TTK4250 Sensor Fusion

Solution to Assignment 5

Task 1: *Number of association events in JPDA*

Take the data association example in figure 8.1. We are wondering how many data association events there are here. There are 3 targets and 4 measurements, where no target gates all the measurements.

(a) If we disregard the validation gates, there is a formula for calculating the number of association events in JPDA. If we have N targets that is going to be associated to m out of a total of M measurements ($m \leq \min(N, M)$), it becomes the number of ways to pick m out of N targets unordered and m out of M measurements undordered before assigning these m targets to the m measurements. This number is given by $\binom{N}{m}\binom{M}{m}m! = \frac{N!M!}{m!(N-m)!(M-m)!}$. However, m can be any number between 0 and $\min(N, M)$ so we have to sum over this range to find the total number of association events:

$$\sum_{m=0}^{\min(N,M)} \frac{N!M!}{m!(N-m)!(M-m)!} \tag{1}$$

What is this number for the example in figure 8.1?

> **Solution:** Inserting $N = 3$ and $M = 4$ gives 73.

(b) We now include the validation gates. There is no simple formula for the number of events in this case. The simplest option is to enumerate the possibilities and count them. What is the number of association events in figure 8.1 considering the validation gates? How many percent of the non gated events is this? Would you say gating is helpful for data association?

> **Solution:** Enumeration gives
>
> $$(t_1, t_2, t_3) = \begin{cases} \{(0,0,0), (0,0,2), (0,1,0), (0,1,2), (0,2,0), (0,4,0), (0,4,1)\}, \\ \{(1,0,0), (1,0,2), (1,2,0), (1,4,0), (1,4,2)\}, \\ \{(3,0,0), (3,0,2), (3,1,0), (3,1,2), (3,2,0), (3,4,0), (3,4,2)\}, \end{cases} \tag{2}$$
>
> which counts 19. This is $\frac{19}{73} \approx 26\%$ of the non gated events, which is a significant decrease.

(c) How many of the events you counted does not have any misdetections?

> **Solution:** Removing any association events that has 0 in the above, we are left with 3: $(t_1, t_2, t_3) = \{(1, 4, 2), (3, 1, 2), (3, 4, 2)\}$.

(d) If the unassociated measurements could also be new targets, there are 2 possible events for each of these. When m out of M measurements are known to be associated to already established targets, it is therefore also 2^{M-m} events for new targets or false alarms in addition to the possible target to measurement associations. The total number of events when not considering validation gates then becomes

$$\sum_{m=0}^{\min(N,M)} \frac{2^{M-m} N! M!}{m!(N-m)!(M-m)!} \tag{3}$$

What is this number for the example? Is there a lot to save by not having to consider these hypotheses (do not consider the validation gate)?

Solution: Inserting $N = 3$ and $M = 4$ we get 304. Having to consider new tracks gives a factor of $\frac{304}{73} \approx 4.16$ increase in the amount hypotheses needed to consider. This can quickly go out of hand when we have many measurements and/or targets.

Task 2: *Finding the most probable associations in JPDA*

You are here to investigate the auction algorithm (Algorithm 5) and Murty's method (Algorithm 6) on the association problem in figure 8.1. A reward matrix (log likelihood ratio matrix) for this association problem is given in equation (8.21). You can use $\epsilon = 0.01$ for this task.

(a) Go through the auction algortihm (by hand) for the problem to show that the most likely association is $a(1) = 3$, $a(2) = 1$ and $a(3) = 2$, where $a(t) = j$ indicates that target t is associated to measurement j.

Solution: We begin with the unassigned queue set to $[1, 2, 3]$, and call it UQ for short. The items (measurements) are not assigned, which we indicate by the variable $\text{IA} = [0, 0, 0, 0]$.

First costumer

1. $t^* = 1$, UQ $= [2, 3]$
2. $i^* = 3$ corresponding to the reward minus price $4.78 - 0$
3. IA = [0, 0, 1, 0, 0, 0, 0]
4. $y = 4.78 - (-0.46 - 0) = 5.24$.
5. Prices = [0, 0, 5.24 + 0.01, 0, 0, 0, 0]

Next costumer

1. $t^* = 2$, UQ $= [3]$
2. $i^* = 1$, corresponding to the reward minus price $5.37 - 0$.
3. IA = [2, 0, 1, 0, 0, 0, 0]
4. $y = 5.37 - (5.36 - 0) = 0.01$
5. Prices = [0.01 + 0.01, 0, 5.25, 0, 0, 0, 0]

Next costumer

1. $t^* = 3$, UQ $= []$
2. $i^* = 2$, corresponding to the reward minus price $6.58 - 0$
3. IA = [2, 3, 1, 0, 0, 0, 0]
4. $y = 6.58 - (-0.6 - 0) = 7.18$

5. Prices = [0.02, 7.58 + 0.01, 5.25, 0, 0, 0, 0]

UQ = [] and we are done with IA = [2, 3, 1, 0, 0, 0, 0] corresponding to the given assignment. The reward is given by $4.78 + 5.37 + 6.58 = 16.73$.

