Carleton University
Department of Systems and Computer Engineering
SYSC 4405 Digital Signal Processing
Fall semester 2014
in-class midterm Exam 1
Date: October 15

Question 1 [10 points]
Consider the analog signal
$x_a(t) = 3\cos(2000\pi t) + 5\sin(6000\pi t) + 10\cos(12000\pi t)$
a) What is the Nyquist rate for this signal?
b) Assume now that we sample this signal using a sampling rate $F_s = 5000$ samples/s. What is the discrete time signal obtained after sampling? What is the period of this discrete time signal?
c) what is the analog signal $y_a(t)$ that we can reconstruct from the samples if we use ideal interpolation?

<u>Solution</u>
a) $f_{Nyquist} = 2 \times max\{1000, 3000, 6000\} = 12000$ samples/s

b) $x(n) = 3\cos\left(2000\pi\frac{n}{5000}\right) + 5\sin\left(6000\pi\frac{n}{5000}\right) + 10\cos\left(12000\pi\frac{n}{5000}\right) =$

$$3\cos\left(\frac{2\pi}{5}n\right) + 5\sin\left(\frac{6\pi}{5}n\right) + 10\cos\left(\frac{12\pi}{5}n\right) = 3\cos\left(\frac{2\pi}{5}n\right) - 5\sin\left(\frac{4\pi}{5}n\right) + 10\cos\left(\frac{2\pi}{5}n\right)$$

The discrete time signal obtained has a period 5 because $x(n+5) = x(n)$ for $-\infty < n < \infty$

c) I deal interpolation produces a continuous time signal with frequencies
$f_1, f_2, f_3 = \frac{1}{5} \times 5000, \frac{2}{5} \times 5000, \frac{1}{5} \times 5000 = 1000, 2000, 1000$ Hz. In other words
$y_a(t) = 3\cos(2000\pi t) - 5\sin(4000\pi t) + 10\cos(2000\pi t)$

Question 2 [10 points]
a) consider the discrete time signal $x(n) = \{2, 3, \underline{4}, 5, 6\}$ where the "_" indicates the $n = 0$ position in the sequence. You are asked to express this signal as a sum of an even and an odd signals in the form $x(n) = x_e(n) + x_o(n)$. Give clearly annotated sketches showing each of $x_e(n)$ and $x_o(n)$.

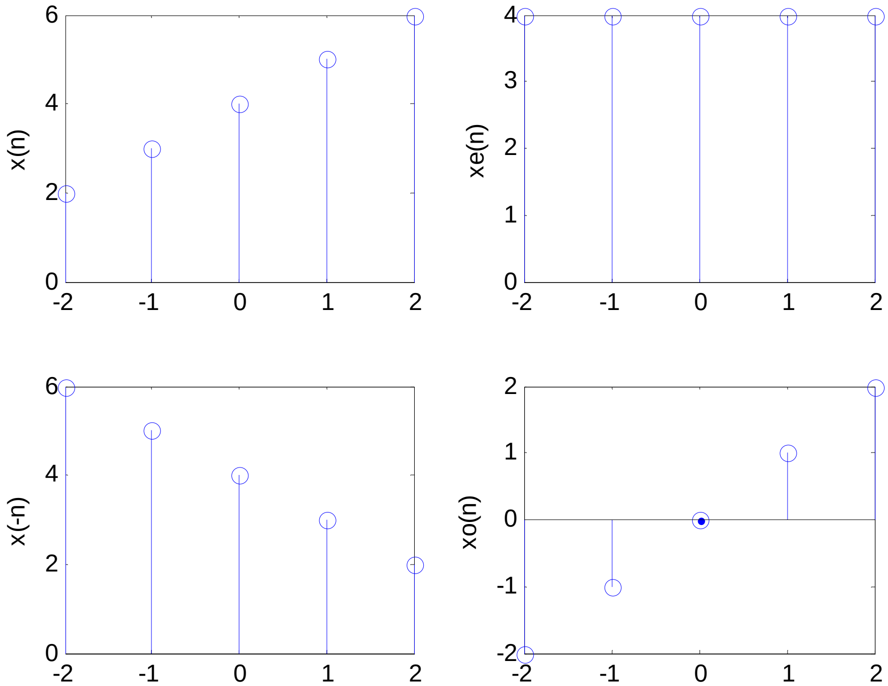
b) A discrete time signal is described by $x(n) = \cos\left(\frac{\pi n}{3}\right)$ for $0 \le n \le 11$, and $x(n) = 0$ for $n \ge 12$. You are asked to give a stem plot of $x(n)$. also, give an equation to express $x(n)$ in terms of the unit step sequence $u(n)$

