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Developing a coherent approach to multiplication and measurement

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Abstract
We examine opportunities and challenges of applying a single, explicit definition of multiplication when modeling situations across an important swathe of school mathematics. In so doing, we review two interrelated conversations within multiplication research. The first has to do with identifying and classifying situations that can be modeled by multiplication, and the second has to do with identifying what is consistently characteristic of the operation when considering nonnegative real numbers. We review seminal lines of research—including those of Vergnaud and Davydov—and highlight ways that these lines do not provide a thoroughly unified view of multiplication. Then we offer our own approach based in measurement. To underscore consequences of the approach we outline, we use rectangular area and division to illustrate that, as a field, we may need to adjust how we think about connections between multiplication equations and at least some problem situations. We close with a set of questions about unified approaches to topics related to multiplication.

Keywords Multiplication · Division · Multiplicative conceptual field · Coherence · Mathematical definitions

1 Introduction

The large body of research on topics in school mathematics related to multiplication has demonstrated unequivocally that the domain is complex for students, teachers, and theorists. Much of the research has highlighted challenges that students experience transitioning from additive to multiplicative reasoning and has identified significant limitations of characterizing multiplication as an abbreviated form of repeated addition (e.g., Greer, 1992; Nunes & Bryant, 1996; Simon, Kara, Norton, & Placa, 2018; Thompson & Saldanha, 2003). What the field has

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paid less attention to is examining whether a single meaning of multiplication can be used to model diverse situations with different types of numbers. There are various possible reasons for this. As one possibility, researchers and teachers might think that such articulation is not especially important when considering how to support students. Perhaps students do not have to think about why the same arithmetic operation can be used with whole numbers, fractions, and decimals in order to become proficient at forming multiplicative comparisons, thinking about equal groups, modeling rectangular areas, reasoning about rates, analyzing proportional relationships, and so on. As a second possibility, researchers and teachers might think that articulating an explicit, consistent perspective on multiplication would necessarily distort or only partially capture the many facets of a complex concept, at least when modeling problem situations. Either one of these possibilities could lead to the conclusion that it is sufficient for students and teachers to know that multiplication appears in diverse situations without searching for what those situations have in common.

In the present article, we examine the possibility that a single meaning of multiplication based in measurement can be used to achieve coherence in this complex domain, at least for nonnegative numbers, and we argue that articulating such coherence is important if we are to help students and teachers better understand an essential swathe of school mathematics. We have developed the meaning for multiplication presented in this article over several years of teaching mathematics to future school teachers in the USA. Our approach to instruction is consonant with McCallum's (2018) discussion of two complementary stances toward mathematics. He used the term *sense making* in reference to processes such as pattern seeking and problem solving and the term *making sense* in reference to actively structuring a domain so that it is comprehensible. Sense making highlights the reasoning of learners, and making sense highlights the reasoning of those with more expertise.[1] McCallum argued that inattention to either stance can be detrimental to supporting mathematics learning. We view the meaning for multiplication presented in this article as an instance of making sense, because we use it to structure a central strand of school mathematics with the goal of making that strand more accessible to teachers and, ultimately, to students. Presenting tasks we design to help future teachers employ their own sense making in ways that lead to a coherent view of topics related to multiplication is beyond the scope of the present article, but readers may consult Beckmann (2017) and Izsák, Kulow, Beckmann, Stevenson, and Ölmez (in press).

Consistent with the notions of sense making and making sense discussed above, we emphasize that we do not see coherence and diverse ways of connecting multiplication to problem situations as mutually exclusive: A person could have multiple ways of thinking about multiplication as modeling problem situations, including as repeated addition in a prescribed set of situations, and also be able to think of an underlying structure that works equally well with whole numbers and with numbers less than 1. Furthermore, we wholeheartedly agree that learners may experience the wide range of topics related to multiplication as initially disjoint and that developing a coherent perspective in which multiplication carries a consistent meaning across topics is a significant accomplishment involving considerable psychological complexity.

[1] The distinction between sense making and making sense is reminiscent of Vygotsky's (1986) discussion of spontaneous and scientific concepts.

2 Comments on coherence

The New Shorter Oxford English Dictionary (Brown, 1993) definition of coherence provides several descriptions of smaller pieces being united into a larger whole that include "sticking together," "logical or clear interconnection," "consistency," and "agreement" (Vol 1., p. 435). These descriptions connote related but not identical perspectives on unification. In our reading, several of these senses of uniting have appeared in mathematics education research, and delineating such different senses is essential for the contribution we make in this article.

Various scholars in mathematics and mathematics education have explicitly stated that coherence is a desirable goal for school mathematics curricula (e.g., Conference Board of the Mathematical Sciences, 2012; Cuoco & McCallum, 2018; Schmidt, Wang, & McKnight, 2005). Schmidt et al. (2005) pointed out different senses in which curriculum standards could cohere. As one example, such standards could align with textbooks and assessments (Schmidt et al., 2005, p. 527), which might be an instance of the "consistency" or "agreement" sense of uniting. As a second example, standards could outline a logical and hierarchical progression that first addresses "particulars" and then moves to "deeper structures" (Cuoco & McCallum, 2018; Schmidt et al., 2005)—for instance, arithmetic with whole numbers and with fractions can lead to deeper structures such as rings, fields, and groups (Schmidt et al., 2005, p. 529). In the USA, the National Mathematics Advisory Panel (2008) provided a similar definition in their report: "By the term coherent, the Panel means that the curriculum is marked by effective, logical progressions from earlier, less sophisticated topics into later, more sophisticated ones" (p. xvi). Such progressions can span multiple years of school mathematics. Characterizing uniting through logical progression is an instance of "logical or clear interconnection." Our approach to multiplication, outlined later in this article, emphasizes both the "consistency" and the "logical or clear interconnection" senses of uniting.

We see several reasons for learners to grapple with challenges that may come with constructing a coherent view of multiplication. First, learning to reason with an explicit meaning for multiplication makes contact with several important aspects of mathematical thinking that have been captured in curriculum documents from different countries. In the USA, the *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) included practices that have to do with reasoning quantitatively, constructing viable arguments, attending to precision, and looking for and making use of structure. In Australia, students "develop an increasingly sophisticated capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying, and generalising" (Australian Curriculum, Assessment, & Reporting Authority, 2010). In Germany, the educational standards for mathematics include general mathematical competencies, such as examining mathematical statements and testing their correctness, recognizing mathematical connections and developing conjectures, and looking for and making sense of explanations (Sekretariat der Ständigen Konferenz der Kulturminister der Länder in der Bundesrepublik Deutschland, 2005, pp. 7–8). When students encounter these practices, they have opportunities to experience sense making authentic to the discipline of mathematics. Second, having an explicit meaning for multiplication provides students a critical resource for deciding for themselves, rather than relying on external authority, which situations can and cannot be modeled by the operation. Third, if students were to experience a consistent meaning for multiplication as they progressed through topics related to multiplication across grades, it might be more apparent to them how their prior experiences are relevant when reasoning in new situations. Fourth, developing a coherent view of diverse topics related to multiplication could help students construct an interconnected knowledge base with which to reason flexibly and reconstruct any forgotten elements.

