Introduction to vectors & linear algebra.

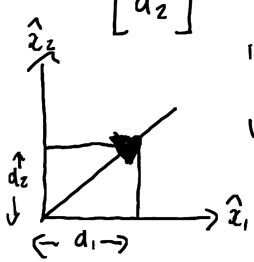
Scalars are simple mathmatical objects with a magnitude Eg. Temperature $T = 10°C$. Another name for such an object is a tensor of rank zero.

Vectors are the next level up in terms of complexity. Vectors like scalars have a magnitude but they also include a direction. Vectors exist in $\mathbb{R}^n$ space, where n is the dimensionality.

A vector in $\mathbb{R}^2$ has two components here is an example:

$$\vec{d} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

where a_1 is some number & a_2 another, they form a coordinate in 2d space. where $\hat{x}_1$ is one axis & $\hat{x}_2$ is 90° rotated.



Vectors are known as rank one tensors as they need an index to identify a coordinate. within the matrix.

Tensors

Matrix Notation:

Row vector/matrix: 1 by N (1, N)

$$\vec{a} = [a_{11} \; a_{12} \cdot \cdot \; a_{1N}]$$ 1 row

N columns of elements

Column vector/matrix: M by 1 (M, 1)

$$\vec{b} = \begin{bmatrix} b_{11} \\ b_{21} \\ \cdot \\ \cdot \\ b_{M1} \end{bmatrix}$$ M rows of elements

1 column

So in general we can have objects with shape

Rows, Columns

(M, N)

We can assign labels to index the matrix components i, j to find them in a matrix

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

where X_{ij} identifies an element. Eg $i=1$, $j=2$ X_{12}

<u>Retangular / square matrix:</u> M by N (M, N)

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \cdots & \Gamma_{1N} \\ \Gamma_{21} & \Gamma_{22} & \cdots & \Gamma_{2N} \\ \vdots & \vdots & \ddots & \\ \Gamma_{M1} & \Gamma_{M2} & & \Gamma_{mn} \end{bmatrix}$$

$$= [\Gamma_{ij}]$$

A matrix is said to be zero if all of its component are zero.

e.g.: $\underset{(2,2)}{A} = [A_{ij}]$ but $A_{ij} = 0$ so $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Γ is a square matrix is $M = N$ e.g. an $N \times N$ where e.g. $n=2$
(N, N)

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix}$$

<u>The identity matrix</u> is a special case of an N by N square matrix, where the diagonals are equal to 1, that is to say given $N=2$

$\underset{(2,2)}{I} = \underset{(2,2)}{[I_{ij}]}$ but $I_{ij} = 1$ where $i=j$ & zero elsewhere

$$\therefore \quad I = \begin{array}{c|c} & \begin{array}{cc} j=1 & j=2 \end{array} \\ \begin{array}{c} i=1 \\ i=2 \end{array} & \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix Operations

Equality

Two matrices A & B are said to be the same if each of their shapes are equal & each of their respective components at the given location are the same.

$A(N, M) \quad B(O, P) \quad O = N \quad P = M$

$A_{ij} = B_{ij}$ for every $i = 1, 2 \dots N$, $j = 1, 2 \dots M$ $\therefore$ Equal!

Trace

The trace of a matrix is only defined for a square matrix. One can obtain the trace which is the sum of the diagonal components:

$$\text{trace } A = tr(A) = \sum_{i=1}^{N} A_{ii}$$

Determinant

The determinant is a function to obtain a scaler value from a square matrix based upon all of its components.

e.g. 2x2 $\quad \det(A) = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

For a 3x3:

$$\det(B) = \det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$\det([x]) = |x|$ mod notation

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \cdot \det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \cdot \det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \cdot \det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= aei + bfg + cdh - afh - bdi - ceg$$

Visual method:

$$\begin{array}{|c|c|c|c|c|} a & b & c & a & b \\ d & e & f & d & e \\ g\,- & h\,- & i\,- & g\,+ & h\,+ \end{array}$$

$+aei + bfg + cdh$

$-ceg - afh - bdi$

Addition / Subtraction.