solution

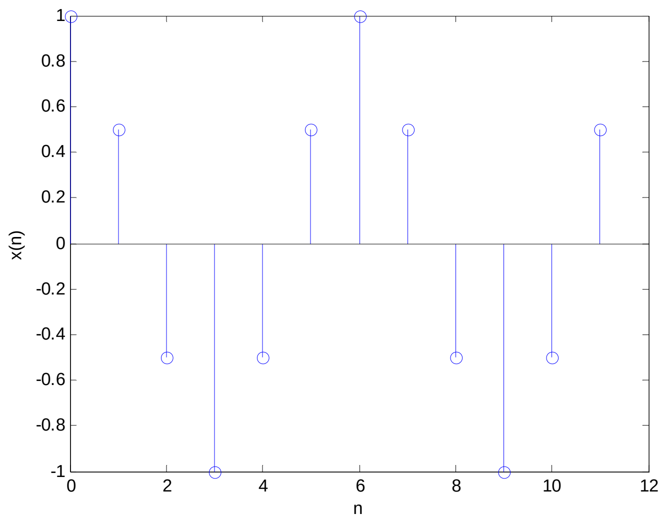
a) we know that $x_e(n) = \frac{1}{2}[x(n) + x(-n)]$ and $x_o(n) = \frac{1}{2}[x(n) - x(-n)]$

Therefore $x_e(n) = \frac{1}{2}[\{2, 3, \underline{4}, 5, 6\} + \{6, 5, \underline{4}, 3, 2\}] = \{4, 4, \underline{4}, 4, 4\}$

and $x_o(n) = \frac{1}{2}[\{2, 3, \underline{4}, 5, 6\} - \{6, 5, \underline{4}, 3, 2\}] = \{-2, -1, \underline{0}, 1, 2\}$



b) Here is a stem plot showing x vs n



$$x(n) = \cos\left(\frac{\pi n}{3}\right)[u(n) - u(n-12)]$$

Question 3 [10 points]

a) The discrete time system

$$y(n) = ny(n-1) + x(n)$$

is at rest [i.e. $y(-1) = 0$]. Check if the system is linear, time invariant and BIBO stable.

b) a first order discrete time system has the impulse response $h(n) = (0.9)^n u(n)$ where $u(n)$ is the unit step sequence. Comment on whether this is an IIR or FIR system and explain why. Also, determine an analytical expression of the system response to a unit step sequence, assuming that the system was initially at rest [i.e. $Y(-1) = 0$].

solution

a) Linearity:

find the system response to an input $x_1(n)$. Since the system is initially at rest, it can be seen that

$y_1(0) = x_1(0)$, $y_1(1) = y_1(0) + x_1(1) = x_1(0) + x_1(1)$,

$y_1(2) = 2y_1(1) + x_1(2) = 2[x_1(0) + x_1(1)] + x_1(2)$

$y_1(3) = 3y_1(2) + x_1(3) = 6[x_1(0) + x_1(1)] + 3x_1(2) + x_1(3)$

similarly, the response to $x_2(n)$ can be developed as

$y_2(0) = x_2(0)$, $y_2(1) = y_2(0) + x_2(1) = x_2(0) + x_2(1)$,

$y_2(2) = 2y_2(1) + x_2(2) = 2[x_2(0) + x_2(1)] + x_2(2)$

$y_2(3) = 3y_2(2) + x_2(3) = 6[x_2(0) + x_2(1)] + 3x_2(2) + x_2(3)$

find the system response to an input $\alpha x_1 + \beta x_2$. Since the system is initially at rest, it can be seen that $y(0) = \alpha x_1(0) + \beta x_2(0) = \alpha y_1(0) + \beta y_2(0)$,
$y(1) = y(0) + \alpha x_1(1) + \beta x_2(1) = \alpha x_1(0) + \beta x_2(0) + \alpha x_1(1) + \beta x_2(1) = \alpha y_1(1) + \beta y_2(1)$,
$y(2) = 2y(1) + \alpha x_1(2) + \beta x_2(2) = \alpha y_1(1) + \beta y_2(1) + \alpha x_1(2) + \beta x_2(2) = \alpha y_1(2) + \beta y_2(2)$

Based on above, the system is linear because $y(n) = \alpha y_1(n) + \beta y_2(n)$

stability:
the system is not stable because its output grows indefinitely, even for simple input sequences. for example, an input $\delta(n)$ gives an output $[1, 1, 2, 2 \times 3, 2 \times 3 \times 4, 2 \times 3 \times 4 \times 5, \ , \ ]$ which is an unbounded response to a bounded input. The system is not BIBO stable.

