,2)}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}$

$|1\rangle\langle 1| = \begin{bmatrix} 0 \\ 1 \end{bmatrix}[0\ 1] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$S = |0\rangle\langle 0| + i|1\rangle\langle 1|$

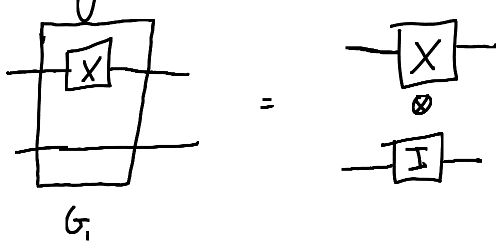
$S\psi = (|0\rangle\langle 0| + i|1\rangle\langle 1|)(\alpha|0\rangle + \beta|1\rangle)$

$= \alpha|0\rangle\underbrace{\langle 0|0\rangle}_{1} + \beta|0\rangle\underbrace{\langle 0|1\rangle}_{0} + i\alpha|1\rangle\underbrace{\langle 1|0\rangle}_{0} + i\beta|1\rangle\underbrace{\langle 1|1\rangle}_{1}$

$= \alpha|0\rangle + i\beta|1\rangle$

$T_{\pi/8} = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix}$

Combining circuit elements in parallel:



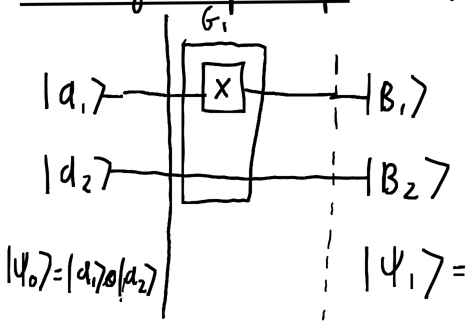
$= X \otimes I = \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}$

Kronecker Product

2,2   2,2

$$G_1 = X \otimes I = \begin{bmatrix} 0\begin{bmatrix}1&0\\0&1\end{bmatrix} & 1\begin{bmatrix}1&0\\0&1\end{bmatrix} \\ 1\begin{bmatrix}1&0\\0&1\end{bmatrix} & 0\begin{bmatrix}1&0\\0&1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}$$

Processing a simple example $|\psi_0\rangle \to |\psi_1\rangle$

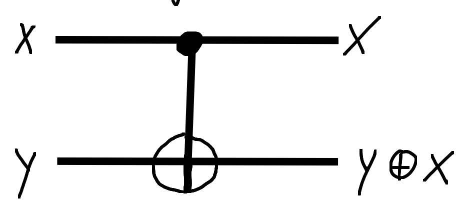


$$|\psi_1\rangle = G_1|\psi_0\rangle = G_1(|a_1\rangle \otimes |a_2\rangle)$$
$$= |B_1\rangle \otimes |B_2\rangle$$

$$|\psi_1\rangle = |B_1\rangle \otimes |B_2\rangle$$

Input			Output		
$\lvert a_1\rangle$	$\lvert a_2\rangle$	$\lvert\psi_0\rangle$	$\lvert B_1\rangle$	$\lvert B_2\rangle$	$\lvert\psi_1\rangle = G\lvert\psi_0\rangle$
$\lvert 0\rangle$	$\lvert 0\rangle$	$[1000]^T$	$\lvert 1\rangle$	$\lvert 0\rangle$	$[0010]^T$
$\lvert 0\rangle$	$\lvert 1\rangle$	$[0100]^T$	$\lvert 1\rangle$	$\lvert 1\rangle$	$[0001]^T$
$\lvert 1\rangle$	$\lvert 0\rangle$	$[0010]^T$	$\lvert 0\rangle$	$\lvert 0\rangle$	$[1000]^T$
$\lvert 1\rangle$	$\lvert 1\rangle$	$[0001]^T$	$\lvert 0\rangle$	$\lvert 1\rangle$	$[0100]^T$

Control gates (dual qubits) CNOT



Input		Output	
X	Y	X	X ⊕ Y
$\lvert 0\rangle$	$\lvert 0\rangle$	$\lvert 0\rangle$	$\lvert 0\rangle$
$\lvert 0\rangle$	$\lvert 1\rangle$	$\lvert 0\rangle$	$\lvert 1\rangle$
$\lvert 1\rangle$	$\lvert 0\rangle$	$\lvert 1\rangle$	$\lvert 1\rangle$
$\lvert 1\rangle$	$\lvert 1\rangle$	$\lvert 1\rangle$	$\lvert 0\rangle$

Convolve states $X \otimes Y$

$\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $|0\rangle \otimes |0\rangle$

$\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $|0\rangle \otimes |1\rangle$

$\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ $|1\rangle \otimes |0\rangle$

$\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ $|1\rangle \otimes |1\rangle$