Given two matricies A & B with shape (N, M) & (Θ, P) respectively, these matricies are said to be conformable for addition / multiplication if $N = \Theta$ & $M = P$. AKA they are the same shape. Otherwise addition / subtraction are not defined.

$$\underset{(n,m)}{C} = \underset{(n,m)}{A} + \underset{(n,m)}{B} = \left[A_{ij} + B_{ij} \right] = \left[C_{ij} \right]$$

Example $\underset{(2,2)}{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\underset{(2,2)}{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

$$\underset{(2,2)}{C} = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{bmatrix} = \begin{bmatrix} (1+5) & (2+6) \\ (3+7) & (4+8) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

Multiplication

Given two matricies A & B of form (n, m) (Θ, P)

The two matricies are said to be conformable for multiplication.

If $k = M = \Theta$ otherwise the matrix product is not defined.

$$\underset{(n,p)}{C} = \underset{(n,k)}{A} \; \underset{(k,p)}{B}$$

e.g. $\underset{(1,3)}{A} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

$$\underset{(3,2)}{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$= \left[(1 \cdot 1 + 2 \cdot 3 + 3 \cdot 5) \; (1 \cdot 2 + 2 \cdot 4 + 3 \cdot 6) \right]$$

$$\underset{(1,2)}{C} = \left[(1+6+15)(2+8+18) \right] = \begin{bmatrix} 22 & 28 \end{bmatrix}$$

Minor of Matrix

Given an n by n matrix, the Minor of the matrix modifies each element by inspecting the matrix, removing the row & column that the element resides within, inspecting the remaining square matrix, performing the determinant of this, finally replace the element with this value.

$$\underset{(3,3)}{A} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then the minor $\underset{(3,3)}{M(A)} = [M_{ij}] = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \det \overset{M_{11}\downarrow}{\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \det \overset{M_{12}\downarrow}{\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \det \overset{M_{13}\downarrow}{\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \det \overset{M_{21}\downarrow}{\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \det \overset{M_{22}\downarrow}{\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \det \overset{M_{23}\downarrow}{\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \det \overset{M_{31}\downarrow}{\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \det \overset{M_{32}\downarrow}{\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \det \overset{M_{33}\downarrow}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}$$

Adjoint/Cofactor

Given an n by n matrix $A = A_{ij} = [a_{ij}]$ and assuming we have already calculated the minor of this matrix $M(A) = M_{ij} = [m_{ij}]$ one can find the cofactor of A given the following expression:

$$\boxed{\operatorname{cof} A_{ij} = (-1)^{i+j} M_{ij}}$$

For a (3, 3) matrix one would apply the following polarity matrix to M_{ij}

Example

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{array} \quad \text{Even} \rightarrow +\text{ve} \quad \text{odd} \rightarrow -\text{ve} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\underset{(3,3)}{\operatorname{cof} A_{ij}} = \begin{bmatrix} +M_{11} & -M_{12} & +M_{13} \\ -M_{21} & +M_{22} & -M_{23} \\ +M_{31} & -M_{32} & +M_{33} \end{bmatrix}$$

Adjoint

$$\operatorname{adj}(A_{ij}) = (\operatorname{cof} A_{ij})^T$$

Inversion / Division

Just like a number has reciprocal e.g $z = 8$ $z^{-1} = \frac{1}{8}$ and we have an identity that any number multiplied by its inverse is unity $zz^{-1} = 1$

For matrices we have a similar rule however replacing unity with the identity matrix.

A is some (n, n) matrix, A^{-1} also (n, n) is defined such that:

$$AA^{-1} = \underset{(n \times n)}{I}$$

Utility from this gives us the concept of division within linear algebra say we have some (n, n) matrices A, B & C. B & C are known, how do we find A?

$AB = C$ it would be nice to divide both sides by B, but how?

$$ABB^{-1} = CB^{-1}$$

$$AI = CB^{-1}$$

$A = CB^{-1}$ and we have A, given that we can find B^{-1}

Methods of calculation

1) Adjoint / Det $\quad adj(A)A = det(A)I$

$$A^{-1} = \frac{adj(A)}{det(A)}$$ where $det(A) \neq 0$

$det(A) = 0$ Singular / non-invertable

2) Row echelon form

Vectors in $\mathbb{R}^n$:

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Addition: $\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ v_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$

Subtraction: $\vec{u} - \vec{v} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ \vdots \\ u_n - v_n \end{bmatrix}$

Scaler Multiplication: $\alpha \vec{u} = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \\ \vdots \\ \alpha u_n \end{bmatrix}$

Vector product(s)

Dot product is the scalar product of the magnitudes of two vectors & the cosine of the angle which subtends one from another. It is also called inner product of the projection product.

Inner / Scaler / Dot product takes two equal dimension vectors & returns a scaler value.

