GPH 259 Design Geometry
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Geometry of Spline Curves

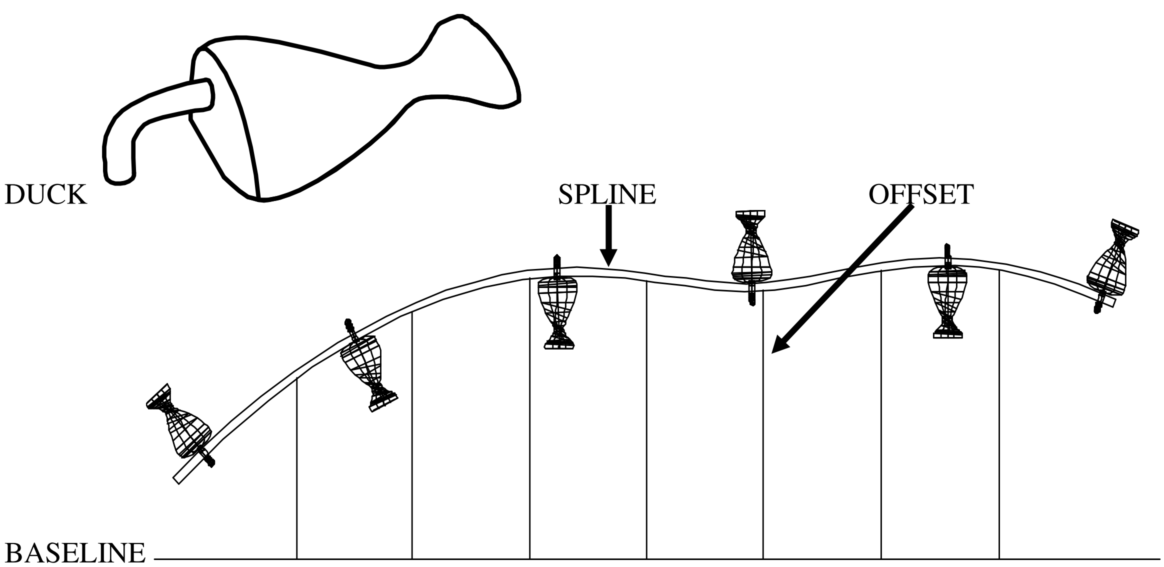
The traditional drawing tools of the draftsman – triangle, straight edge, t-square and compass are especially suited for architectural and mechanical drawing. However, such tools poorly represented the smoothly flowing curves found in nature and human designs inspired from nature.

I. <u>Splines</u>

In the shipbuilding industry draftsmen were frequently called upon to draw these types of smooth curves. Their solution was to represent these curves by tracing along thin lengths of wood or metal, supple enough to bend into streamlined arcs. These drawing aids, called splines, were held in place by lead weights, called ducks due to their resemblance to this aquatic fowl.

Interpolation

A spline is an interpolation curve: its function is to track a smooth line through a series of given points.



The classic method for drawing a spline was to specify a baseline and a series of points offset 90º at intervals along the line. The spline, held in place on the drafting table by the weight of the ducks, would snake through the sequence of points in a continuous curve. This is the origin of the phrase “putting all of one's ducks in a row”

