GPH 259 Design Geometry
Instructor: Stephen Luecking

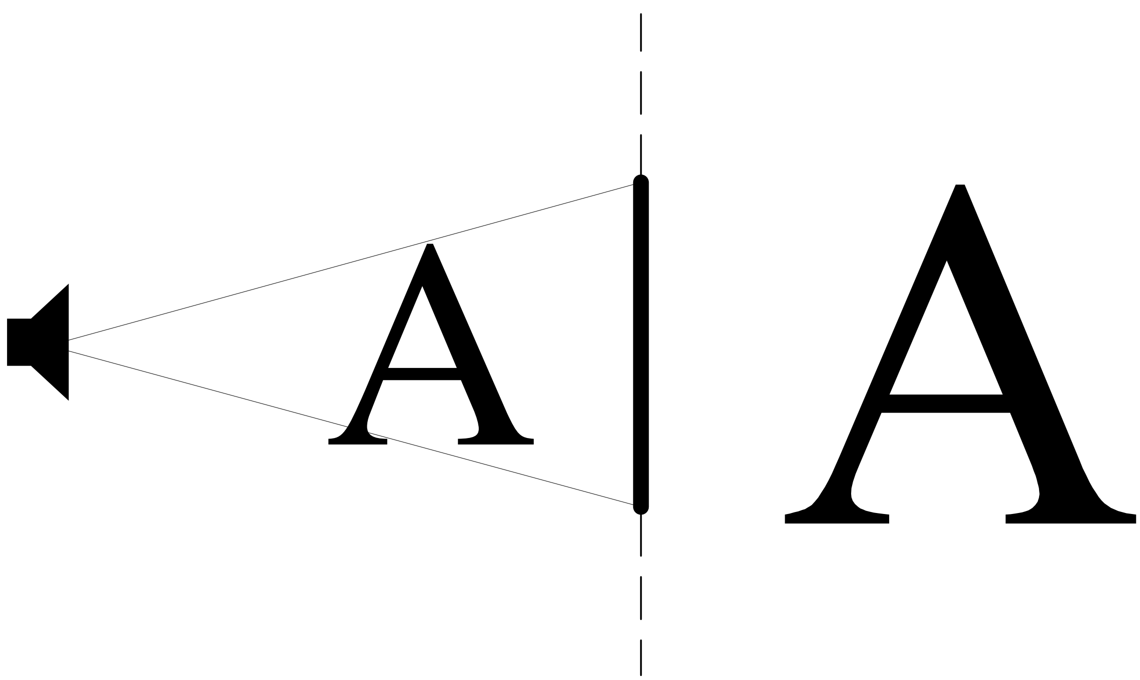
Projection Geometry

Projective geometry studies the transformations of form that occur when an entity in a higher dimension is projected onto a space in a lower dimension. The most common projections involve the projection of a 3-space object onto a 2-space surface. Every time we observe the shadow of a tree fall on the street we are seeing the 2-space projection of the tree. Every time we look at a photograph, a painting, a movie or a monitor image of a solid object we are seeing projection at work.

There are three basic elements of a projection from 3-space to 2-space: the projected object, the projective plane and the convergence point, also called the center of projection (COP). Depending on how these are arranged, projection may occur as shadow projection, photographic projection or perspective projection. Computer graphics concerns itself primarily with perspective projection, but it will increase our understanding to look at the other two types.

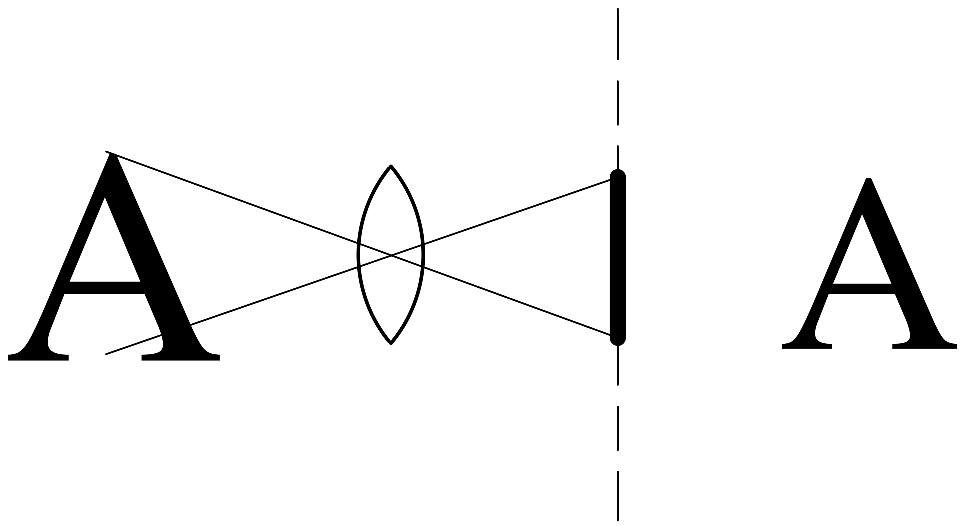
Shadow Projection

Shadow projection places the COP as the source of light and the object intersperses between the COP and the screen. While shadows are the most common result of this projection, film and slide projectors also use this ordering of the three elements of projection. The projector bulb, the COP, shines through the transparent film, the object, and onto the screen, projective plane.



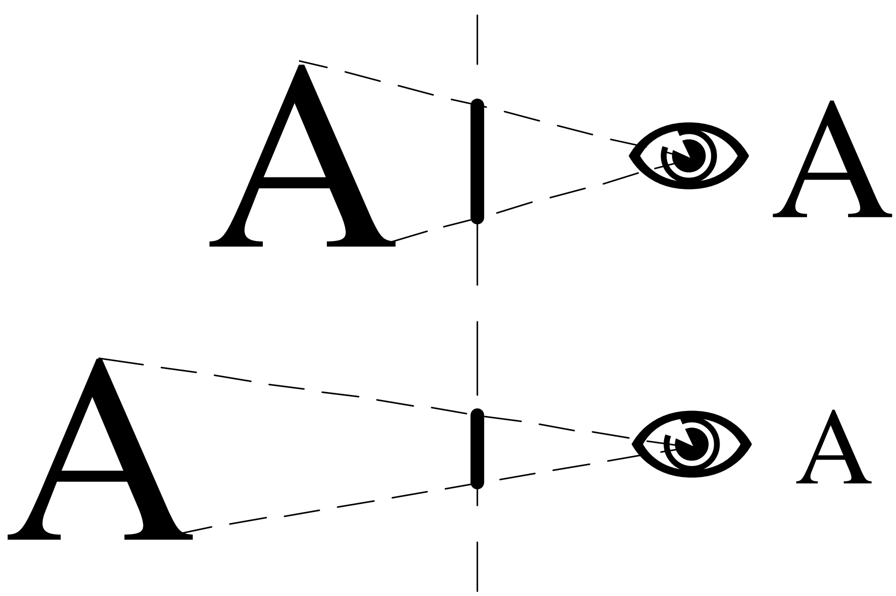
Photographic Projection

Cameras and our own eyes create projection by setting the COP in the lens: rays from the object converge in the lens where it refracts to project on the film plane. In this case the COP lies between the object and the plane.



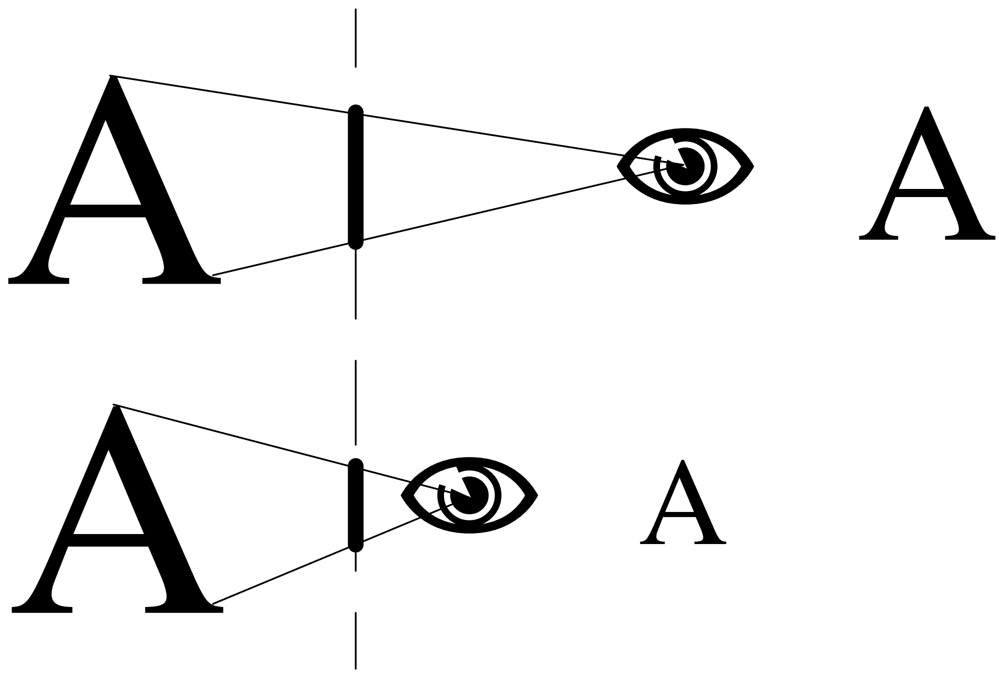
Perspective Projection

In perspective projection the projective plane is placed between the object and the COP. Here the COP is the human eye. The rays from the object pass through the plane to meet at the eye. If the projective plane is a window and one were to trace objects as they appear through the window the result would be a drawing of the scene outside the window.



