Automatically Generating Puzzles of Different Complexity

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1 Introduction

2 General Approach

Definition 1. *We define a puzzle board $\mathcal{P}$ to be a two dimensional $n \times m$ board. Let V denote a function that assigns values to the squares on the puzzle board such that the value of the square (i, j) is defined by $V(i, j)$, where $1 \leq i \leq n$, $1 \leq j \leq m$, $V : \mathbb{N} \times \mathbb{N} \to \mathcal{D}$, and $\mathcal{D}$ denotes the set of all possible values the puzzle squares can take.*

The main components of our technique are:

- A declarative definition of puzzle D
- A Complexity Function : C
- A set of Transformation Functions : $\tilde{T}$

Definition 2. *A declarative definition of puzzle D defines constraints over the set of valid values $V(i, j)$ that puzzle squares can take.*

Definition 3. *The complexity function $C : \mathcal{P} \to H$ takes a puzzle board puzzleboard as input and maps it to a finite class of hardness levels denoted by H.*

Definition 4. *A transformation function $T : \mathcal{P} \to \mathcal{P}$ takes a puzzle board as input and transforms it another puzzle board such that the new puzzle board also satisfies the puzzle constraints. The set of all transformation functions are denoted by $\tilde{T}$.*

```
The General Puzzle Creation Algorithm
P_I = GetRandomPuzzle(P_D)

While isRemoveValid(P_c,i,j):
  P_c = P_c - {i,j}
  For T \in T_P:
    R = R U T(P_c)

For P \in R:
  D[F_c(P)] = P
```

The algorithm first uses an off-the-shelf constraint solver to solve the puzzle constraints P_D to get an initial random board configuration. It then starts removing the value at a square (i,j) until there are no more square values remaining to be removed. The algorithm uses the isValidSquare function to check if certain puzzle constraints hold after removing the square value (i,j). An example isRemoveOK function is that the Number of solutions to the puzzle after removing the square value is only 1. Let P_c denote the puzzle obtained after removing a valid square. We then apply a set of transformation functions T_P to get a set of new puzzles.

The difficulty of a generated puzzle is determined by characteristics of the unsolved board.

3 CaseStudies

3.1 Sudoku

After a Sudoku puzzle is generated, we determine its difficulty using machine learning. For each puzzle, a vector is generated with components describing the unsolved board. These characteristics are:

- Number of solutions
- Number of empty squares
- Number of rows with at least seven blank squares
- Number of columns with at least seven blank squares

- Number of 3x3 grids with at least seven blank squares
- Number of occurrences of each digit (9 components)
- Standard deviation of number of occurrences of each digit from the mean number of occurrences

The SVM library by scikit-learn then uses the vector to categorize the puzzle into one of four difficulties: (1) Easy, (2) Medium, (3) Hard, and (4) Evil.

3.1.1 Declarative Definition

We use the python frontend of z3 constraint solver to specify the 9X9 sudoku puzzle declaratively. As can be noticed from the encoding it can be easily generalized to other sudoku sizes such as 16X16 or 25X25.

We first define 81 different integer variables (X[0][0], X[0][1], , X[8][8]), where X[i][j] denotes the value of the sudoku cell (i,j). We also define the valid set of values each element can take, i.e. 1 ¡= X[i][j] ¡= 9 (valid values).

```
X = [[Int('x%d%d' % (i,j)) for i in range(9)] for j in range(9)]
valid_values = [And( X[i][j] >= 1, X[i][j] <= 9) for i in range(9)
for j in range(9)]
```

We now add sudoku constraints that the values in each row should be distinct (rowdistinct), values in each column should be distinct (colsdistinct), and that each 3X3 square should have distinct values (threebythreedistinct).

```
row_distinct = [Distinct(X[i]) for i in range(9)]
cols_distinct = [Distinct([X[i][j] for i in range(9)]) for j in
range(9)]
three_by_three_distinct = [ Distinct([X[3*k + i][3*l + j] for i in
range(3) for j in range(3)]) for k in range(3) for l in range(3)]
```

To encode partially filled sudoku board (where a 0 value denotes an empty space), we simply add the constraint X[i][j] == board[i][j] when board[i][j] != 0.

```
already_set = [X[i][j] == board[i][j] if board[i][j] != 0 for i in
range(9) for j in range(9)]
```

The complete set of constraints sudoku constraint is obtained by combining all previous constraints:

```
sudoku_constraint = valid_values + row_distinct + cols_distinct +
three_by_three_distinct + already_set
```

3.1.2 Creating the Initial Puzzle

To accomplish this, we use a set of equations using the python z3 constraint solver. This would generate a sudoku board such as the one below.

