BLDC system

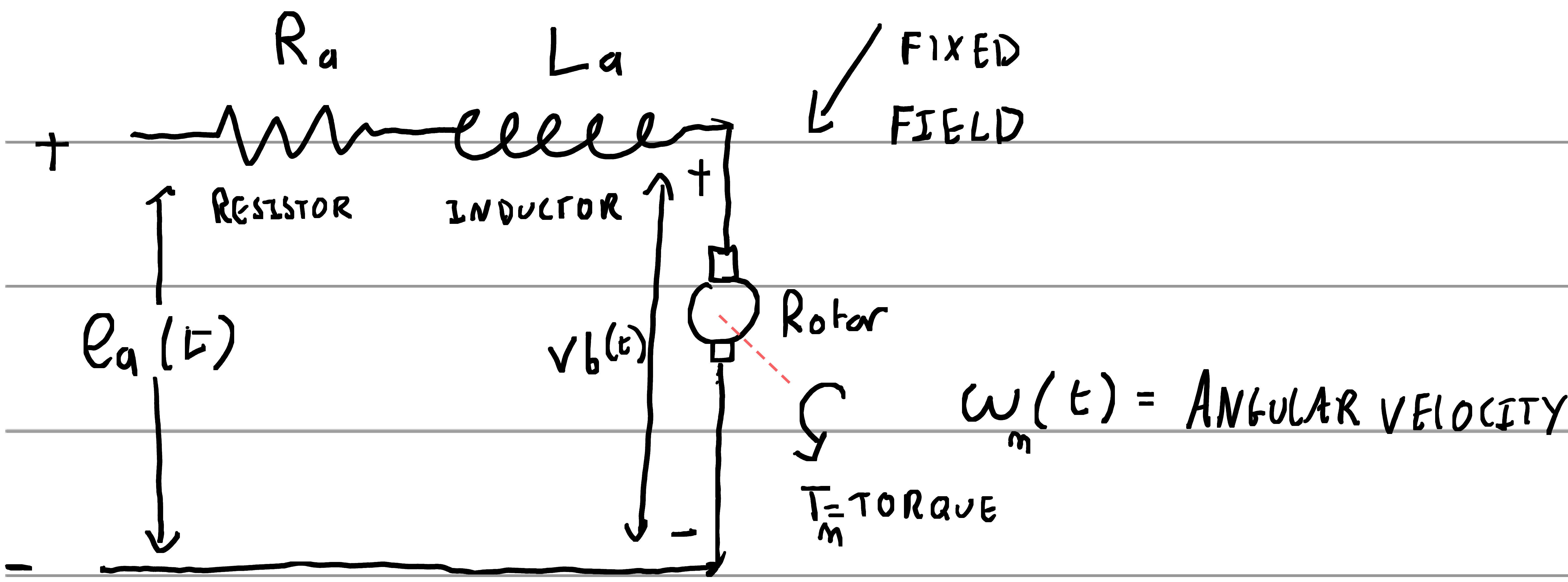
Time domain

INPUT $\qquad\qquad\qquad\qquad$ OUTPUT

$$e_a(t) \longrightarrow \boxed{\text{MOTOR}} \longrightarrow \omega_m(t)$$

Voltage of armature $\qquad\qquad\qquad$ Angular speed of motor

Approximate physical system circuit



1) When the rotor is spinning it interacts with the fixed field at 90° & generates a voltage V_b accross the +/- terminal, this is the back emf.

$$V_b(t) \propto \omega_m(t)$$

$$\therefore \; V_b(t) = K_b \, \omega_m(t)$$

2) Kirchoff's law applies within the armature

$$e_a(t) = \underbrace{i_a(t)}_{\substack{\text{armature} \\ \text{current}}} \underbrace{R_a}_{\text{resistance}} + \underbrace{L_a}_{\text{inductance}} \frac{di_a}{dt} + \underbrace{V_b(t)}_{\text{back emf}}$$

3) Put $i_a(t)$ in terms of $\omega_m(t)$. The torque T_m depends on the armature current i_a.

$$T_m(t) \propto i_a(t)$$

$$\therefore \; T_m(t) = \underbrace{K_t}_{\text{torque constant}} i_a(t)$$

$$i_a(t) = \frac{T_m(t)}{K_t}$$

4) Consider newton $F=ma$, the torque depends on the inertia of the motor & also friction which is a function of speed. Put T_m interms of $W_m(t)$. The motor must overcome both the inertia & the friction of the system to achieve a given speed.

$$\underset{Nm}{T_m(t)} = \overbrace{\underbrace{J_m}_{\text{moment of inertia}} \underbrace{\alpha_m(t)}_{\text{acceleration}}}^{\text{INERTIA}} + \overbrace{\underbrace{D_m}_{\text{Damping coefficient}} \underbrace{W_m(t)}_{\text{speed}}}^{\text{FRICTION}}$$

$$= \underset{[kg\,m^2]}{J_m} \underset{[rad/s^2]?}{\frac{d\,W_m(t)}{dt}} + \underset{\frac{kg\,m^2}{s}}{D_m} \underset{[rads/s]?}{W_m(t)}$$