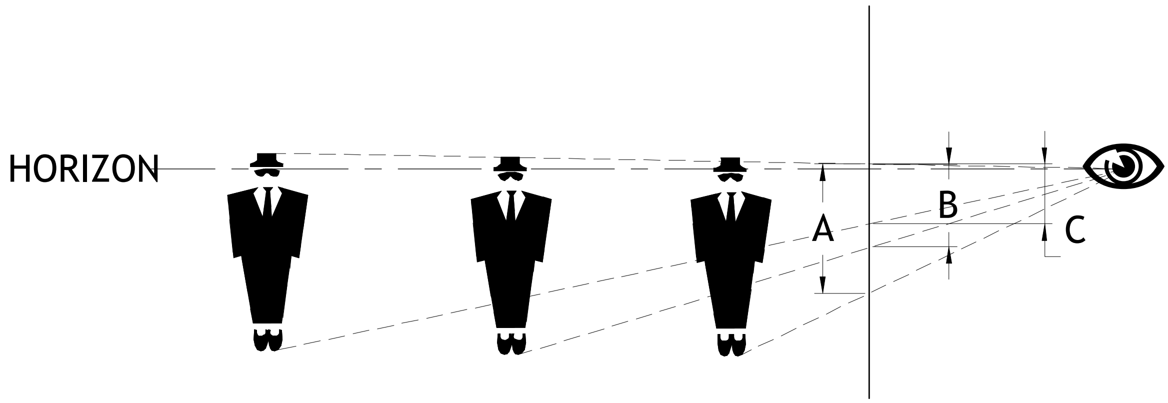
In perspective more distant objects appear smaller than nearer objects. This is due to a principle called angular intercept. When the rays from the top and bottom of the object meet at the eye they form an angle that is intercepted by the projective plane. Given two objects of equal size the more distant object will yield a narrower angle and a smaller intercept. Should the object move to an infinite distance away, then the angle becomes negligible and the intercept shrinks to a point.

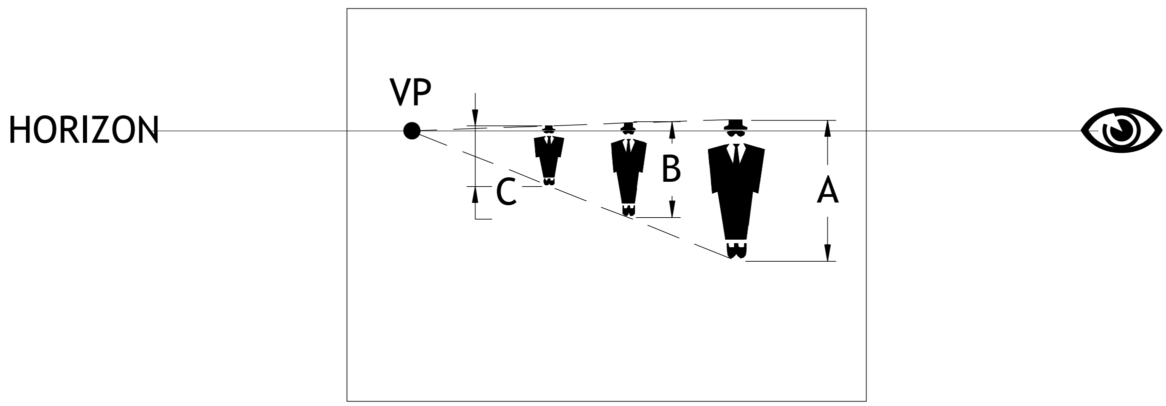
Moving the eye closer or further from the plane will also affect the angle intercept. As the eye nears the plane the image shrinks relative to the size of the plane's surface; withdrawing from the plane enlarges the intercept and the appearance of the object on the plane.



A perspective painting is essentially an imaginary window with the imaginary scene behind it drawn and colored as it projects on the window. Since perspective artists cannot necessarily see the scene they are painting, they require a method that will allow them to duplicate the effect of diminishing size. They do this by drawing parallel straight lines heading back into space as if they were converging to a point at infinity. Since infinity is impossibly far away they placed the point, called the vanishing point, at the horizon. Not quite infinity, but far enough so that even an elephant will be a mere speck - if it can be seen at all.

The convention of placing the vanishing point on the horizon developed scientifically. The great Italian perspective painter Alberti worked this out in the 15th century by tracing scenes on a glass plate marked off in a grid. The German printer and mathematician Durer refined Alberti's experiments with special instruments he built just for the purpose of investigating perspective.





Though developed scientifically, the conventions of perspective imaging are just that - conventions: they work well to imitate human vision in a relatively narrow range of object scales and within a fairly small portion of the visual field. The human eye is circular as is its lens and straight lines in the environment project as curves on the retina. On wide screens perspective causes objects to distort severely. Cinematographers use complex lenses designed to reduce this distortion.

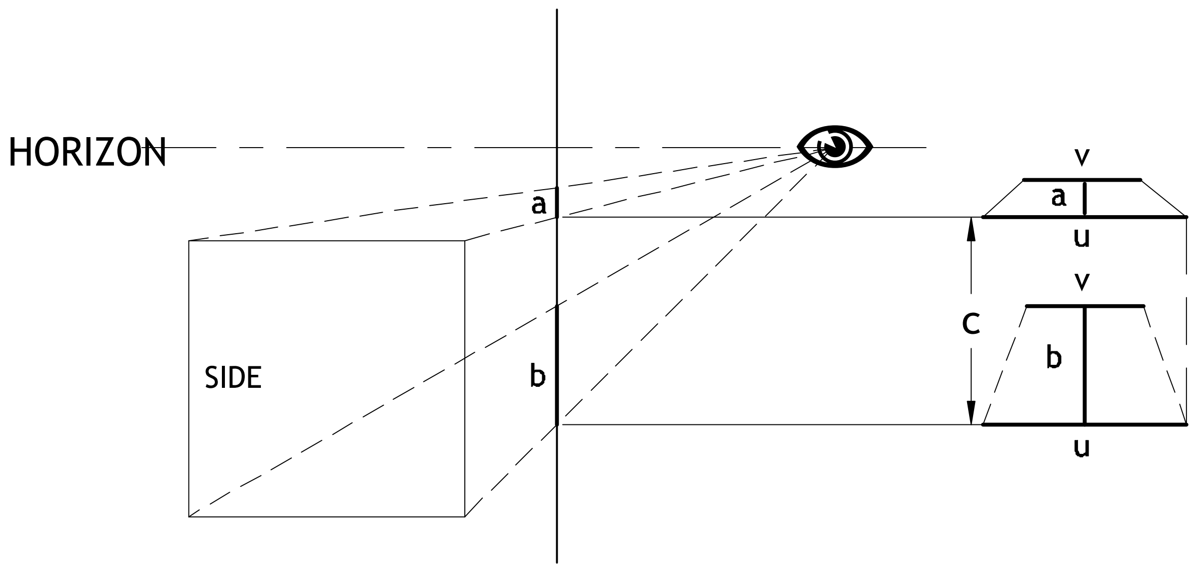
The eye is also not a fixed point. It is always moving and scanning its environment. In doing so it is rapidly, fluidly and subtly changing the position of the vanishing point and in the process correcting distortions.