3 Perspectives on multiplication

Our search for coherence engages two interrelated conversations in multiplication research. The first of these conversations has to do with identifying and classifying diverse situations that can be modeled with the operation; the second has to do with articulating what is consistently characteristic of the operation. We discuss several lines of multiplication research in some detail to make two main points that will set up our contribution. First, researchers have connected multiplication as a numerical operation to analyses of units in problem situations in ways that are often inconsistent or incomplete. Second, although researchers have identified characteristics that distinguish multiplication from addition, those characteristics do not define multiplication. Identifying characteristic features that distinguish multiplication from addition is not the same as providing a means for classifying what is and what is not multiplication. A few researchers have explicitly addressed coherence when examining whether a single meaning of the operation can be used both with whole numbers and with fractions (e.g., Boulet, 1998; Greer, 1994; Simon et al., 2018) but these efforts have not succeeded across as wide a range of situations as we consider below. We begin by reviewing how both conversations are evident in Vergnaud's (1983, 1988, 1994) seminal contributions.

3.1 Vergnaud's theory

Vergnaud (1988) defined the *multiplicative conceptual field* (MCF) as "all situations that can be analysed as simple and multiple proportion problems and for which one usually needs to multiply or divide" (p. 141). He then argued that mathematics education researchers should take a comprehensive approach to studying learning of the complex system of situations, interconnected topics, and symbolic representations related to multiplication. We agree both with Vergnaud's fundamental premise that situations, topics, and notations related to multiplication form a complex system and with his recommendation to build comprehensive approaches to studying such a system.

Vergnaud's (1983, 1988, 1994) theoretical and empirical approach to studying the MCF has been based on identifying the range of situations in which one multiplies or divides, classifying those situations based on different underlying mathematical structures, and describing solution methods that children use when solving problems with those structures (1988, p. 149). We highlight three aspects of Vergnaud's contributions. First, with respect to topics, he has consistently included whole-number multiplication and division, fractions, ratios, and linear functions, among others. Second, he has identified subtypes of situations, each with its own mathematical structure. Across the three publications referenced above, the number and names of the subtypes have varied. In 1983, he identified isomorphism of measures, product of measures, and multiple proportion. In 1988, he discussed just two subtypes—simple proportion and multiple proportion (1988, pp. 150–151). The quote above references both of these. In 1994, he identified four subtypes—simple proportion, concatenation of simple proportion, double proportion, and comparison of rates and ratios—but he also referred the reader back to his 1983 analysis (1994, p. 48). This suggests that his 1983 analysis, the one we emphasize in this article, was still an important part of his thinking 10 years later. Third, to analyze students' reasoning about situations in the MCF, Vergnaud introduced the notion of theorems-in-action (1988). These are “mathematical relationships that are taken into account by students when they choose an operation or a sequence of operations to solve a problem” (1988, p. 144).

With respect to the second conversation, Vergnaud (1983) characterized multiplication as a bilinear composition, and his analysis of the MCF highlights three ways that students might take mathematical relationships into account when reasoning across different problem subtypes. He identified the first two approaches when analyzing isomorphism of measures. Vergnaud defined isomorphism of measures as “a structure that consists of a simple direct proportion between two measure-spaces M_1 and M_2” (p. 129). In one problem that he gave to illustrate this subtype, four cakes cost 15 cents each and the question asks how much do the cakes cost all together (pp. 129–130). Vergnaud stated that children might see numerically that 4 • 15 or 15 • 4 answers the question but be confused by the units: “It is not clear why 4 cakes • 15 cents yields cents and not cakes” (p. 129). As a consequence, children tend to use one of two unary operations. In the first approach, multiplication is viewed as operating within one measure space at a time (cakes or cents): One could reason that four cakes is four times as many as one cake, so the answer should be four times as much 15 cents. Here Vergnaud viewed multiplication by 4 as a scalar operator: 1 cake • 4 = 4 cakes and 15 cents • 4 = 60 cents. In the second approach, multiplication is viewed as a function operator. Using the same cake problem, one could reason that 1 (cake) • 15 (cents per cake) = 15 (cents) and so 4 (cakes) • 15 (cents per cake) = 60 cents. Here multiplication by 15 goes from one measure space to the other (from cakes to cents), and the number 15 has units attached. More specifically, the units for 15 are the quotient of the units for the two original measure spaces.

Vergnaud (1983) described a third way that multiplication can model situations when analyzing product of measures. He defined product of measures as “a structure that consists of the Cartesian composition of two measure-spaces, M_1 and M_2, into a third, M_3” (p. 134) and claimed such structures could *not* be represented as isomorphisms of measures (p. 134). He further stated that products of measures should be analyzed as double proportions (p. 135) and explained such situations as products “both in the dimensional and in the numerical aspects” (p. 136). He used both combinatorial and rectangular area problems to illustrate this subtype. Vergnaud gave the combinatorial example in which four girls and three boys are at a dance and the question asks how many boy-girl couples can be formed. In this case, he stated that the product of one boy and one girl was one couple (p. 134). Thus, from the dimensional aspect, 1 (boy) • 1 (girl) = 1 (couple), and, from the numerical aspect, 3 • 4 = 12. Vergnaud viewed rectangular area analogously and considered one unit of area to be the product of two units of length. So if one had a rectangle that was 3 cm long and 4 cm wide, the area would be conceived of as multiplication through 1 (cm) • 1 (cm) = 1 (cm^2) and 3 • 4 = 12. In both the dance and rectangle examples, the units for the product (couples or cm^2) are different in kind from units for either of the factors (boys, girls, or cm).

From our point of view, if Vergnaud’s (1983, 1988, 1994) analysis of the MCF were taken as a description of coherence, then that coherence would be based on the sticking together sense of uniting. In particular, Vergnaud (1983) did not discuss how the three approaches to modeling with multiplication he described could either reflect or lead to a single meaning of the operation that applied across the different structures he identified. The three approaches could be summarized as one in which units are not transformed (scalar operator), one in which units are transformed through a quotient (function operator), and one in which units are transformed through a product (product of measures). For us, this diversity of approaches to relating the operation of multiplication to units in problem situations raises the following question about coherence in the consistency sense: How is it that a single arithmetic operation can model situations (and the units in those situations) in such diverse ways? One answer to this question could be that the operation of multiplication is best conceived of as a complex, multi-faceted concept different aspects of which are evident in different parts of Vergnaud's analysis. An alternative answer to this question is that there is a common, or shared, underlying structure across Vergnaud's subtypes that he did not highlight.