[[4, 9, 7, 1, 8, 2, 5, 3, 6] [1, 5, 2, 3, 6, 4, 8, 9, 7] [8, 6, 3, 5, 7, 9, 4, 1, 2], [7, 3, 4, 6, 9, 1, 2, 5, 8], [2, 8, 9, 4, 3, 5, 7, 6, 1], [5, 1, 6, 7, 2, 8, 9, 4, 3], [3, 2, 5, 9, 1, 7, 6, 8, 4], [9, 7, 1, 8, 4, 6, 3, 2, 5], [6, 4, 8, 2, 5, 3, 1, 7, 9]]

3.1.3 Emptying Squares

The next step is to start emptying this board. Our method for selecting the next square to remove included a number of substeps. First, we calculate the percentage of squares filled in each row (number of squares in row that are full/9). We then randomly select one of these percentages and generate a random number between 0 and 1. If this number is greater than the percentage, we find a new percentage and a new decimal between 0 and 1. Once we have a number between 0 and 1 that is less than the percentage calculated we keep this row. We then follow the same process to find which column in the row we should empty. We set the square in the selected row and column equal to zero, signifying emptiness.

After we have found a square to empty, we generate a temporary board with the selected square having a value of 0. We also a new set of constraints to go along with this board using z3. If z3 can solve the new set of equations in less than K ways, we have a desired result and we keep this board. If z3 can generate a number of solutions that is greater than or equal to K, the chosen square is added to a list of squares that do not work and should not be tried again. This temporary board is discarded and we find another square using the board we had before the temporary board was generated. Once we find a square that, when removed, results in a desired board, we find another square to empty. If no other squares work, meaning that the number of squares emptied plus the number of squares in the list of squares that do not work equals 81, we stop looking for another square to empty. If the algorithm does not stop in the first case, it stops once we reach the number of squares we wish to empty, although this second case is less likely if the desired number of empty squares is greater than 60.

3.1.4 Transformations

We are able to quickly generate more full Sudoku boards for emptying without using SAT solvers. To do this, we apply the following mathematically symmetrical transformations on an already-created Sudoku board:

1. Relabeling the nine digits
2. Permuting the three 3x9 stacks
3. Permuting the three 9x3 bands
4. Permuting the three rows within a stack
5. Permuting the three columns within a band
6. Reflecting about the axes of symmetry in a square
7. Rotation by 90 degrees

We chose a random transformation and applied it to an already existing board 1000 times to create a new Sudoku board satisfying the original constraints. Out of about 6 x 1021 total unique 9x9 Sudoku boards, these transformations can generate about 3 x 106 new unique Sudoku boards from an existing one [9].

We can apply most or all of these transformations to already-created 12x12, 15x15, 16x16, and 25x25 boards. On a 12x12 board, for instance, the following transformations are analogous:

1. Relabeling the twelve digits
2. Permuting the four 3x12 stacks
3. Permuting the three 12x4 bands
4. Permuting the three rows within a stack
5. Permuting the four columns within a band
6. Reflecting about the horizontal and vertical axes in a square

This method of quickly generating new boards generalizes to any higher-numbered Sudoku board.

The last step of our algorithm is quickly generating more sudoku puzzles from the board that was emptied. This task is accomplished by performing transformation on the existing board as described above.

3.2 Fillomino

Using machine learning, we determine the difficulty of an unsolved fillomino puzzle based on the following characteristics:

- Number of cells
- Number of empty squares
- Number of regions (connected areas of the same value)
- Optimality (whether the puzzle can be emptied further)

3.2.1 Declarative Definition

Using the python frontend on z3, we can declaratively create a Fillomino puzzle. First, we define an NxN board and assert that only the values between 1 and N can be on the board:

```
cells = [[Int("x%d%d" % (i,j)) for i in range(1,N+1)] for j in
                range(1,N+1)]

valid_cells = [And(cells[i][j] <= N, cells[i][j] >=1) for i in
range(N) for j in range(N)]
```

Now we must assert that, for each region on the board, the value of all of the squares in a specified region must be the same as the number of squares in that region.