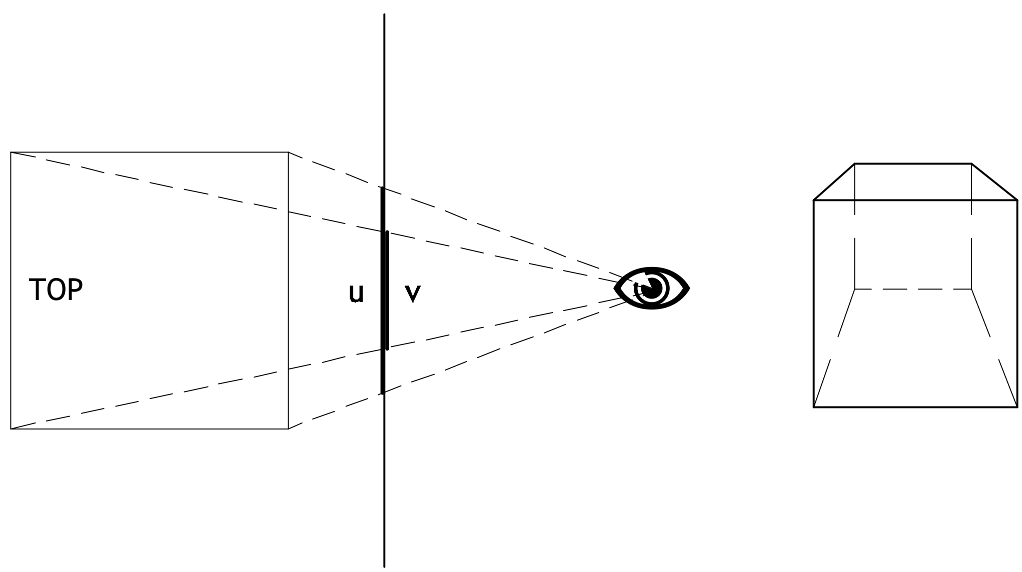
On the other hand, treating the eye as a fixed point and the projective surface as a plane works pretty good overall, especially in the aspect ratio (height:width) of most computer monitors. Best of all it is geometrically consistent and therefore highly computable.

Shape Distortions

One reason our eyes see realistic space in a perspective image is that perspective distortion for the most part matches the projective distortion of our own eyes. The angles of the edges of solid objects, for example, distort in projection and this is one of the perceptual cues by which the brain perceives space. In perspective these distortions, like scale change, occur due to angular intercept.

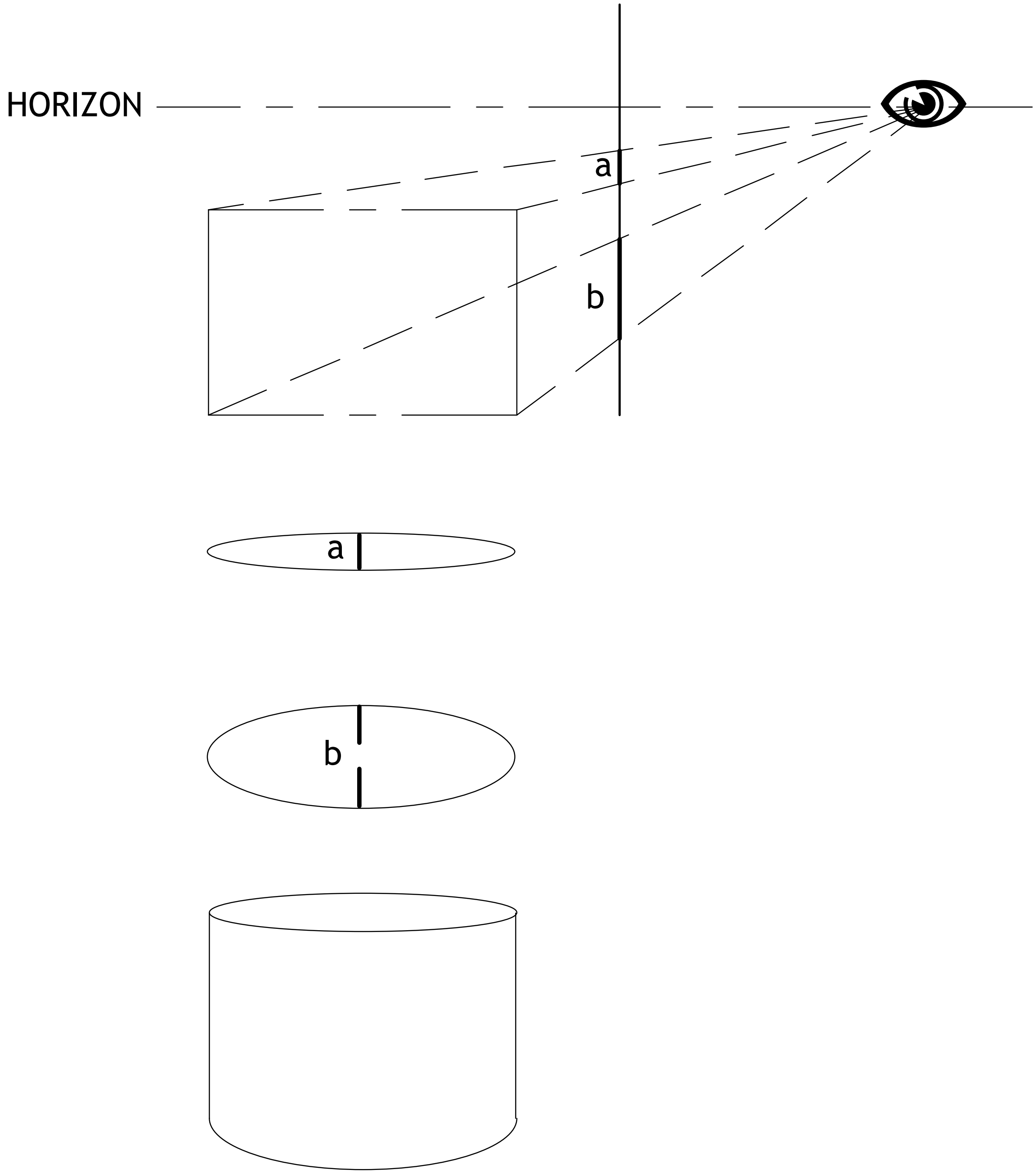
The images below diagram how angular intercept determines the horizontal and vertical spacing of the edges of a cube as they are projected on the image plane.



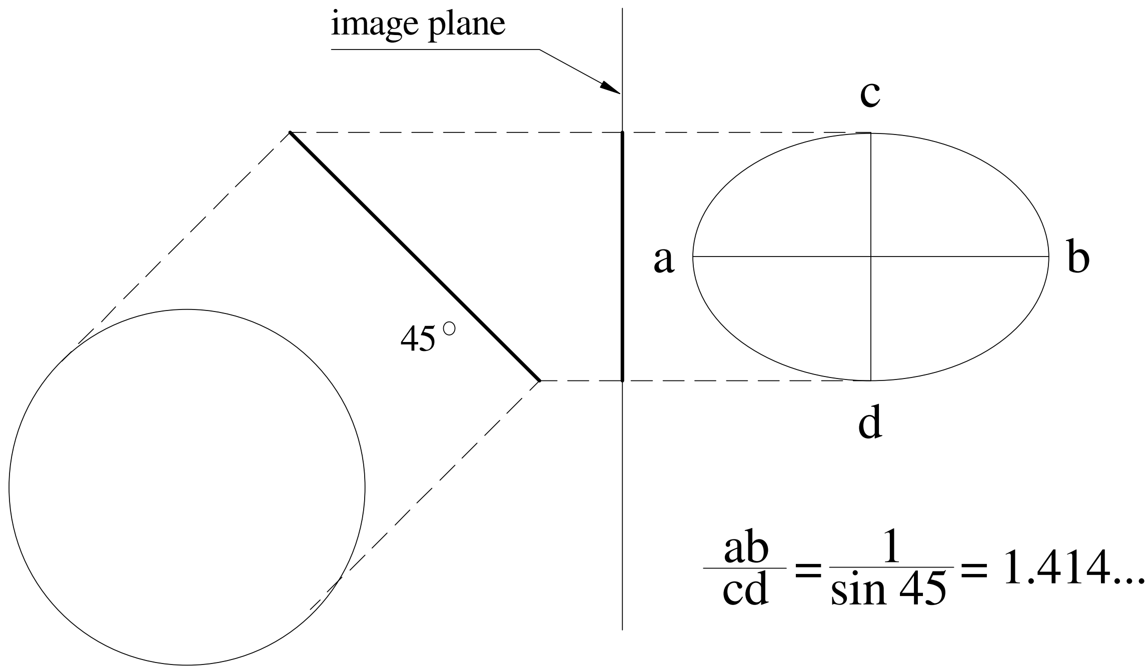


Another important shape distortion is the transformation of a circle into an ellipse. The following illustrations show how angular intercept determines the elliptical distortion of the end circles of a cylinder. The horizontal and vertical positions of the quadrants of a circle determine the major and minor axes of the ellipses.

