GPH 259 Design Geometry
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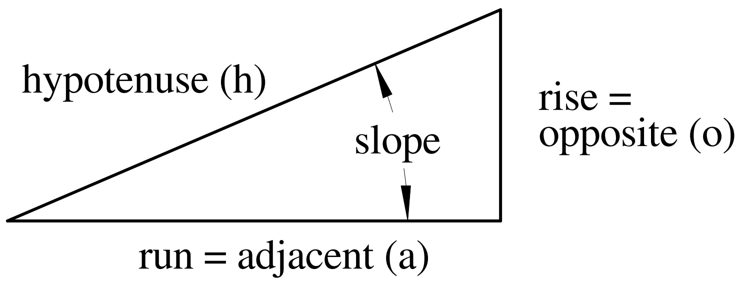
Trigonometry Basics

Trigonometry is the earliest form of computational geometry. Its use dates back over 2000 years and some of its ideas extend back perhaps 6000 years. It is computational, because, in contrast to constructive geometry, it can be used to calculate geometric relationships from numerical values. In vector programs like CAD, where all elements are specified numerically, most of the computing uses trigonometry to determine vector relationships. One advantage of CAD programs is that they can be used as "visual" calculators to solve problems traditionally solved by use of trigonometric formulas.



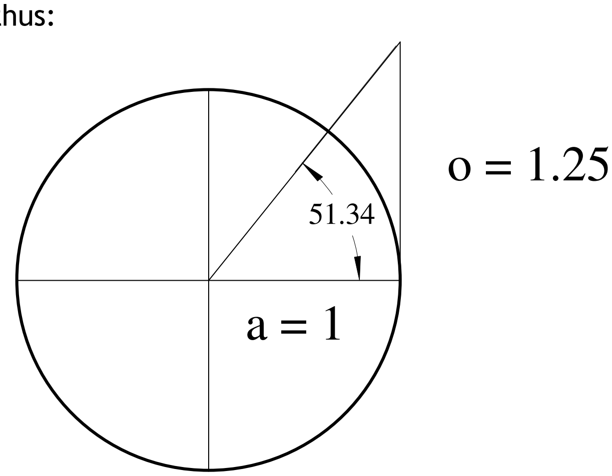
The idea of specifying a slope as the proportion ***rise/run*** is the earliest concept in trigonometry. This method to determine the direction of a line off of the horizontal baseline defines the slope as the hypotenuse of a right triangle: the run is the base of the triangle and rise is the perpendicular side of the triangle. In standard terminology the sides of a right triangle are labeled as follows: hypotenuse (h) = the sloped side, opposite (o) = the vertical side, and adjacent (a) = the horizontal base. Under this terminology a slope is equal to ***opposite/adjacent*** or ***o/a***. In trigonometry this relationship is called ***tangent*** or ***tan***. Trigonometric relationships are based on the ratios between two sides of a right triangle.

The Unit Circle

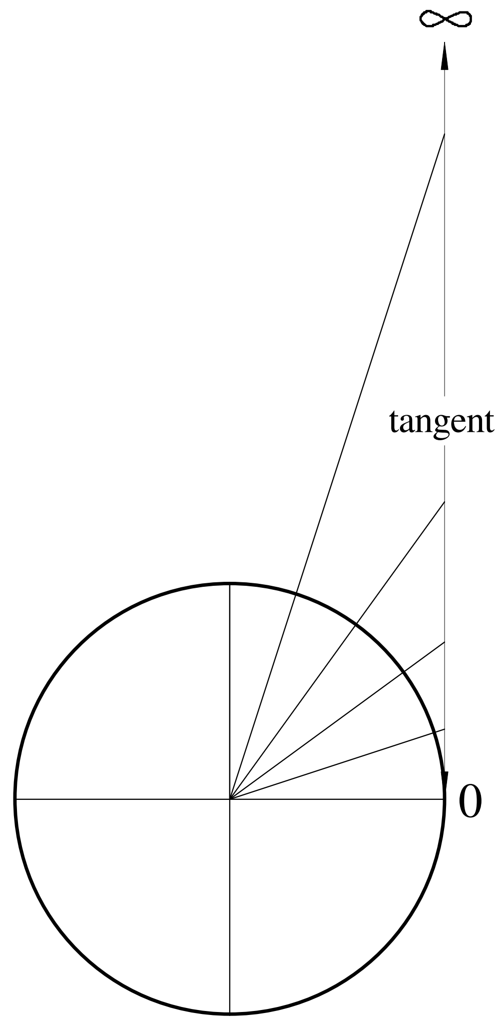
What does the slope of a line have to do with tangency? The answer begins with an ancient Greek invention for measuring angles called the unit circle. The unit circle is a circle whose radius is 1. In order to simplify the proportion ***o/a***, ***a*** is also set equal to 1 and the value of tan is therefore equal to the length of side ***o***. Drawn out in reference to the unit circle the proportion appears thus:

When placed on the unit circle the slope of the angled line is equal to the length of the tangent line o. For every unit, 1, the line moves to the right it moves upward 1.25 units. The angle measure of this line is 51.34°. Therefore:

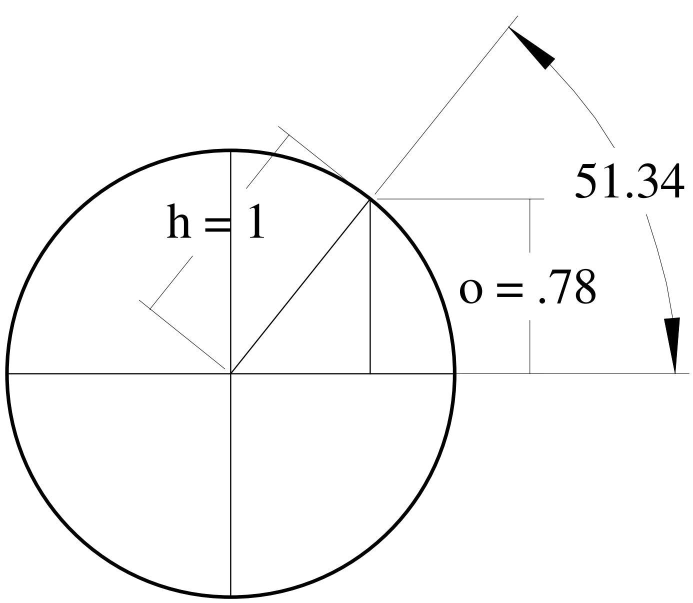
tan 51.34° = 1.25



The tan function worked well for calculating angles close to earth, like those used in architecture. However, at angles near 90° the values for tan approached infinity and became unwieldy. This became a problem for Hipparchus, the great Greek astronomer of the 2nd century BCE, when he wanted to record values for the vertical angles of stars very high overhead.



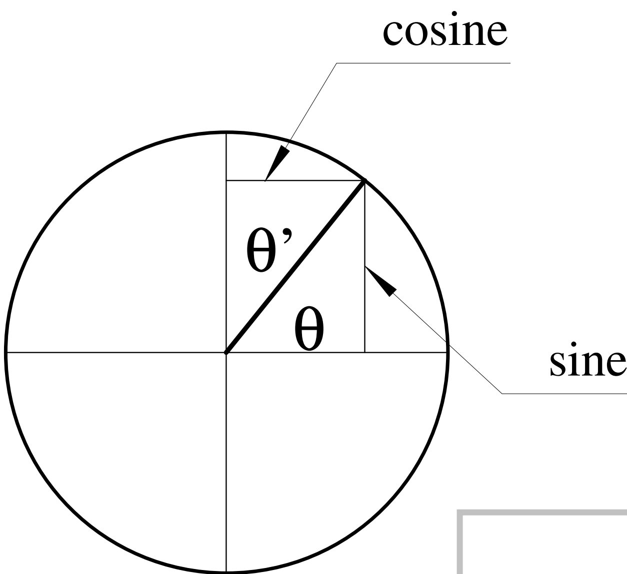
To remedy this Hipparchus created a method based on half chords – also called sines – of a unit circle (see diagram below). Hipparchus stipulated the sloped line as a radius of the unit circle, thereby holding its value to 1. From the point where the sloped line intersects the unit circle he dropped a vertical chord down to meet the horizontal baseline at a right angle. This formed a right triangle whose hypotenuse (h) was the sloped line, whose adjacent side (a) was the baseline, and whose opposite side (o) was the half chord or sine. The angle of the sloped line could be specified by the ratio ***o/h***. Since h remains 1 in the unit circle the value of ***o/h*** is equal to the length of the sine, or, in the notation of trigonometry: ***sin θ = o/h***, where ***θ*** is understood to be any angle.