To do this, we first create a variable edgeval that can either be one, representing that an outgoing edge to an adjacent square exists or 0, showing that the edge does not exist. An outgoing side should only exist between two cells that are in the same region. The variable for an edge is edgevar(i,j,k,l), where (i,j) is the location of the original square and (k,l) is the location of the adjacent square.

```
for i in range(N):
    for j in range(N):
        for (k,l) in getAdjacent1(i,j):
            edge_var[(i,j,k,l)] = Int("e%d%d%d%d" % (i,j,k,l))
edge_val_constraints = [Or(edge_val==0,edge_val==1) for edge_val in
edge_var.values()]
```

For our method to work, we only want one outgoing edge from every square, meaning that the value of edgevar will always be ¡= 0.

```
for i in range(N):
        for j in range(N):
            for (k,l) in getAdjacent1(i,j, N):
                if lessThan(i,j,k,l):
                    sum_edges = edge_var[(i,j,k,l)] + edge_var[(k,l,i,j
                    edge_val_constraints.append(sum_edges <= 1)
```

Now we create a new variable for each cell, incell, that is equal to the number of edges going into the cell.

```
in_cell = [[Int("in%d%d"%(i,j)) for i in range(N)] for j in range(N)]

    in_cell_constraints=[]
    for i in range(N):
        for j in range(N):
            in_cell_constraints.append(And(in_cell[i][j]>=0,
            in_cell[i][j]<=1))
            sum_incoming_edges = Sum([edge_var[(k,l,i,j)] for (k,l) in
            getAdjacent1(i,j, N)])
            in_cell_constraints.append(in_cell[i][j]==sum_incoming_edges
```

Next we say that if there is an edge between two cells, those two cells are in the same regions, and therefore, they should have the same value.

```
same_value_constraint = []
    for i in range(N):
        for j in range(N):
            for (k,l) in getAdjacent1(i,j, N):
                if lessThan(i,j,k,l):
                    same_value_constraint.append(Implies(Or(edge_var[(i
                    edge_var[(k,l,i,j)]==1), cells[i][j] ==
                    cells[k][l]))
```

Finally, we define a cells size as being the sum of the sizes of the adjacent cells + 1. We want there to be exactly one cell whose size is the same as its value.

```
size_region_constraint=[]
size_cell = [[ Int("size%d%d" % (i,j)) for i in range(N)] for j in
```

```
                                range(N)]
size_region_constraint=[And(size_cell[i][j] >=1,
size_cell[i][j]<=N) for i in range(N) for j in range(N)]


for i in range(N):
    for j in range(N):
        size_adjacent = Sum([If(edge_var[(i,j,k,l)]==1,
        size_cell[k][l],0) for (k,l) in getAdjacent1(i,j, N)])
        size_region_constraint.append(size_cell[i][j] ==
        (size_adjacent+1))
        size_region_constraint.append(Implies(in_cell[i][j]==0,
        size_cell[i][j] == cells[i][j]))
```

3.2.2 Creating the Initial Puzzle

We have two option to create the initial puzzle. We can run the declarative definition of a Fillomino puzzle, but this code is slow and will only get as puzzle because z3 does not come up with a different solution each time the code is run, it is the same one every time. Our second option is to randomly create a completed Fillomino puzzle using python without any z3. The python code works by selecting a starting square, randomly selecting a sequence length from a list of valid sequence lengths, and then setting a list of squares that are sequence length long that all have the value sequence length. We continue like this until the board is completely filled.

3.2.3 Choosing a Valid Square

Choosing a Valid Square Our first step is define what we mean by a valid square. For most cases, we want to find a square that, when removed, results in a board that has exactly one solution, but in general, we want it to have less than K solutions. Our first step is to randomly choose a square on the board, with the only restriction being that every region must have at least one cell that is not emptied, as in the rules of Fillomino.

In Sudoku, a square in a row, column, or three by three grid that had less emptied squares in it already was more likely to be chosen. This is not the case in Fillomino because the number of squares in a row or column do not

matter, the only constraint is that there must be at least one value in every region that is not emptied.

As in Sudoku, we create a temporary board that has the selected square removed (the square has a value of 0). We then apply the z3 declarative definition on this temporary board, and if z3 can find K or more ways to solve the puzzle, we know that we should not remove the square. We resort back to the old board, add the square that we tried to a list of squares that we know do not work, and try again with a new random square. This process continues until we have a desired number of squares emptied.

3.2.4 Transformations

Because of the randomness of the different regions, there are less transformations that can be applied to Fillomino than Sudoku, but a number still exist. These are:

1. Rotation
2. Vertical reflection
3. Horizontal reflection

Applying these transformation to the original z3 declarative definition puzzles as well as the randomly generating python generated puzzles result in a fast way to create many more boards. The transformation can also be applied to an emptied board to get more solutions.