(b) The last part basically solved lines 2-4 of Murty's method. Go through lines 5-19 of Murty's method to find the second best association. State the solution and reward for each solution. You do not need to go through more than one more run of the auction algorithm as you should be able to see the solutions straight away in this problem.

As there are some minor flaws i Murtys algorithm in the book, we restate how to perform it here.

1. Solve the initial problem and store the problem and results in a list, L.
2. Find the solution with highest reward in L and save it as the next best solution. Call the corresponding problem P and remove it from L.
3. Partition P into two subproblems:
   (a) one with the first track's measurement assignment in P made impossible (corresponding entry in reward matrix set to $-\infty$). Solve this problem and put it in L if it gives a valid solution (eg. finite reward).
   (b) one with the first track's measurement assignment in P enforced (corresponding to removing the track and measurement from the problem P (column and row in the reward matrix, respectively)). Set P to be this problem.
4. Repeat the partitioning process (step 3) untill it is not possible to partition more. This happens when all tracks have their assignments enforced.
5. Go back to finding the next best solution (step 2) in L until L is empty or N solutions has been found.

In our case, the first partition will make the first subproblem by setting $R(3,1) = -\infty$ and solve that problem, and the second subproblem by removing column 1 and row 3 from R, making R_1. The second partition will then take this second subproblem with reduced reward matrix size and set $R_1(1,1) = -\infty$ corresponding to setting $R(1,2) = -\infty$ in the original reward matrix, for creating the first subproblem of this problem partitioning. The second subproblem of this partitioning is then created by removing row 1 and column 1 in R_1 to create R_2, wich is the removal of column 1 and 2 and rows 1 and 3 in the original R.

Solution: Our problem solution pair sets only contains the original problem and the optimal solution, so that is selected as our solution to proceed with in line 6. We skip the book keeping and termination part of lines 7-11. We then proceed by going through the tracks and setting the reward of the measurement it is associated to in the selected solution to $-\inf$ in the reward matrix of the selected problem (note that we have picked a solution-problem pair), and solve that using the auction method.

Track 1 is associated to measurement 3, so we first set $R(3,1) = -\inf$ in our working copy of the reward matrix. The auction algorithm is as before

First costumer

1. $t^* = 1$, UQ = $[2,3]$
2. $i^* = 5$ corresponding to the reward minus price $-0.46 - 0$
3. IA = [0, 0, 0, 0, 1, 0, 0]

4. $y = -0.46 - (-5.69 - 0) = 5.23$.
5. Prices = [0, 0, 0, 0, 5.23 + 0.01, 0, 0]

The rest is as for the last run of the auction method. The association is IA = [2, 3, 0, 0, 1, 0, 0], with reward $-0.46 + 5.37 + 6.58 = 11.49$.

Track 2 is associated to measurement 1, so the second problem is found by setting $R(1, 2) = -\inf$ in our working copy of the selected problems reward matrix. This gives the association is IA = [0, 3, 1, 2, 0, 0, 0], with reward $4.78 + 5.36 + 6.58 = 16.72$.

Track 3 is associated to measurement 2, so the third problem is found by setting the $R(2, 3) = -\inf$ in our working copy of the selected problems reward matrix. This gives the association IA = [2, 0, 1, 0, 0, 0, 3], with reward $4.78 + 5.36 + -0.6 = 9.55$.

Going back to line 6, the solution to the second problem gives the best reward and is hence the second best association.

(c) Say that the best solution to the last problem was found to come from the solution of making the association $a(2) = 1$ impossible. What would the reward matrix look like for the selected problem-solution pair in line 6?

Solution: This will be the original reward matrix with $R(1, 2) = -\inf$ (the first row in the second column set to negative infinity).

