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is a quadratic programming problem, although we see in Section 3.4.4 that efficient algorithms are available for computing the entire path of solutions as λ is varied, with the same computational cost as for ridge regression. Because of the nature of the constraint, making t sufficiently small will cause some of the coefficients to be exactly zero. Thus the lasso does a kind of continuous subset selection. If t is chosen larger than $t_0 = \sum_1^p |\hat{\beta}_j|$ (where $\hat{\beta}_j = \hat{\beta}_j^{\text{ls}}$, the least squares estimates), then the lasso estimates are the $\hat{\beta}_j$'s. On the other hand, for $t = t_0/2$ say, then the least squares coefficients are shrunk by about 50% on average. However, the nature of the shrinkage is not obvious, and we investigate it further in Section 3.4.4 below. Like the subset size in variable subset selection, or the penalty parameter in ridge regression, t should be adaptively chosen to minimize an estimate of expected prediction error.

In Figure 3.7, for ease of interpretation, we have plotted the lasso prediction error estimates versus the standardized parameter $s = t/\sum_1^p |\hat{\beta}_j|$. A value $\hat{s} \approx 0.36$ was chosen by 10-fold cross-validation; this caused four coefficients to be set to zero (fifth column of Table 3.3). The resulting model has the second lowest test error, slightly lower than the full least squares model, but the standard errors of the test error estimates (last line of Table 3.3) are fairly large.

Figure 3.10 shows the lasso coefficients as the standardized tuning parameter $s = t/\sum_1^p |\hat{\beta}_j|$ is varied. At $s = 1.0$ these are the least squares estimates; they decrease to 0 as $s \to 0$. This decrease is not always strictly monotonic, although it is in this example. A vertical line is drawn at $s = 0.36$, the value chosen by cross-validation.

3.4.3 Discussion: Subset Selection, Ridge Regression and the Lasso

In this section we discuss and compare the three approaches discussed so far for restricting the linear regression model: subset selection, ridge regression and the lasso.

In the case of an orthonormal input matrix $\mathbf{X}$ the three procedures have explicit solutions. Each method applies a simple transformation to the least squares estimate $\hat{\beta}_j$, as detailed in Table 3.4.

Ridge regression does a proportional shrinkage. Lasso translates each coefficient by a constant factor λ, truncating at zero. This is called "soft thresholding," and is used in the context of wavelet-based smoothing in Section 5.9. Best-subset selection drops all variables with coefficients smaller than the Mth largest; this is a form of "hard-thresholding."

Back to the nonorthogonal case; some pictures help understand their relationship. Figure 3.11 depicts the lasso (left) and ridge regression (right) when there are only two parameters. The residual sum of squares has elliptical contours, centered at the full least squares estimate. The constraint

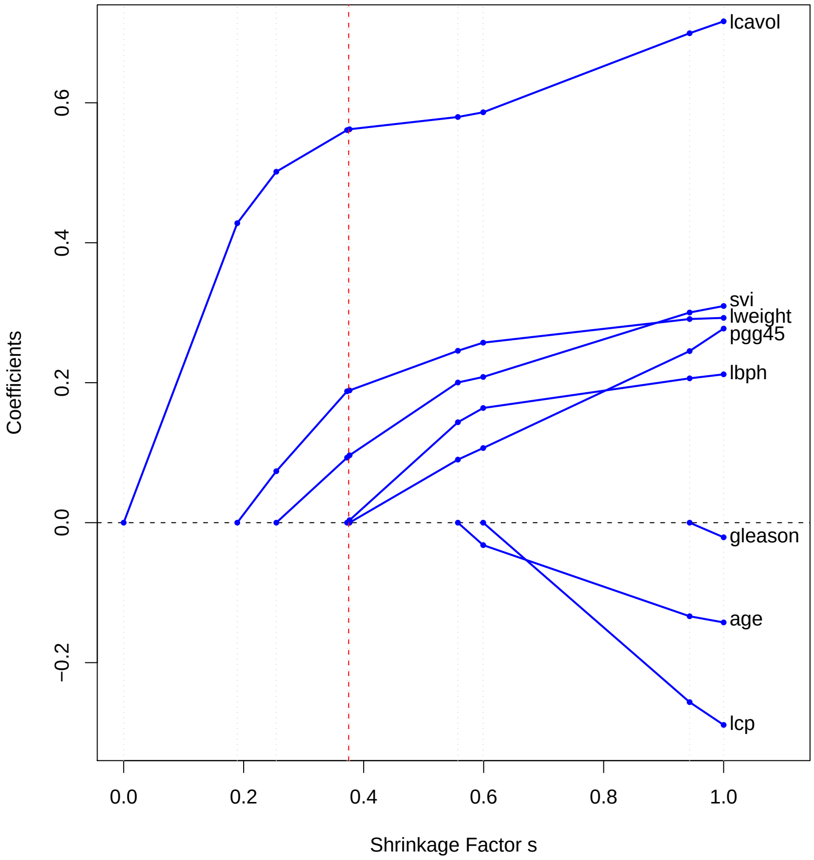
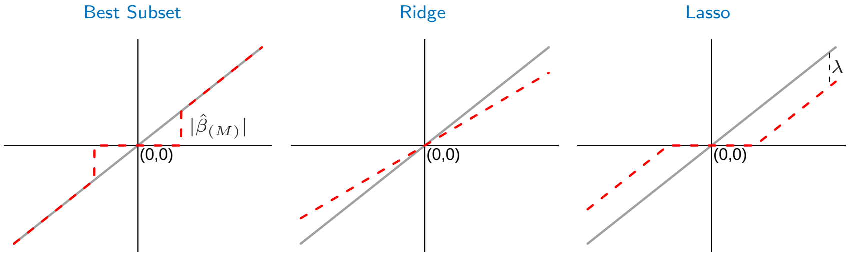


