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Group theory students' perceptions of binary operation

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Abstract
Binary operations are one of the fundamental structures underlying our number and algebraic systems. Yet, researchers have often left their role implicit as they model student understanding of abstract structures. In this paper, we directly analyze students' perceptions of the general binary operation via a two-phase study consisting of task-based surveys and interviews. We document what attributes of binary operation group theory students perceive as critical and what types of metaphors students use to convey these attributes. We found that many students treat superficial features as critical (such as element-operator-element formatting) and do not always perceive critical features as essential (such as the binary attribute). Further, these attributes were communicated across three metaphor categories: arithmetic-related, function-related, and organization-related.

Keywords Binary operation · Group theory · Metaphors · Variation

Binary operations are one of the fundamental structures underlying our number and algebraic systems. Yet, researchers have often left their role implicit as they model student understanding of abstract structures. In this paper, we directly analyze students' perceptions of the general binary operation via a two-phase study consisting of task-based surveys and interviews. We document what attributes of binary operation group theory students perceive as critical and what types of metaphors students use to convey these attributes. We found that many students treat superficial features as critical (such as element-operator-element formatting) and do not

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always perceive critical features as essential (such as the binary attribute). Further, these attributes were communicated across three metaphor categories: arithmetic-based, function-based, and organization-based.

Binary operations are a foundational topic throughout the mathematics curriculum, spanning areas such as arithmetic, functions, and matrices (see Bergeron & Alcántara, 2015). Although binary operations are profuse through all levels of mathematics, they are not formally defined until advanced undergraduate courses such as group theory. In this setting, a binary operation is defined as:

*A binary operation * on a set S is a function mapping S x S into S.*[1]

This general treatment of binary operation can serve to highlight the structural similarities across the many operations that exist in our mathematical systems. Despite the prevalence of operations in our mathematical systems, little attention has been given to student understanding of the general binary operation. This is disconcerting for two reasons: (1) binary operation is one of the core concepts that can be connected from group theory to the secondary level (e.g., Novotná & Hoch, 2008); (2) according to a survey of expert group theory instructors, binary operation is one of the most important topics in group theory (Melhuish & Fasteen, 2016).

This survey of expert instructors also revealed a possible cause for the neglect. Through several rounds of ratings (via a Delphi study consensus protocol), the experts identified binary operation as one of the least difficult topics for students (Melhuish & Fasteen, 2016). This Delphi study served to launch a large-scale exploration of student understanding of the identified important group theory topics. We found that student understanding of binary operation influenced performance on tasks targeting topics ranging from the associative property to subgroups (Melhuish & Fasteen, 2016; Melhuish, 2018). Such results align with Larsen (2009) and Wasserman (2017) who have noted students struggle when engaging with binary operations in group theory. To address students' struggle with binary operations, we designed a qualitative study to explicitly probe student perceptions of binary operations. We use a variation theory lens (Marton & Booth, 2013) to focus on students' varying perceptions of binary operations in terms of attributes and communicated metaphors. Our research questions are:

1. What attributes do students treat as critical when engaging with binary operation tasks?
2. What metaphors do students leverage when communicating about operations and their attributes?

1 Theoretical framing

In order to analyze students' perceptions of operations, we first provide a grounding theory on how students learn, variation theory. We treat learning as an individualized experience through which variations in concept instantiations provide the primary context for learning (Marton & Booth, 2013). From this view, learning is defined as the perception of new attributes of a

[1] We include closure in our binary operation definition, but acknowledge the inclusion of closure is not universal.

phenomenon or experience of a phenomenon in a new way. For a given individual, their understanding of a concept reflects which attributes of the concept are foregrounded for them. Through variation in instantiations, an individual may become aware of different properties. By comparing examples and non-examples, students may perceive attributes as either *critical aspects* or *permissible variations* (Marton & Pang, 2006). A critical aspect is an attribute of a given concept that is invariant across all examples. In the case of binary operation, students may not discern the critical aspect of two inputs if they never encounter a non-example with a different number of inputs. Permissible variations in contrast are features that can vary across examples of a given concept. If a student only experiences instantiations with a particular variation, they may overgeneralize that attribute to be a critical aspect. In the case of binary operation, a student may perceive a feature such as a "+" symbol as being critical if they never experience contrasting notation.

Variation theory provides a complementary tool to cognitive or social investigations of learning via shifting from cognition to perception (Dahlin, 2007). From a variation theory stance, there is a positivist underlying assumption indicating that an expert could identify critical aspects and permissible variations for a given mathematical concept. It is then through a student's interaction with the concept that opportunity to learn occurs. These opportunities may or may not lead to students perceiving the desired attributes. Variation theory posits that in learning situations, mathematical objects exist in three stages*: intended object of learning* (teacher or curricular goal of learning), the *enacted object of learning* (what is actually provided in a learning situation), as well as the *lived object of learning* (Marton & Pang, 2006). The lived object reflects what students learn, that is what they perceive as critical after their experiences. The lived object for each individual may vary as their experiences (even within the same classroom) lead to different perceptions. In fact, it is not just variations observed in a given lesson or course at the same time (synchronic), but also their earlier related experiences that can provide the backdrop for perceiving variations (diachronic simultaneity, Vikström, 2008). In many ways, the lived object of learning parallels constructs such as Tall and Vinner's (1981) *concept image*—a students' concept image and surrounding cognitive structures. A student's knowledge of a concept goes beyond a memorized formal definition to all sorts of surrounding attributes. Students' experiences with examples play a large role in their concept understanding. As such, even in the formal setting of group theory, one would expect students to have perceived varied critical aspects of binary operation beyond a given formal definition.