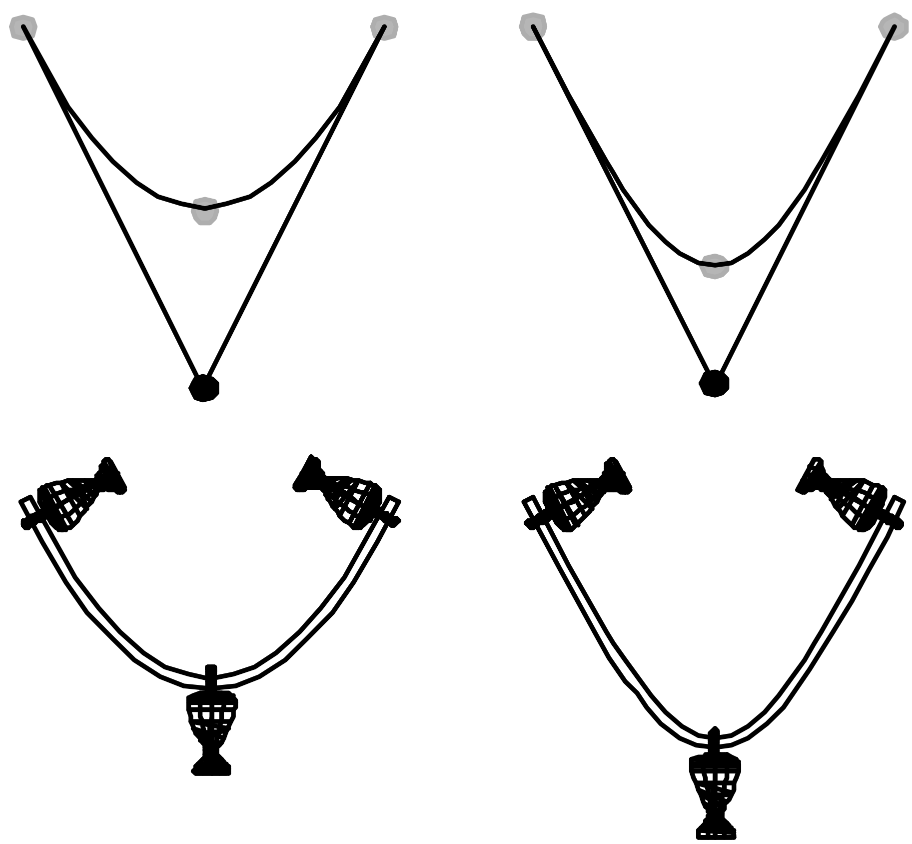
The ducks could be moved or more could be added in order to smooth the curve while still maintaining a path through the points. This process was called fairing, because its usual purpose was to improve the sleekness and visual appeal of the line (think fair as in "fair maiden"). Shipbuilders, and subsequently airplane and automobile designers, applied the term fairness to describe the degree of this appeal.

Mathematical Splines

While fairness is a technical term, it is a human judgment that cannot be quantified mathematically. However, in computer graphics mathematics can be used to define splines as well as aid in fairing and the visual elegance it engenders. Much of the appeal of spline geometry is its coordination of visual and mathematical goals.

Around 1940 mathematicians adopted the idea of the draftsman's spline in a quest to develop a single geometry of curves that could approximate all curves plotted onto a coordinate grid from polynomial equations. This host of curves includes conic sections, polar curves like spirals and conchoids, and others.

Mathematicians started with the equation that the great 18th century geometer Euler created for describing the physical deformation of a weighted beam. The strip of wood comprising a spline was, in effect, an extremely thin beam. Although completely useless as a beam, it did behave physically like a beam. In the illustration above the curves of the spline result from the deformations caused by the pulling and pushing forces of the ducks.

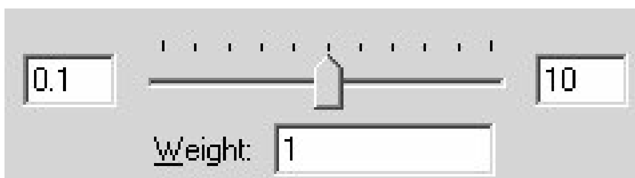


Three points, called control points, determine the geometry required in flexing a basic spline. Two of the points are endpoints of the spline arc and the third lies off of the line. The spline bends towards the off-line point, which exerts a "weight" on the curve that determines how much it flexes. This is equivalent to increasing the weight supported by a beam.

