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Book Review: The philosophy of mathematics education today. Paul Ernest (Ed.) (2018)

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The Philosophy of Mathematics Education Today (hereafter *PMET*) is a significant product of the ICME-13 congress, which took place in July 2016 in Hamburg. Given that the scope of the conference presentations collected in this book includes philosophical discussions about mathematics that have occurred over the last two and a half thousand years, those arising from an event 3 years ago may indeed feel like "today"! This book should be a valuable reference for PhD fellows and their supervisors—and anyone interested in the nature and impact of mathematics education.

The book opens with a provocative prologue by Luis Radford, "A plea for a critical transformative philosophy of mathematics education", arguing from his reading of chapters within the book. Radford's opening is the only explicit attempt to draw out a unifying message from the book, and he states:

> I would like to contend that we must go one step further and *act*, *take action*, so that our analyses, questions and critiques come to make, through concerted movements, a *transformation* of mathematics education as it is practiced today. … we need … a critical and transformative philosophy of mathematics education. (p. 8)

This feels more like a political manifesto than a treatise on philosophy.

Paul Ernest, in Part 1 of *PMET*, *Introduction to the Field*, appears to agree with Radford's *sentiment*; however, Ernest distinguishes between "philosophy (theory) and practice, … the role of philosophy of mathematics education is to analyse, question, challenge, and critique the claims of mathematics education practice, policy and research" (p. 33). *PMET* thus raises the question whether philosophy offers only the tools to "analyse, question, challenge, and critique"? Or does philosophy, as Radford requires, embrace the action as well—is (or should)

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philosophy (be) "transformative", and does philosophy lead to a theoretically grounded transformation of mathematics education in which the imagined outcomes are rationally and ethically grounded?

My initial reaction, on reading Radford's assertion, was to disagree. I felt more comfortable with a rather restricted interpretation of Paul Ernest's observation: "… philosophy enables us to 'question the unquestionable', including the problematising of some of the sacred Shibboleths of mathematics education … " (pp. 15–16).

Ernest emphasises the role of questions in his opening chapter that constitutes Part 1, "The philosophy of mathematics education: An overview", by posing, rhetorically, over 130 questions for which there are no definite answers (yet). As Ernest observes: "… the questions, when suitably refined, could form the bases for hundreds of PhDs in mainstream mathematics education" (p. 23).

Perhaps for these questions alone, there is a place for this book on the (virtual) shelf of every PhD supervisor and research project proposer.

My disagreement with Radford was resolved through reflecting on why one should engage with philosophy at all. Whereas I had assumed the *basic question* was "how should I *think*?", after reading the opening sentence in Michael Otte's chapter in Part 2, *The Nature of Mathematics*, "The basic question in philosophy is: how should I live?" (p. 61), I began to be persuaded to agree with Radford's assertion. By the time I had read through to the end of Part 3, *Critical Mathematics Education*, I was fully persuaded by Radford's position.

The chapters collected in this book have the power to persuade as well as inform, and they will reward careful reading and reflection. Many years' experience of teaching doctoral courses in mathematics education tells me that a shared critical reading of any of the chapters included in *PMET* could be used to stimulate great discussion amongst PhD fellows.

I continue this review by addressing several broad issues that I believe are either directly addressed or implicit within the chapters included in *PMET*. There are three central themes. First, the nature of mathematics education as a field of inquiry; second, the consideration of educating students in the discipline of mathematics; and third, the consideration of mathematics as making a distinct and essential contribution to education, that is the enculturation of future citizens. Then, following the structure of the book, I include a section on a philosophy of mathematics education as a research field, and a section on the philosophy of mathematics as approached in classrooms. Finally, in a postscript, I will make some observations about the production of the book. First, I will say something about the overall structure and content of *PMET*.

1 About the book

Following Radford's prologue, *PMET* is organised into five parts, each addressing a broad theme:

1. Introduction to the Field (comprising one chapter by Paul Ernest)
2. The Nature of Mathematics (4 chapters)
3. Critical Mathematics Education (6 chapters)
4. Philosophical Theory in Mathematics Education Research (4 chapters)
5. Philosophy of/in Teaching, Learning and Doing Mathematics (8 chapters)

The book is thus composed of 24 chapters, including Radford's prologue. The chapters are, for the most part, written by senior, experienced academics, who have been engaged with the philosophy of mathematics education for many years. Thus, the book makes a magisterial contribution to the field. I would prefer to refer to the book as a collection of papers rather than chapters because they are written independently, largely without reference to each other (except for Radford's prologue), and there is a very small number, two or three, of references to other articles in the book. There is no "red thread" running through the book, which has the advantage that the chapters can be read quite independently of one another; one does not need to engage with large chunks of the book to make sense of each chapter. On the other hand, reading from cover to cover does require some discipline, and each chapter provides very little with which to organise ideas before moving on to the following one. The composition of the chapters into each of the five parts makes sense (although I question the position of two in Part 5), and there is a coherence of *focus* within each part, but less so of authors' perspectives. From a reviewer's perspective, it is challenging to write a comprehensive and coherent review of a book that is composed of a collection of chapters that address such a wide collection of themes.