Marton and Booth (2013) have described this learning process as both discerning features and structuring the features into a larger image of a given concept. Vikström (2008) has expanded variation theory to incorporate metaphor as a way to make sense of this structuring. Metaphors provide a tool for concretizing abstract mathematical concepts (Lakoff & Núñez, 2000). We adopt Vikström's view that metaphors serve as the means for students to structure and communicate their lived objects of learning. When students engage in tasks, they rely on metaphors to connect a given situation to their greater understanding of a concept. Generally, metaphors are defined as a projection from a source domain onto a target domain. For example, the familiar context of object collections (source) may serve as a metaphor for the mathematical context of arithmetic (target). Lakoff and Núñez distinguish between *grounding metaphors*, which are metaphors that connect mathematics to everyday experience, and *linking metaphors*, those metaphors that connect different mathematical domains. For the scope of this paper, we focus on students' grounding metaphors for understanding binary operations.

Using Vikström's (2008) theory linking variation and metaphors, we can examine a students' lived object of learning by attending to both their grounding metaphors and the critical aspects communicated with these metaphors.

2 Literature review

In order to situate our work, we turn to the literature to describe research that has been done related to binary operation. In general, researchers make occasional nods to its importance or difficulty such as in Larsen's (2009) group reinvention study identifying that students required significant support "to formulate a definition for operation" (p. 125). However, researchers have not yet explored student conceptions of binary operation beyond theoretical cognitive models (e.g., Brown, DeVries, Dubinsky, and Thomas's, 1997 *action-process-object* decompositions) or illustrations for more overarching theories (e.g., Hazzan's, 1999 *reducing abstraction*). We synthesize this work, other studies related to arithmetic (the most studied specific type of binary operations), and literature related to function, as binary operation is a specialized case of function. Such work backgrounds our analysis of binary operation and contributes potential attributes and metaphors that may carry over to the general operation setting.

2.1 The process-object duality of operations in arithmetic

The K-12 literature has documented important aspects of understanding arithmetic operations including shifting away from a need for concrete referents (e.g., Slavit, 1998) and seeing an operation expression as both a process and object (e.g., Gray & Tall, 1994). These aspects can be similarly seen in genetic decompositions of the general operation construct. Brown et al. (1997) and Wasserman (2017) provide two such decompositions as theoretical tools for their larger studies of student understanding in group theory. These decompositions illustrate progressively more sophisticated conceptions of the operation concept. An *action* conception is reflected in ability to "evaluate the operation in a specific instance, frequently one step at a time and often requiring some visible physical manipulations" (p. 184). For example, adding two elements may require the merger of two collections of items. Once students no longer need to have concrete referents and can think of the operation in its totality, they have a *process* conception. An action can be anticipated without having to carry it out. Returning to our addition example, a student can conceive of adding two numbers without needing individual numbers to act on. If this process is encapsulated, that is, if a student can see the operation as something that can be manipulated, the student has an *object* conception. Wasserman explains a student with an object conception of operation would see the expression 2 + 3 as equivalent to 5. When prior conceptions are coordinated and can be used in a problem-solving situation, a student then has a *schema* conception. Researchers have documented that students may be limited to action or process conceptions of binary operation particularly when working with unfamiliar operations in the context of group theory (e.g., Larsen, 2009).

2.2 Activity related to unfamiliar operations

Another set of group theory studies has explicitly focused on the role of unfamiliar operations in either uncovering student struggles or making increases in sophistication. Novotná and

Hoch (2008) argued that recognizing familiar operations (such as addition) and then recognizing unfamiliar binary operations (such as $x + y + 4$ over $\boldsymbol{Z}$) are markers of structure sense in abstract algebra. Hazzan (1999) argued that students without such sophistication might not be able to engage with binary operation constructs (like inverses) in unfamiliar settings. For example, she documented a student providing ½ as the inverse of 2 for the operation addition mod 3. Rather than engaging with an unfamiliar operation, the student reverted to the familiar operation of multiplication. Melhuish (2018) similarly documented that providing the unfamiliar context of modular arithmetic can unearth potential issues related to binary operation. In her large-scale study, she found many students did not recognize addition mod 3 and addition mod 6 as different operations (over $\{0, 1, 2\}$ and $\{0, 1, 2, 3, 4, 5\}$, respectively), which was problematic for determining whether $\boldsymbol{Z}_3$ or $\{0, 1, 2\}$ forms a subgroup of $\boldsymbol{Z}_6$ with over half of a sample of 800 students responding incorrectly.[2] Novotná and Hoch (2008), Hazzan (1999), and Melhuish (2018) evidenced that for a student to have a complete understanding of operation, they must be able to recognize familiar operations, and productively engage with less familiar operations. The group theory students across their studies may not have perceived all the critical aspects of operation needed to address unfamiliar situations.

Zaslavsky and Peled (1996) provided an alternate look at unfamiliar operations via having pre-service teachers (PSTs) generate examples of binary operations that are commutative, but not associative. As most standard operations do not meet this requirement, the PSTs could not rely on familiar examples. Their examples included non-binary operations, and operations that did not have the relevant property combination. The results of this study led us to develop researchable questions including whether students perceive important definitional aspects of binary operation (such as the need for binary), and how their perceptions of operation may account for incoherence around operation properties (such as moving parentheses without attention to the operation in the case of associativity).