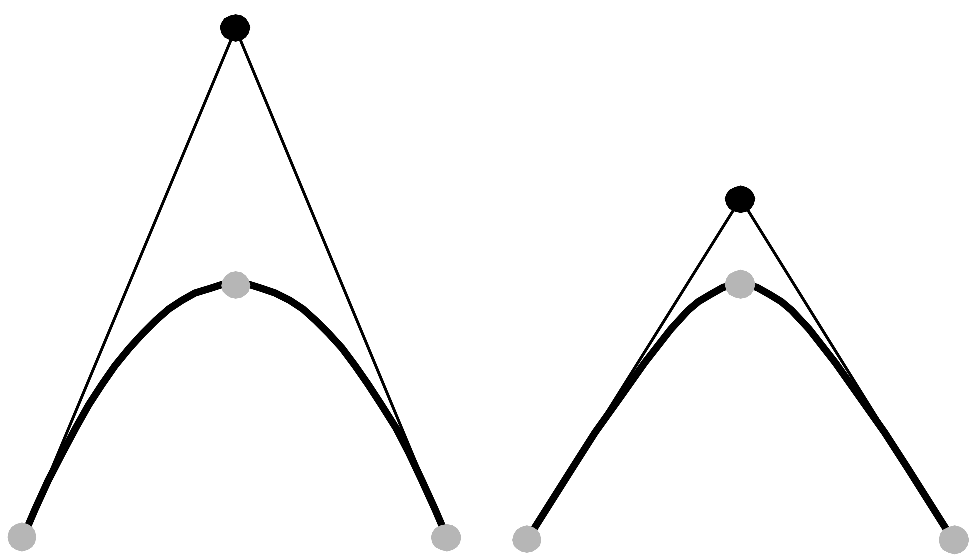
Both spline curves above possess congruent control points, but the curves themselves are not congruent and interpolate different points. This is due to the change in weight assigned to the off-line points of the two curves. The off-line point for the curve on the left is weighted at 1.0; the curve on the right is weighted at 2.0. This increased weight is equivalent to pulling harder on the center duck, to increase the force on the spline. The outcome is a sharper curve. In both curves the endpoints remain tangent to the lines that connect them to the off-line point.

What the numbers mean…

The mathematical “weights”, called *rho* numbers, are arranged on a two-part scale. The first half of the scale below gradates from 0 to 1 and the second half gradates from 1 upward. In the slider scale below the upper end is set at 10 as a practical limit. This division of the scale is due to the fact that numbers below 1 produce arcs of conic sections (see below). Also the rho number of 1 will deform the curve half the distance to the control point. Higher numbers will pull the curve toward the control point in decreasing increments approaching infinity.



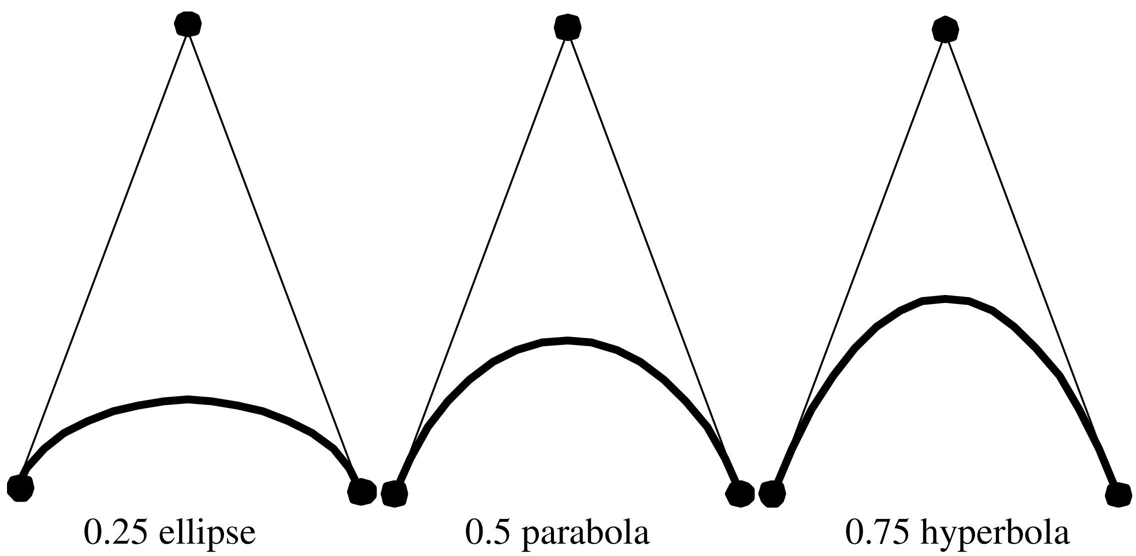
The two splines below interpolate through three congruent points even though their off-line control points are spaced at different heights. The curve with the higher point has a weight of 1.0 tugging on it, while the curve with the lower point has a force of 3 bending it. This is equivalent to a draftsman using a thinner and shorter spline.