time invariance
The system is not time invariant in general terms due to the dependence of its coefficients on time.

b) The impulse has an infinite duration because it follows a real exponential function. Therefore the system is IIR
The system response to a unit step sequence is given by

$$y(n) = x(n) \otimes h(n) = \sum_{k=0}^{n} (0.9)^k = \frac{1-(0.9)^{n+1}}{1-0.9}$$

Question 4 [12 points]
Consider a system described by the difference equation

$$y(n) + \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n)$$

and the system initial conditions are $y(-1) = y(-2) = \alpha$
a) Determine the initial condition response of this system in terms of the constant α
b) Give a block diagram realization of this system, using the smallest number of unit delays
c) If the input was given by $x(n) = \cos\left(\frac{\pi n}{16}\right)u(n)$, give only the steps necessary to find the particular solution of this system

Solution
a) The system's characteristic roots are defined by $\lambda^2 + \frac{3}{4}\lambda + \frac{1}{8} = 0 = \left(\lambda + \frac{1}{4}\right)\left(\lambda + \frac{1}{2}\right)$

$\lambda_1, \lambda_2 = -\frac{1}{4}, -\frac{1}{2}$ and the initial condition response is of the form

$y(n) = c_1\left(-\frac{1}{4}\right)^n + c_2\left(-\frac{1}{2}\right)^n$ where

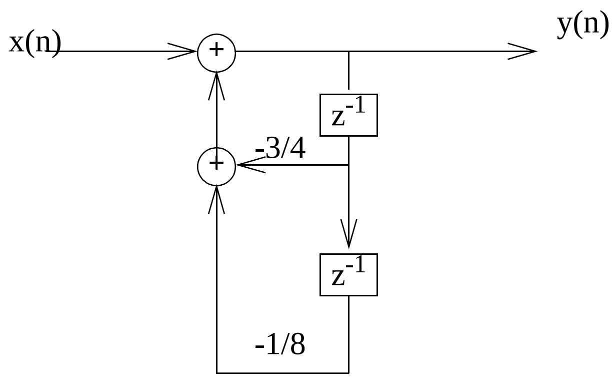
$y(0) = -\frac{7}{8}\alpha = c_1 + c_2$

and $y(1) = -\frac{3}{4}y(0) - \frac{1}{8}y(-1) = \frac{17}{32}\alpha = c_1\left(-\frac{1}{4}\right) + c_2\left(-\frac{1}{2}\right) = -\frac{c_1 + 2c_2}{4}$ i.e.

$c_1 + 2c_2 = -\frac{17}{8}\alpha$

solve to find $c_2 = -\frac{5}{4}\alpha$ and $c_1 = \frac{3}{8}\alpha$

b)



c) The general form of the particular solution is

$y(n) = c_3\cos\left(\frac{\pi n}{16}\right)u(n) + c_4\sin\left(\frac{\pi n}{16}\right)u(n)$

where c_3, c_4 are unknown constants.

To find the constants 1) substitute the general form of the solution in the difference equation

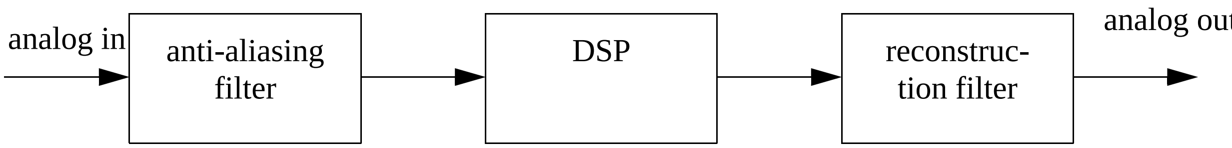
2) simplify the difference equation for $n \geq 2$

3) solve to find the two constants c_3, c_4

Question 5 [8 points]

a) Consider a DSP system intended for audio processing as shown below. The system uses a sampling frequency 44 k sample s/s.

Comment on each of the anti aliasing and the reconstruction filters indicating 1) the function the filters serve, 2) The implementation of the filter (analog or digital), 3) The type of response (low pass, high pass or band pass), and 4) The cut off frequencies (in kHz) of each filter.



b) Consider a discrete time system described by the difference equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

Give a brief, and clear explanation how you can test such a system for stability.

solution

a) The anti aliasing filter is an analog component, needed to suppress signal components with frequencies higher than 22 kHz, It has a low pass frequency response with a cutoff frequency less than 22 kHz.