Note that the bottom circle of the cylinder distorts into a fatter ellipse than did the top circle. This is because the point of the eye is at a greater angle to the plane of the lower circle. At an angle of 0° to the plane of the circle we are seeing the circle end-on, so that it appears to only be a line. As our eye is raised or the circle is lowered the angle of the eye to the circle's plane increases and ellipse thickens. Once the line of sight is directly above, or 90°, to the circle the ellipse has waxed into a full circle.

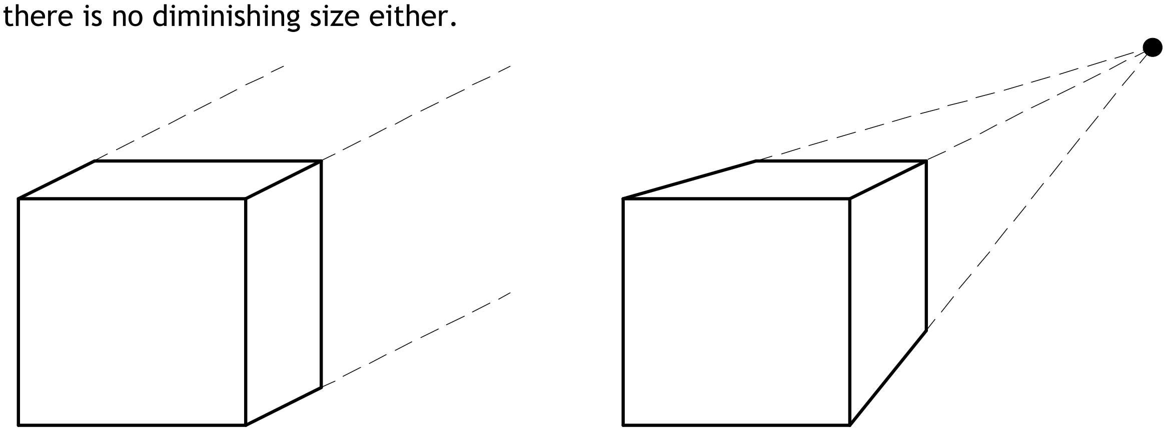


One method of specifying an ellipse is by the angle of the circle to the line of sight. Thus a 45° ellipse is equal to the distortion of the circle as seen from a 45° angle. A second method is to specify the ellipse according to the ratio of the major (longest) diameter to the minor (shortest) diameter. A 45° ellipse is equal to a 1.414 ellipse. This number is equal to 1/sin 45°. In practice we most often want to know how much we need to flatten a given circle to get the right ellipse. In this case the major diameter is the diameter of the original circle and the minor diameter is that distance multiplied by sin of the sight angle.



Orthographic Projection

The above diagram illustrates this 1/sin x relationship in orthographic projection, where it is easier to see. The distortion of the circle is similar to perspective. In orthographic projection objects project on the image plane at 90°, so that there is no COP as in perspective. The line of sight, too, is orthogonal to the image plane. Consequently there is no diminishing size either.

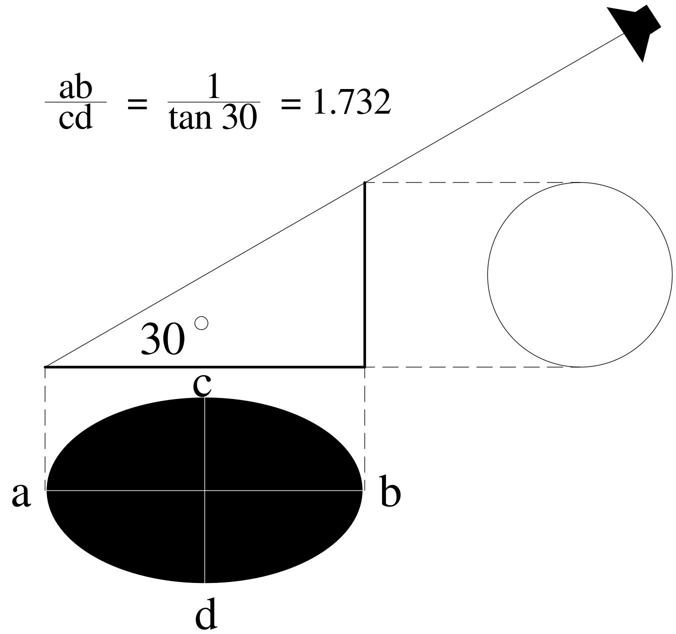


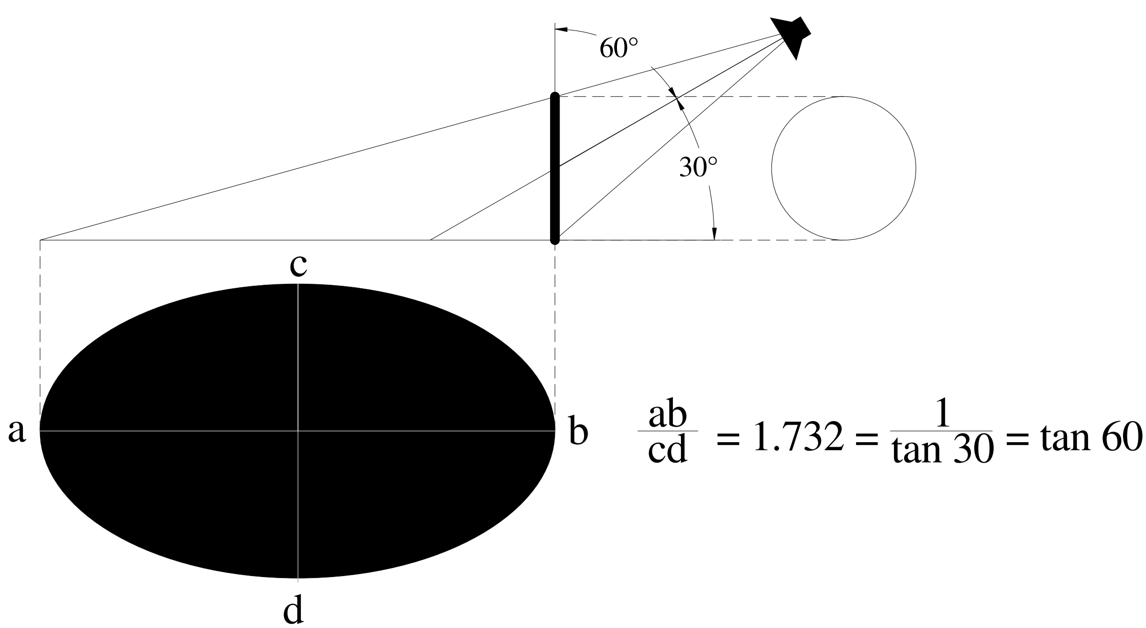
Orthographic projection indicates depth by virtue of oblique parallel lines. These lead to the other terms for this projection: oblique projection and parallel projection. This projection is useful in technical drawing, but does not convey the realism of perspective.

Shadows

Shadows, as we see them in our surroundings, are projections. The projective plane may be vertical as with the wall of a building, horizontal as with the ground, or at other angles as with the plane of a roof. The shadow may even fall on a surface, rather than a plane, for example, a shadow falling on a car or on a beach ball.

The diagram on the right shows the calculation for the shadow of a vertical circle cast onto the ground. The sun angles at 30° to the horizontal and its shadow is a circle elongated into a 30° ellipse. This elongation is expressed as a major/minor ratio of 1/tan 30. When expressed in relation to the angle of the sun to the vertical plane of the circle this ratio is tan 60. The tangents of complementary angles are inverse values.





The sun is an example of a parallel light source and its shadows are equivalent to the parallel projection version of shadow projection: there is no COP. (The sun is so far away - 93 million miles that it rays are, for all practical purposes, parallel) Suppose, however, that we replace the sun with a light bulb. Now we have a radiating light source, which also functions as a COP.

The diagram above depicts the light at 60° to the plane of the circle and 30° off of the horizontal. Note how much larger the elliptical shadow is due to the radiating of the light. Note, also, how the ellipse is similar in proportion to the shadow projected by the parallel light source. The radiating light widens the shadow as well as lengthens it.