3.2 Further classifications of situations

Other researchers (e.g., Anghileri, 1989; Greer, 1992; Nunes & Bryant, 1996; Schwartz, 1988) have also analyzed subtypes of situations. Greer identified 10 subtypes and argued that some of those, including equal-group and rate situations, can be conceived in similar terms with clear asymmetry between the multiplier and multiplicand. Meanwhile other subtypes, including Cartesian products and rectangular areas, lack such asymmetry: One can interchange the role of the length and the width of a rectangle when computing its area (p. 277). Schwartz distinguished between extensive and intensive quantities. Extensive quantities refer to "amounts," and numbers associated with such quantities are often viewed as counts. Intensive quantities characterize "qualities" and can be thought of as "a generalization of the notion of an attribute density" (1988, p. 43). A common indicator that a quantity is intensive is the appearance of the word "per" in the associated units (1988, p. 42). He then analyzed multiplication situations in terms of different combinations of extensive and intensive quantities. What Vergnaud (1983) characterized as isomorphism of measures, Schwartz (1988) characterized as I • E = E' (15 cents/cake • 4 cakes = 60 cents), and what Vergnaud characterized as product of measures, Schwartz characterized as E • E' = E" (3 cm • 4 cm = 12 cm^2). Schwartz also identified situations that could be classified as I • I' = I" and gave the example of 33.3 mi/gal • 1.5 gal/h = 49.95 mi/h (pp. 50–51).

3.3 Further characterizations of multiplication

Research on multiplication contains various proposals for what is characteristic of the operation. One strand of this conversation has sought to identify psychological primitives that provide a basis for children's early notions of multiplication. As examples, researchers have proposed repeated addition (e.g., Fischbein, Deri, Nello, & Marino, 1985), a mental operation based on forming one-to-many relationships called splitting (e.g., Confrey, 1994; Confrey & Smith, 1995; Nunes & Bryant, 1996), and a mental operation based on the coordination of two composite units called units coordination (e.g., Steffe, 1988, 1994; Steffe & Olive, 2010). Vergnaud's (1988) theorems-in-action also fit here in as much as he describes them as "intuitive strategies" (p. 149) based on (bi)linearity.

A second strand of this conversation contains several conceptual analyses of multiplication as an operation. These analyses have emphasized distinctions in the role of units when modeling situations with addition and with multiplication that preclude reducing multiplication to repeated addition. We organize these analyses into ones that highlight *unit transformation* and ones that highlight *coordinated measurement*.

Unit transformation approaches Conceptual analyses of multiplication proposed by Schwartz (1988) and by Thompson and Saldanha (2003) are characteristic of the unit transformation perspective. As part of his discussion of extensive and intensive quantities reviewed above, Schwartz argued that when addition is used to model a situation, both addends and the sum refer to the same unit (e.g., apples + apples = apples), but when multiplication is used to model a situation, the units of the factors are transformed into units of the product (e.g., miles per hour • hours = miles). He characterized addition as referent preserving and multiplication as referent transforming (pp. 47–48), and his classification of situations in terms of I • E = E', E • E' = E", and I • I' = I" highlights different units attached to each of the factors and to the product (see discussion above). In their characterization of multiplication, Thompson and Saldanha (2003) echoed Vergnaud's (1983) discussion of double proportion: The measure of the product is proportional to that of each of the quantities that correspond to the two factors. In particular, they discussed a new measurement unit formed by the product of units for each factor: "By convention, the standard unit of the product quantity is defined as that amount made by one unit of each constituent quantity" (p. 103). They illustrate their perspective with an example in which an area unit is formed by the product of two length units.

From our point of view, a main challenge for unit transformation approaches is that they often do not address how exactly one might derive, or determine, units of the product given units of the factors. To illustrate, how would a student who does not already know how to measure area determine what the product of two length units ought to be? Why does 1 cm • 1 cm create 1 cm^2 and not k cm (where k is a positive constant) or 1 cm^3? In order to make sense of an equation like 1 cm • 1 cm = 1 cm^2, it seems that a student would either have to accept a convention without justification for determining the units of a product or already know something about tiling rectangles with square centimeters. At the same time, coordinating tiling with multiplication, in principle, is not straight forward. How would a student differentiate the affordances of tiling a rectangle with square centimeters, rectangles (that are not squares), or other polygons when connecting measures of length and area? Would it be just as sensible to multiply lengths of sides of "unit triangles?" Students could gain initial experience using units to tile a rectangle without any reference to multiplication and only in subsequent work consider how that tiling might be modeled with an arithmetic operation.

Coordinated measurement approaches Davydov (1992) provided an alternative to the unit transformation characterization of multiplication in which he emphasized indirect measurement (see also Boulet, 1998; Simon et al., 2018; Simon & Placa, 2012). For reasons that will become clear in the next section of the article, we characterize Davydov's approach slightly differently as coordinated measurement. He focused on whole numbers and motivated his approach using situations in which direct measurement with a given unit is not practical. For instance, one might have the ultimate goal of measuring some volume of liquid in terms of cups. If the volume were large enough, it would be easier to measure with gallons first. Then, one could measure 1 gal using 1 cup to see how many cups were in each gallon and use the result to compute the measure of the large volume in terms of cups (e.g., cups in 1 gal • gallons in the total volume = cups in the total volume). Davydov interpreted such an indirect approach to measuring the large volume in terms of cups as a change in measurement units (p. 19) and emphasized multiplication as combining smaller into larger units. (See Simon et al., 2018, for recent work building directly on Davydov's approach to multiplication.) In contrast to the unit transformation perspectives discussed above, units of the factors are not transformed into units of the product. Rather, the product shares units (cups in this example) with one of the factors.

Our search for coherence in the MCF is influenced most directly by the work of Davydov (1992) discussed above, and Boulet's (1998) extension of that work. In particular, Boulet argued that a single "concept" of multiplication was foundational for understanding the nature of multiplicative thinking and that "without a uniform theory, studies on the learning of multiplication result in scattered bits of information dealing with specificities rather than being an elucidation of the learning of the concept in general" (p. 17). She extended Davydov's (1992) notion of combining smaller into larger units; argued for consistent application of the distinction between the multiplier and multiplicand across positive integers, negative integers, and positive rational numbers; and demonstrated how this distinction could be applied across the 10 subtypes of situations discussed by Greer (1992). At the same time, however, she discussed four different types of "multiplication formulas" depending on whether the multiplier and multiplicand were whole numbers or unit fractions ($a \cdot b$, $a \cdot 1/b$, $1/a \cdot b$, and $1/a \cdot 1/b$; where a and b are natural numbers), and she discussed multiplying by unit fractions as division (p. 15). Thus, in our reading, she associated iterating with multiplication and partitioning with division and, as a consequence, did not fulfill completely her quest for a uniform theory. Later in the article, we explain that both actions of iterating and partitioning are associated with our meanings for multiplication and for division. Thus, for us, iterating is not identified with multiplication and partitioning is not identified with division.

4 Multiplication as coordinated measurement

The two conversations summarized in the preceding section lead us to ask whether it might be possible to articulate a meaning of multiplication that could be applied consistently across situations (e.g., isomorphism of measures, product of measures, and multiple proportion) and topics (e.g., whole-number multiplication and division, fractions, ratios, and linear functions, among others) within the multiplicative conceptual field. In this section, we present a perspective on multiplication that builds on Davydov's (1992) and that can be used to achieve a greater degree of unification across the MCF than described by Greer (1994), Boulet (1998), or Simon et al. (2018). At the same time, we are not claiming that our approach is the only viable path to achieve unification across the MCF in the consistency or logical interconnection senses.