The visual consequence of manipulating weights of the off-line control point is that the artist/geometer can draw curves of very different character through the same set of points.

Conics

The weight of a control point also has a geometric consequence in that this attribute allows spline curves to replicate arcs of conic sections. The weights, also called the *rho* values, below 1.0 will produce conic curves. Values less than 0.5 yield elliptical arcs; a value of 0.5 will yield a parabola; and values between 0.5 and 1 will yield a hyperbola.

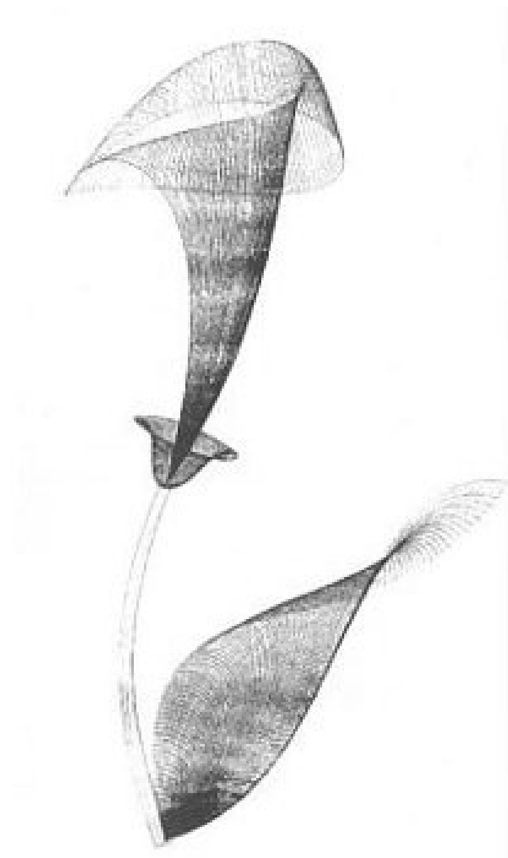


Splines in Computing

By 1960 research was underway to develop methods for computing splines and by the 1970's splines began to appear in commercial design programs. Between the late 1957 and 1975 the designer-engineer Pierre Bezier worked on incorporating splines into design programs for Renault, the noted French automaker. Bezier's chief research was developing numerical controls for cutting and drawing machines. One of his drawings is reproduced to the right. The Bezier curve is still a standard tool in CAD and graphics programs.

The Bezier curve turned out to be one of a number of mutations of a more general spline, the B-spline, which stands for basis-spline (the splines described above are simple B-splines). While the Bezier curve was and still is a very effective design tool, the B-spline could adapt to describe virtually any geometric shape or line, including straight lines and angles. The most adaptable and stable version of this curve is the non-uniform rational B-spline, or NURBS. Non-uniform refers to its ability to take on free-form shapes and rational refers to its mathematical stability under transformation. This made it as computable as vectors.

Splines could represent anything that vectors could and more. Programs based on NURBS geometry opened up a new world of elegant, natural appearing form. Designers were now limited less by geometry, than by their ability to draw.

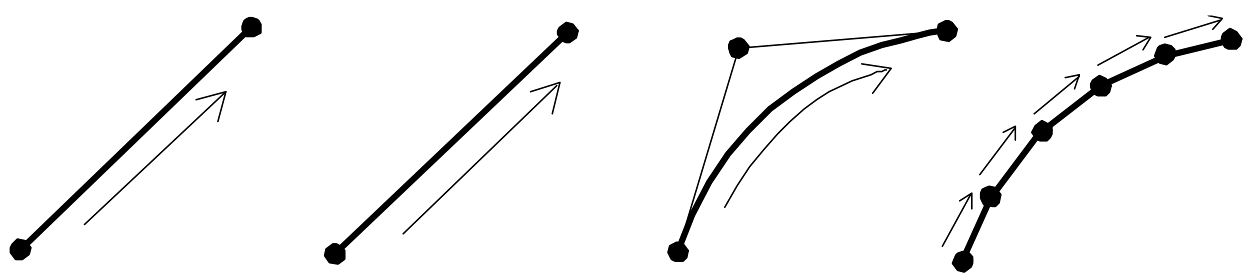


II. NURBS Geometry

The NURBS space is very like vector space, with the exception that its basic unit of form is a NURBS rather than a ray segment. NURBS geometry manifests an effectiveness similar to vector geometry in computing transformations, but it is significantly more efficient in describing form.