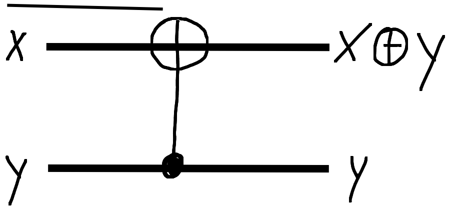
Input	Output
$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$
$\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$
$\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$
$\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$	$\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$

Input	Output
$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$
$\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$
$\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$
$\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$	$\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$

what is the matrix that describes such a transformation

$$\begin{array}{l} \text{row 1} \\ \text{row 2} \\ \text{row 3} \\ \text{row 4} \end{array} \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$$

Upside down:

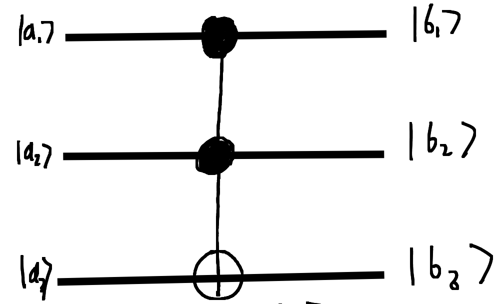


input		output	
X	Y	$x \oplus y$	Y
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|0\rangle$
$\|1\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$

Input	Output
$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$
$\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$
$\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$
$\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$	$\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$

$$\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}$$

Toffoli gate (CCNOT) 3 qubits



$|b_3\rangle = |a_3 \oplus a_1 a_2\rangle$

	Inputs a_1	a_2	a_3	Outputs b_1	b_2	b_3
1)	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
2)	$\|0\rangle$	$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|0\rangle$	$\|1\rangle$
3)	$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|0\rangle$	$\|1\rangle$	$\|0\rangle$
4)	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
5)	$\|1\rangle$	$\|0\rangle$	$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|0\rangle$
6)	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
7)	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$	$\|1\rangle$
8)	$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$

$|a_1\rangle \otimes |a_2\rangle \otimes |a_3\rangle \quad |b_1\rangle \otimes |b_2\rangle \otimes |b_3\rangle$

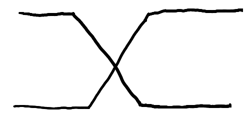
$\vert a_1\rangle\ \vert a_2\rangle\ \vert a_3\rangle$	$\vert a_1\rangle \otimes \vert a_2\rangle \otimes \vert a_3\rangle$
$\vert 0\rangle\vert 0\rangle\vert 0\rangle$	$\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}$
$\vert 0\rangle\vert 0\rangle\vert 1\rangle$	$\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{bmatrix}$
$\vert 0\rangle\vert 1\rangle\vert 0\rangle$	$\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{bmatrix}$
$\vert 0\rangle\vert 1\rangle\vert 1\rangle$	$\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\1\\0\\0\\0\\0\end{bmatrix}$
$\vert 1\rangle\vert 0\rangle\vert 0\rangle$	$\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{bmatrix}$
$\vert 1\rangle\vert 0\rangle\vert 1\rangle$	$\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{bmatrix}$
$\vert 1\rangle\vert 1\rangle\vert 0\rangle$	$\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\0\end{bmatrix}$
$\vert 1\rangle\vert 1\rangle\vert 1\rangle$	$\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}\otimes\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{bmatrix}$

Inputs = outputs apart from state 7&8 which are swapped so the matrix is:

$$\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&0\end{bmatrix}$$

Combining circuit elements in series

Swap gate



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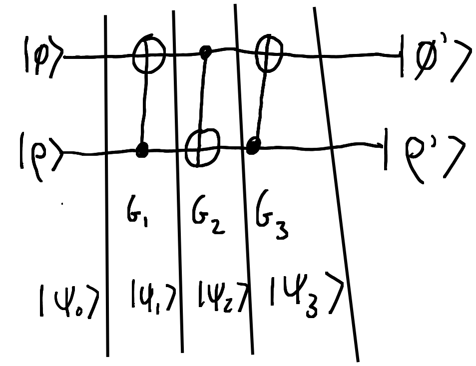

$$\vert\psi_3\rangle = G_3\, G_2\, G_1\, \vert\psi_0\rangle$$

$$G_1 = G_3 = \begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix} \quad G_2 = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$$

$$G_2 G_1 = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0\end{bmatrix}$$

$$G_3 G_2 G_1 = \begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$$

In put		Output	
$\vert 0\rangle\vert 0\rangle$	$[1000]^T$	$\vert 0\rangle\vert 0\rangle$	$[1000]^T$
$\vert 0\rangle\vert 1\rangle$	$[0100]^T$	$\vert 1\rangle\vert 0\rangle$	$[0010]^T$
$\vert 1\rangle\vert 0\rangle$	$[0010]^T$	$\vert 0\rangle\vert 1\rangle$	$[0100]^T$
$\vert 1\rangle\vert 1\rangle$	$[0001]^T$	$\vert 1\rangle\vert 1\rangle$	$[0001]^T$