Figure 1 summarizes our definition of multiplication based in measurement. The definition applies to situations in which there is a quantity (the product amount) that is simultaneously measured with two different units that we call *base units* and *groups*. If N is how many base units make one group exactly, and M is how many groups make the product amount exactly, then we define the product $N \cdot M$ to be how many base units make the product amount exactly. If we let P stand for the product, then we can summarize our definition of multiplication using the equation $N \cdot M = P$.[2] Our definition functions as a means for classification: Situations are multiplicative if and only if one can identify a base unit and a group such that the equation fits the situation. In contrast to Davydov (1992), who emphasized a change of measurement units from smaller to larger, we characterize our definition as one about coordinated measurement. The distinction is subtle and, from our point of view, has to do with whether one is emphasizing solution methods or mathematical structure. Figure 2 uses double number lines to represent that structure in the case where N, M, and P are greater than 1: One number line is dedicated to measuring in base units and the other is dedicated to measuring in groups. The

[2] This definition of is a refinement of the one that appeared in Beckmann and Izsák (2015).

N	$\bullet$	M	$=$	P
How many base units make one group exactly?		How many groups make the product amount exactly?		How many base units make the product amount exactly?

!

Fig. 1 Summary of a definition for multiplication based in coordinated measurement

alignment of M groups and P base units is central to the structure because they are two different measures of the same quantity, but, as we will demonstrate, the left-to-right order of N, M, and P can vary depending on the situation.

We make seven points about this definition. First, from our point of view, the order of N and M is less important than consistently assigning well-defined roles for numbers on the left-hand side of the multiplication sign and on the right-hand side of the multiplication sign. Such consistent assignment can help reveal common underlying structure across situations in the MCF (see also Beckmann & Izsák, 2015).

Second, both multiplication and measurement are based on partitioning quantities into equal-sized parts. For this reason, they are intertwined throughout a central swathe of school mathematics. The definition summarized in Fig. 1 helps explain why the phrase "times as many" is connected to determining how many units make a given quantity, the basis of multiplicative comparison. Thus, explicitly connecting measurement to the definition of multiplication is a promising start for unifying the MCF.

Third, N, M, and P are numbers, each of which answers a measurement question. This notion of number as the outcome of measurement is consistent with Davydov (1992, p. 21) and recent adaptations of his work (e.g., Simon & Placa, 2012, p. 36; Simon et al., 2018). N is the measure of one group in terms of base units, M is the measure of the product amount in terms of groups, and P is the measure of the product amount in terms of base units.

Fourth, similar to Davydov and Tsvetkovich (1991), we emphasize a measurement meaning not just of whole numbers but also of fractions. As shown in Fig. 3, fixing a measurement unit and asking how many of these make that exactly—the very same questions as in Fig. 1—makes sense both when the answer is greater than 1 and when the answer is less than 1. We acknowledge that applying the phrase "how many" in the latter case can take getting used to and claim that this is a small but consequential adjustment in our usage. We call the answers to such measurement questions measures. To define fractions, take a quantity to be a unit of measure and consider that quantity partitioned into b equal-sized parts (where b is a natural number). By $1/b$ we mean the measure of any one of the b equal-sized parts. By a/b we mean the measure of a quantity formed from any a parts of measure $1/b$.[3] Notice that this definition works equally well when $b = 1$ and thus can also be used to interpret whole numbers in terms of measurement.

Fifth, we use the terms "base units" and "groups" as generic terms. Each new problem situation requires identifying what is 1 base unit, 1 group, and 1 product amount. Moreover, in any given situation, there can be more than one reasonable way to make such assignments.

[3] Our definition for fractions refines that stated in the *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, p. 24).

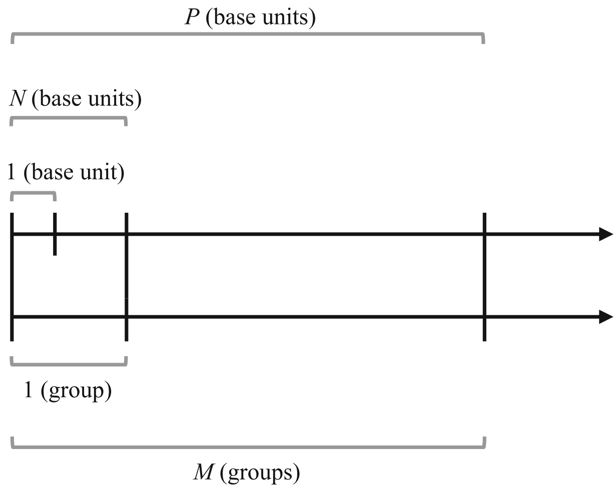

Fig. 2 Using double number lines to represent coordinated measurement, one number line for measuring in terms of base units and one for measuring in terms of groups

Thus, there is no a priori equal-group structure in a situation. Such structures are constructed by people. Different assignments of base units and groups lead to different values for N, M, and P. For example, in a situation where a potter uses 3 oz of clay to make each of 7 plates, we could take a lump of clay that is 1 oz to be 1 base unit, 1 plate to be 1 group, and the total amount of clay to be 1 product amount. In practice, we often rely on the context to make clear that we are taking 1 plate to be a unit of weight. The total amount of clay could be measured in groups—there are 7 plates in the total (so $M = 7$)—or in base units—there are 21 oz in the total (so $P = 21$). The multiplicand N, which is the number of ounces to make 1 plate, relates the two measurement units. In this case, the equation 3 (ounces to make 1 plate) • 7 (plates to make the total) = 21 (ounces to make the total) models the situation (see Fig. 4, not to scale). As a second perspective on the same situation, one could take the weight of 1 plate to be 1 base unit, 1 oz to be 1 group, and the total weight of clay to be 1 product amount. In this case, the equation 1/3

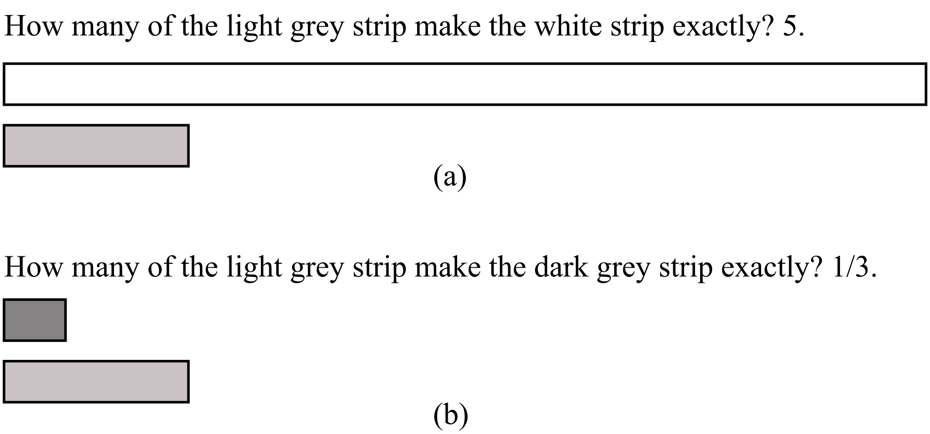

Fig. 3 A measurement sense of number. **a** Measurement greater than 1. **b** Measurement less than 1