The NURBS Unit

A programmer can efficiently define a NURBS unit by a) two control points, b) a direction and c) a weighted value. A NURBS straight line has two control points, one at each end, a direction and with no off-line control it has no weighted value. It can be computed as efficiently as a vector.

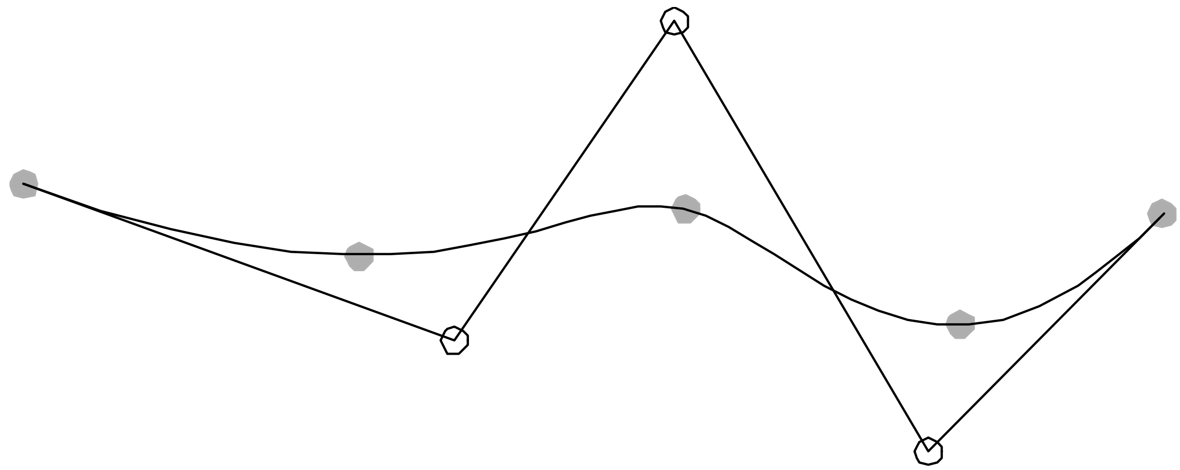


vector ***NURBS line*** ***NURBS curve*** ***vector curve***

The real efficiency comes with computing curves. Here NURBS surpass vectors. In vector geometry the program must control a series of many vectors in order to define a curve. In NURBS geometry just two added factors, a control point and a weight, will create a single curved unit.

Piece-Wise Curves

A real spline, though, is more than a single bowed curve: it can spiral, snake from side to side and even loop back on itself. A single NURBS curve, too, can imitate its physical counterpart in how it distributes any number of control points on the same and/or opposite sides of the curve. Each control point affects a different piece of the curve, with each piece continuous to the next. This piece-wise construction is essential to the success of NURBS geometry as a drawing and modeling environment.