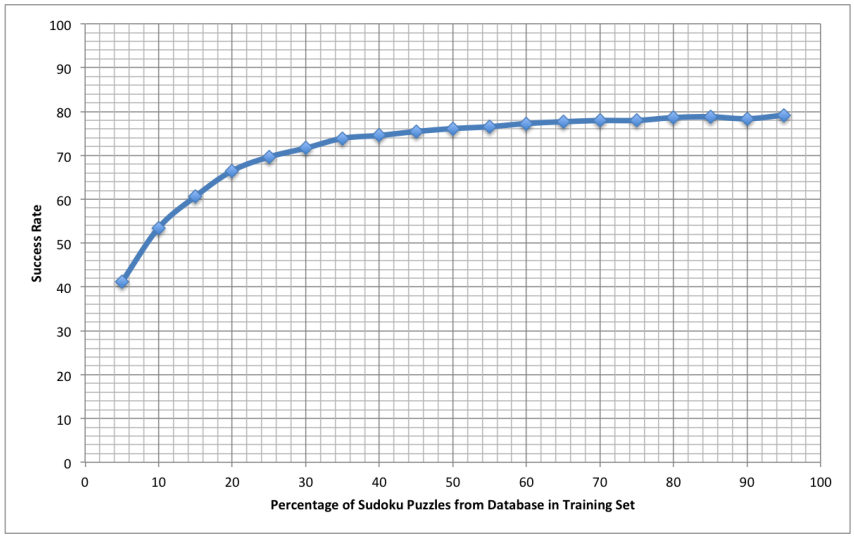
4 Experiments

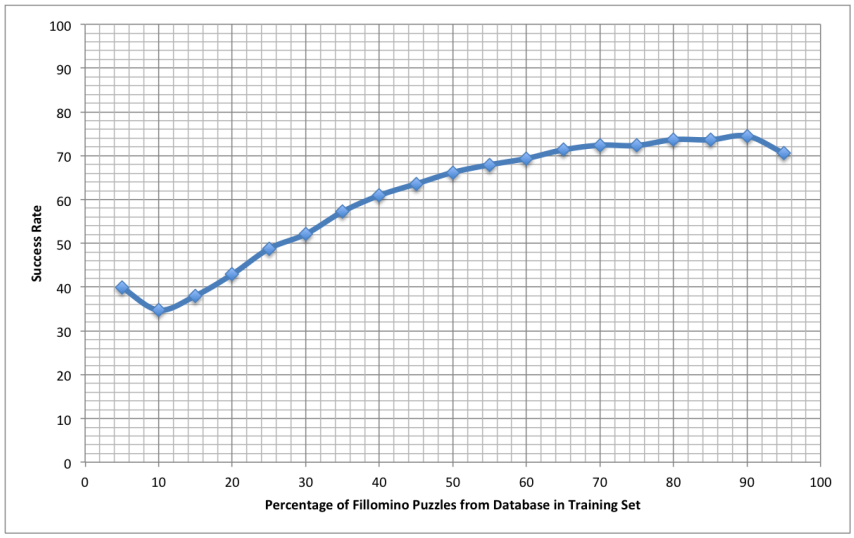
4.1 Machine Learning

Our method of determining puzzle difficulty was to use machine learning to categorize a puzzle's characterizing vector. To test the reliability of this approach, we recorded published puzzles and their respective difficulty levels from online puzzle providers. This database of puzzles was randomly divided into two sets: a training set and a testing set. The training set was used to generate the SVM's categorization function, and the testing set was used to generate the "success rate": the percentage of puzzles in the testing

set whose SVM-determined difficulty matched the difficulty assigned by the puzzle providers.

For our sudoku database, we recorded 206 puzzles from Web Sudoku, the largest online sudoku puzzle provider. For our fillomino database, we recorded 40 puzzles from Math In English, a puzzle website that had categorized puzzles by difficulty levels similar to those of Web Sudoku: (1) Easy, (2) Moderate, (3) Challenging, and (4) Super Difficult. Below are graphs of the average success rates of 500 trials as the percentage of puzzles in the training set was increased.





Our results show that as we increased the number of puzzles in the training set, the success rate increased to 80%.

4.2 Scalability

To test the scalability of our puzzle generation algorithm, we generated 16x16 and 25x25 sudoku boards to compare with the standard 9x9 boards and all square fillomino boards from 2x2 to 16x16.

When we empty a puzzle with dimensions NxN, we set as a parameter the target number of cells that the program would aim to empty from an initially full board. When the program reaches a point where it cannot further empty any more cells (i.e. the emptying of any remaining cell would cause the board to have more than the maximum number of allowable solutions), the emptying process will be considered finished. Because of this, we observe stagnation in our run times as the target number of empty cells is increased beyond a certain threshold.