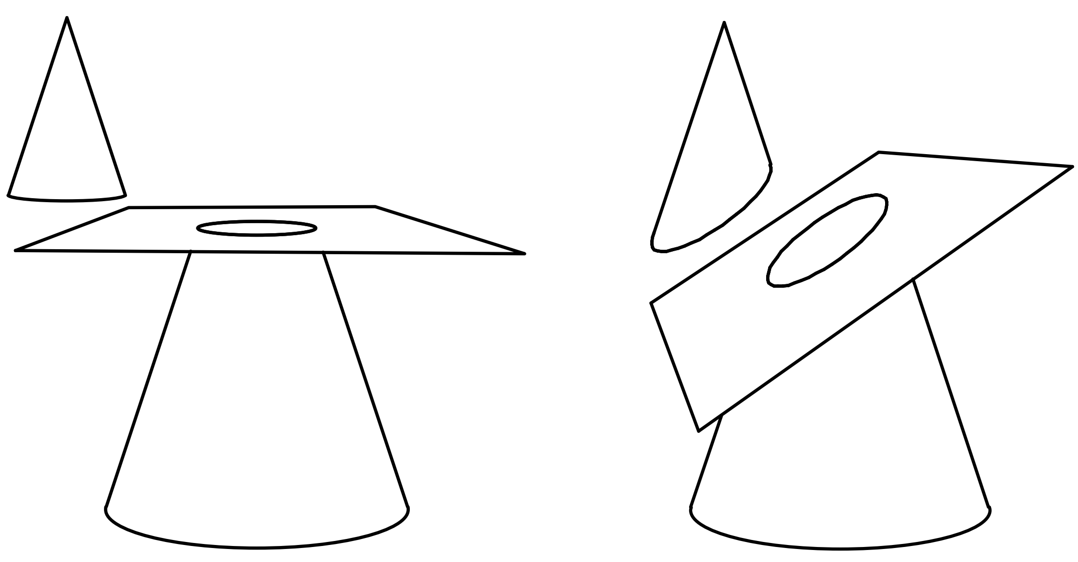
Conic Sections

The circle in the previous diagram creates a dark cone as it blocks the radiating light. Anything inside of this cone will be cloaked in darkness. The portion of the ground inside of this dark cone is the cast shadow of the circle. We can think of the elliptical cast shadow as a slice of the cone taken by the plane of the ground passing through it.

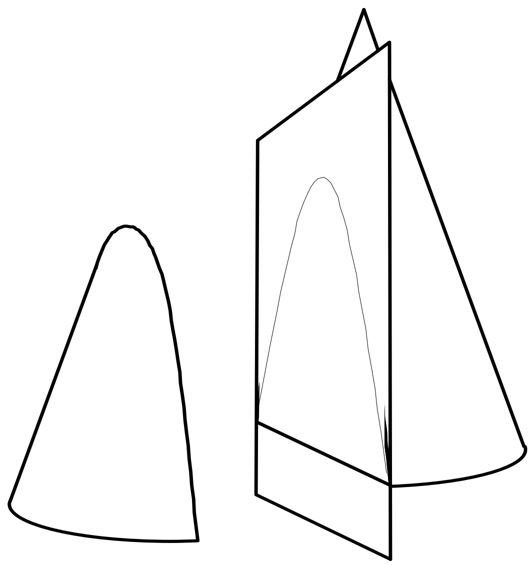
An ellipse is a slice - a section - of a cone, the result of a plane passed through a cone. As such it is classified as a conic section, one of a group of four curves produced by slicing a cone. The others in the group are the circle, the parabola, and the hyperbola.

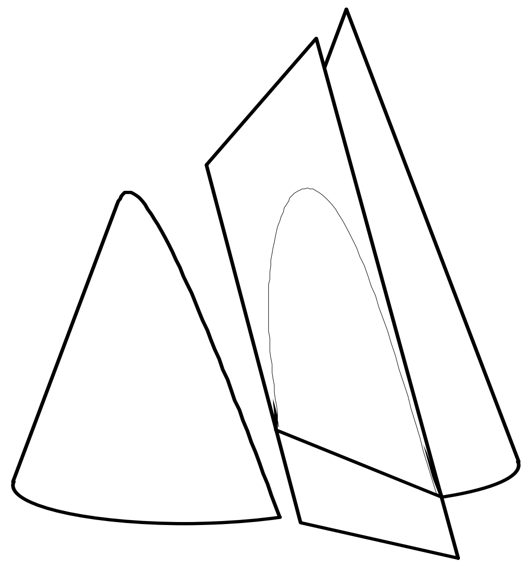
Conic sections came under close scrutiny during the development of perspective geometry in the Renaissance. They began to show up in the development of perspective for painting and architectural drawings and in shadow projection with advancements in the design of sundials. The artist Albrecht Durer entered mathematics history when he published a treatise on conic sections that resulted from his experiments with perspective.

An orthogonal slice of a cone will produce a circle and a transverse angled slice yields an ellipse. A parabola appears when the slice lies parallel to a side of the cone. The hyperbola arises when the slice passes through the axis at an angle less than the angle of the side of the cone to its axis.



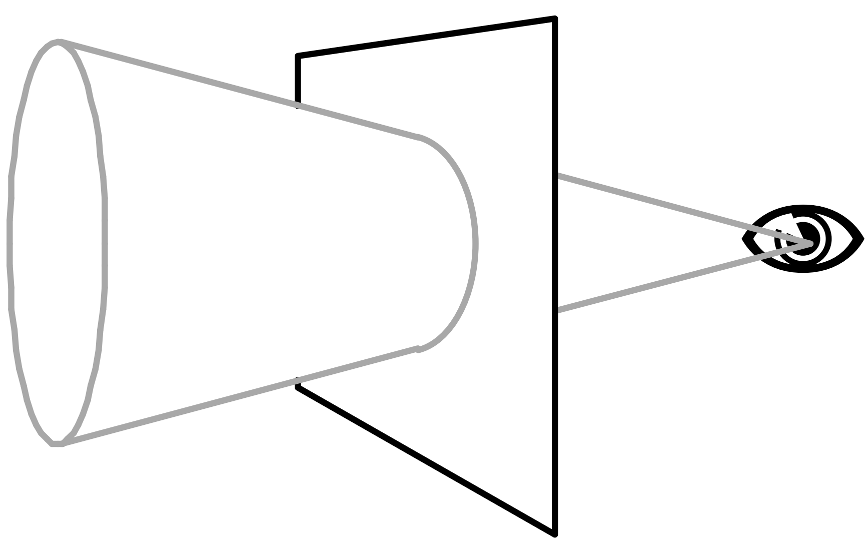
CIRCLE ELLIPSE



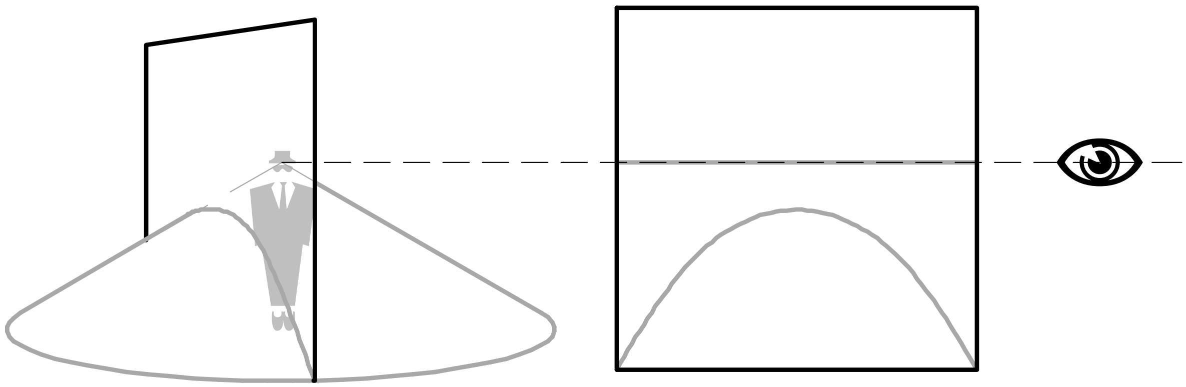


HYPERBOLA PARABOLA

According to the geometry of perspective projection the rays of a circle converge on the COP of our eye to form a virtual cone. As that cone passes through the image plane, a.k.a. the projective plane, it projects an image of the circle that is a slice of that virtual cone. The intersection of the image plane and the converging rays from the circle is a conic section, usually an ellipse if the circle is at an angle to the image plane or a circle if the circle is parallel to the image plane.



There are two special situations in which the circle projects on the image plane as a hyperbola or a parabola. The first occurs when the viewer stands inside a circular space, like a ringmaster at the center of the ring of a circus. As the ringmaster looks out a portion of the ring appears in his field of vision, but the rest of the ring swings around behind him. The ringmaster's eye is the apex the virtual cone formed by the ring and the COP. In this situation the image plane slices the cone at an angle needed to yield a hyperbola, as in the drawing below.