2.3 Representations of operation

Representations provide another lens for parsing students' understanding of mathematical concepts. If a student understands a concept, she should be able to flexibly engage with different representations. In terms of binary operations, an explicit symbolic rule and operation tables are the dominant representations. Researchers have shown that many, but not all students, are able to reason about a binary operation from a tabular representation (Bagni, 2000; Hazzan, 2001). We conjectured that students may not perceive tabular representations as an attribute of a binary operation since not all students were able to reason with a table.

Because so little has been explored related to binary operation representations, literature on function provides insight into student treatment of representations. Researchers have shown that students struggle to move between various function representations (e.g., Akkoç & Tall, 2002) or that they conceive of different representations of a given function as entirely different functions (Elia, Panaoura, Eracleous, & Gagatsis, 2007). One of the most robust findings from this literature is that students often require or prefer explicit symbolic rules even when other representations would be advantageous (e.g., Knuth, 2000). Thompson (1994) argued that this preference remains through the undergraduate level with the "predominant image evoked in students by the word 'function' is of two written expressions separated by an equal sign" (p.

[2] A small proportion of students addressed the isomorphic copy of $\boldsymbol{Z}_3$ in $\boldsymbol{Z}_6$. This group is not included in the "incorrect" category. See Melhuish (2018) for a more nuanced treatment of this idea.

5). As such, we conjectured students might similarly perceive an explicit symbolic rule as a critical aspect of binary operation.

2.4 Metaphors for operation

As in the case of representations, literature on relevant metaphors exists in K-12 contexts. Notably, there are series of metaphors that may ground arithmetic operations such as the dominant collection metaphor (e.g., Lakoff & Núñez, 2000). Addition can be conceived as combining two *collections of objects*. Lakoff and Núñez term such metaphors as grounding metaphors, taking an abstract domain (addition) and grounding it in an embodied experience (collection of objects). Operation metaphors may be specific to a certain operation such as scaling for multiplication (Davis & Renert, 2013), specific to a class of operations such as collections for arithmetic, or more general such as operations as *object construction* (Lakoff & Núñez). That is, a binary operation takes two elements and *constructs* a third. Much of this work is theoretical in nature, but we conjecture that such metaphors may exist empirically as students engage with the general binary operation.

The function literature also provides insight into a number of metaphors that students could potentially leverage if attending to the function portion of the binary operation definition. Lakoff and Núñez's (2000) categories include *function is a machine* and *function is a collection of objects with directional links*. Recently, Zandieh, Ellis, and Rasmussen (2017) categorized function metaphors used by linear algebra students including input/output, travelling, mapping, morphing, and machine. The commonality across these categories is "an entity 1, an entity 2, and a description about how these two are connected" (p. 28). In terms of binary operation, this may look like going from entity 1 (two elements in the set) to entity 2 (one element in the set). We conjectured that some of these metaphors might carry over to the general binary operation setting.

2.5 Conclusion

From this research body, we can deduce that despite the prevalence of binary operations from elementary level onward, understanding this topic is nontrivial for students. Further, work done by researchers has started to unpack some of the intricacies of understanding the concept through genetic decompositions and documenting student struggles with unfamiliar operations. We aim to contribute to this foundation through a complementary exploration of students' perceived critical aspects (representational and definitional) and metaphors that organize and communicate their perceptions. Specifically, we seek to answer the following two questions:

1. What attributes do students treat as critical when engaging with binary operation tasks?
2. What metaphors do students leverage when communicating about operations and their attributes?

3 Methods

In alignment with our theoretical perspective, we designed a two-part study that would engage students with examples and non-examples of binary operation through surveys and interviews.

We worked with the underlying assumption that different students would have different experiences and perceive binary operation differently in terms of metaphors and critical aspects.

3.1 Context and data collection

The first part of our study consisted of a survey with various tasks meant to target students' lived object of learning. The survey was administered to a pilot class (not reported here), refined, and then administered to two introductory, undergraduate level abstract algebra classes from two universities in the USA (n = 12, n = 12). Both classes focused on group theory and provided students with a definition of binary operation that included closure.[3] We designed our survey based on the literature search and on conjectures from our prior work done with group theory tasks (e.g., Melhuish, 2018; Melhuish & Fasteen, 2016). Students had opportunities to engage with binary operation examples and non-examples through the following activities:

1. Determining if a given instantiation is an example of a concept (e.g., Ehmke, Pesonen, & Haapasalo, 2011),
2. Determining if two examples are mathematically the same (e.g., Novotná, Stehlíková, & Hoch, 2006),
3. Determining what properties an example may or may not have (e.g., Novotná & Hoch, 2008),
4. Generating an example meeting some criteria (e.g., Zaslavsky & Peled, 1996).

We conjectured that activity 1 would uncover critical aspects related to students' personal definition (designing tasks that varied on the feature of two inputs), activity 2 would uncover critical aspects that determined how students' perceived the nature of particular operations (designing tasks with some operations defined holistically and some at the element level), activity 3 would uncover critical aspects related to notation (moving between a property notated generically and an operation notated specifically, not in element-operator-element formatting), and activity 4 would uncover implicit aspects treated as critical that can constrain example generating (via building an operation on a set without a familiar explicit symbolic rule available). Every activity was open-ended and provided a prompt for the students to explain their reasoning. Furthermore, our goal was not to evaluate the correctness of students' responses, but rather elicit reasoning that may illustrate what binary operation attributes students were foregrounding.[4] The surveys contained twelve common questions; two additional tasks (content) were included for the second institution. In this report, we focus on the common questions (Table 1).