The reconstruction filter is needed to smooth the output of the D/A converter, it is an analog component with a low pass frequency response, and a cutoff frequency equal to the bandwidth of the desired output signal.

b) to test the system for stability one may 1) find the characteristic roots of the system and compare their absolute values to unity. The system is stable if all roots have a magnitude less than unity. 2) Or one may determine the impulse response of the system. The system is stable if the impulse response is absolutely summable.

Question 2

Consider the analog signal $x_a(t) = \cos(500\pi t) + 3\sin(800\pi t)$.

a) Determine the Nyquist sampling rate for $x_a(t)$.

b) A discrete-time signal $x(n)$ is obtained by sampling the analog signal $x_a(t)$ at the rate $F_s = 600$ *samples per second*. Write an expression for $x(n)$ and find its period. What are the cyclic frequencies contained in $x(n)$.

c) Give an expression for an analog signal $z_a(t)$ that would lead to a sequence $z(n)$, when sampled at the rate $F_s = 600$ *samples per second*, which is identical to $x(n)$ of part b) above.

a) The Nyquist sampling rate is $f_N = 2 \times max\{250, 400\} = 800$ sample/sec

$$\text{b)}x(n) = \cos\left(500\pi\frac{n}{600}\right) + 3\sin\left(800\pi\frac{n}{600}\right) = \cos\left(2\pi \times \frac{5}{12}n\right) + 3\sin\left(2\pi \times \frac{4}{6}n\right) =$$

$$\cos\left(2\pi \times \frac{5}{12}n\right) + 3\sin\left(2\pi \times \frac{-2}{6}n\right) = \cos\left(2\pi \times \frac{5}{12}n\right) - 3\sin\left(2\pi \times \frac{2}{6}n\right)$$

with a period $N = 12$ and cyclic frequencies $\frac{5}{12}, \frac{2}{6}$ cycles/sample

c) A possible signal is $z_a(t) = \cos(500\pi t) - 3\sin(400\pi t)$

Question 3

a) The discrete time system

$$y(n) = ay(n-1) + bx(n)$$

is at rest [i.e. $y(-1) = 0$]. What are the conditions (on a and b) that are necessary for this system to be time invariant, linear and BIBO stable. Explain the reasons for your answers.

Time invariance: requires that a and b be constants. It places no restriction on their actual values.

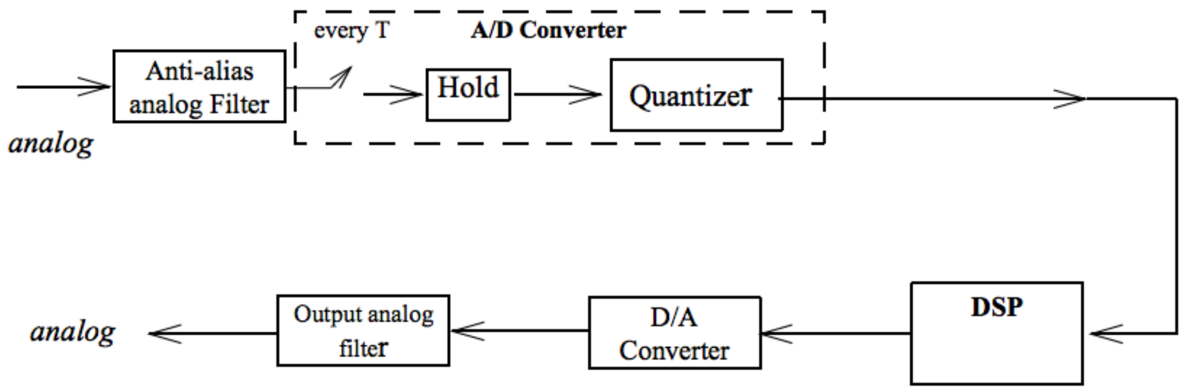
c) Determine an analytical expression for the system response when the input is

$$x(n) = \left(\frac{1}{2}\right)^n u(n)$$

and the initial conditions are $y(-1) = y(-2) = 0$

BIBO stability: The system is a 1st order recursive system, with one c/c root at $\lambda = a$. For stability we require that $|a| < 1$; no restrictions are imposed on b.
Linearity: For this system to be linear, a and b must not depend on x(n) or y(n). Their values are not constrained to be real or complex.

b) A simplified block diagram of a DSP system is shown below. The input and output signals of the system are both analog.



Write a brief statement explaining the function performed by each of the building blocks shown on the diagram.