FIGURE 3.10. *Profiles of lasso coefficients, as the tuning parameter t is varied. Coefficients are plotted versus $s = t/\sum_1^p |\hat{\beta}_j|$. A vertical line is drawn at $s = 0.36$, the value chosen by cross-validation. Compare Figure 3.8 on page 65; the lasso profiles hit zero, while those for ridge do not. The profiles are piece-wise linear, and so are computed only at the points displayed; see Section 3.4.4 for details.*

TABLE 3.4. *Estimators of β_j in the case of orthonormal columns of $\mathbf{X}$. M and λ are constants chosen by the corresponding techniques;* sign *denotes the sign of its argument (± 1), and x_+ denotes "positive part" of x. Below the table, estimators are shown by broken red lines. The 45° line in gray shows the unrestricted estimate for reference.*

Estimator	Formula
Best subset (size M)	$\hat\beta_j \cdot I(\lvert\hat\beta_j\rvert \geq \lvert\hat\beta_{(M)}\rvert)$
Ridge	$\hat\beta_j/(1+\lambda)$
Lasso	$\text{sign}(\hat\beta_j)(\lvert\hat\beta_j\rvert - \lambda)_+$



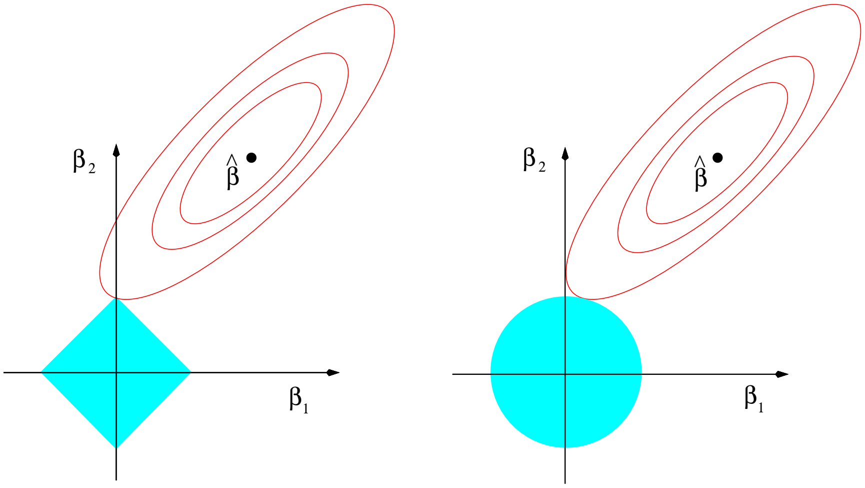


FIGURE 3.11. *Estimation picture for the lasso (left) and ridge regression (right). Shown are contours of the error and constraint functions. The solid blue areas are the constraint regions $|\beta_1| + |\beta_2| \leq t$ and $\beta_1^2 + \beta_2^2 \leq t^2$, respectively, while the red ellipses are the contours of the least squares error function.*

region for ridge regression is the disk $\beta_1^2 + \beta_2^2 \leq t$, while that for lasso is the diamond $|\beta_1| + |\beta_2| \leq t$. Both methods find the first point where the elliptical contours hit the constraint region. Unlike the disk, the diamond has corners; if the solution occurs at a corner, then it has one parameter β_j equal to zero. When $p > 2$, the diamond becomes a rhomboid, and has many corners, flat edges and faces; there are many more opportunities for the estimated parameters to be zero.