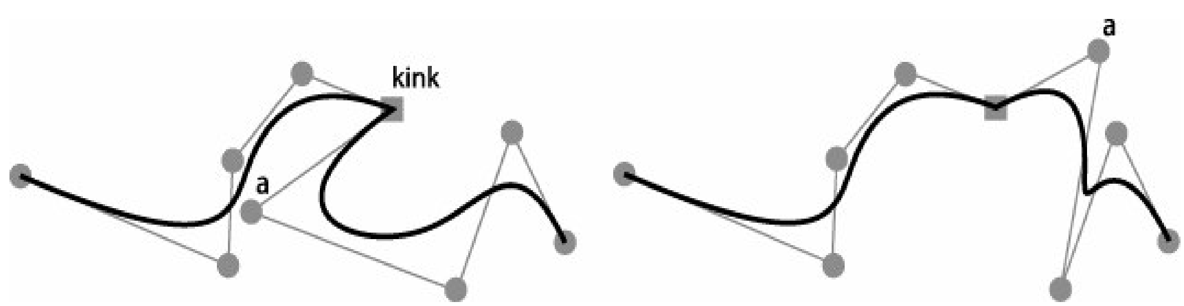


The pieces are tied together by a series of *knots*, linking points that maintain the curve as a continuous whole. Imagine tying a number of shorter ropes together. They will behave like one longer rope. Now replace the knots with splices so elegantly crafted that they are practically invisible to the human eye. In NURBS geometry knots are the mathematical versions of these splices. In the figure below the gray dots indicate knots and the white dots are the control points. The angled lines are referred to as the spline polygon.

Continuity

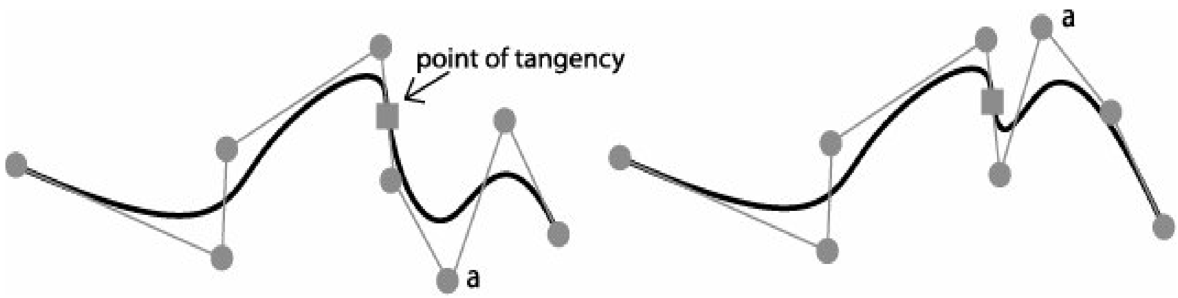
Although they are constructed piece-wise, NURBS are continuous. The NURBS geometry in most programs distinguishes between three degrees of continuity:

- C1: Also called ***0*** continuity. The pieces meet, but do not flow. Instead they form a "corner", called a *kink*. A kink is like joining two pieces of rope with a hinge.



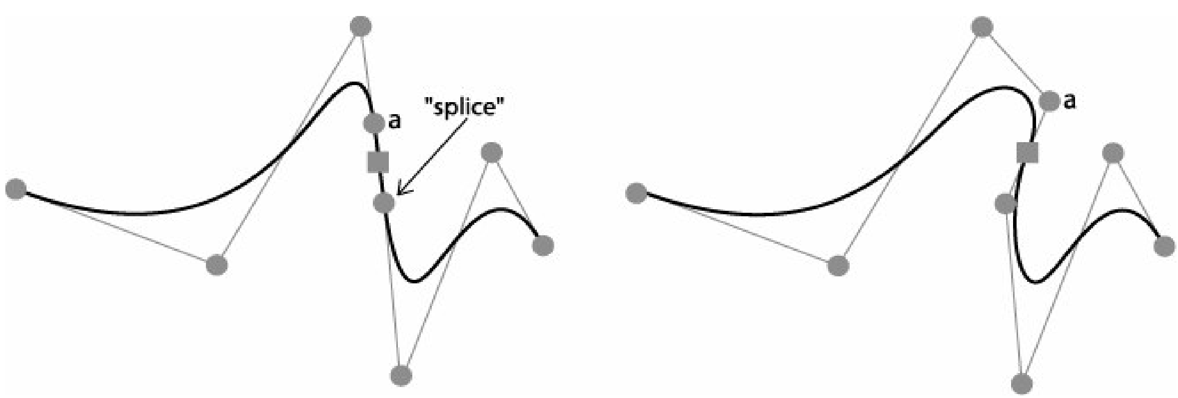
The kink acts as the mutual endpoint of the two NURBS pieces it connects. Moving the kink will change the curvature on each side of the kink. Moving a control point on one side of the kink will change the piece it controls as well as the neighboring pieces, but this change will not affect the kink, nor will it affect the curve piece situated across the kink. In the example above note how moving control point **a** affects the piece on the right of the kink, but not to the left. Change will flow through a knot, but not through a kink.