In addition to the surveys, we conducted semi-structured interviews with each of six participants to gain deeper insight into the students' mathematical thinking as they worked through the survey. Four students volunteered from the survey rounds. Two additional students were recruited from the first institution (based on responses to a shortened survey) during a

[3] A binary operation on set G is a function that assigns each ordered pair of elements of G an element of G. In both classes, notation typically used a generalized arithmetic notation (multiplication or addition).

[4] This is particularly the case for the sameness prompts, as the construct of sameness is not well-defined. Mathematically normative approaches to sameness include identical inputs producing identical outputs (potentially with restricted domains such as with subgroups having the same operation as the parent group) or as inducing isomorphic structures (magmas). See Melhuish and Czocher (2019) for a detailed discussion of the multitude of reasonable approaches to sameness of operations.

Table 1 Survey tasks

<table>
<tr><th>Activity</th><th>Examples</th><th>Activity</th><th>Examples</th></tr>
<tr>
<td>Determine If a Binary Operation</td>
<td>Task 1a: $\diamond(a) = a^2$ on $\boldsymbol{R}$
Task 1b: $\diamond(a,b)=\sqrt{a}$ on $\boldsymbol{Z}$
Task 1c: $\diamond(a)=\sqrt[3]{a}$ on $\boldsymbol{R}$
Task 1d: 1+1=1, 2+2=1, 1+2=2, 2+1=2 on the set {1,2}
Task 1e: Addition mod 3 on the set {0,1,2}</td>
<td>Same or Different</td>
<td>Task 2a: Multiplication and Division on $\boldsymbol{R}$
Task 2b: Addition mod 4 on the set {0,1,2,3} and Composition on the Rotations on a Square
Task 2c:
$\begin{array}{c|cc} \diamond & a & b \\ \hline a & a & b \\ b & b & a \end{array}$ $a+a=a, a+b=b, b+a=b, b+b=a.$
Task 2d:
Op 1: $\begin{array}{c|ccc} \diamond & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 2 & 0 & 1 \end{array}$ Op 2: $\begin{array}{c|cccccc} \star & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 3 & 4 & 5 & 0 \\ 2 & 2 & 3 & 4 & 5 & 0 & 1 \\ 3 & 3 & 4 & 5 & 0 & 1 & 2 \\ 4 & 4 & 5 & 0 & 1 & 2 & 3 \\ 5 & 5 & 0 & 1 & 2 & 3 & 4 \end{array}$</td>
</tr>
<tr>
<td>Property Check</td>
<td>Is the operation: $a\diamond b = \frac{1}{2}(a+b)$ on $\boldsymbol{R}$
Task 3a: Associative?
Task 3b: Commutative?</td>
<td>Generate Example</td>
<td>Task 4a: Define a binary operation on {1,2,4}. (Variation: define a binary operation {1,2,4} that forms a group)</td>
</tr>
</table>

different semester to capture a wider range of survey response types. For each attribute identified from the survey analysis, we interviewed at least two students treating the attribute as critical, and two students treating the attribute as variable. During the interview, students were asked to explain their understanding of binary operation, and share their reasoning for each survey task. The survey tasks and their responses were provided to the students during the interview. The purpose of the interviews was twofold: (1) to validate our interpretations of written responses and (2) to allow for a closer inspection of metaphors that were leveraged in verbal communication.

3.2 Data analysis

Our analyses were driven by methods of phenomenographic analysis (e.g., Trigwell, 2006) and was conducted in two phases. First, we wanted to identify a set of perceived critical aspects that varied across the twenty-four survey responders. Through the literature review and our prior studies, we identified the following attributes: binary, explicit-symbolic rules, and element-operator-element formatting. We designed a set of tasks that may elucidate whether students were treating these attributes as critical or variable. However, we were open to other attributes that might emerge from the task responses. We analyzed the surveys for the prior set of attributes, and identified additional attributes that were treated as critical by some students and variable by others. Using this set of attributes, two researchers independently coded all surveys identifying each student as having treated each attribute as a critical aspect (critical), permissible variation (variable), or inconsistently (as critical or variable) across tasks. There was good agreement across the two researchers with an average κ = 0.704 across the resulting attribute categories (See Table 2).

To situate students' perceived critical aspects and permissible variations, we conducted a more thorough analysis of the six interviews using Larsson and Holmström's (2007) phenomenographic process which relies on a primary reader and co-reader to develop metaphor profiles. Both readers read through the entirety of the transcribed interviews, identified instances where binary operation was addressed, and then analyzed these excerpts for