(plates to make 1 oz) • 21 (ounces to make the total) = 7 (plates to make the total) models the situation (see Fig. 5, not to scale). As a third perspective on the same situation, one could take 1 oz to be 1 base unit, the total weight of clay to be 1 group, and the weight of 1 plate to be 1 product amount. In this case, the equation 21 (ounces to make the total) • 1/7 (total weight to make 1 plate) = 3 (ounces to make 1 plate) models the situation. Here, 1/7 of the total weight of clay makes (corresponds to) 1 plate exactly (see Fig. 6, not to scale). The last two perspectives illustrate how our measurement meaning for fractions can be applied either at the level of base units when applied to N or at the level of groups when applied to M. Figures 4, 5, and 6 use double number lines to represent the common structure of measuring in base units and in groups for the three interpretations of the pottery situation.

Sixth, in contrast to Greer's (1992) discussion of situations in which multiplication is asymmetric and other situations in which the operation is symmetric, N and M play distinct roles in measurement. Thus, there is a consistent, inherent asymmetry in our approach. (On this point, our own perspective aligns with Boulet's, 1998.) Put another way, multiplication is commutative when calculating but not when modeling problem situations (e.g., the structure of 3 oz of clay to make each of 7 plates is not the same as the structure of 7 oz of clay to make each of 3 plates, even though there are a total of 21 oz of clay in both situations). At the same time, in contrast to analyses like those of Schwartz (1988), the asymmetry comes not from two distinct *types* of units, one for intensive quantities and one for extensive quantities, but rather from two different measurement questions, one for N and one for M.

Finally, we emphasize that the definition in Fig. 1 applies to nonnegative numbers. We imagine that teachers and students would modify this definition as needed. The critical point is that such modifications should be consistent with the definition in Fig. 1. One modification would be to replace "make" with "correspond." In the clay situation, the notion of ounces making plates is sensible and sufficient for thinking about the total weight of clay being measured in terms of two different units. In a uniform speed situation, however, such as traveling at 5 km per hour for 4 h, we could take the product amount to be the complete trip, and view it as measured in kilometers and in hours. Here it makes less sense to think about 5 km making 1 h and more sense to think about 5 km corresponding to 1 h. Then our $N \bullet M = P$ equation becomes 5 (kilometers correspond to 1 h) • 4 (hours correspond to 1 trip) = 20 (kilometers correspond to 1 trip). Of course, one could also talk about how many ounces correspond to one plate, but the motivation for doing so might not be apparent until one encountered a situation where notions of "make" did not fit particularly well. When extending multiplication to negative numbers, we think it reasonable to use the distributive property as the basis for determining the sign of the product (as opposed to using a situation with signed quantities as the basis for such determination). Certainly the distributive property is central to more general notions of product that apply to mathematical objects such as matrices and vectors.

5 Coordinated measurement and coherence

For the balance of the article, we use rectangular area and division to illustrate how the meaning of multiplication shown in Fig. 1 holds promise for realizing a coherent approach to the MCF, coherent both in the consistency and in the logical or clear interconnection senses. At the same time, the examples illustrate how our approach will likely require

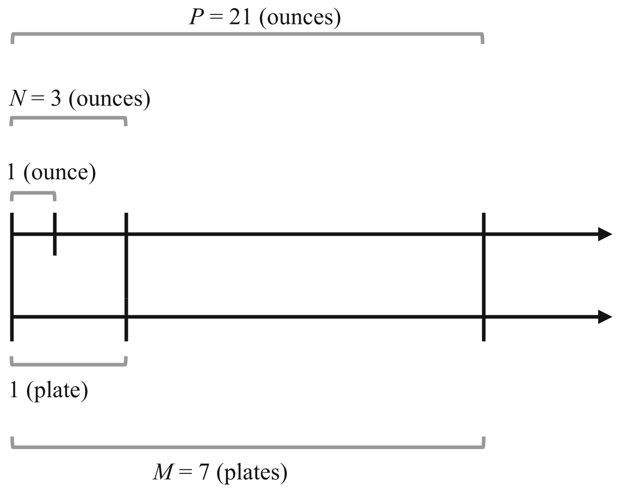

Fig. 4 3 (ounces to make 1 plate) • 7 (plates to make the total) = 21 (ounces to make the total)

some teachers and researchers, alike, to adjust how they think about multiplication as a model of problem situations.

5.1 Rectangular area

Recall that Vergnaud (1983) identified rectangular area as an example of the product-of-measures subtype, characterized situations within this subtype as "Cartesian composition" (p. 134), and stated that this structure could *not* be represented as an isomorphism of measures (p. 134). We begin our application of coordinated measurement to product-of-measures situations by returning to the example of boy-girl couples at a dance. We can apply the meaning of multiplication in Fig. 1 to the couples situation by letting 1 couple correspond to 1 base unit and 1 group correspond to all couples that can be formed with 1 girl. In this case, the number

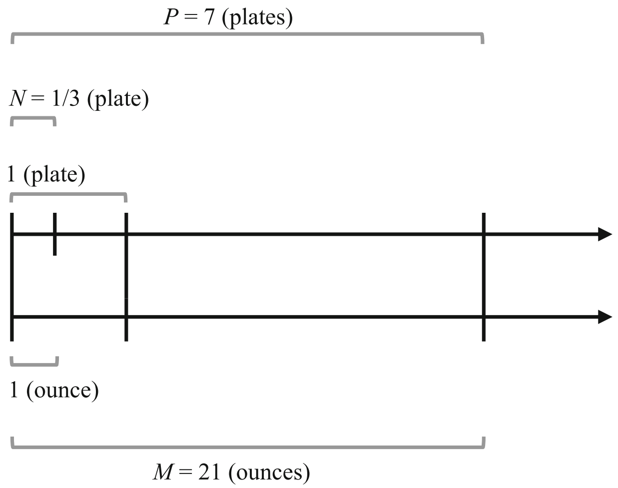

Fig. 5 1/3 (plate to make 1 oz) • 21 (ounces to make the total) = 7 (plates to make the total)

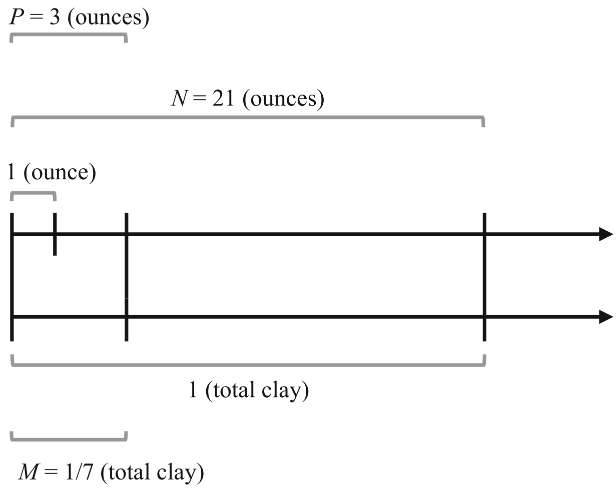

Fig. 6 21 (ounces to make the total clay) • 1/7 (total clay to make 1 plate) = 3 (ounces to make 1 plate)

of boys is the same as the number of couples in 1 group, and the number of girls is the same as the number of groups in 1 product amount. In this situation, the equation 3 (couples make 1 group) • 4 (groups make all the combinations) = 12 (couples make all the combinations) models the situation. To underscore the contrast between this analysis of multiplication and Vergnaud's, recall that he discussed the products of 1 boy • 1 girl = 1 couple and of 3 (boys) • 4 (girls) = 12 (couples) as parallel aspects of multiplication (p. 136).