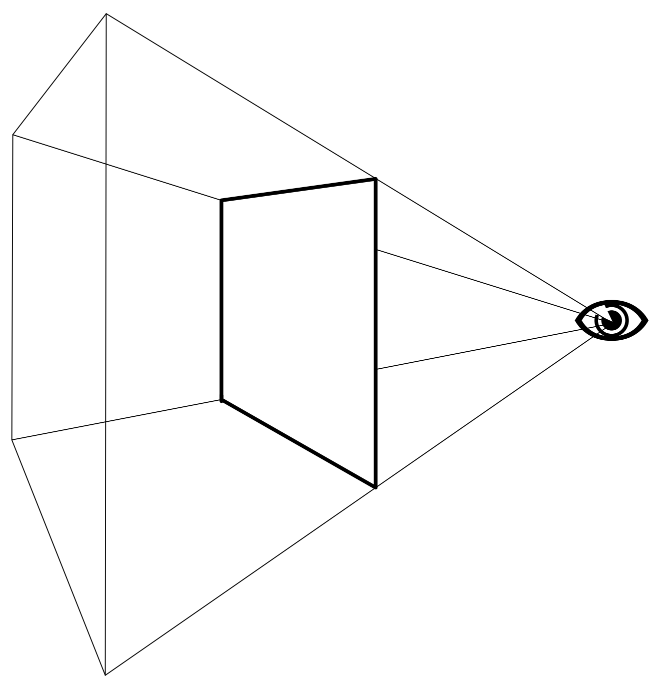


The second situation appears when the viewer stands on the perimeter of the circle, looking out along a diameter of the circle space. Drawing a cone from the circle to the eye of the viewer and it becomes apparent that the side of the cone is vertical to the viewer's eye. The image plane is also vertical, and thus parallel to the side of the perspective cone of the circle. The intersection of the image plane with this cone yields a parabola.

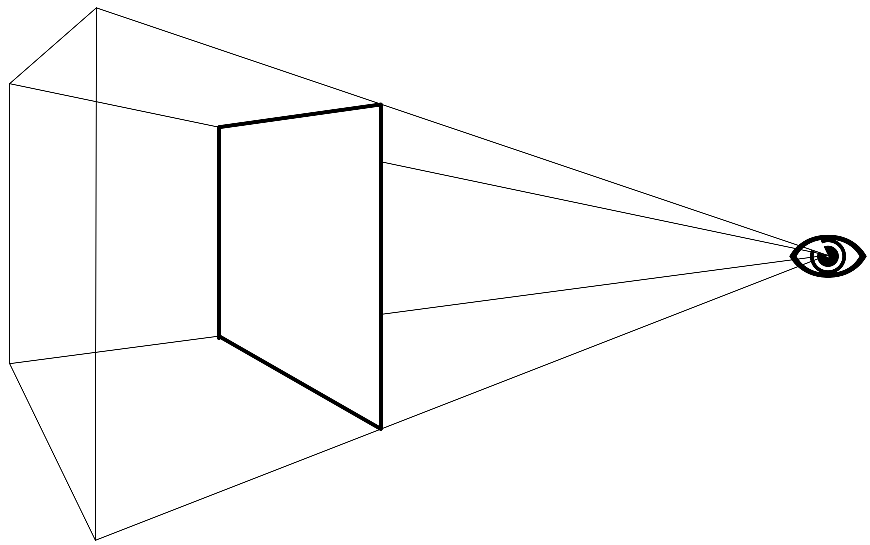
Wide Angle and Telephoto

In the ringmaster's view of the ring shown above, the effect looks exaggerated. When this occurs we say that the perspective is forced. This effect shows up when the COP is placed close to the projective plane. In the illustration of the example of the ringmaster he stands very near the plane, so that his eye, the COP, is very near, too.

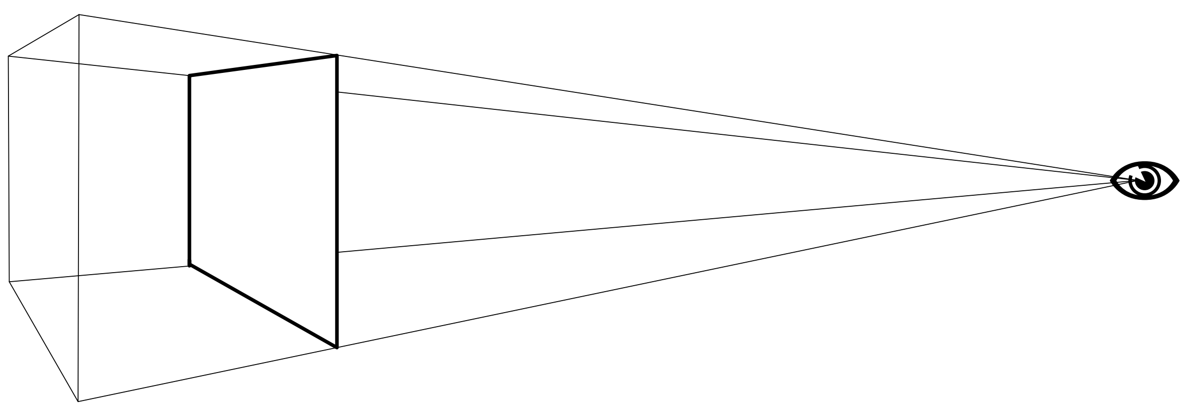
In the illustration to the left rays are drawn from the eye to the corners of the image frame, and then continued into a widening pyramid. The viewer can see only those objects that fall within the spread of the pyramid.



When the focal length - the distance from the COP to the projective plane - is shortened, the spread of the angle is widened. If this spread is greater than that permitted by the optics of the human eye, then we call this a wide-angle view. In the next two images the focal length has increased and thus narrowed the angle of vision.



Traditional 35mm film cameras have a default focal length of 50mm from the focal point of the lens, which is the camera's COP, to the film plane, which is the camera's projective plane. This approximates the viewing angle of the human eye and produces the most natural looking images. A typical wide-angle lens shortens this focal length to about half that length. Longer focal lengths are found in telephoto lenses: these start at about 75mm.



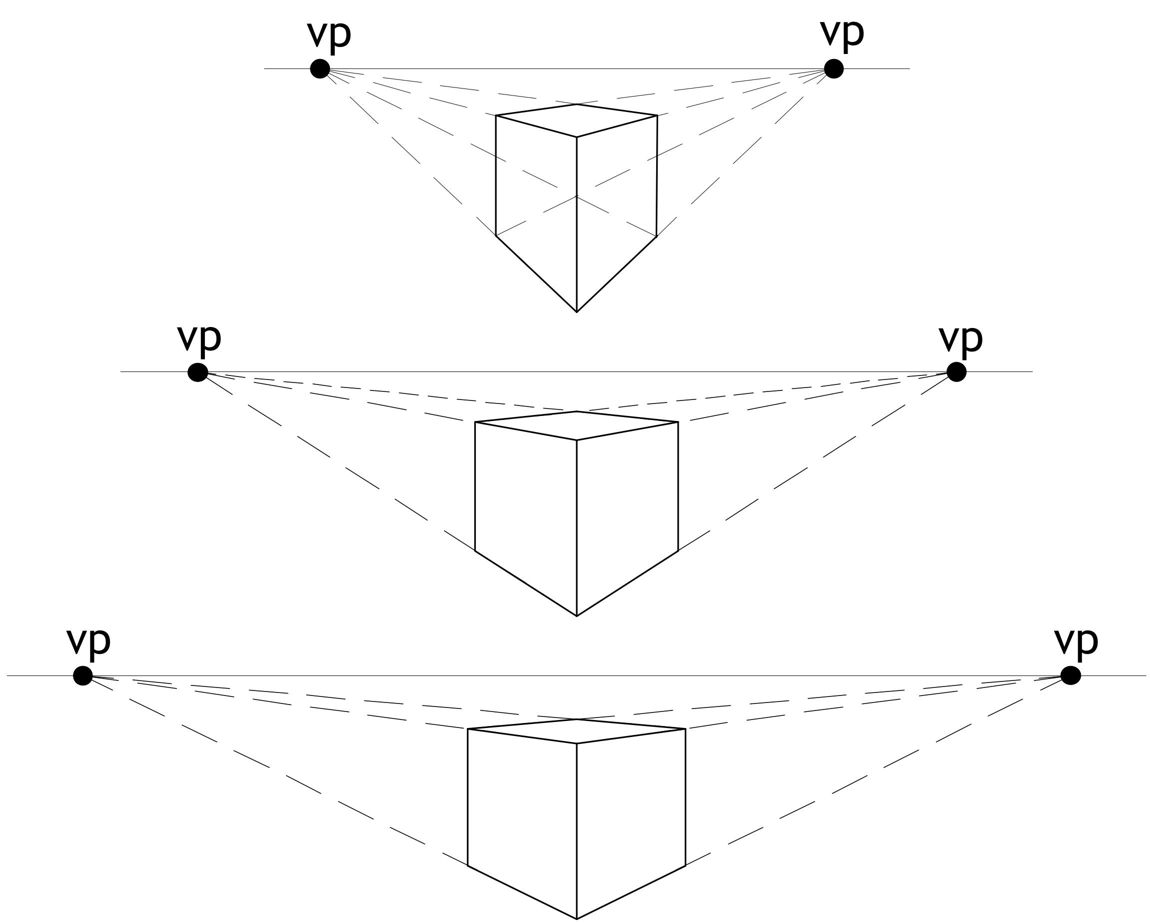
The objects in images shot through a telephoto lens appear larger because the longer focal length permits the lens to only capture a small area of the overall scene. However, this small area is projected on the same area of film as images shot with shorter focal lengths that encompass a wider field of view. Consequently the objects appear proportionally larger, since the smaller area of view must expand to fill the frame.