Table 2 Frequency of critical aspects and permissible variation from the survey

Attribute	Description		Indicator in response	Frequency (n=24)	
Critical aspect: *closure* (κ=0.69)	Closure reflects that the image of a function is a subset of a domain	Task 1	Used at least once to evaluate whether a function was a binary operation	*Treated as critical	n=16
			Never used to evaluate whether a function was a binary operation	Treated as variable	n=8
Critical aspect: *binary* (κ=0.86)	Binary reflects that the function is defined on two variables or an ordered pair	Task 1	Used at least once to evaluate whether a function was a binary operation	*Treated as critical	n=11
			Never used to evaluate whether a function was a binary operation	Treated as variable	n=13
Critical aspect: *element-wise-defined* (κ=0.71)	An operation is determined based on function value corresponding to each input pair	Task 2	Determined if operations were the same via attending to individual elements	*Treated as critical	n=7
			Allowed for operations to be the same even if individual elements operated to produce different outputs	Treated as variable	n=15
Permissible variation: *explicit- symbolic rule* (κ=0.67)	An operation may or may not have an explicit symbolic rule	Task 4	Could not produce a binary operation on set {1,2,4} (instead checked symbolic rules)	Treated as critical	n=7[a]
			Produced or argued for a binary operation using a non-explicit symbolic rule representation	*Treated as variable	n=10
Permissible variation: *element-operator-element formatting* (κ=0.58)	An operation may or may not be in the format $a * b$	Task 3	Did not engage with operation $\frac{1}{2}(a+b)$ in totality, focused on portions that appeared in element-operator-element format	Treated as critical	n=14
			Treated operation $\frac{1}{2}(a+b)$ in totality, as one operation	*Treated as variable	n=10

*reflects a mathematically normative way of treating the aspect

[a] An additional seven students left this item blank; based on interviews, we hypothesize students also treated this variation as critical; however, we have insufficient information to positively classify them this way

perceived critical aspects and metaphors. This analysis led to the identification of three metaphor categories: arithmetic, function, and organization (expanded in the results). For each interview participant, the primary reader classified the subject's dominant metaphor category, with the co-reader serving to validate or challenge these interpretations. Through this process, we further operationalized dominant metaphor to allow for an individual to have more than one. By dominant metaphor we mean a metaphor that was communicated across various tasks, rather than a metaphor that was evoked by a particular task type or context. We then identified additional non-dominant metaphors, that is, metaphors that a student communicated in relation to a particular task type (such as organization when comparing operations) or context (such as arithmetic only when dealing with arithmetic). For each interview participant, we created profiles including descriptions of their dominant metaphor categories and perceived critical aspects and permissible variations for binary operation.

4 Results

Through our analysis, we found that students exhibited different treatment of five key attributes: closure, binary, element-wise-defined, element-operator-element formatting, and explicit-symbolic rule. Table 2 contains the descriptions of each attribute, and the frequency of students treating the attribute as critical (a consistent attribute of operation) or variable (an attribute that an operation may or may not have). For each of these attributes, at least a third of student responses diverged in their treatment of features as critical or variable. In the next subsections, we share interview data illustrating variable and critical treatment of each of these attributes.

Additionally, during the follow-up interviews, we identified three categories of metaphors students communicated related to binary operation and its attributes: function-based, arithmetic-based, and organization-based. We focus primarily on function and arithmetic as these metaphor types occurred across most interviews. We acknowledge the critical aspect *element-wise-defined* may be critical depending on context. That is, this aspect is critical when determining whether a subgroup has the same operation as a parent group. See the "Limitations" section for more discussion of this issue. Table 3 provides an overview of the metaphor types and attribute profiles of the interview participants.

4.1 Function-based metaphors

The first category we unpack is that of function-based metaphors. Function-based metaphors leverage notions of mappings and directionality indicating that two elements are mapped to a third. Lakoff and Núñez (2000) have similarly identified function as a machine and function as a collection of objects with directional links as grounding metaphors for the function concept. Language indicators included "inputs" "outputs" and "map."

Function-based metaphors case Johnny's case provides insight into how this metaphor category plays out. He explained a binary operation as, "the binary operation, as far as I'm concerned, *takes* two numbers and *uses* them in some way, to *generate* a unique *output*..." Johnny's description of a binary operation uses language from the realm of functions. In terms of critical aspects, his primary definitional property is binary (two inputs). Johnny's overall profile was: closure: variable (CV), binary: critical (2C), element-wise-defined: critical (EC), element-operator-element-format: variable (FV), and explicit-symbolic-rule: variable (SV).

Table 3 Profiles of interview students

	Cheryl	Erin	Johnny	Kyle	Phoenix	Roberta
Critical aspect						
Closure	Y	Y	N*	Y	Y	N*
Binary	N	N	Y	Y	N	Y
Element-defined	N	N*	Y	Y	N	N
Permissible variation						
Element-operator-element	N	N	Y	N	N	N
Explicit Symbolic rule	Y	N	Y	Y	N	Y
Metaphor category[1]						
Arithmetic	Dominant	Dominant		Dominant	Non-dominant	Non-dominant
Function	Non-dominant	Non-dominant	Dominant	Dominant		Dominant
Organization		Dominant		Non-dominant		

Y: Treated critical aspect as critical or permissible variation as permissible; N: Inappropriate treatment of attribute: *Inconsistent treatment as critical

[1] Empty cells indicate that metaphor was not evoked by the student during the interview

Johnny used the binary critical aspect to rule out cases like $\diamond(a) = a^2$. In a more ambiguous case, $\diamond(a, b) = \sqrt{a}$ on $\mathbf{Z}$, he remained concerned about two inputs and communicated this concern using function language:

> This stood out immediately, as not binary operation, because root a has nothing to do with b and so even though it's *taking* two numbers in the *input*, the *output* doesn't depend on b, so I guess that didn't register as a binary operation to me [2C].

