GPH 259 Design Geometry
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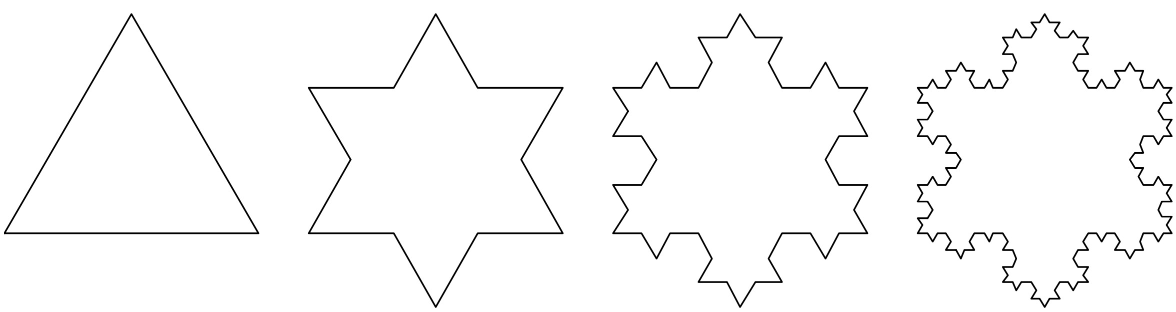
Fractal Geometry

Most of the geometries covered thus far came into being before computing, but have adapted nicely to computing applications. One recent development lay in wait for the inception of computing before entering the mainstream of mathematics. That was fractal geometry.

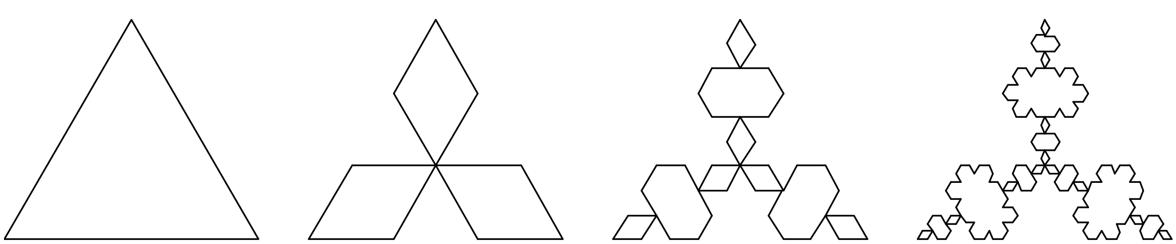
Fractal is a term coined to describe one essential character of this geometry: the breaking, or fracturing, of large pieces into smaller and smaller pieces with each level of breakage resulting in pieces that resemble the original form. A peek at a tiny portion of a fractal form is like another look at the form as a whole.

I. Infinite Patterns

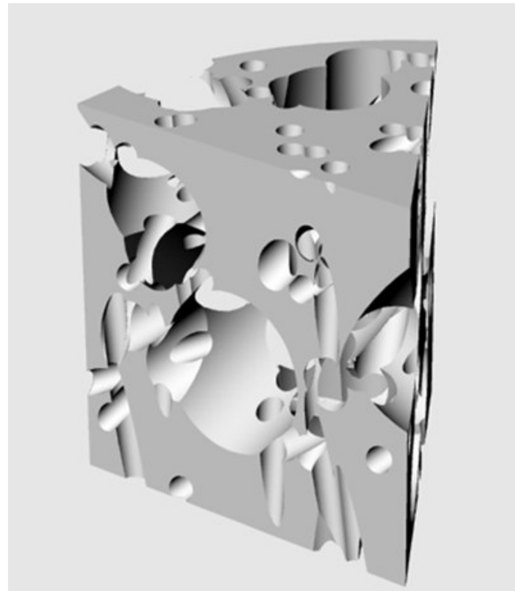
One of the best-known examples of a fractal curve is the Koch snowflake. This figure begins as an equilateral triangle and then develops through an indefinite number of stages. The center third of each side kinks out into two sides of an equilateral triangle that is one-third the scale of the original. This new stage is a dodecagon, a 12-sided figure in the form of a six-pointed star. Each of the 12 new sides is kinked again yielding a 48-sided figure. This process continues to be repeated, or, in fractal terminology, self-iterated. Each iteration multiplies the number of line segments in the curve by four and increases the perimeter by 25%. The area enclosed by the curve continues to grow, although by less and less with each iteration.