To discuss rectangular area, we will use two rectangles—a 3 cm-by-4 cm rectangle and a 1/3 cm-by-1/4 cm rectangle—to compare the units transformation and coordinated measurement perspectives on multiplication. Our discussion of the units transformation approach reflects analyses presented by Vergnaud (1983, pp. 136–137), Schwartz (1988, p. 51), and Thompson & Saldanha (2003, p. 103). From a units transformation approach, multiplication combines the units of each factor, centimeters in this example, to create a new kind of unit, square centimeters. At the same time, the numerical aspect of multiplication expresses a double proportion (e.g., Vergnaud, 1983, p. 134 ff.). In this case, the area of a rectangle is directly proportional to both its length and its width. With respect to the 3 cm-by-4 cm rectangle, the double proportion and the alignment of dimensional and numerical aspects of multiplication can be seen in the following series of equations:

$$\begin{aligned} 1\ (\mathrm{cm}) \bullet 1\ (\mathrm{cm}) &= 1\ \left(\mathrm{cm}^2\right) \\ 3\ (\mathrm{cm}) \bullet 1\ (\mathrm{cm}) &= 3\ \left(\mathrm{cm}^2\right) \\ 1\ (\mathrm{cm}) \bullet 4\ (\mathrm{cm}) &= 4\ \left(\mathrm{cm}^2\right) \\ 3\ (\mathrm{cm}) \bullet 4\ (\mathrm{cm}) &= 3 \bullet 4\ \left(\mathrm{cm}^2\right) \end{aligned}$$

Vergnaud (1983) did not discuss explicitly what a fraction might mean, but he did discuss the "/*b*" notation as the "mental inversion of the relationship "x *b*" in the case of the scalar operation meaning of multiplication (p. 131) and as inverting both the numerical operation and the units in the case of the function operator meaning of multiplication (p. 132). Nevertheless, with respect to the 1/3 cm-by-1/4 cm rectangle, the double proportion and the alignment of dimensional and numerical aspects of multiplication can be seen in the following series of equations:

$$\begin{aligned} 1\ (\mathrm{cm}) \bullet 1\ (\mathrm{cm}) &= 1\ \left(\mathrm{cm}^2\right) \\ 1/3\ (\mathrm{cm}) \bullet 1\ (\mathrm{cm}) &= 1/3\ \left(\mathrm{cm}^2\right) \\ 1\ (\mathrm{cm}) \bullet 1/4\ (\mathrm{cm}) &= 1/4\ \left(\mathrm{cm}^2\right) \\ 1/3\ (\mathrm{cm}) \bullet 1/4\ (\mathrm{cm}) &= 1/12\ \left(\mathrm{cm}^2\right) \end{aligned}$$

From a coordinated measurement perspective, the analysis of rectangular area looks somewhat different. We make no presumption about the role of multiplication in rectangular area. Rather we seek base units and groups that will determine if the meaning of multiplication shown in Fig. 1 can fit. We return to the 3 cm-by-4 cm rectangle and view the unit square (of area 1 cm^2) as a unit that can be used to tile, and hence measure, the area of the rectangle. We take the unit square as 1 base unit and can choose either 1 row or 1 column of unit squares to be 1 group. Here we choose 1 row as a second unit of area (Fig. 7a). The product amount is the entire 3 cm-by-4 cm rectangle. Because we can measure the area simultaneously with base units (unit squares) and with equal-sized groups (rows), we determine that the situation can be modeled by multiplication. In particular, with these choices for 1 base unit, 1 group, and 1 product amount, we have the following equation:

$$\begin{aligned} &3\ (\text{unit squares make 1 row exactly}) \bullet 4\ (\text{rows make 1 rectangle exactly}) \\ &= 12\ (\text{unit squares make 1 rectangle exactly}) \end{aligned}$$

Figure 7b shows the product amount measured simultaneously in base units and in groups.

Notice that our analysis of the whole-number case is similar to our analysis of the dance situation: We specified a unit of measure for the product amount that is different from those attached to the 3 and the 4 in the description of the situation. In the dance situation, we introduced 1 couple, and in the area situation, we introduced 1 unit square. In essence, we are asking what unit would be appropriate for measuring the product amount early in our analysis and using that unit to measure equal-sized groups as well. Thus, we approach the task as one about measurement and seek the underlying structure that would allow us to fit the meaning of multiplication shown in Fig. 1. Our approach contrasts with forming units as the *outcome* of

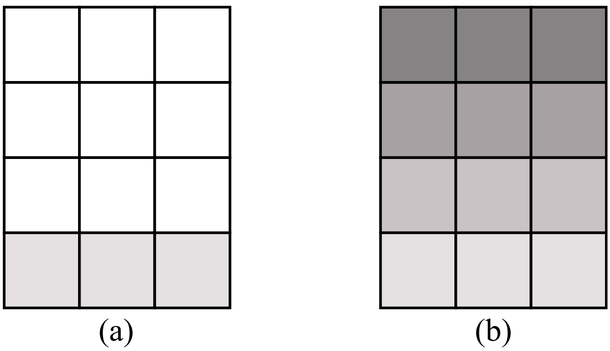

Fig. 7 Coordinated measurement approach to multiplication in the context of area. **a** 3 unit squares make 1 group (row) exactly. **b** 4 groups (rows) make 1 product amount exactly and 12 unit squares make 1 product amount exactly

multiplication as an operation. Furthermore, the familiar length • width = area formula emerges not as the justification for multiplication but as a consequence of fitting the coordinated measurement perspective. In particular, the number of unit squares in the bottom row is the same as the number of centimeters along the bottom edge of the rectangle, and the number of rows is the same as the number of centimeters along the vertical side. Thus, the length • width = area formula can be viewed as an abbreviation that elides, or omits, details of a coordinated measurement argument. A similar comment applies to the dance situation and the equation 3 (boys) • 4 (girls) = 12 (couples).