5) Combining ① + ② + ③ + ④ we can obtain a differential equation of $e_a(t)$ in terms of $W_m(t)$ and its first & second derivatives wrt t:

$$e_a(t) = \underbrace{i_a}_{\text{armature current}}(t) \underbrace{R_a}_{\text{resistance}} + \underbrace{L_a}_{\text{inductance}} \frac{di_a}{dt} + \underbrace{V_b(t)}_{\text{back emf}} \quad K_b w(t)$$

$$\text{Torque} \quad T_m = \underbrace{J_m}_{\text{inertia}} \alpha_m(t) + \underbrace{D_m}_{\text{viscous damping friction}} w_m(t) = \underbrace{K_T}_{} i_a(t)$$

$$i_a(t) = \frac{1}{K_T}\left[J_m \alpha_m(t) + D_m w_m(t)\right]$$

$$\frac{di_a(t)}{dt} = \frac{1}{K_T}\left[J_m \frac{d\alpha_m(t)}{dt} + D_m \frac{dw_m(t)}{dt}\right]$$

$$i_a(t) = \frac{1}{K_T}\left[J_m \frac{dw_m(t)}{dt} + D_m w_m(t)\right]$$

$$\frac{di_a(t)}{dt} = \frac{1}{K_T}\left[J_m \frac{d^2 W_m(t)}{dt^2} + D_m \frac{dW_m(t)}{dt}\right]$$

$$e_a(t) = \frac{R_a}{K_T}\left[J_m \frac{dw_m(t)}{dt} + D_m W_m(t)\right] + \frac{L_a}{K_T}\left[J_m \frac{d^2 W_m(t)}{dt^2} + D_m \frac{dW_m(t)}{dt}\right]$$

$$+ K_B W_m(t)$$

$$= W_m(t)\left[K_B + \frac{D_m R_a}{K_T}\right] + \frac{dW_m(t)}{dt}\left[\frac{L_a D_m}{K_T} + \frac{R_a J_m}{K_T}\right]$$

$$+ \frac{L_a J_m}{K_T} \frac{d^2 W_m(t)}{dt^2} \quad \text{of form} \quad e_a(t) = a\frac{d^2 w}{dt^2} + b\frac{dw}{dt} + cw$$

2nd order ode - linear

Example values

$K_T = 0.02 \ \frac{Nm}{A}$ $\quad$ $K_B = 0.22 \ \frac{V}{rad/s}$

$R_a = 1\,\Omega$ $\quad$ $L_a = 2H$ $\quad$ $J_m = 0.005 \ Kg\,m^2$

$D_m = 0.001 \ Kg\,m^2 s^{-1}$?

Solve ODE via sympy python package:

```
import sympy as sp
t = sp.symbols("t")
e_a=sp.symbols("e_a", cls=sp.Function) # voltage
w_m=sp.symbols("w_m", cls=sp.Function) # angular velocity
dw_mdt = w_m(t).diff(t)
d2w_mdt2 = dw_mdt.diff(t)
J_m, R_a, K_b, K_t, D_m, L_a = sp.symbols("J_m R_a K_b K_t D_m L_a", real=True, positive=True)
# motor inertia/torque coeff, armature resistance, back-emf/angular velocity coeff, torque/current coeff, viscous damping term, armature inductance
a = L_a * J_m / K_t
b = (L_a * D_m + R_a * J_m) / K_t
c = (K_b + ((D_m * R_a) / K_t))
kirchoff_newton_ode = sp.Eq(a*d2w_mdt2 + b*dw_mdt + c*w_m(t), e_a(t))
# solve
sol = sp.dsolve(kirchoff_newton_ode,w_m(t), ics={w_m(0):0, sp.diff(w_m(t), t).subs(t,0): 0})
print(sol)
```

$\int_t^0$ $\int_0^t$??