Two-Point Perspective

As noted earlier wide-angle views force the perspective, that is, they extend the sense of depth; by contrast telephoto views will flatten the perspective, that is, they diminish the sense of depth. This effect can best be demonstrated in two-point perspective.

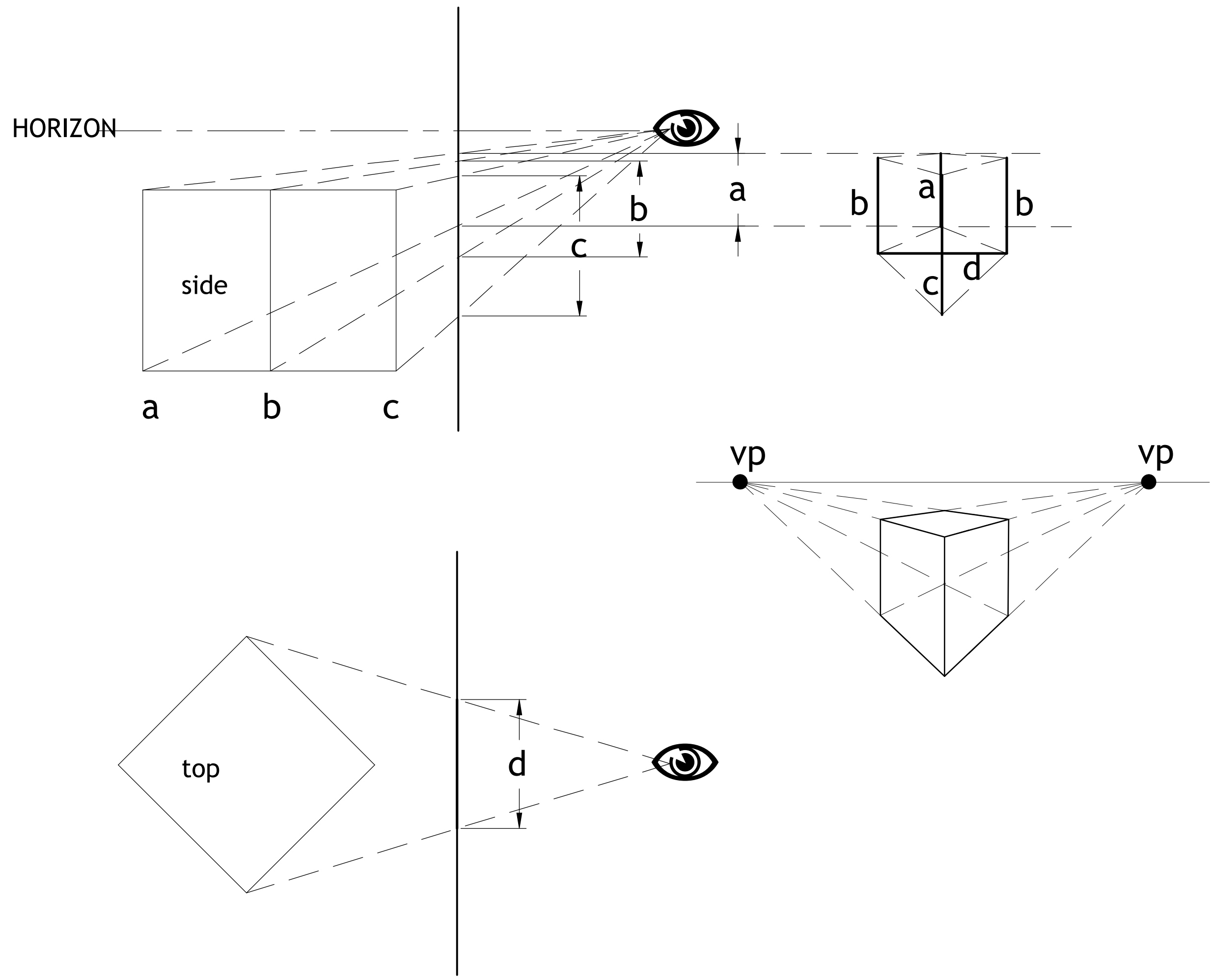
Two-point perspective requires a second vanishing point (vp), because the projected object needs to show two sets of parallel lines converging. If we observe a cube from just one side, then only one set of parallel edges moves back into space. If, however, we rotate the cube so that we view it from the corner edge, we can see two sets of parallel lines angling back into space.

The illustrations below depict three examples of two-point perspective: forced (wide-angle), normal and flattened (telephoto):



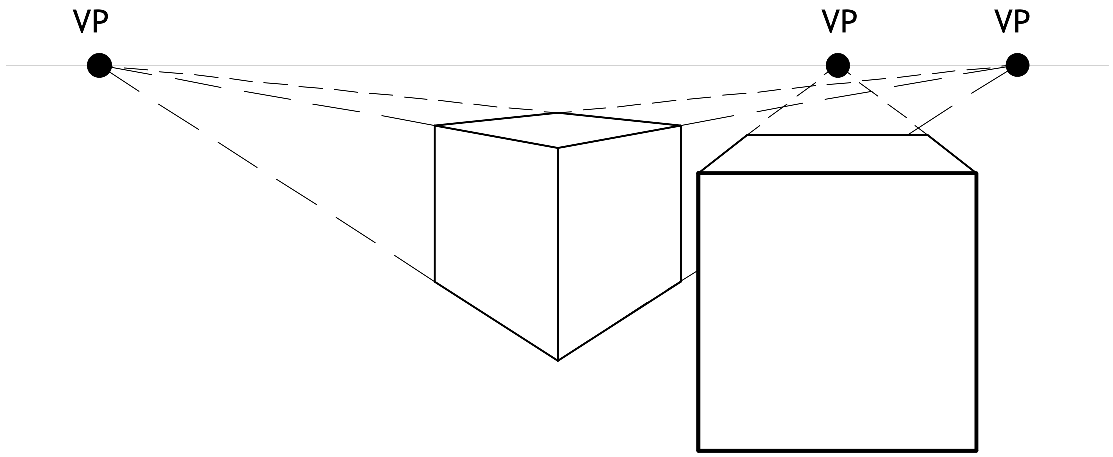
Spreading the two vanishing points further apart will flatten the perspective. This corresponds to the effect of stretching out the focal length. More to the point of the geometry of perspective projection is the fact that by spreading the vanishing points we are simultaneously increasing the distance from the object to the vanishing point as we increase the distance of the focal point. In perspective the positioning of the vp in the image space correlates with the position of the COP in 3-space.

The following diagrams apply the angle intercept method to the projection of a rotated cube and show how the horizontal and vertical intercepts determine the angular distortion of the cube. The further the eye moves back from the image plane, the flatter are the angles of the converging rays. This means that the difference in size between the front and back edges grows smaller and less obvious. The result is such that the angles of convergence will flatten and cause the image as a whole to flatten.



Multi-Point Perspective

Every time an object like a cube is rotated a new set of parallel lines comes into being along with its own vanishing points. Some famous paintings by Renaissance artists feature as many as 20 vanishing points. Such multi-point perspectives confer a sense of more active viewing, since each vanishing point establishes a direction of perceptual movement back into space. When viewing the world, the eye does not hold still, but darts over a scene and constantly changes the direction of the vanishing point. Multiple vanishing points help to duplicate this natural way of seeing.