We can apply the coordinated measurement perspective to determine the area of a 1/3 cm-by-1/4 cm rectangle as well. In this case, we take the unit square (1 cm^2) as the base unit once more. If we consider 1/4 to be how many base units make 1 group exactly, then the measurement definition of fractions applied to 1 base unit tells us to partition the unit square into 4 equal-sized parts (Fig. 8a). We take the bottom row as a second unit of area. If we consider 1/3 to be how many groups make 1 product amount exactly, then the measurement definition of fractions applied to 1 group tells us to partition the bottom row into 3 equal-sized parts (Fig. 8b). Finally, we see that the product amount partitions the base unit into 12 equal-sized parts (Fig. 8c). Again, because we can measure the darkly shaded area simultaneously with base units (1 cm^2) and with equal-sized groups (1 row), the situation can be modeled by multiplication. In particular, with these choices for 1 base unit, 1 group, and 1 product amount, we have the following equation:

$$1/4\ (\text{unit square makes 1 row exactly}).1/3\ (\text{row makes 1 mini rectangle exactly})$$
$$= 1/12\ (\text{unit square makes 1 mini rectangle exactly})$$

Once again, the familiar length • width = area formula can be derived from coordinated measurement. Figure 9 (not to scale) shows a double number line representing the same coordinated measurement structure.

Notice that across the pottery situation (an instance of isomorphism of measures) and both the dance and rectangular area situations (instances of product of measures), we have interpreted every whole number and every fraction in terms of measurement (how many of this make that exactly). Furthermore, we did not presume that any of the problem situations we

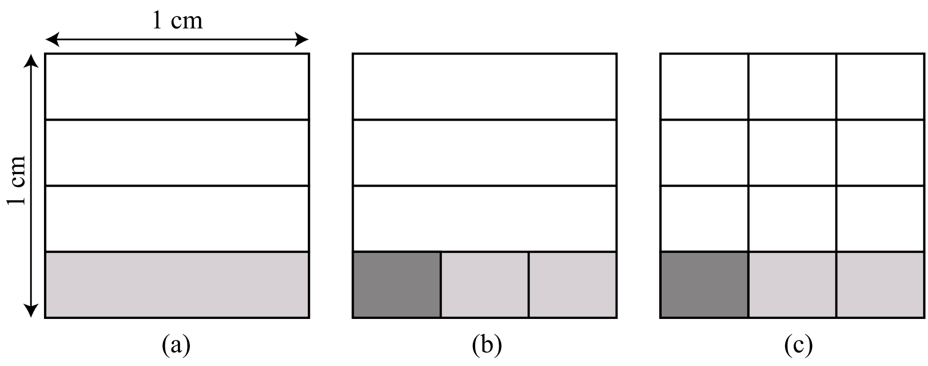

Fig. 8 Coordinated measurement approach to multiplication in the context of area. **a** 1/4 of 1 cm^2 makes 1 group exactly. **b** 1/3 of 1 group makes 1 product amount exactly. **c** 1/12 of 1 cm^2 makes 1 product amount exactly

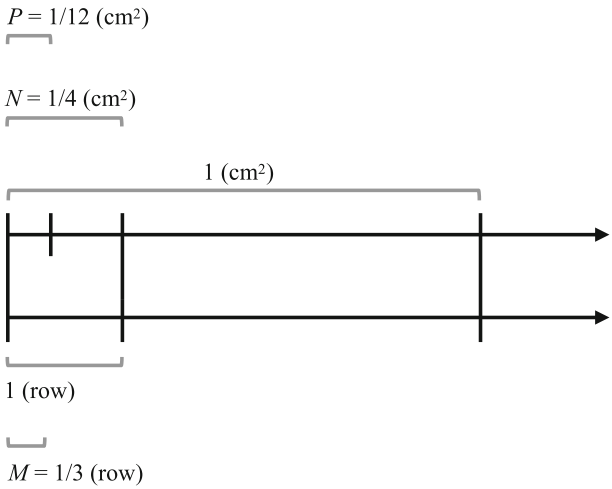

Fig. 9 1/4 (cm^2 makes 1 row) • 1/3 (row makes 1 mini rectangle) = 1/12 (cm^2 makes 1 mini rectangle)

analyzed were multiplicative. Rather, we used the definition of multiplication in Fig. 1 to decide whether they were multiplicative. Our discussion of these examples illustrates the consistency sense of coherence across Vergnaud's isomorphism of measures and product of measures subtypes: One definition applies consistently to all these cases.

5.2 Division

Our approach to multiplication—in which N, M, and P are consistently interpreted as results of measurement—has significant consequences for thinking about division. We emphasize that we view division in terms of the $N \bullet M = P$ equation and that we do *not* identify the action of iterating with multiplication or the action of partitioning with division. When combining copies of a given quantity, we use the term iterating (not multiplication), and when breaking a given quantity into equal-sized parts, we use the term partitioning (not division). In general, iterating and partitioning are actions that contribute to determining an outcome of measurement, especially when fractions are involved. When these actions are in service of determining P, they are connected with multiplication; when they are in service of determining N or M, they are connected with division.

From a coordinated measurement perspective, division situations are characterized as multiplication with an unknown factor. In particular, what makes a situation a division situation, given the use of base units and groups to measure a product amount, is either a question about the measure of 1 group in terms of base units—a question about the value of N—or a question about the measure of the product amount in terms of groups—a question about the value of M. The former is often referred to as partitive or sharing division, and the latter is often referred to as quotitive or measurement division (e.g., Greer, 1992). Notice that, whereas sharing is often associated with partitive division and measuring is often associated with quotitive division, in our discussion, measuring is associated with both types of division. The distinction between partitive and quotitive division lies in what is being measured—1 group or 1 product amount—and what is being used as the unit of measure—1 base unit or 1 group.

Viewing both N and M in terms of measurement has consequences for how one thinks about division longitudinally across topics, starting with whole-number situations. We return to the situation in which a potter has 21 oz of clay, and we take 1 base unit to be 1 oz, 1 group to be the weight of 1 plate, and 1 product amount to be the total weight of clay (Fig. 4). We can ask how many ounces are used for 1 plate if the potter uses all the clay to make 7 identical plates. Here we are asking how many base units make one group exactly ($N \bullet 7 = 21$). We can also ask how many identical plates can be made if the potter uses 3 oz to make 1 plate and uses all the clay. Here we are asking how many groups make the product amount exactly ($3 \bullet M = 21$).

Interpreting both N and M in terms of measurement has consequences for thinking about further topics. We illustrate this point by returning to the distinction between extensive and intensive quantities (e.g., Schwartz, 1988; Simon & Placa, 2012). Schwartz motivated the distinction between intensive and extensive quantities by considering 5 pounds of coffee beans that costs 15 dollars. He explained that if two piles of such beans were combined, the weight and the cost of the combination would increase but that the price per pound would stay the same. From Schwartz's perspective, N in Fig. 1 would correspond to an intensive quantity, while M and P would correspond to extensive quantities that counted pounds and dollars, respectively (3 dollars per pound • 5 pounds = 15 dollars).