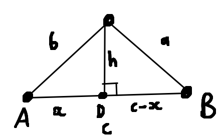
Algebraically the inner product is the sum of the product of the respective elements of each vector:

$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{N} a_i b_i = a_1 b_1 + a_2 b_2 + \dots + a_N b_N = \vec{a} \vec{b}^T = \langle a | b \rangle$$

Geometrically it is the products of the Eucledean magnitudes of the two vectors & the cosine of the angle between them.

$$\vec{a} \cdot \vec{b} = |a||b| \cos\theta$$

Cosine rule



Pythagoras

① ΔBCD $\quad a^2 = h^2 + (c-x)^2$

$a^2 = h^2 + c^2 - 2cx + x^2$

② ΔACD $\quad b^2 = x^2 + h^2$

$h^2 = b^2 - x^2$

③ = ② & ① $\quad a^2 = b^2 \cancel{- x^2} + c^2 - 2cx \cancel{+ x^2}$

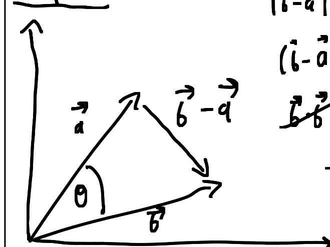
$a^2 = b^2 + c^2 - 2cx$ (need to get rid of x)

④ Trig ΔACD $\quad C\frac{A}{H}$ $\quad \cos\theta = \frac{x}{b}$ $\quad x = b\cos\theta$

⑤ = ④ & ③

$a^2 = b^2 + c^2 - 2cb\cos\theta$

Dot product



$|\vec{b} - \vec{a}|^2 = |\vec{b}|^2 + |\vec{a}|^2 - 2|\vec{a}||\vec{b}| \cos\theta$

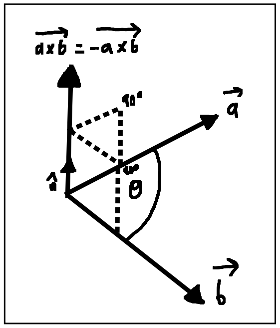
$(\vec{b} - \vec{a}) \cdot (\vec{b} - \vec{a}) = \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} - 2|\vec{a}||\vec{b}| \cos\theta$

$\cancel{\vec{b} \cdot \vec{b}} - \vec{b} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \cancel{\vec{a} \cdot \vec{a}} = \cancel{\vec{a} \cdot \vec{a}} + \cancel{\vec{b} \cdot \vec{b}} - 2|\vec{a}||\vec{b}| \cos\theta$

$-\cancel{2}(\vec{a} \cdot \vec{b}) = -\cancel{2}|a||b| \cos\theta$

$$\boxed{\vec{a} \cdot \vec{b} = |a||b| \cos\theta}$$

Cross / Vector / Directed area Product



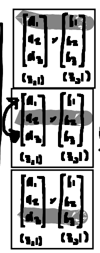
Given two vectors $\vec{a}$ & $\vec{b}$, the cross product $\vec{a}\times\vec{b}$ yields a vector which is perpendicular to both $\vec{a}$ and $\vec{b}$ & thus the normal to the plane which they are embedded within.

If two vectors have the same direction or the exact opposite, they are said to not be linearly independent. If a & b are not or one or more of the vectors is zero then the cross product is zero.

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$$

Definition of cross product in $\mathbb{R}^3$

$$\underset{(3,1)}{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}} \times \underset{(3,1)}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$



REVERSE ROWS BEFORE DET

$$\begin{bmatrix} \det\begin{bmatrix} a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \\ \det\begin{bmatrix} a_3 & b_3 \\ a_1 & b_1 \end{bmatrix} \\ \det\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \end{bmatrix}$$

$\vec{a}\times\vec{b}$

$$\underset{(3,1)}{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}} \times \underset{(3,1)}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{c}$$

$$||\vec{a}||^2 = a_1^2 + a_2^2 + a_3^2 \qquad |\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$||\vec{c}||^2 = c_1^2 + c_2^2 + c_3^2 \qquad |\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$$

$$= (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2$$

$$= a_2^2 b_3^2 + a_3^2 b_2^2 - 2 a_2 a_3 b_2 b_3$$
$$+ a_3^2 b_1^2 + a_1^2 b_3^2 - 2 a_1 a_3 b_1 b_3$$
$$+ a_1^2 b_2^2 + a_2^2 b_1^2 - 2 a_1 a_2 b_1 b_2$$