The diagram at the left displays a radius of the unit circle whose angular direction is 51.34° and whose sine line is equal to 0.78. Thus:
sin 51.34° = 0.78

As the radius increases its slope through 90° the maximum length of the sine cannot exceed the length of 1 on the unit circle. Thus all values of sine lay between 0 and 1. The tan ratio's problem of the indeterminate number is then surmounted.

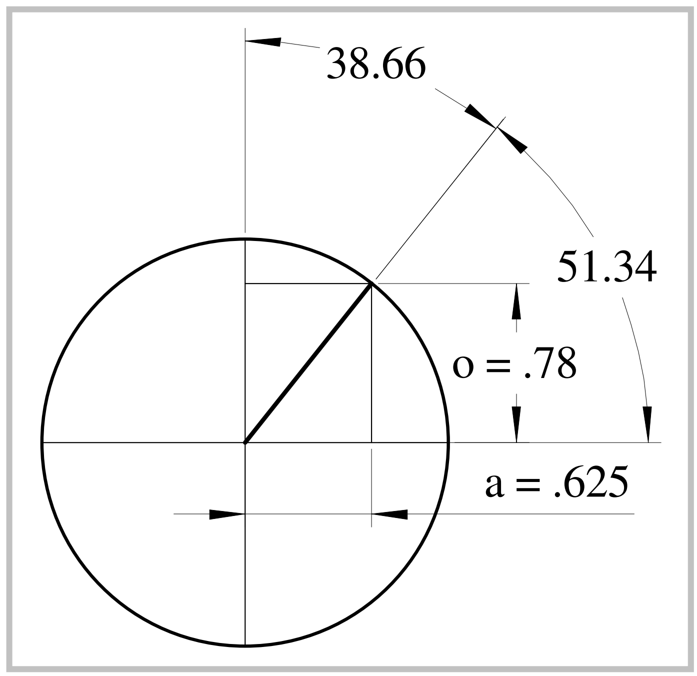
Note that the angle formed between the sloped line and the



vertical axis of the unit circle is the complement of θ. A sine drawn to this axis will be the sine of the complement of θ, or θ'. The sine of this complementary angle is also known as the cosine of θ. Note from the diagram that the cosine is equal to the base of the right triangle defined by the sloped line and its sine. Thus the value of cosine θ is expressed as: ***cos = a/h***. On the unit circle ***h = 1***, so that the value of cosine θ is equal to ***a***.

The cosine of an angle is equal to the sine of its complement. Based on the diagram to the right the following can be concluded:

sin 51.34° = 0.78
sin 38.66° = 0.625
cosin 51.34° = 0.625
cosin 38.66° = 0.78
or
$\sin\theta = \cos\theta'$
and
$\cos\theta = \sin\theta'$