Task 3: *Pure quaternion powers and exponential*

For simplicity, we let the quaternion $\begin{bmatrix}\eta & \epsilon^T\end{bmatrix}^T$ be equivalently written as $\eta + \epsilon$ in this task. Here ϵ is of course the imaginary (hyper imaginary, if you prefer) part of the qaternion and η is the real part. All the rules for complex numbers can be shown to hold for quaternions, except commutativity of the imaginary parts in the product of two quaternions. For instance, we have that the quaterion product can be written as $(\eta_1 + \epsilon_1)(\eta_2 + \epsilon_2) = \eta_1\eta_2 + \eta_1\epsilon_2 + \eta_2\epsilon_1 + \epsilon_1\epsilon_2$ using the current notation. The imaginary product, $\epsilon_1\epsilon_2$, written out in terms of vector products gives $\epsilon_1\epsilon_2 = -\epsilon_1^T\epsilon_2 + \epsilon_1 \times \epsilon_2$, where the first term is real while the latter term is imaginary. Since the cross product is anti commutative ($v \times w = -w \times v$), the imaginary product is not commutative unless $\epsilon_1 \times \epsilon_2 = 0$. $v \times w = 0$ only holds when the vectors are parallell or at least one is zero.

We state two handy facts of imaginary numbers (complex numbers with zero real part).

- The powers of the imaginary numbers can be written as

$$
\begin{aligned}
(\alpha i)^{2n+1} &= \alpha^{2n+1} i^{2n} i = \alpha^{2n+1}(-1)^n i, && \text{(odd powers)} \\
(\alpha i)^{2n} &= \alpha^{2n} i^{2n} = \alpha^{2n}(-1)^n. && \text{(even powers)}
\end{aligned}
$$

- The exponential function relates imaginary numbers to rotations through

$$
\begin{aligned}
e^{\alpha i} &= \sum_{n=0}^{\infty} \frac{(\alpha i)^n}{n!} = \left[\sum_{n=0}^{\infty} \frac{(\alpha i)^{2n}}{(2n)!}\right] + \left[\sum_{n=0}^{\infty} \frac{(\alpha i)^{2n+1}}{(2n+1)!}\right] = \left[\sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n}}{(2n)!}\right] + \left[\sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{(2n+1)!}\right] i \\
&= \cos(\alpha) + \sin(\alpha) i.
\end{aligned}
$$

(a) Show that the powers of a pure quaternion (zero real part), αv with $|v| = 1$, acts like the powers of imaginary numbers, only with v substituted for i. That is, show that $(\alpha v)^{2n+1} = \alpha^{2n+1}(-1)^n v$ and $(\alpha v)^{2n} = \alpha^{2n}(-1)^n$.

Hint: Write out the powers up to 2 and see if you can write the higher powers in terms if these, and note that $q^0 = 1$. Equation (10.21) and (10.34) can be useful.

Solution: We follow the hint:

$$(\alpha v)^0 = 1, \qquad (\alpha v)^1 = \alpha v,$$
$$(\alpha v)^2 = \alpha^2(-v^T v + v \times v) = -\alpha^2, \qquad (\alpha v)^3 = (\alpha v)^2(\alpha v) = -\alpha^3 v,$$
$$(\alpha v)^{2n} = ((\alpha v)^2)^n = (-1)^n \alpha^{2n}, \qquad (\alpha v)^{2n+1} = ((\alpha v)^2)^n(\alpha v) = (-1)^n \alpha^{2n+1} v.$$

The left corresponds to the even powers and the right corresponds to the odd powers. It can clearly be seen that this generalizes the powers of the imaginary numbers since the formulas are exactly the same only with i substituted with the imaginary vector v. It should also be clear that the imaginary powers are a special case of this more general formula with $v = i$.

(b) Show that similarly to the imaginary numbers, the relelationship $e^{\alpha v} = \cos(\alpha) + \sin(\alpha)v$ also holds for a pure quaternion αv, with $|v| = 1$ and α real. Briefly discuss how this relates to theorem 10.1.2.

Solution: Since the powers match up to the imaginary numbers we can exchange i with v in the derivation to get $e^{\alpha v} = \cos(\alpha) + \sin(\alpha)v$. For a rotation α around the axis v, this result relates this rotation to its quaternion representation in theorem 10.1.2 through $q = e^{\frac{\alpha v}{2}} = \cos\left(\frac{\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right) v$.