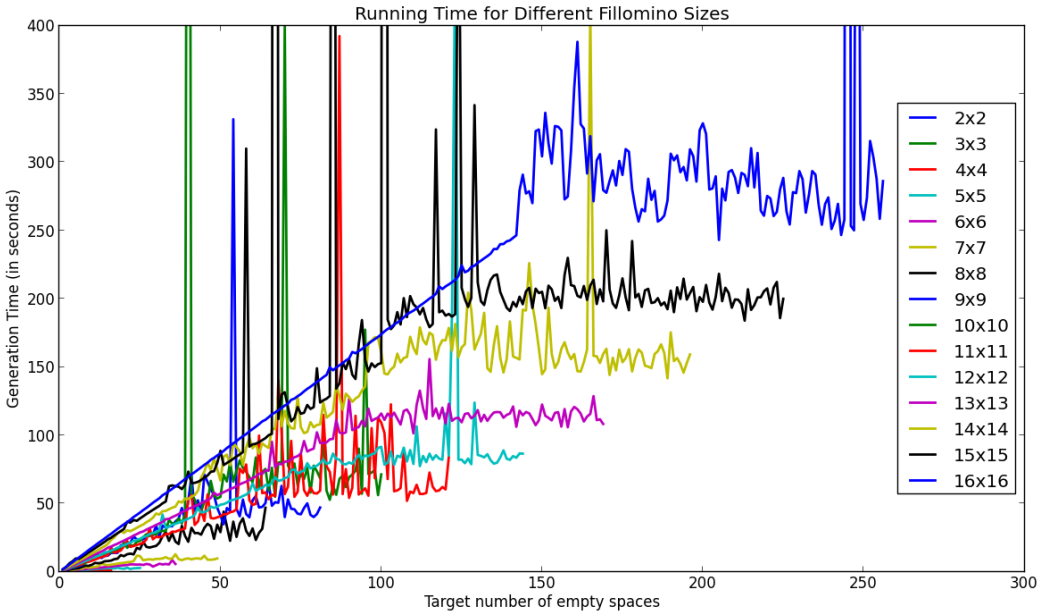
The graph below shows program running times to create and empty 9x9, 16x16, and 25x25 Sudoku puzzles as the target number of empty squares is increased. There is an exponential increase in run time as the number of possible empty spaces increased. Even for 25x25 sudoku puzzles, we find that the time required to generate a sudoku puzzle is quite reasonable (around

500 seconds).

[insert Sudoku Running Time Graph here]

We can see in the graph that the generation time for 9x9 puzzles does not continue to increase after the target number of empty cells was raised beyond 60 because the emptying had stopped before the target of 60+ empty cells had been reached.

A similar experiment with Fillomino puzzles demonstrates a similar pattern of stagnation after a threshold. Unlike the run times of Sudoku puzzles, however, the run times for fillomino puzzles follow a linear trend, not an exponential trend, as the target number of empty spaces is increased.



5 Related Work

With the advent of recent online education initiatives, we are seeing an ever increasing number of students enrolling in online classrooms. Some of the popular courses such as Introduction to Programming and Introduction to Machine Learning routinely reports more than 100,000 student enrollment. This large scale of students has forced us to develop new automated technologies to solve problems such as automated feedback generation [4] and solution generation[5]. Another important problem that comes up because of this scale is that of automated problem generation to cater to the practice needs of different students as well as for providing different exams to students of a given difficulty level.

There has been some previous work on generating new problems in various domains namely algebra and programming. The work on generating new algebra problem has mostly been looked upon using two main approaches. In the first approach, a teacher is provided a certain set of parameter values that are fixed for a given domain [2]. For example, for generating a quadratic equation, the parameters can be the number of roots, difficulty of factorization, whether there is an imaginary root, the range of coefficient values etc. Given a set of feature valuations, the tool generates the corresponding quadratic equations. The second approach takes a particular proof problem, and tries to learn a problem template from the problem which is then instantiated with different concrete values [3]. The system first tries to learn a general query from a given proof problem, which is then executed to generate a set of proof problems. Since the query is only a syntactic generalization of the original problem, only a subset of them are valid problems, which are identified using polynomial identity testing. Our approach, however, creates different versions of the same problem by introducing different number of holes in the original problem based on a parametric complexity function.

More recently, a technique was proposed to generate fill-in-the-blank Java problems where certain keywords, variables and control symbols are removed randomly from a correct solution [1]. The technique blanks variables using the condition that at least one occurrence of each variable remains in the scope and blanks control symbols such that at least one occurrence of a paired symbol (such as brackets) remains. Our technique, on the other hand, is

more general since it is constrained-based and can check for more interesting constraints such as unique solutions and an arbitrary complexity function that it takes as an additional parameter.

6 Conclusion