$$w_m(t) = \frac{K_t e^{\frac{t\left(-\frac{D_m}{J_m} - \frac{R_a}{L_a} + \frac{\sqrt{D_m^2L_a^2 - 2D_mJ_mL_aR_a + J_m^2R_a^2 - 4J_mK_bK_tL_a}}{J_mL_a}\right)}{2}} \int e_a(t) e^{\frac{D_m t}{2J_m}} e^{\frac{R_a t}{2L_a}} e^{-\frac{t\sqrt{D_m^2L_a^2 - 2D_mJ_mL_aR_a + J_m^2R_a^2 - 4J_mK_bK_tL_a}}{2J_mL_a}}\,dt}{\sqrt{D_m^2L_a^2 - 2D_mJ_mL_aR_a + J_m^2R_a^2 - 4J_mK_bK_tL_a}} -$$
$$\frac{K_t e^{\frac{t\left(-\frac{D_m}{J_m} - \frac{R_a}{L_a} + \frac{\sqrt{D_m^2L_a^2 - 2D_mJ_mL_aR_a + J_m^2R_a^2 - 4J_mK_bK_tL_a}}{J_mL_a}\right)}{2}} \int^0 e_a(t) e^{\frac{D_m t}{2J_m}} e^{\frac{R_a t}{2L_a}} e^{-\frac{t\sqrt{D_m^2L_a^2 - 2D_mJ_mL_aR_a + J_m^2R_a^2 - 4J_mK_bK_tL_a}}{2J_mL_a}}\,dt}{\sqrt{D_m^2L_a^2 - 2D_mJ_mL_aR_a + J_m^2R_a^2 - 4J_mK_bK_tL_a}} -$$
$$\frac{K_t e^{-\frac{t\left(\frac{D_m}{J_m} + \frac{R_a}{L_a} + \frac{\sqrt{D_m^2L_a^2 - 2D_mJ_mL_aR_a + J_m^2R_a^2 - 4J_mK_bK_tL_a}}{J_mL_a}\right)}{2}} \int e_a(t) e^{\frac{D_m t}{2J_m}} e^{\frac{R_a t}{2L_a}} e^{\frac{t\sqrt{D_m^2L_a^2 - 2D_mJ_mL_aR_a + J_m^2R_a^2 - 4J_mK_bK_tL_a}}{2J_mL_a}}\,dt}{\sqrt{D_m^2L_a^2 - 2D_mJ_mL_aR_a + J_m^2R_a^2 - 4J_mK_bK_tL_a}} +$$
$$\frac{K_t e^{-\frac{t\left(\frac{D_m}{J_m} + \frac{R_a}{L_a} + \frac{\sqrt{D_m^2L_a^2 - 2D_mJ_mL_aR_a + J_m^2R_a^2 - 4J_mK_bK_tL_a}}{J_mL_a}\right)}{2}} \int^0 e_a(t) e^{\frac{D_m t}{2J_m}} e^{\frac{R_a t}{2L_a}} e^{\frac{t\sqrt{D_m^2L_a^2 - 2D_mJ_mL_aR_a + J_m^2R_a^2 - 4J_mK_bK_tL_a}}{2J_mL_a}}\,dt}{\sqrt{D_m^2L_a^2 - 2D_mJ_mL_aR_a + J_m^2R_a^2 - 4J_mK_bK_tL_a}}$$

Wolfram check:

$$w(t) = \exp\left(\frac{1}{2} t \left(\frac{\sqrt{-4 B J L T + D^2 L^2 - 2 D J L R + J^2 R^2}}{J L} - \frac{D}{J} - \frac{R}{L}\right)\right)$$
$$\int_1^t -\left(\left(\exp\left(\frac{\zeta D}{J} + \frac{\zeta R}{L} + \frac{1}{2} \zeta \left(-\frac{D}{J} - \frac{R}{L} - \frac{\sqrt{(D L - J R)^2 - 4 B J L T}}{J L}\right)\right)\right.\right.$$
$$T \sqrt{(D L - J R)^2 - 4 B J L T}\ e(\zeta)\Bigg) \Bigg/$$
$$\left.\left(-D^2 L^2 + 2 D J R L + 4 B J T L - J^2 R^2\right)\right) d\zeta +$$
$$\exp\left(\frac{1}{2} t \left(-\frac{\sqrt{-4 B J L T + D^2 L^2 - 2 D J L R + J^2 R^2}}{J L} - \frac{D}{J} - \frac{R}{L}\right)\right)$$
$$\int_1^t \left(\exp\left(\frac{\xi D}{J} + \frac{\xi R}{L} + \frac{1}{2} \xi \left(-\frac{D}{J} - \frac{R}{L} + \frac{\sqrt{(D L - J R)^2 - 4 B J L T}}{J L}\right)\right)\right.$$
$$T \sqrt{(D L - J R)^2 - 4 B J L T}\ e(\xi)\Bigg) \Bigg/$$
$$\left(-D^2 L^2 + 2 D J R L + 4 B J T L - J^2 R^2\right) d\xi +$$
$$c_1 \exp\left(\frac{1}{2} t \left(-\frac{\sqrt{-4 B J L T + D^2 L^2 - 2 D J L R + J^2 R^2}}{J L} - \frac{D}{J} - \frac{R}{L}\right)\right) +$$
$$c_2 \exp\left(\frac{1}{2} t \left(\frac{\sqrt{-4 B J L T + D^2 L^2 - 2 D J L R + J^2 R^2}}{J L} - \frac{D}{J} - \frac{R}{L}\right)\right)$$

https://www.wolframalpha.com/input?i=%28d%5E2w%28t%29%2Fdt%5E2%29%28%28LJ%29%2FT%29%2B%28dw%28t%29%2Fdt%29%28%28LD%2BRJ%29%2FT%29%2Bw%28t%29*%28B%2B%28%28DR%29%2FT%29%29%3De%28t%29