The Koch snowflake is clearly computer friendly. A very simple algorithm that is continuously reapplied to the figure can generate it with the barest of code. At millions of calculations per second the segments of the curve can be quickly become far, far smaller than an atom, even if it started out at the diameter of the universe. In theory the process can be iterated an infinite number of times at which point even the smallest triangle will reach an infinite length. It can, however, enclose only a finite area of surface.

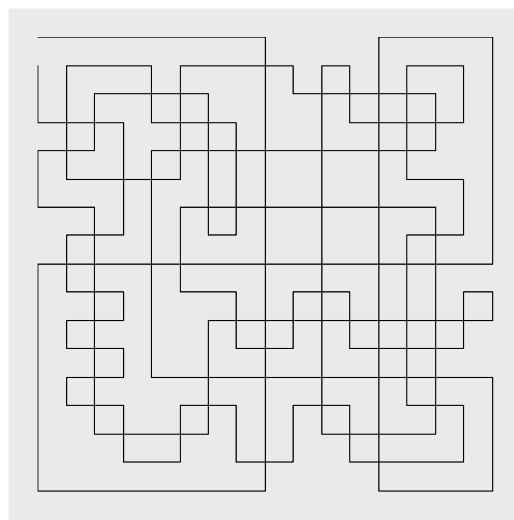


A question posed by fractal geometers might be: what is the limit of area enclosed by a Koch snowflake? Will it be that of the hexagon enclosing the star? If the curve had kinked inward the volume would be much smaller. Would it reach zero? Not in this case, but in fractal geometry there exists infinitely long closed curves that enclose no area and closed infinite surfaces that containing no interior volume. Eating a wedge of fractal Swiss cheese could mean eating a wedge of air.



Fractal Dimensions

Fractal curves present geometers with another problem. What dimension does one assign a curve that wanders through a 2D area on an infinitely long path? On the one hand it is infinitely thin so it can never fill in the space; on the other hand it is in finitely long so that there is plenty of line available for the task. The curve to the right is an example of this problem.



The rules of this curve are that it follows the lines of a grid without retracing a line. When no other path is available the curve begins to follow a grid with double the frequency of the grid it has been following. This grid began on a 2 x 2 grid and then switched to a 4 x 4 grid after six segments were drawn. After the possible paths on the 4 x 4 grid were exhausted the grid doubled to 8 x 8 and finally to 16 x 16. Continuous doubling of the grid will keep filling in unmarked territory.

An infinitely thin line cannot fill a space no matter how long it grows (a whole lot of nothing is still nothing), but fractal geometers found it useful to rate curves on their tendency to wander enough to cover a surface. To do this they created a mathematical valuation called *fractal dimension*. For a curve this is a number greater than 1.0 and less than 2.0. A curve can never reach a full dimensional value of 2.0, for then it would be a surface, but it could be assigned a dimension greater than 1.0 to assess its ability to "fill" a surface with its meander. The Koch snowflake's tendency to cluster along a perimeter would give it a lower dimension than the grid curve's tendency to cover the surface.

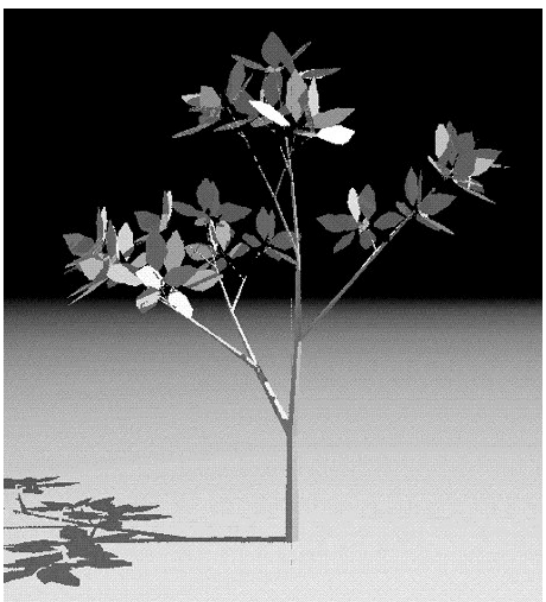
By the same token fractal surfaces are assigned dimensions between 2.0 and 3.0 to measure their degree of folding throughout a third dimension.