The interviewer followed up by modifying the task slightly from $\sqrt{a}$ to $\sqrt{a+b}$ in order to test whether Johnny was focusing on two inputs as the only definitional property to check. Johnny explained that this new function would "totally" be a binary operation, reflecting treatment of closure as a potential variable aspect [CV].

At other times, he appeared to appreciate the role of closure; when addressing addition mod 3, he noted, "it does explicitly *take* two numbers," and "you will *return* a number generated from a and b, that is within that set, 0, 1 and 2." When asked explicitly if he thought closure was necessary (due to his reference "within that set"), he answered, "Yeah. I think so. Right?" This evidenced more hesitancy around the closure aspect than other interviewees who used closure as one of their primary critical definitional aspects.

In the sameness prompts, Johnny continued using function language and was able to attend to the critical aspect element-wise-defined. When addressing Task 2a, he explained, "If you have the same *inputs*, and the *outputs* are different, then completely you're dealing with a different operation." After prompting for an example, he elaborated, "Five and 2. a times b is going to be 10 and a divided by b is going to be 2.5. You've got the same *inputs*, that you can *put in*, but clearly the *outputs* are going to be different, so that would make me think that the operation is different" [FV]. In his explanation, it is evident that Johnny treated a binary operation as being determined by what it does to any *particular* elements.

When asked about the operation $\frac{1}{2}(a+b)$, Johnny explained he "*plugged in* some random numbers," successfully producing the counterexample in Fig. 1. In contrast to other participants, Johnny was able to assess the operation in its totality rather than focusing on the portions in element-operator-element formatting. As such, we see that Johnny treated element-operator-element formatting as a permissible variation allowing for operations in alternate formats.

Johnny also treated a symbolic form as a permissible variation. This was most clearly evidenced when addressing the prompt to create a binary operation for $\{1, 2, 4\}$. He explained that he wrote the Cayley table representation for an arbitrary diamond and that it was not like an operation that "I'm familiar with," but:

> We don't need to think about that as any sort of pre-set operation. I don't need to specify how it combines the *inputs* to *produce* that single *output*, but it does *take* two of the *inputs*, *produces* an *output*… [SV].

Johnny's focus on inputs and outputs seemed to allow for several permissible variations, but ultimately did not seem to include closure as a critical aspect.

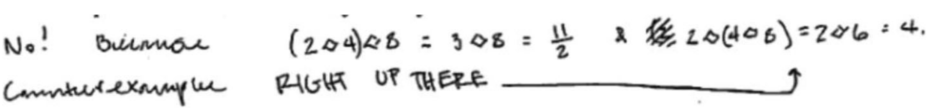

Fig. 1 Johnny's response to Task 3a

4.2 Arithmetic-based metaphors

The second category we unpack is arithmetic-based metaphors. These metaphors naturally extend the grounding metaphors from familiar arithmetic operations indicating that two objects are being formed into a third object. This metaphor category is reflected in Lakoff and Núñez's (2000) arithmetic metaphors, particularly the metaphor *arithmetic as construction*. We identified language indicators for this category including "combines" "constructs" "answer" "put together." While we are not claiming mutual exclusivity between arithmetic-based and function-based metaphors, we identified the key difference as whether two elements are internally combined (arithmetic) or if they are inputted and outputted or mapped to a different element (function).

Arithmetic-based metaphors case Erin's case instantiates one image of a dominant arithmetic metaphor. Arithmetic language around "answer" emerged in reflections such as, "That was the other part of that definition I never remembered. It has to give you back one unique *answer*?" Later, she articulated additional arithmetic imagery such as, "If you could *combine* other numbers and still get the same *answer,* then it wouldn't give you something unique." The use of combining language in conjunction with answer language was a strong indicator that Erin was leveraging arithmetic metaphors. In terms of critical aspects, her primary definitional property is closure. Erin's overall profile was: closure: critical (CC), binary: variable (2V), element-wise-defined: variable (EV), element-operator-element-format: critical (FC), and explicit-symbolic-rule: critical (SC).

In terms of definitional aspects, Erin treated closure as critical, but binary as a permissible variation. She used this feature to permit examples like addition mod 3 by explaining, "*take* any of those numbers and add them *together*" to check that they remain in the set [CC], and reject examples like the square root function on $\boldsymbol{Z}$ "because if I take an integer such as negative one, and try to *take* the square root of that, it's gonna *give* me a complex number, not an integer" [CC]. Similarly, when addressing Task 1a, Erin did not attend to the binary aspect, although she registered concern about uniqueness: "[Y]ou couldn't *take* two of them [-3 and 3] and *get* the same *answer* out of it" [2V] evidencing binary as a permissible variation. Although the uniqueness property may suggest a connection to functions, Erin's language remained consistent with arithmetic metaphors.

Arithmetic language appeared in later tasks as well when evaluating if two operations were the same, taking on language like "combination" when operating two elements. In general, Erin voiced uncertainty about what attributes of an operation determined if they were the same. When approaching Task 2d, she explained she was looking for a pattern and focused on what happens when squaring an element: "...2 *times* 2 itself is *giving* me back... One is 2 ... Okay, ...but... 2 *times* 2 *is* 4." At this point, she identified that 2 squared produced different answers but did not treat this as a critical difference explaining that she still had not found the "pattern" and "I just don't know. I mean, that's what I would look for" [EV]. When the interviewer provided that the operations were addition mod 3 and addition mod 6, Erin remained unsure noting that they may both be "addition."