$$= a_1^2(b_2^2 + b_3^2) + a_2^2(b_1^2 + b_3^2) + a_3^2(b_1^2 + b_2^2)$$
$$-2(a_2 a_3 b_2 b_3 + a_1 a_3 b_1 b_3 + a_1 a_2 b_1 b_2)$$

$$\vec{a}\circ\vec{b} = |a||b|\cos\theta$$

$$||\vec{a}\circ\vec{b}||^2 = ||a||^2\,||b||^2\cos^2\theta = (a_1 b_1 + a_2 b_2 + a_3 b_3)^2$$

$$\vec{a}\cdot\vec{b} = |a||b|\cos\theta$$

$$\|\vec{a}\cdot\vec{b}\|^2 = \|a\|^2\|b\|^2\cos^2\theta = (a_1b_1+a_2b_2+a_3b_3)^2$$

$$= (a_1b_1+a_2b_2+\underline{a_3b_3})(a_1b_1+a_2b_2+a_3b_3)$$

$$a_1^2b_1^2 + 2a_1a_2b_1b_2 + 2a_1a_3b_1b_3$$

$$+ \cancel{a_1a_2b_1b_2} + a_2^2b_2^2 + 2a_2a_3b_2b_3$$

$$+ \cancel{a_1a_3b_1b_3} + \cancel{a_2a_3b_2b_3} + a_3^2b_3^2$$

$$= a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2 + \overbrace{2(a_1a_2b_1b_2 + a_2a_3b_2b_3 + a_1a_3b_1b_3)}^{①}$$

$$\|c\|^2 = \|a\times b\|^2 = a_1^2(b_2^2+b_3^2) + a_2^2(b_1^2+b_3^2) + a_3^2(b_1^2+b_2^2)$$
$$\underbrace{-2(a_2a_3b_2b_3 + a_1a_3b_1b_3 + a_1a_2b_1b_2)}_{②}$$

$\|\vec{a}\cdot\vec{b}\|^2 + \|\vec{a}\times\vec{b}\|^2 = x$ ① + ② cancel

$$x = a_1^2(b_1^2+b_2^2+b_3^2) + a_2^2(b_1^2+b_2^2+b_3^2) + a_3^2(b_1^2+b_2^2+b_3^2)$$

$$= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$$

$$= \|\vec{a}\|^2\|\vec{b}\|^2 = \|\vec{a}\|^2\|\vec{b}\|^2\cos^2\theta + \|\vec{a}\times\vec{b}\|^2$$

$$\|a\times b\|^2 = \|\vec{a}\|^2\|b\|^2 - \|\vec{a}\|^2\|b\|^2\cos^2\theta$$

$$= \|\vec{a}\|^2\|b\|^2(1-\cos^2\theta) = \|a\|^2\|b\|^2\sin^2\theta$$

$\sin^2\theta + \cos^2\theta = 1$

$$|a\times b| = |a||b|\sin\theta$$

Kronecker Product $\otimes$

Given $A_{(m,n)}$ & $B_{(p,q)}$ matrices

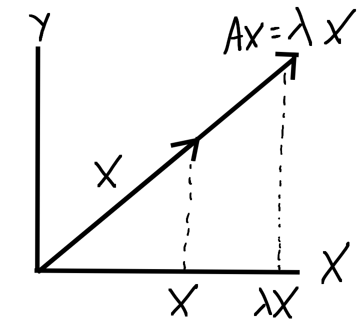
$$A\otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix}$

$$A\otimes B = \begin{bmatrix} 1\begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} & 2\begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} \\ 3\begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} & 4\begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1\cdot 0 & 1\cdot 5 & 2\cdot 0 & 2\cdot 5 \\ 1\cdot 6 & 1\cdot 7 & 2\cdot 6 & 2\cdot 7 \\ 3\cdot 0 & 3\cdot 5 & 4\cdot 0 & 4\cdot 5 \\ 3\cdot 6 & 3\cdot 7 & 4\cdot 6 & 4\cdot 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 5 & 0 & 10 \\ 6 & 7 & 12 & 14 \\ 0 & 15 & 0 & 20 \\ 18 & 21 & 24 & 28 \end{bmatrix}$$