- C2: Also called first order continuity. The pieces meet such that they are tangent at the same point. A tangent joining is like joining two pieces of rope by lashing their ends to a splint.



This produces visual flow, but will not represent the physics of curvature in a material spline. There is no representation of physical flow, because change will not flow across the meeting point of the two pieces. Rather, each piece will preserve tangency as it changes, but this change will not affect the piece on the opposite side of the tangent meeting point. Note in the example above how moving control point **a** affects the curve to the right of the point of tangency, but not to the left.

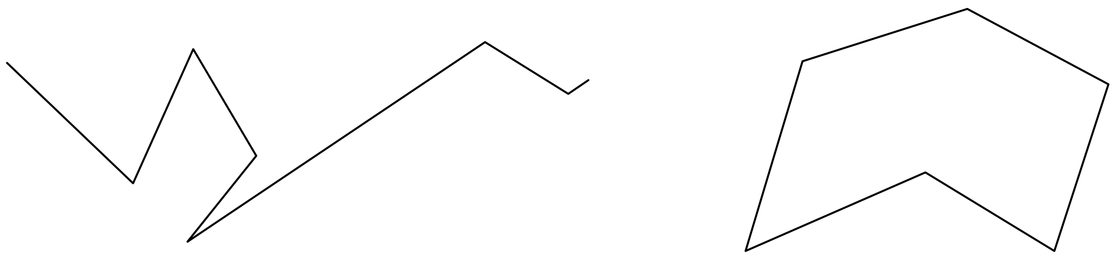
- C3: This level of continuity allows change to flow across the mating of the two curve pieces. The spline can then fully represent the behavior of a physical spline. This is like joining two pieces of rope with a splice.

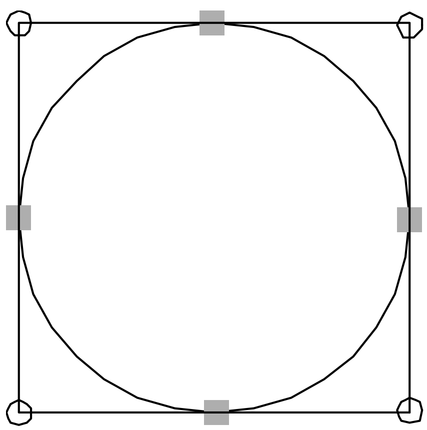


Ovals and Polygons

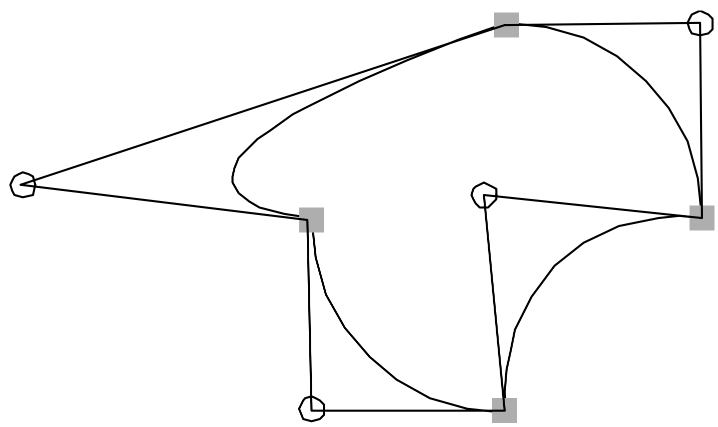
The ability of a NURBS to piece together with varying levels of continuity allows it to accurately depict conventional geometry. Kinked B-splines with 0 weight form polygonal curves. In NURBS terminology all lines and plane figures are constructed as curves and consequently are referred to as curves.