We can generalize ridge regression and the lasso, and view them as Bayes estimates. Consider the criterion

$$\tilde{\beta} = \underset{\beta}{\operatorname{argmin}} \left\{ \sum_{i=1}^{N} \Big(y_i - \beta_0 - \sum_{j=1}^{p} x_{ij}\beta_j \Big)^2 + \lambda \sum_{j=1}^{p} |\beta_j|^q \right\} \tag{3.53}$$

for $q \geq 0$. The contours of constant value of $\sum_j |\beta_j|^q$ are shown in Figure 3.12, for the case of two inputs.

Thinking of $|\beta_j|^q$ as the log-prior density for β_j, these are also the equi-contours of the prior distribution of the parameters. The value $q = 0$ corresponds to variable subset selection, as the penalty simply counts the number of nonzero parameters; $q = 1$ corresponds to the lasso, while $q = 2$ to ridge regression. Notice that for $q \leq 1$, the prior is not uniform in direction, but concentrates more mass in the coordinate directions. The prior corresponding to the $q = 1$ case is an independent double exponential (or Laplace) distribution for each input, with density $(1/2\tau)\exp(-|\beta|/\tau)$ and $\tau = 1/\lambda$. The case $q = 1$ (lasso) is the smallest q such that the constraint region is convex; non-convex constraint regions make the optimization problem more difficult.

In this view, the lasso, ridge regression and best subset selection are Bayes estimates with different priors. Note, however, that they are derived as posterior modes, that is, maximizers of the posterior. It is more common to use the mean of the posterior as the Bayes estimate. Ridge regression is also the posterior mean, but the lasso and best subset selection are not.

Looking again at the criterion (3.53), we might try using other values of q besides 0, 1, or 2. Although one might consider estimating q from the data, our experience is that it is not worth the effort for the extra variance incurred. Values of $q \in (1, 2)$ suggest a compromise between the lasso and ridge regression. Although this is the case, with $q > 1$, $|\beta_j|^q$ is differentiable at 0, and so does not share the ability of lasso ($q = 1$) for

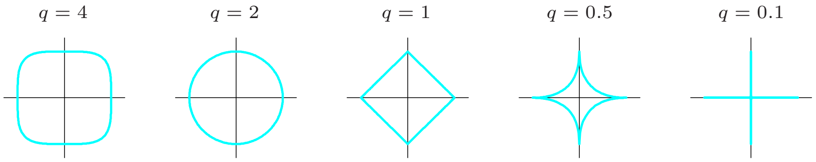


FIGURE 3.12. *Contours of constant value of $\sum_j |\beta_j|^q$ for given values of q.*

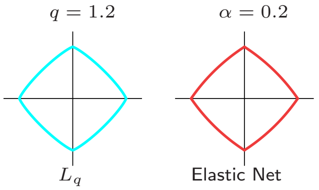


FIGURE 3.13. *Contours of constant value of* $\sum_j |\beta_j|^q$ *for* $q = 1.2$ *(left plot), and the elastic-net penalty* $\sum_j (\alpha\beta_j^2 + (1-\alpha)|\beta_j|)$ *for* $\alpha = 0.2$ *(right plot). Although visually very similar, the elastic-net has sharp (non-differentiable) corners, while the* $q = 1.2$ *penalty does not.*

setting coefficients exactly to zero. Partly for this reason as well as for computational tractability, Zou and Hastie (2005) introduced the *elastic-net* penalty

$$\lambda \sum_{j=1}^{p} \bigl(\alpha\beta_j^2 + (1-\alpha)|\beta_j|\bigr), \tag{3.54}$$

a different compromise between ridge and lasso. Figure 3.13 compares the L_q penalty with $q = 1.2$ and the elastic-net penalty with $\alpha = 0.2$; it is hard to detect the difference by eye. The elastic-net selects variables like the lasso, and shrinks together the coefficients of correlated predictors like ridge. It also has considerable computational advantages over the L_q penalties. We discuss the elastic-net further in Section 18.4.

3.4.4 Least Angle Regression

Least angle regression (LAR) is a relative newcomer (Efron et al., 2004), and can be viewed as a kind of "democratic" version of forward stepwise regression (Section 3.3.2). As we will see, LAR is intimately connected with the lasso, and in fact provides an extremely efficient algorithm for computing the entire lasso path as in Figure 3.10.

Forward stepwise regression builds a model sequentially, adding one variable at a time. At each step, it identifies the best variable to include in the *active set*, and then updates the least squares fit to include all the active variables.

Least angle regression uses a similar strategy, but only enters "as much" of a predictor as it deserves. At the first step it identifies the variable most correlated with the response. Rather than fit this variable completely, LAR moves the coefficient of this variable continuously toward its least-squares value (causing its correlation with the evolving residual to decrease in absolute value). As soon as another variable "catches up" in terms of correlation with the residual, the process is paused. The second variable then joins the active set, and their coefficients are moved together in a way that keeps their correlations tied and decreasing. This process is continued