Eigenvectors/values



$AX = \lambda X$

$\det(A - \lambda I)$

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\lambda I = \begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 1 & -2 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix} \qquad \det(A - \lambda I) = 0$$

$$\det(A - \lambda I) = (2-\lambda)\big(\overbrace{(-\lambda)(-\lambda)}^{\lambda^2} - \cancel{1 \cdot 0}\big)$$
$$- 1 \cdot \big(\overbrace{(-\lambda)(1)}^{-\lambda} - \cancel{0 \cdot 0}\big)$$
$$- 2 \cdot \big(\overbrace{(1)(1)}^{1} - (\cancel{-\lambda})(0)\big)$$

$$= (2-\lambda)(\lambda^2) + \lambda - 2$$

$$= (2-\lambda)\lambda^2 - (-\lambda + 2)$$

$$= (2-\lambda)(\lambda^2 - 1) = 0 = (2-\lambda)(\lambda+1)(\lambda-1)$$

$\lambda = 2, -1, +1$ $\boxed{\text{EIGEN VALUES}}$

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \lambda = 2, -1, 1$$

$Ax = \lambda x \quad (A - \lambda I)x = 0$

$$\underset{3\times 3}{\begin{bmatrix} 2-\lambda & 1 & -2 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{(3,1)}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

Null space

$x_1 = \lambda x_2$

$x_2 = \lambda x_3$

$$x_1(2-\lambda) + x_2(1) - 2(x_3) = 0$$

$$x_1(1) + x_2(-\lambda) + x_3(0) = 0 \qquad x_1 - \lambda x_2 = 0$$

$$x_1(0) + x_2 - \lambda x_3 = 0$$

1. $x_1(2-\lambda) + x_2 - 2x_3 = 0$ where $\lambda \in \{2, -1, 1\}$
2. $x_1 = \lambda x_2$
3. $x_2 = \lambda x_3$

① $x_1(2-\lambda) + x_2 - 2x_3 = 0$ where $\lambda \in \{2, -1, 1\}$

② $x_1 = \lambda x_2 \therefore x_2 = \frac{1}{\lambda} x_1$

③ $x_2 = \lambda x_3 \therefore x_3 = \frac{1}{\lambda} x_2$

This system of equations has an infinite set of solutions, affix $x_1 = 1$ or $x_2 = 1$ or $x_3 = 1$ to attempt to get any particular eigenvector

- set $\lambda = 2$ $x_1 = 1$ what do eq ① → ③ look like?

① $x_2 = 2x_3$ ② $x_2 = \frac{1}{2}$ ③ $x_3 = \frac{1}{2} x_2$

① + ③ $x_3 = \frac{1}{4}$ stays consistent with ① good test!

Thus we have $X_{\lambda=2} = \begin{bmatrix} 1 \\ 1/2 \\ 1/4 \end{bmatrix}$ which we can scale $X_{\lambda=2} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ for an integer solution

- set $\lambda = 1$ $x_1 = 1$ what do eq ① → ③ look like?

① $1 + x_2 = 2x_3$ ② $x_2 = 1$ ③ $x_3 = 1$

② + ③ consistent with ① solution: $X_{\lambda=1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

- set $\lambda = -1$ $x_1 = 1$ what do eq ① → ③ look like?

① $3 + x_2 = 2x_3$ ② $x_2 = -1$ ③ $x_3 = 1$

② + ③ consistent with ① solution $X_{\lambda=-1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

So in conclusion the Eigenvectors are: $\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Eucledeen Transformations

Translations:

Consider for a minute a vector in $\mathbb{R}^2$ with components (x_1, x_2) we can take it to its new location (x_1', x_2') with a displacement (h, k) thus:

$x_1' = x_1 + h$ $x_2' = x_2 + k$ $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_1 + h \\ x_2 + k \end{bmatrix}$