Question 4

A 1st order LTI system has an impulse response $h(n) = \alpha^n u(n)$ where $0 < \alpha < 1$. The system is initially relaxed. An input of the form $x(n) = u(n) - u(n-N)$ is applied to the system.
a) Explain why this is a causal, stable IIR system.
b) Find an expression for the system output as a function of n and α.
a) causal because $h(n) = 0$ for $n < 0$

stable because $\sum\limits_{n=0} |h(n)| = \dfrac{1}{1-|\alpha|} = finite\ \ number$

It is an IIR system because the impulse response has a duration $0 \le n < \infty$ which is infinite
b) The system's response to a unit step input is:

$$y_1(n) = conv(u(n), h(n)) = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}u(n)$$

The system is linear, and time invariant. Therefore, its response to $x(n) = u(n) - u(n-N)$ can be written as

$$y(n) = y_1(n) - y_1(n-N) = \frac{1-\alpha^{n+1}}{1-\alpha}u(n) - \frac{1-\alpha^{n+1-N}}{1-\alpha}u(n-N)$$

c) solution II

$$\lambda = \frac{1 \pm \sqrt{1-2}}{2} = \frac{1 \pm j}{2} = \frac{1}{\sqrt{2}} e^{\pm j\frac{\pi}{4}}$$

find the particular solution assuming an input $x(n) = a$

$y_p(n) = Ca^n u(n)$

substitute in the difference equation:

$Ca^n u(n) = Ca^{n-1} u(n-1) - 0.5Ca^{n-2} u(n-2) + 0.5$

solve for c, when $n \geq 2$

$C(a^2 - a + 0.5) = 0.5a^2$

substitute $a = \frac{1}{2}$

$$C = \frac{0.5a^2}{a^2 - a + 0.5} = \frac{1}{2}$$

$y_p(n) = \frac{1}{2} a^n u(n)$

Express the homogeneous solution in a general form

The impulse response can be written as:

$h(n) = c_1(\lambda_1)^n + c_2(\lambda_2)^n$

To find the constants c_1, c_2

$h(0) = -h(-1) - \frac{6}{25}h(-2) + \delta(0) - \delta(-1) = 1 = c_1 + c_2$

$h(1) = -h(0) - \frac{6}{25}h(-1) + \delta(1) - \delta(0) = -2 = -\frac{2}{5}c_1 - \frac{3}{5}c_2$

$c_1, c_2 = -7, 8$

Therefore, the impulse response is given by:

$$h(n) = \left[-7\left(-\frac{2}{5}\right)^n + 8\left(-\frac{3}{5}\right)^n\right] u(n)$$

$y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n$

$y(n) = C_1\lambda_1^n + C_2\lambda_2^n + \frac{1}{2}a^n u(n)$

Determine C_1, C_2 using the given initial conditions of the system

$y(0) = C_1 + C_2 + \frac{1}{2}$---------from the analytical solution

$y(1) = C_1\lambda_1 + C_2\lambda_2 + \frac{1}{2}a$--------from the analytical solution

$y(0) = 0 - 0 + 0.5 = 0.5$--------from the difference equation

$y(1) = 0.5 - 0 + 0.5a = 0.5(1 + a)$---------from the difference equation

Therefore

$(c_1 + c_2 + 0.5 = 0.5) \Rightarrow c_2 = -c_1$

$(0.75 = c_1\lambda_1 + c_2\lambda_2 + 0.25) \Rightarrow c_1\lambda_1 + c_2\lambda_2 = 0.5$

$(c_1(\lambda_1 - \lambda_2) = 0.5) \Rightarrow c_1 = \frac{0.5}{\lambda_1 - \lambda_2}$ & $c_2 = -\frac{0.5}{\lambda_1 - \lambda_2}$

$$c_1 = \frac{0.5}{\frac{1}{\sqrt{2}}\left(e^{j\frac{\pi}{4}} - e^{-j\frac{\pi}{4}}\right)} = \frac{0.5}{j\sqrt{2}\sin\left(\frac{\pi}{4}\right)} = -0.5j = -c_2$$

Now substitute for c_1, c_2 in the general form of $y(n)$

$$y(n) = (-0.5j)\left(\frac{1}{\sqrt{2}}\right)^n\left(e^{j\frac{\pi}{4}}\right)^n + (0.5j)\left(\frac{1}{\sqrt{2}}\right)^n\left(e^{-j\frac{\pi}{4}}\right)^n + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^n$$

$$y(n) = (-0.5j) \times \left(\frac{1}{\sqrt{2}}\right)^n \times \left(j2\sin\frac{\pi n}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^n = \left(\frac{1}{\sqrt{2}}\right)^n\left(\sin\frac{\pi n}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^n$$