Task 4: *Quaterion kinematics*

We are here going to derive equation (10.43) while showing that the first order approximation done in the book is actually valid. (10.43) states that $\dot{q} = \frac{1}{2} q\omega$, where the factor $\frac{1}{2}$ can be attributed to the quaternion only represtenting half the rotation, whereas ω is the full rotational rate. Similarly to the book we are going to use

$$q(t + \Delta t) = q(t)\Delta q(t, t + \Delta t).$$

However, we are going to use the result of the previous task to state that

$$\Delta q(t, t + \Delta t) = \cos\left(\frac{\Delta\alpha(t, t + \Delta t)}{2}\right) + \sin\left(\frac{\Delta\alpha(t, t + \Delta t)}{2}\right) \Delta v(t, t + \Delta t) = e^{\frac{\Delta\theta(t,t+\Delta t)}{2}}$$

for the pure quaternion $\Delta\theta(t, t+\Delta t) = \Delta\alpha(t, t+\Delta t)\Delta v(t, t+\Delta t)$. Here $\alpha(t, t+\Delta t)$ is0 the angle of rotation between time t and $t + \Delta t$, and $v(t, t + \Delta t)$ is the pure quaternion of unit length (ie. a unit vector) which specifies the axis this rotation is about. We also use (10.42) in the sense that $\lim\limits_{\Delta t \to 0} \frac{\Delta\theta(t,t+\Delta t)}{\Delta t} = \omega(t)$.

(a) What is the interpretation of $\omega(t) = \lim\limits_{\Delta t \to 0} \frac{\Delta\theta(t,t+\Delta t)}{\Delta t} = \omega(t)$, as it is stated here and in equation (10.42) in the book? Specifically mention which coordinate systems it relates and is specified in.

Solution: Before the limit is taken, it is how much rotation, and around wich axis, happens per time unit between t and $t + \Delta t$. In the limit this becomes the instantaneous amount of rotation aournd a specified axis happening at time t. Since $\Delta\theta$ is specified as the right hand side perturbation (being on the local body coordinate side) ω specifies the instantaneous rotation between the body coordinate system and an inertial coordinate system measured by a gyro fixed in the body coordinate's origin (that is, in body coordinates).

(b) Show that the quaternion time derivative $\dot{q}(t) = \lim\limits_{\Delta t \to 0} \frac{q(t+\Delta t) - q(t)}{\Delta t}$ can be written as $q(t) \lim\limits_{\Delta t \to 0} \frac{e^{\frac{\Delta\theta(t,t+\Delta t)}{2}} - 1}{\Delta t}$, and that it gives equation (10.43).

Hint: The quaternions are distributive ($a(b + c) = ab + bc$). Constants (here a quaternion) in terms of the limiting variable (here Δt) can always be taken outside the limit. Also, using the series representation of the exponential function one can write

$$\begin{aligned}\Delta q(t,t+\Delta t) = e^{\frac{\Delta\theta(t,t+\Delta t)}{2}} &= 1 + \frac{\Delta\theta(t,t+\Delta t)}{2} + \sum_{n=2}^{\infty}\frac{\left(\frac{\Delta\theta(t,t+\Delta t)}{2}\right)^n}{n!} \\ &= 1 + \frac{\Delta\theta(t,t+\Delta t)}{2} + \frac{\Delta\theta(t,t+\Delta t)}{2}\sum_{n=2}^{\infty}\frac{\left(\frac{\Delta\theta(t,t+\Delta t)}{2}\right)^{n-1}}{n!} \\ &= 1 + \frac{\Delta\theta(t,t+\Delta t)}{2}[1 + O(\Delta\theta(t,t+\Delta t))],\end{aligned}$$

where $O(\Delta\theta(t,t+\Delta t)) \overset{\Delta t\to 0}{\longrightarrow} 0$.

Solution: We take small steps for the sake of clarity.

$$\dot{q}(t) = \lim_{\Delta t\to 0}\frac{q(t+\Delta t) - q(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{q(t)\Delta q(t,t+\Delta t) - q(t)}{\Delta t},$$