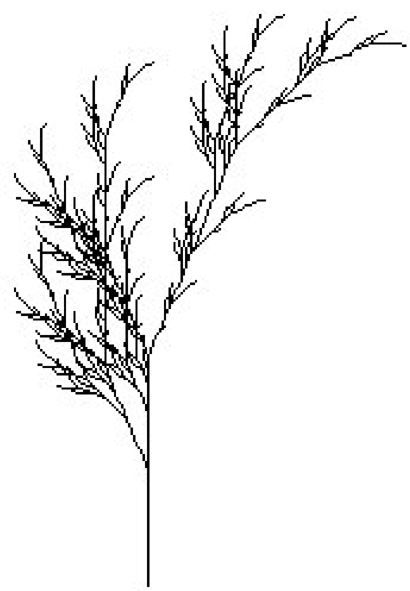
II. Natural Order

Fractal geometry is nothing new to nature. Nature employs its principles to structure everything from the froth of foam to the fronds of ferns. Most of the rich complexity and elegant patterns in nature owe their being to the hierarchical ordering of fractal geometry.

Self-Iteration

The branching of trees is an example of nature's hierarchical patterning at work. A single trunk diverges into two or more smaller versions and these in turn divide several more times until they become leaf-bearing twigs at their farthest extensions. A tree's DNA need not specify the form of each individual branch: it needs only to specify the form of one branching along with the instruction to iterate that branching at the leading end of each new branch. Computer programs for modeling plants use a system of algorithms developed by the bio-mathematician Aristid Lindenmayer to specify branching and growth patterns of plants. L-systems, as these algorithms are called, boil the branching patterns of all plant species down to a relative handful of programming instructions.





Chaos

In representing nature artists were arguably the first to notice this inherent structure, especially in portraying the dynamics of physical processes. In his untitled image of a storm-tossed sea, for example, the great Japanese printmaker and painter Hiroshige rendered the froth of a giant wave with fractals. The leading edge of the arcing wave has been subdivided into a series of similar arcs. These new edges are then further fringed with new subdivisions of the same arc. Hiroshige's systematic fracturing of the form of the wave stylishly replicates the physical dispersion of turbulent liquid into streaming droplets.

Because of the vagaries of physical processes in nature, real waves would behave more randomly than portrayed by Hiroshige. The principle, though, remains the same. In physics such processes -- those featuring randomized fractal structure -- are termed chaotic. In this technical sense *chaos* is not the completely lack of order, but rather a form of order in that it can be characterized mathematically. In the language of the geometry of chaos this structure is *dynamical* and *non-linear*.

Dynamical refers to the fact that the structure unfolds through time, the result of active physical movement. Non-linear is also a temporal term. It refers to the unpredictability of the exact sequence, or linearity, of the emergence of forms in such a process. The general appearance and pattern of the emerging forms will be consistent, but, like human fingerprints, will never be exactly the same.

One chaos experiment involved injecting a stream of dyed liquid into a vat of clear liquid. Upon entering the vat the colored stream split off into separate spirals, which in turned divided into smaller and smaller whorls as the energy of the incoming stream dissipated. Though this general pattern appeared with each repetition of the experiment, the exact size, position and number of diverging spirals could never be predicted. Sometimes such patterns are preserved in their erosive effects on the landscape. This is the case of the Mars photograph above. The combination of wind, water, volcanic flow and meteor strikes from the planets distant past are preserved in this landscape fragment.

The principle of non-linearity means that, though these systems are random, they are computable and can therefore be modeled. The problem is that a particular computer model will never exactly conform to its correlation in actuality. This is because extremely slight perturbations can dramatically affect chaotic systems, especially if they occur early on. Even the most thorough and powerful computer models cannot account sufficient effects to predict an outcome very far in the future of a dynamical system. The predictive accuracy of weather models, for instance, depreciates rapidly after less than a day.

In computer graphics predictive accuracy is less important than convincing appearance. There non-linear geometry is especially effective for simulating the apparent behavior of physical phenomena -- the flow of water, the behavior of mist, an avalanche of snow, the cracking of ice. Like their natural counterparts no two occurrences, virtual or real, will be identical. The most common application of these principles is particle modeling in which the program treats all of these phenomena as huge sets of particles that behave, in effect, like molecules or sand or dust, etc.

III. Graphic Applications

The L-systems used in plant modeling and the chaotic simulations of particle modeling are examples of fractal related applications in computer graphics. Two other important applications are fractal enhancement and fractal generation.