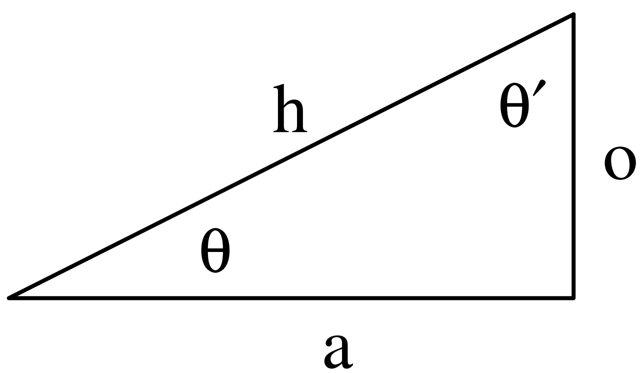
Hipparchus went on to create a chord table to stipulate the horizontal bearing (azimuth) and vertical angle (declination) of all the stars that he could observe with out the aid of the yet to be invented telescope. By doing so on different dates and by keeping prodigious records he tabulated the data from which Claudius Ptolemy calculated his model for planetary motion. In these ancient times trigonometry was not considered a separate field of mathematics, but an aid to astronomers. Hipparchus, however, did use his chord tables to solve problems in plane geometry.

During the Middle Ages Arabic mathematicians merged trigonometry with algebra, so that it could be expressed as functions with unknowns. This greatly increased its power as a computational tool and in Renaissance times it contributed to the development of numeric approaches to geometry. This approach was growing as an alternative to and an augmentation of constructive geometry. This paralleled the evolution of mathematics as a whole from a system of practical methodologies to an intellectual field governed by numerical analysis.

The Right Triangle

All of the constructions of trigonometric functions on the unit circle take the form of a right triangle. While the unit circle does a good job of explaining the origins, naming and development of these functions, in practical application the focus is on the right triangle.

The three sides of the right triangle can be used to define six different ratios depending on which side takes which position in the ratio:

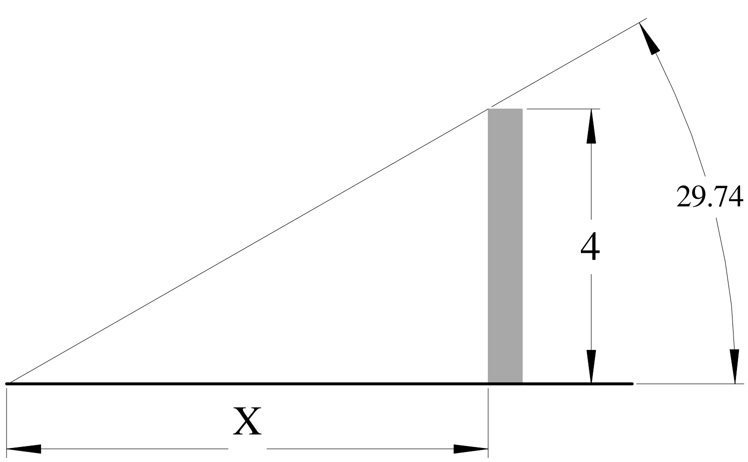


$$\sin\theta = o/h$$
$$\cos\theta = a/h$$
$$\tan\theta = o/a$$
$$\cot\theta = a/o$$
$$\sec\theta = h/a$$
$$\csc\theta = h/o$$

The three most commonly used are those introduced earlier: cosine, sine and tangent. The remaining three are the reciprocals of the first three: cotangent, secant and cosecant. While all the functions are useful, the first three are the most common by far. This introduction to trigonometry will concentrate on those three.

The acute angles of the right triangle are complementary, so the relationship of $\boldsymbol{\sin\theta = \cos\theta'}$ holds true as it did in the unit circle.