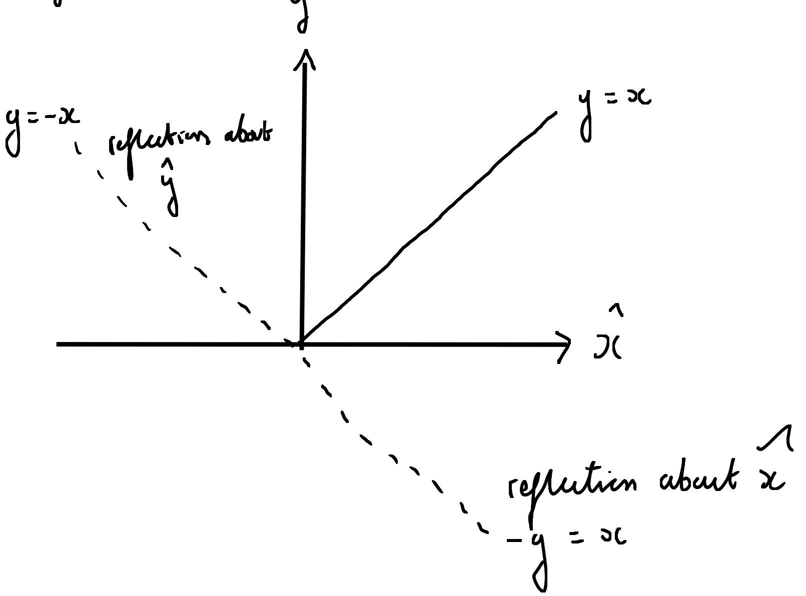
Lets transform into the form $A\vec{x} = \vec{b}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \begin{matrix} 1 \cdot x_1 + 0 \cdot x_2 \\ 0 \cdot x_1 + 1 \cdot x_2 \end{matrix}$$

In order to add (h, k) to (x_1, x_2) we need a new dimension, introducing homogeneous coordinates:

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot x_1 + \cancel{0 \cdot x_2} + 1 \cdot h \\ \cancel{0 \cdot x_1} + 1 \cdot x_2 + 1 \cdot k \\ \cancel{0 \cdot x_1} + \cancel{0 \cdot x_2} + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} x_1 + h \\ x_2 + k \\ 1 \end{bmatrix}$$

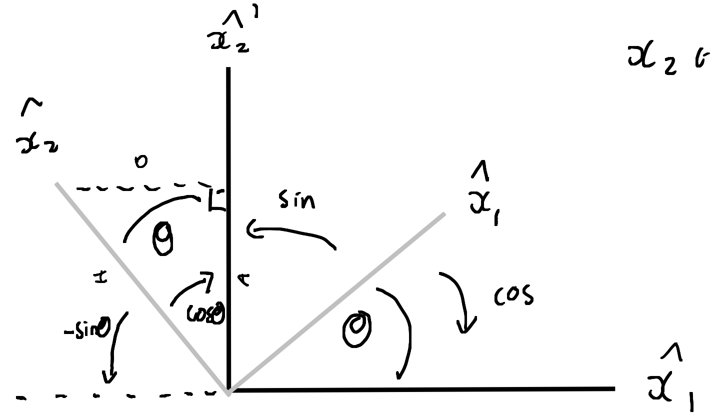
Reflections



Rotations

Rotate x_1 to x_1' by θ degrees

x_2 to x_2'



$$x_1' = x_1 \cos\theta - x_2 \sin\theta$$

$$x_2' = x_1 \sin\theta + x_2 \cos\theta$$

Reflection about $\hat{x}$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \cdot x + 0 \cdot y \\ 0 \cdot x - 1y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

Reflection about $\hat{y}$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \cdot x + 0 \cdot y \\ 0 \cdot x + 1y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

Clockwise

Now in the form $Ax = y$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

Anticlockwise

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Affine transformations

Are ones which preserves angles

Scaling

To do this we need a factor for each dimension so for $\mathbb{R}^2$ lets have p & q for axis $\hat{x}_1$ & $\hat{x}_2$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$p x_1 + 0 x_2$$
$$0 x_1 + q x_2$$

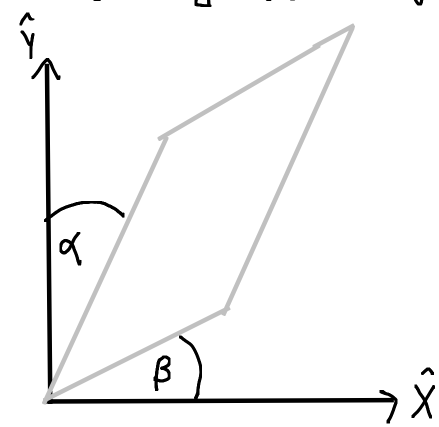
If $p=q$ then the scaling factor is simply a scaler which can be factored out.

$$\begin{bmatrix} x_1' \\ x_2 \end{bmatrix} = \mu \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{where } \mu = p = q$$

Skew

$$\begin{bmatrix} 1 & S_x \\ S_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$S_x = \tan \alpha$
$S_y = \tan \beta$



$$1 \cdot x + S_x \cdot y = x'$$
$$S_y \cdot x + 1 \cdot y = y'$$