Fractal Enhancement

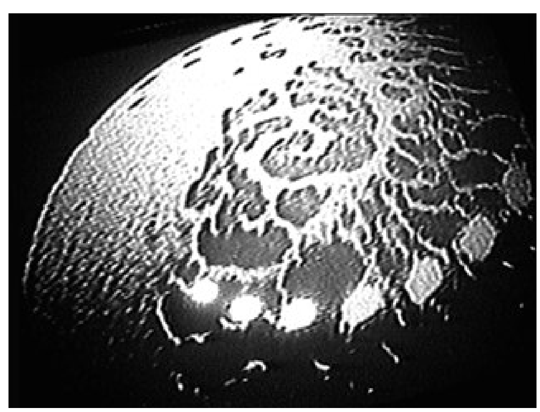
Despite what television police and spy dramas would lead us to believe a photograph or video can only carry so much visual information. Details cannot be manufactured through enhancement unless some significant trace of the detail is already in the image. If so the detail may be amplified and sharpened through fractal enhancement.

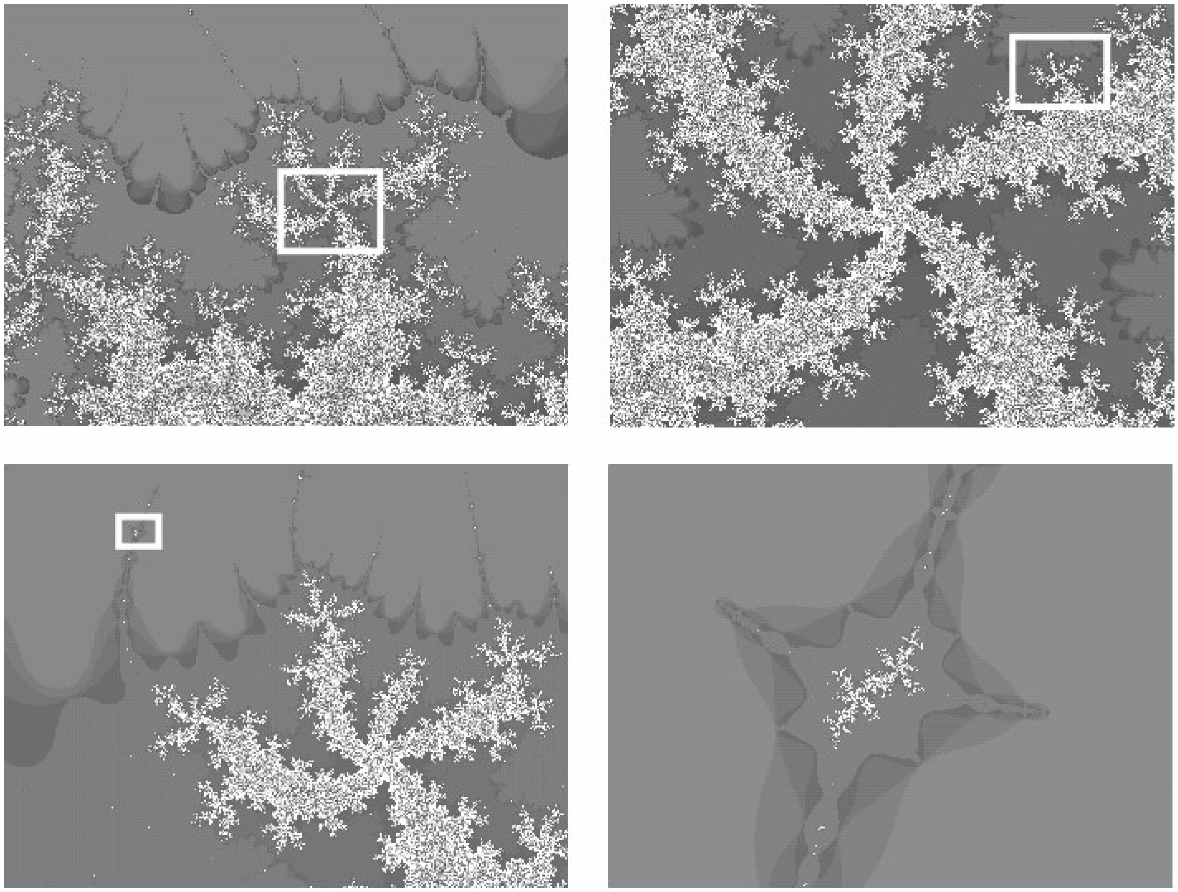
Usually increasing the pixel count of an image adds no detail. The result is the same information translated with the same level of detail blurred into a larger version. Fractal enhancement can eliminate much of this blur by iterating pixel patterns within small portions of the image into smaller neighboring versions of that same pattern. By doing so it multiplies the detail into "sub-details" that preserve clarity of edges and textures in the enlarged image. This multiplication of detail can, within statistical limits, fill in small patches of missing information.