In terms of representation, Erin's task engagement reflected treatment of an explicit-symbolic-rule and element-operator-element formatting as critical. For example, she struggled with tasks that did not provide a representation that could not easily be converted to an explicit symbolic rule such as in Task 1d [SC] (elaborated in section

5.3) or Task 2d above; she appeared to be searching for a "pattern" which we interpreted as looking for an explicit rule for the table. Even when dealing with an explicit rule, Erin struggled to make sense of an operation not in element-operator-element formatting. When addressing the operation $\frac{1}{2}(a+b)$, in another format, Erin asked, "What is the [operation] rule?" [FC]. She focused on the individual operations (the parts in element-operator-element formatting), moving parentheses to check associativity, explaining: "from the a and the b to the one half, and *multiplying* the one half in front of it. Is that the same as *taking* one half a and then adding it to the b?" The metaphor language of "taking" is ambiguous here, but it is reasonable to conjecture that the arithmetic focus may align with a focus on the element-operator-element formatted aspects.

Overall, Erin voiced concerns about the abstract concept of binary operation and what aspects were critical, describing, "Yeah, you tend to focus on one little thing that stands out, and you miss the other one, and you're just not..." Her arithmetic metaphors may provide some insight into her struggles when engaging with tasks that were not aligned with traditional arithmetic features.

4.3 Organization-based metaphors

The third type of metaphor category to emerge was organizational metaphors. This metaphor category captures language reflecting that the operation is *organizing* a set. Indicators included language around "behaving" "pattern" and "structure." This metaphor is in some ways parallel to a process-conception of operation in that operation is appreciated holistically. However, using a metaphor about organizing or patterning does not ensure a more sophisticated conception.

Organization-based metaphors examples This metaphor type was observed less frequently than function or arithmetic, but served a dominant role in Erin's case and a non-dominant role for Kyle. In Erin's case, the organization focus varied in terms of productivity (i.e., sometimes it served as a hindrance and other times as a support for answering prompts), and in Kyle's case this focus was productive (i.e., it allowed him to address the sameness prompts appropriately). We briefly share two excerpts to illustrate how this type of metaphor was communicated.

When Erin approached Task 1d, she began by engaging with the operation through an organization lens. She explained, "I was looking at it going, 'Well, wait a minute, what *pattern* do we have going on there?' And I just thought I was gonna run out of time" [SC]. Here, we see her patterning metaphor did not appear to allow for a structure that did not have a 'nice' expressible pattern, indicating explicit-symbolic rule as a critical aspect. This is further evidenced when she expressed a general concern about how she was thinking during the interview, saying "What are my rules that I'm trying to use? What am I trying to fit this into?"

While Erin brought up ideas of patterns in multiple tasks, Kyle only used an organization metaphor in the context of comparing whether two operations were the same. When comparing the operations in Task 2b, he explained, "I really like this question because the two sets are fundamentally very different," then noted, "the [operations] *behave* rather similarly." This focus on behavior seemed to support Kyle in treating sameness as isomorphism seeing as he considered the operations in their totality.

4.4 No dominant metaphor

Finally, we would be remiss not to address a case in which we did not identify a dominant metaphor. This participant used some language reminiscent of arithmetic (such as "answer") but did not incorporate any surrounding metaphors for binary operation. For example, when asked to describe binary operation, Phoenix explained, "Isn't it just an operation or two, not elements but two ... Well I'd say the elements, but two groups, elements, stuff like that?" When further prompted to explain what she meant by the term "operation," she explained, "It's whatever it's defined as. It can be a product, sum, division stuff."

Without clear grounding metaphors, Phoenix appeared to struggle to identify binary operations in given situations. For example, in Task 3, she identified "the plus sign or addition" [FC] as the operation when prompted. In Task 1d, she struggled to see an operation at all: "To be quite honest, I'm not all too sure what that's saying in terms of the binary operation." She went on to explain that she did not see how 2 + 1 could equal 1 because it was not a familiar operation like modular arithmetic. Phoenix was limited to tasks with easily available procedures and familiar contexts, not allowing for permissible variations of element-operator-element formatting, non-symbolic representations, and not attending to aspects of binary and defined element-wise.

5 Discussion

This paper produced two primary contributions. First, we illustrated that at the end of a group theory course, many students did not address two critical aspects of the binary operation definition: binary and closure. Furthermore, many students did not see binary operations being defined at the element-level as critical. Such inattention to this attribute may account for students commonly seeing $\{0, 1, 2\}$ as forming a subgroup of $\mathbf{Z}_6$ with the *same* operation (Melhuish, 2018), despite $1 + 2 = 3$ in one setting and $1 + 2 = 0$ in another. Students also treated representational elements as critical, even though they are permissible variations: explicit-symbolic rule (paralleling the function literature, e.g., Thompson, 1994), and element-operator-element formatting. A number of binary operations, such as function composition $f\circ g$ which is frequently notated $f(g(x))$, do not come in that format. Without flexible treatment of symbolic forms, addressing important properties such as the associative property becomes challenging.