Polygonal Curves: Open and Closed





Circles and ovals:
4 kinks and four control points



An "edited" circle

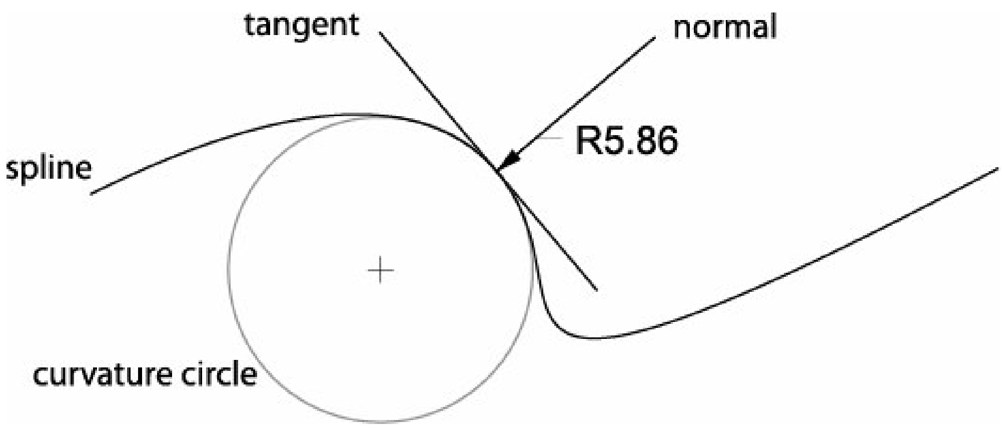
Ovals and circles are pieced together from conic segments joined by clamped knots – the number depends on the program. Clamped knots behave similarly to kinks, but affect the curve differently. Clamping consists of fixing a group of knots together as one. While continuity will flow through a single knot it will not flow through a coincident group of, for example, three knots. The space between the knots is 0 and, although the continuity passes through to the next piece of the curve, that piece has no length.

Even though it has nowhere to go, the fact that the clamped knot does not block continuity permits the arcs to be elliptical. Note how each arc of the circle remains elliptical as the control point moves.

Curvature Circles and Normals

The most common measure of curvature is the radius dimension. In a circle this is always the same, but in a standard spline curve there are an infinite number of radii, one for each point on the curve. Excluding circular arcs and straight lines, B-splines are curves of continuously changing radii. The circle formed by the radius at a particular point is called the *curvature circle* of that point.

A radius of a circle is said to be a *normal* of the circle: it is a line perpendicular to the circle at the point it intersects the perimeter, or, more precisely, perpendicular to the tangent at that same point.



Spline curves have normals, too. Each point on the curve has a unique line, which is perpendicular to the tangent at that point. This line is the unique radial of the curvature circle corresponding to that point.

Spline Aesthetics

Living form in nature is built from curved lines and curved surfaces. Polygons need not apply.

These forms evolve due to stresses within and without the form. The curvature of bones, for example, evolved as the most efficient way of depositing material to resist the forces placed on the bone at the joints. Hydrostatic pressure, also accounts for some of nature's most elegant stress curves. This is the internal pressure of our bodies and cells – like water in a water balloon – that stretches skin and muscles into sleek curves. More marked in our teens and twenties, this pressure reduces as we age and the younger, sleeker splines take on more knots and kinks, which we call wrinkles or character, depending on your point of view.

Much of the beauty and uniqueness of form we sense in nature is the consequence of the physical, stressed curves that. In computer graphics, we represent through splines.

- ***Spline Exercises***

These exercises will be executed in a single document. They stress problem solving by applying ideas and methods used in vector and constructive geometry to spline curves. Try to master these exercises without seeking help. Seek your solutions from the following Rhino menus: Line, Curve, Dimension and Analyze.

Using Rhino draw two given control point curves:

Curve A has the following sequence of control points –

(-20,6), (1,10), (15,6), (9,0), (0,6), (7,-12)

Curve B has the following sequence of control points –

(10,10), (4,5), (6,-5)

Create a document that provides the following information:

1) **The coordinates of the points of intersection of the two curves.**
2) **The radius of curve A at the points where it intersects curve B.**
3) **The common normal of the two curves.**
4) **The radius of curve A at its highest x value.**
5) **The angular direction of curve A at y=-10.**