Notice, however, that interpreting N, M, and P in terms of measurement blurs the distinction between intensive and extensive quantities, because the units attached to all three numbers are of similar form. This blurring occurs precisely because the definition in Fig. 1 explicitly treats the product amount as something that is being measured both in base units and in groups. If one modified the coffee situation so that the beans were in bags each weighing 5 pounds and costing 15 dollars, and one took 1 base unit to be 1 dollar, 1 group to be 1 pound, and 1 product amount to be 1 bag, then one could fit the measurement definition of multiplication as follows:

$$3\ (\text{dollars correspond to 1 pound exactly})\bullet 5\ (\text{pounds correspond to 1 bag exactly})$$
$$= 15\ (\text{dollars correspond to 1 bag exactly})$$

If one were to combine two identical bags of coffee, similar to Schwartz's combining two piles, then—making different choices for base unit, group, and product amount—one could fit the measurement definition of multiplication in at least two ways:

$$5\ (\text{pounds correspond to 1 bag exactly})\bullet 2\ (\text{bags correspond to 1 total exactly})$$
$$= 10\ (\text{pounds correspond to 1 total exactly}).$$
$$15\ (\text{dollars correspond to 1 bag exactly})\bullet 2\ (\text{bags correspond to 1 total exactly})$$
$$= 30\ (\text{dollars correspond to 1 total exactly}).$$

Thus, whether or not one interprets the 5 and 15 as referring to extensive or intensive quantities is sensitive to how one views the coffee bean situation. From the coordinated measurement perspective on multiplication, the coffee bean situation fits Schwartz's (1988) I • I' = I" classification. Put another way, we see coordinated measurement as subsuming the distinction between intensive and extensive quantities, at least for homogeneous situations like the one involving coffee beans, a point to which we return in the concluding discussion.

Finally, interpreting what traditionally have been termed partitive and quotitive division in terms of measurement can facilitate attending explicitly to division when reasoning about proportional relationships. In fact, there are two distinct possibilities (see also Beckmann & Izsák, 2015). First, one can imagine holding N fixed and allowing M and P to covary. In this case, N describes a fixed quotient, M describes a varying number of groups, and P describes the varying number of base units in the M groups. In terms of the coffee bean situation, this corresponds to fixing the price per pound and letting the number of pounds (M) and the total price (P) covary. Descriptions of "building up" methods (e.g., Hart, 1981; Kaput & West, 1994; Lobato & Ellis, 2010; Tourniaire, 1986) in the literature on proportional relationships provide further examples of this approach to proportional relationships. Second, one can imagine holding M fixed and allowing N and P to covary. In this case, M describes a fixed quotient, N describes a varying number of base units in one group, and P describes the varying number of base units in the M groups. In terms of the coffee bean situation, this corresponds to fixing the number of pounds and letting the price per pound (N) and the total price (P) covary. This perspective on proportional relationships, which Beckmann and Izsák termed the variable-parts perspective, is largely absent from the literature on proportional relationships.

Our discussion illustrates how a consistent measurement meaning for multiplication can span whole-number division situations and proportional relationships. At the same time, doing so provides a different perspective on the distinction between partitive and quotitive division and on the distinction between intensive and extensive quantities. The advantage is that the same measurement questions for N and for M support the "logical or clear interconnection" sense of coherence in our approach to the MCF. In this case, the interconnections span topics that likely would be taught at different grade levels.

5.3 Reconsidering a unified approach to the MCF

In our reading, prior theoretical approaches to the MCF that have explicitly discussed coherence have either done so using the sticking together sense or have only partially succeeded using the consistency or logical interconnection senses of uniting smaller pieces into a larger whole. We think that mathematics education as a field should seek more completely worked out, coherent approaches to the MCF based on consistency and logical interconnection. The absence of such articulation may be constraining our capacity to help students and teachers use prior knowledge and experience to effectively relate topics and construct interconnected bodies of knowledge. It is one thing to know that multiplication can be used to model a variety of situations and another to perceive a common underlying structure. We are not claiming that coordinated measurement is clearly the optimal approach: There might be other approaches, perhaps based on bi-linearity, that work at least as well.

As we have emphasized, our coordinated measurement approach to the MCF is based on interpreting numbers as outcomes of measurement. This is not the same as interpreting numbers as cardinalities of sets. We wonder whether the transition from additive to multiplicative reasoning that researchers have repeatedly identified as challenging for students and sometimes for teachers might be, at least in part, about a transition from interpreting numbers as cardinalities to numbers as measurements. These two senses of number may be hard to distinguish in the case of whole numbers, especially when the items being counted and the units being used to measure are all the same size: Consider the difference between asking what is the count or cardinality of ounces in 1 plate and asking how many ounces make one plate exactly? (see Beckmann & Izsák, 2018a, b). We also wonder whether the distinction between intensive and extensive quantities that we discussed above is in part about different interpretations of numbers, as measurements for intensive quantities and as cardinalities for extensive quantities. Interpreting numbers consistently in terms of measurement may contribute to the blurred distinction we discussed above.

We close with a set of questions about broad, unified approaches to the MCF. First, with respect to coordinated measurement, we have sketched how the meaning of multiplication shown in Fig. 1 can be applied consistently across many of the situations Vergnaud (1983) classified into different subtypes, and we have shown how interpreting numbers in terms of measurement contributes to perceiving the same underlying structure even as those numbers migrate from whole numbers to fractions. At the same time, we do not think coordinated measurement is a panacea for organizing the MCF, at least for learners. Searching for appropriate base units and groups that are not always mentioned explicitly in problem statements is a significant challenge, and the substrate of knowledge that supports proficient determination of base units and groups, especially when they are not indicated explicitly, is likely complex. More generally, developing the habit of searching for a specific structure that fits a definition for multiplication is not often emphasized in school curricula with which we are familiar. Furthermore, we wonder how well the coordinated measurement perspective fits with uses of multiplication in other disciplines that deal with quantities, such as physics, where the use of multiplication is intertwined with creating and evaluating models of physical phenomena (e.g., Schwartz, 2009).

Second, with respect to the units transformation approach to unifying the MCF, we wonder how the fundamental transformation of unit (e.g., units of boys and units of girls into units of couples and units of length into units of area) can make sense to children without making the same move we have made in our discussion of coordinated measurement, which is to examine the product amount directly and ask what sort of units make sense for measuring this quantity? If one does not do this, then how does one motivate statements like 1 cm • 1 cm = 1 cm^2? Why do products of dimensions and of numbers co-occur? Would a statement like 1 cm • 1 cm = 1 cm^3 also be sensible? If you characterize multiplication as a function from $R \times R \rightarrow R$, how do you distinguish those functions that are bi-linear from others that are not? How would you connect measurement, multiplication, and number, especially given multiple uses of multiplication like those delineated by Vergnaud (1983)?

Finally, although learners may experience the wide range of topics related to multiplication and measurement as initially disjoint, developing a coherent view of such topics is a desirable educational goal both because it can support and reinforce an interconnected knowledge base and because seeking and identifying common structure across diverse situations reflects a core value of the mathematics community that school children should experience in age appropriate forms.

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