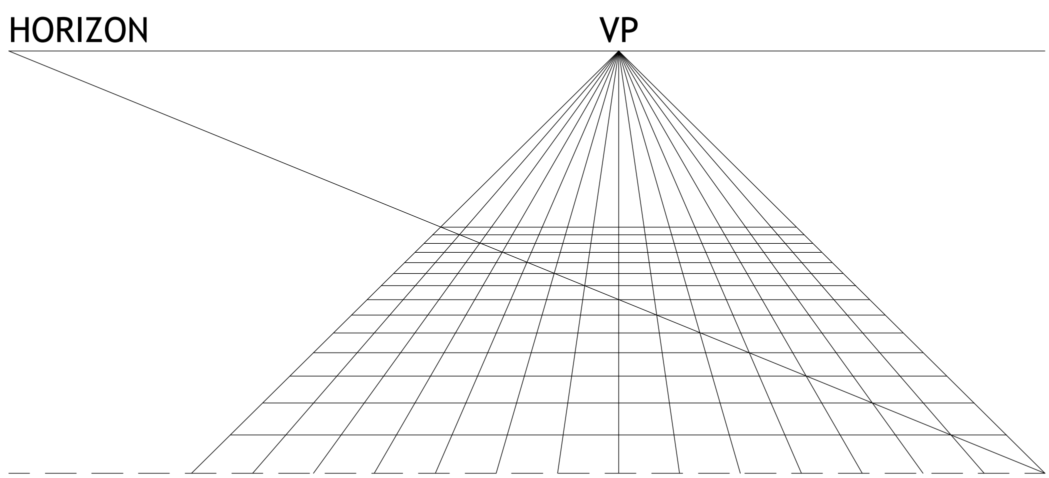


Projection Grids: Perspective

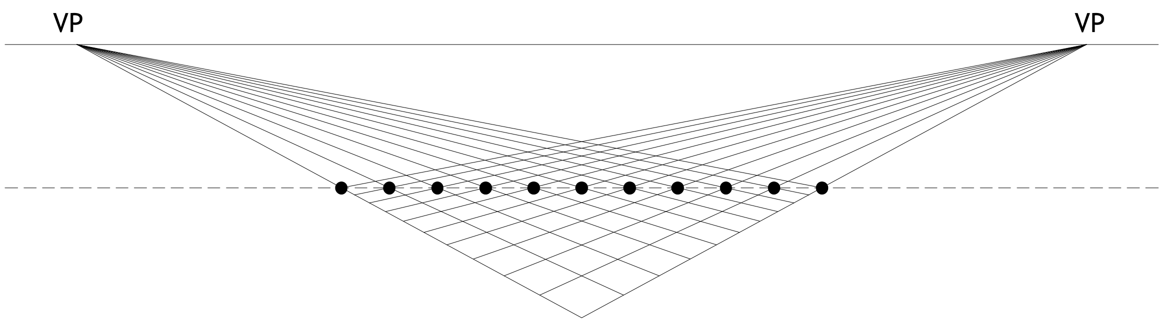
3D modeling programs will have one window that renders the image in perspective. Typically that window will have a grid, distorted in perspective, which references the grid in the orthographic top view windows. Laying out a grid in perspective was one of the most practical skills of the Renaissance painter. Theoretically, too, this is an important concept, because it extends the projection principles applied to individual object to the projection of an entire region of space.

The simplest form of perspective requires only one vanishing point. If we lay an orthogonal grid on the ground plane in one-point perspective, then all of the y co-ordinate lines will converge in the distance while the x co-ordinate lines will remain horizontal and parallel. The size of the grid squares will, however, diminish with the result that the x co-ordinate lines will space closer and closer together.

The diagram below illustrates a method for determining the vertical spacing of y co-ordinates in this perspective. Note how the intersection of the diagonal line with the perspective lines determines the positions of the horizontal lines. Continuing the diagonal to the horizon will determine its vanishing point. Just as was the case in two-point perspective, moving this point closer to the main vanishing point will force the perspective and moving away will flatten the perspective.



In two-point perspective both the x and y lines converge, but to separate vanishing points. In the diagram below the diagonal of the grid is horizontal. Equally spaced points on that line determine the intersection and consequently the placement of the grid lines.

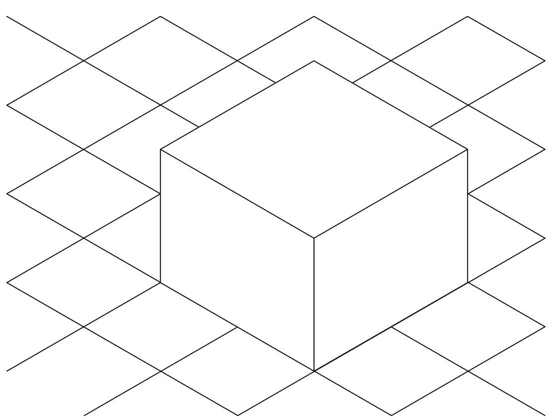


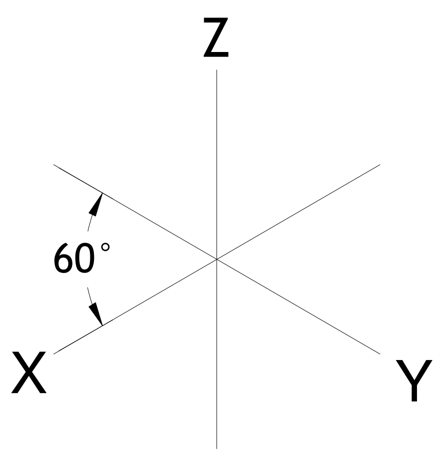
Projection Grids: Orthographic (Parallel)

In orthographic projection the grid distorts from squares into parallelograms. This means that as the grid moves back into space the grid units remain the same size. Objects drawn over a parallel grid will remain the same size as they recede into space.

In technical drawing most common version of parallel projection are axiometric. This means that the dimensions of the drawn object are proportionally equal along all three axis of space. In contrast to perspective projection's naturalistic feel, objects depicted axiometrically appear awkward. However, a curious engineer can measure a dimension in an axiometric drawing and, unlike perspective, rest assured that the dimension is correct.

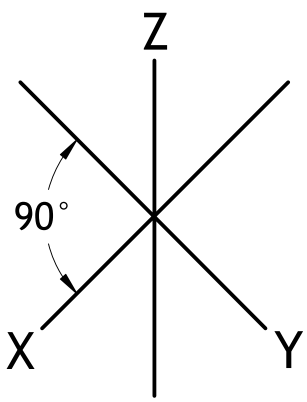
It is important to note that there are many projection systems and that the selection of a particular system is predicated on the function it is to serve. Axiometric orthographic systems are the most systematic in that they constrain the oblique lines, which indicate space, to specific angular arrangements that can be laid over simple grids.

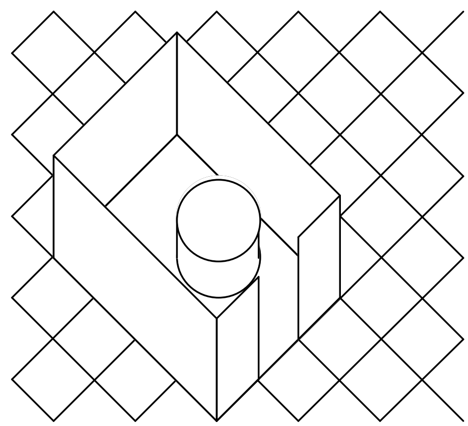




Isometric grids are the most widely applied for parallel projection. They comprise rhomboids of 60º and 120º angles. These are the angles that result when the x, y and z axes cross at the center of a three-dimensional space. An isometric grid derives easily from a net of equilateral triangles. Note the box in the following drawing. It is clear that the box is 2 units by 2 units and 1.5 units high. If each unit equals a fixed measure, say 3 inches, then the box can be seen to be 6 inches x 6 inches by 4.5 inches.

Another common axiometric grid sets the x and y lines 90º to one another, as if the orthogonal grid has been rotated 45º. This projection is useful in situations where the world plane layout is most important, such as architectural drawings where the floor layout needs to be emphasized within a three-dimensional view.





- ***Major Exercises***

- Major 1: Using CAD, create a perspective composition of three interacting rectangular solids placed on a table. Indicate the horizon, all vanishing points and all perspective lines as medium gray lines.

- Major 2: A circle distorts proportionally into the same ellipse whether in orthographic or perspective projection. Despite this apparent similarity the ellipses are different in the two projections. How? Why? Using CAD or Illustrator, create an instructional diagram to demonstrate the answer to these questions. (Hint: Be sure to include the center of the circle.)