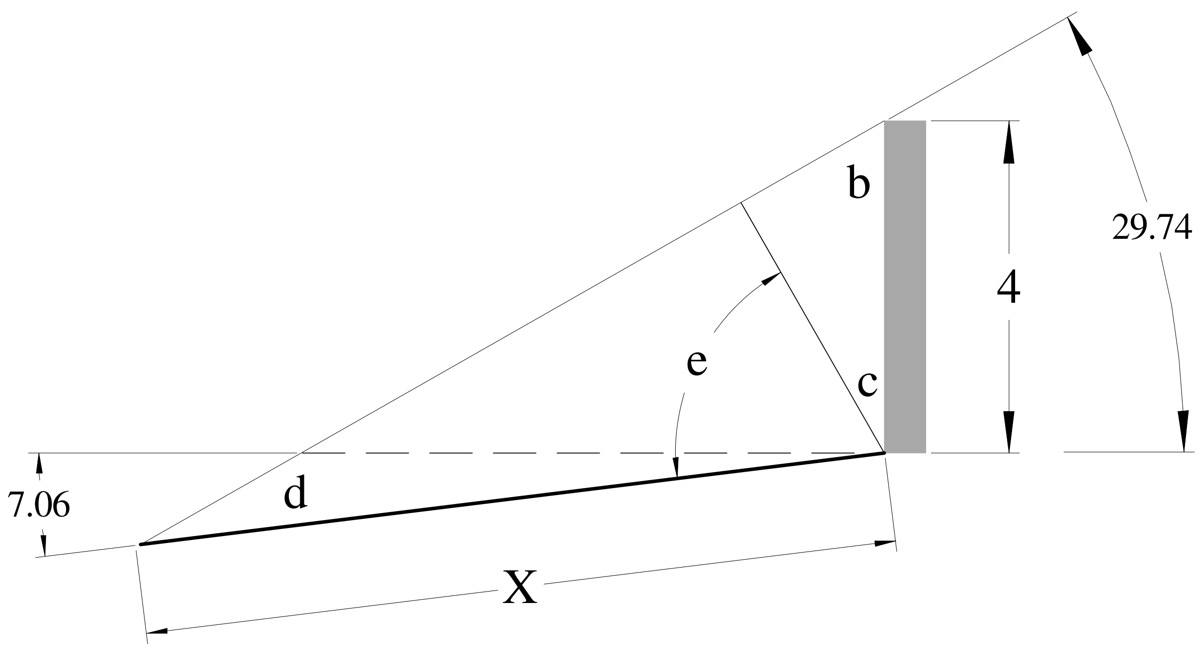
Solving Triangles



Most design applications of trigonometry deal directly or indirectly with solving triangles. This means that the designer visualizes the problem in such a fashion that the unknown line or angle as a component of a triangle whose unknowns can be determined trigonometrically.

One example is the length of a shadow when the height of the object casting the shadow and the angle of the sun are known. The diagram above depicts this situation with the angle of the sun determining the hypotenuse line of the right triangle, a post determining the opposite side and the shadow determining the adjacent side. The angle is known and the rise of the post, too, is known, but the run of the shadow is unknown. The tan function, based on rise/run, and its reciprocal cot look like good candidates for solving this triangle:

Write the equation: **tan 29.74 = 4/X**
Substitute the value of tan: **0.5713 = 4/X**
Multiply both sides by X: **0.5713X = 4**
Divide both sides by 0.5713: **X = 4/0.5713 = 7.0015**
Round off to the nearest 1/100: **X = 7.**



This formula can be shortened to ***X* = *4*/tan 29.74** and generalized to **a = o/tanθ**.
More complex problems require solving multiple triangles. The second example above asks for the length of the shadow if the post is set on a slope declining in the direction of the shadow. With this adjustment the triangle (bcd) created by the sun angle the post and the ground is no longer a right triangle. However, drawing an altitude from the lone of the sun angle to the base of the post will divide that triangle into two right angles. By doing so the shadow becomes the hypotenuse of a right triangle.

The horizontal line is included in the drawing because this is the line from which the solar angle is measured. Angle b is its complement, and so equals 60.26 (90 – 29.74). Angle e then equals b + 7.06, or 67.32. Angle d = 90 – 67.32 or 22.68. Angle d also equals 29.74 – 7.06. Thus all of the angles can be determined. Only one side of the larger right triangle is needed to solve that triangle.

The easiest side to determine is the altitude of triangle bcd. The formula **4 x sin b** will yield the length of this altitude: 4 x 0.868275 = 3.4731. This is because post/altitude is, relative to angle b, the same as **o/h**, or sin. We now have enough information to solve for x:

Write the equation: **sin 22.68 = 3.4731/X**
Substitute and multiply by X: **0.3856X = 3.4731**
Divide by sin 22.68: **X = 3.4731/0.3856 = 9.007**

This formula can be shortened to **X = 3.4731/sin 22.68** or, more generally, **h = o/sinθ**.