The second contribution was an exploratory analysis of what metaphors may organize and communicate these attributes. We identified function-, arithmetic-, and organization-based metaphors. From this analysis, we hypothesized connections between attributes and metaphors. First, the three students with dominant function metaphors all attended to binary as a critical aspect. We hypothesize that connecting binary operation to function may foreground binary, as this feature is atypical of most functions encountered outside of this context (diachronic simultaneity, Vikström, 2008). Second, students who leveraged arithmetic as their dominant metaphor attended to closure as a critical aspect. We conjecture this attention may reflect experience with arithmetic settings which assumes most operations are closed on the real numbers. We have fewer explanatory hypotheses regarding the other critical and variable aspects. We do note that two of the three students exhibiting function dominant metaphors allowed for binary operations to exist without explicit-symbolic rules. It may be the case that their understanding of function was more robust (allowing for permissible variations) and potentially precipitated their leaning towards function metaphors. Finally, we note that the students who engaged with the organization metaphor gained insight in some settings (such as comparing binary operations), but struggled in others (such as engaging with an operation without a pattern). While it may appear that function dominant students had a more robust attribute profile, we also documented a great deal of productive mathematical thinking across metaphors and suggest that each of the metaphor types can be supportive in different contexts.

5.1 Connecting to prior research

This research is intended to complement prior work related to the abstract concept of binary operations. The action, process, object, schema decomposition provides one way to dissect students' conceptions of binary operations (e.g., Brown et al., 1997). Looking at critical aspects and metaphors provides a lens that may further parse students' conceptions and shape the type of evidence needed to argue for a particular understanding. For example, if a student has an object conception of binary operations, we argue they need to develop awareness of permissible representation variations. Even if they can treat a particular binary operation as an object when presented in certain ways (such as seeing modular arithmetic as an operation in element-operator-element format), a true object conception would require the flexibility to see the binary operation as an object across representations.

In terms of familiarity, our analysis reflects additional aspects that may relate to how familiar or unfamiliar a given binary operation is to a student. The students in our study struggled with unfamiliarity in ways parallel to Zaslavsky and Peled's pre-service teachers (1996). By moving beyond example generating, we explicitly evidenced that students struggled with a number of aspects, both definitional and representational. These aspects may provide insight into what constitutes "unfamiliar" in Novotná and Hoch's (2008) framework. From a variation viewpoint, an unfamiliar binary operation would be one possessing variations that the student did not perceive as permissible. For example, it is not just unfamiliar symbolic rules, but also unfamiliar representations associated with familiar binary operations that can provide a setting for students to demonstrate their level of sophistication around binary operation.

5.2 Limitations

As in any small-scale study, we do not make claims of generalizability. We did survey across two institutions that used different curricula, in an attempt to capture student perceptions of binary operations beyond a particular setting. To that end, we identified all the perceived critical aspects and permissible variations in both classes. We note that there were less prevalent attributes (such as associativity and 1-1) present in our data that may become prominent if a larger population were surveyed.

We also acknowledge the role of task design. We entered the project with a number of conjectured important attributes based on the literature and our prior research. As such, the examples and non-examples in the survey had intentional variations on these features. However, unintended uniformity across task types could have masked other important perceived aspects of binary operations. For instance, all of the prompts in Task 1 are functions, and so including a non-function example could unearth whether students view the function aspect as critical. Further, attention to critical aspects such as binary may have been engendered by the survey itself, that is, this feature may not have been foregrounded prior to taking the survey containing both binary and unary operations.

Finally, we acknowledge that our lens on what is critical (or not) may be context dependent and shaped by our view as researchers. For example, element-wise defined would be a critical aspect of binary operation for determining sameness in the typical treatments of sameness in abstract algebra (subgroups having the same operation as the parent group, or isomorphic structures). However, a students' notion of same operation may differ in reasonable manners such as addition mod 3 and addition mod 6 as the same operation, addition.

5.3 Implications

The immediate implications of this research are for education at the abstract mathematics level. As students engage with mathematical structures that rely on sets and operations (such as those found in abstract algebra and linear algebra), we must emphasize the role of binary operations. The construct comes pre-loaded with ideas from arithmetic and function settings, and now takes on the additional role of providing the structure of objects like groups. As students grapple with formal definitions, both their current and prior experiences impact their lived objects of learning. Both researchers and instructors may benefit by attending explicitly to binary operations and providing students with experiences where they can discern important critical and variable aspects via variation.

As the role of binary operations spans the K-12 level, this research also has implications for earlier experiences with operations. If we want students to develop a sense of structure (such as appreciating the associative property for multiplication and addition as the same property), we may want to treat binary operation as an explicit structure itself. Such treatment may mitigate a number of issues documented in the literature. For example, students often treat expressions as left-to-right procedures (e.g., Herscovics & Kieran, 1980). If binary operation is seen as a function of two inputs rather than a direction to compute, the operations in an expression can be treated in different orders (based on properties such as the associative property) rather than a string of numbers and operator symbols. A generalized understanding of binary operation may also support students in making sense of common notation such as superscript "-1" which denotes inverse for both multiplication and function composition, a connection infrequently made by students (e.g., Kontorovich, 2017). Binary operation is a structure that is constant across our number and algebraic systems; however, we rarely engage students with connected notions of binary operation.

In summary, the results from this study illustrate that students' implicit experiences from K-12 and explicit experiences from undergraduate courses may not be foregrounding critical aspects of binary operations. We suggest follow-up studies to analyze how students arrived at their lived object of learning. This could include analysis of the metaphors and example variation in textbooks and instruction. Furthermore, interventions could be developed that leverage sets of examples to highlight critical aspects and permissible variations in binary operations, and provide opportunity to connect the concept of binary operation to the metaphor categories.

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