This process constitutes a sort of mathematical "best guess" of that information. While there is no guarantee of its accuracy, what it does add can be quite useful and it does a very good job of enhancing the visual appearance of the magnified image

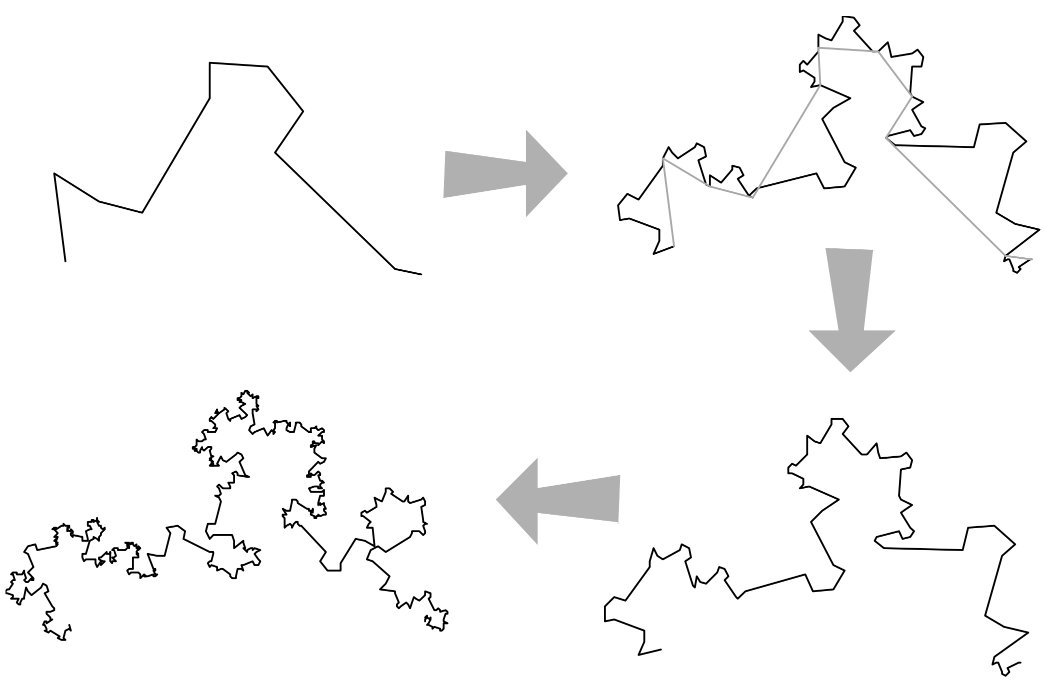
Fractal Depth

Non-linear fractal algorithms can generate rich textural effects with just a few lines of code written to include iteration instructions. It is possible to zoom such textures without the blurring of detail incurred with bitmapped textures. This is because the fractal structure can iterate to smaller and smaller levels of detail. The number of levels down to which this iterative detail descends is termed its *fractal depth*.





Fractal generation has been popular for creating imaginary plants and landscapes, even planets, in film and graphic scenarios. Rocky formations, dunes and coastlines (see below) are some examples of 3D models where initially simple surfaces and curves were iterated and randomized to convincingly replicate physical patterns in nature.



A coastline of an imaginary continent might begin as a curve of a few random line segments. If each segment is replaced with a copy of the curve as a whole it takes on some of the broken effect of a coastline. Replacing each segment of the new curve with a copy of the original further enhances the effect. A 3D example is that of a mountain that begins as simple pyramid whose polygons are multiplied and shifted in direction according to an iterating algorithm. The final iteration results in a mountain, roughly pyramidal in shape, but with the craggy planes, edges and fractures of a natural landscape.

Since fractal geometry takes copies of a figure and maps it back onto itself, it can be represented by *affine transformation*. This transformation requires a positional transformation, or translation, from the whole to its parts, but it must also incorporate scalar and rotational operations in order to map the whole to the parts. This can be seen in the curve above. Transformation from whole to part must account for the changing length and directions of the curve's various line segments. Note that in some cases the curve was also reflected.