Using the distributive law to factor out $q(t)$ and putting it outside the limit due to it being constant in terms of Δt,

$$= q(t)\lim_{\Delta t\to 0}\frac{\Delta q(t,t+\Delta t) - 1}{\Delta t}$$

Letting $\Delta q(t,t+\Delta t) = e^{\frac{\Delta\theta(t,t+\Delta t)}{2}}$ gives

$$= q(t)\lim_{\Delta t\to 0}\frac{e^{\frac{\Delta\theta(t,t+\Delta t)}{2}} - 1}{\Delta t}$$

where using the hint about series gives

$$\begin{aligned}&= q(t)\lim_{\Delta t\to 0}\frac{1 + \frac{\Delta\theta(t,t+\Delta t)}{2}[1 + O(\Delta\theta(t,t+\Delta t))] - 1}{\Delta t} \\ &= q(t)\lim_{\Delta t\to 0}\frac{\frac{\Delta\theta(t,t+\Delta t)}{2}[1 + O(\Delta\theta(t,t+\Delta t))]}{\Delta t} \\ &= \frac{1}{2}q(t)\omega(t)[1 + \underbrace{O(\Delta\theta(t,t))}_{=0}] = \frac{1}{2}q(t)\omega(t).\end{aligned}$$

Task 5: *The error state dynamics*

In this task we are going to take a closer look at the proof of theorem 10.3.1 for the position, velocity and orientation error dynamics.

(a) Which, if any, approximations are made to arrive at the linearized position error state dynamics $\delta\dot{\rho} = \dot{\rho}_t - \dot{\rho} = \delta v$?

Solution: No approximations are made since velocity relates linearly to position.

This is seen from $\delta\dot{\rho} = \dot{\rho}_t - \dot{\rho} = v_t - v = \delta v$. The first and last equality is per definintion, and the second equality follows from the true and nominal dynimcs.

(b) Derive the linearized velocity error state dynamics $\delta\dot{v} = \dot{v}_t - \dot{v} = -R(q)S(a_m - a_b)\delta\theta - R(q)\delta a_b - R(q)a_n$ by using $R(\delta q) = R(e^{\frac{\delta\theta}{2}}) = e^{S(\delta\theta)}$ and its series. Which, if any, approximations are made?

Solution: We use

$$\delta\dot{v} = \dot{v}_t - \dot{v} = R(q_t)(a_m - a_{bt} - a_n) - R(q)(a_m - a_b)$$

We factor out the nominal rotation matrix $R(q)$ and use $a_{bt} = a_b + \delta a_b$

$$= R(q)(R^T(q)R(q_t)(a_m - a_b - \delta a_b - a_n) - (a_m - a_b))$$

rearrange terms and use the rotation matrix of the error quaternion $R^T(q)R(q_t) = R(\delta q)$

$$= R(q)(R(\delta q) - I)(a_m - a_b) - R(q)R(\delta q)(\delta a_b + a_n)$$

using $R(\delta q) = e^{S(\delta\theta)} = \sum_{n=0}^{\infty} \frac{S(\delta\theta)^n}{n!} = I + S(\delta\theta) + \sum_{n=2}^{\infty} \frac{S(\delta\theta)^n}{n!}$

$$= R(q)S(\delta\theta)(a_m - a_b) - R(q)\delta a_b - R(q)a_n$$
$$+ R(q)\left(\sum_{n=2}^{\infty} \frac{S(\delta\theta)^n}{n!}\right)(a_m - a_b) - R(q)\left(\sum_{n=1}^{\infty} \frac{S(\delta\theta)^n}{n!}\right)(\delta a_b + a_n)$$

The first line gives the first order terms after the series expansion, while the second line are all higher order terms in the error variables and noise. Removing higher order terms we get

$$\approx R(q)S(\delta\theta)(a_m - a_b) - R(q)\delta a_b - R(q)a_n$$

the cross product is anti comutative so we can write $S(\delta\theta)(a_m - a_b) = -S(a_m - a_b)\delta\theta$

$$= -R(q)S(a_m - a_b)\delta\theta - R(q)\delta a_b - R(q)a_n$$

This concludes the derivation. The approximations that are done here are that we use the linearized version of the error rotation matrix as well as assuming $\sum_{n=1}^{\infty} \frac{S(\delta\theta)^n}{n!}(\delta a_b + a_n) \approx 0$.

(c) Which, if any, approximations are made in order to derive the linearized orientation error state dynamics $\delta\dot{\theta} = -S(\omega_m - \omega_b)\delta\theta - \delta\omega_b - \omega_n$?

Solution: The approximations used are the linearization of the error quaternion $\delta q \approx \begin{bmatrix} 1 & \frac{1}{2}\delta\theta^T \end{bmatrix}^T$ and neglecting the second order and noise times error terms $S(\delta\omega_b + \omega_n)\delta\theta \approx 0$. Since the unit quaternion only has 3 degrees of freedom we do not actually loose any degrees of freedom by neglecting the differential equation for the real part. But it is an approximation nevertheless.