In the preceding paragraph, I described the book as "magisterial", which I believe to be accurate. At the same time, I believe it to be far from the comprehensive review of the field that one might expect to emerge from an ICME group. There are 32 authors who are working in Belgium, Brazil, Canada, Czech Republic, Denmark, Germany, Israel, New Zealand, the Netherlands, Sweden, UK, and USA. It will be immediately appreciated that there are many significant countries (e.g., France) and continents (e.g., Africa and Asia) that are not represented. I will merely add a further question to Paul Ernest's collection in Part 1: Why are these mathematics education research communities not represented in this book? There may be a simple answer that they were not represented in the participation or contributions in the discussions at ICME-13, but that merely prompts the follow-on question, why?

Nevertheless, the book is a substantial collection of chapters that address many themes and provide a range of starting points for philosophical engagement in mathematics education. A few suffer from brevity, which probably arises from the instructions given to the authors. One notable example, in Part 2, is Michael Otte's chapter: "The philosophy of mathematical education between Platonism and the computer". To provide an account of two and a half thousand years of thought in fewer than 20 pages is a formidable challenge. Otte attempts a meaningful discussion of some major problems confronting the philosophy of mathematics education, from the notions of ideal and generalisation, through meaning and formalisation, semantics and syntax, reference and meaning, formal and algorithmic thinking development and application, mathematics as problem solving, explanation and technique, to theory and algorithms. A relative newcomer to the philosophy of mathematics (education) could reasonably engage with this chapter and tease out the contrasting positions, and thus gain some directions for further reading. I am not sure that it could be a starting point for an absolute newcomer—maybe Davis, Hersh, and Marchisotto's (2012) The Mathematical Experience, Study Edition could serve as an "advance organiser".

2 Mathematics education as a "field" of research, knowledge/knowing and practice

The Philosophy of Mathematics Education Today supports an assertion that mathematics education, as a field, is concerned with education in general as well as teaching and learning mathematics.

Mathematics education as a field of study and research looks in two directions. First, it is concerned with the teaching and learning of mathematics and the contexts in which mathematics is taught and learned. Mathematics education is thus concerned with the development/acquisition/construction/appropriation/internalisation/etc. of mathematical knowledge/knowing/competence/behaviours.[1] The philosophy of mathematics education thus engages with questions of ontology, epistemology, and methodologies of mathematics. Second, mathematics education addresses issues surrounding mathematics as part of a societal educational endeavour. Mathematics makes an important contribution to reaching the general goals of education; mathematics is a component of desirable approaches to and outcomes from education and can be used as an educational device for the reproduction or radical transformation of society.

A philosophy of mathematics education that looks towards mathematics may be content with the formal theory-oriented role for philosophy that Ernest delineates. Part 2 of *PMET*, which focuses on *The Nature of Mathematics*, addresses this perspective. Part 3, *Critical Mathematics Education*, looks in the other direction, towards the educational role of mathematics. It demonstrates the essential inclusion of a philosophy of transformation, action, and practice informed by ethics, politics, economics, and religion (cf. Andrade-Molina, Valero, & Ravn, in Part 2: "The amalgam of faith and reason: Euclid's *elements* and the scientific thinker").

Briefly, the philosophy of *mathematics* education is primarily concerned with ontology, epistemology, and mathematical products and practices. The philosophy of mathematics *education* is concerned with ethics, values, politics, and mathematical competencies and power (to inform, critique, analyse, liberate, and enable). Embracing these is a philosophy of *mathematics education* concerned with the ontology, epistemology, and methodology of our research field, which is addressed in Part 4: Philosophical Theory in Mathematics Education Research. Part 5 departs from the consideration of macro issues of mathematics and education to take a micro focus on classroom level issues of research and practice.

3 Concerning educating about/in *mathematics*

If one is to be clear about what an education in mathematics means, attention to the nature of mathematics is essential. Part 2 considers this theme, which is opened by Jean Paul Van Bendegem: "The who and what of the philosophy of mathematics education". Van Bendegem sets out the complexity encountered by those seeking to understand mathematical practices by outlining eight approaches:

- The Lakatosian approach, also referred to as the "maverick" tradition,
- the descriptive analytical naturalising approach,
- the normative analytical naturalising approach,
- the sociology of mathematics approach,
- the mathematics educationalist approach,
- the ethnomathematical approach,

[1] Note how I have felt obliged to insert so many alternative metaphors for learning—a challenge in our field is that there is no commonly accepted theory of learning; there is not even any agreement about what mathematics is. The absence of agreement demonstrates the necessity of all working in mathematics education to have a sound grasp of the philosophical foundations of the field.

- the evolutionary biology of mathematics,
- the cognitive psychology of mathematics. (p. 46)

I am reminded of the report from an ICMI study published about 20 years ago: *Mathematics Education as a Research Domain: A Search for Identity* (Sierpinska & Kilpatrick, 1998). Reviewing the report, the mathematician Lynn Arthur Steen (1999) was strongly critical, not of the report per se, but of mathematics education as a research field:

> … perhaps the phrase 'research in mathematics education' is simply a pretension, if not an oxymoron. Is mathematics education even susceptible to research? What might be the aims of such research? What are the objects of study? What are the chief questions? What are the main theories? What are the key results? What are the criteria? What are the important applications? Critics of research in mathematics education point to the paucity of answers to these questions as evidence that what we are dealing with here is more like a political movement than a scientific discipline. (p. 235).

Steen claimed that "to a mathematician, mathematics is singular — a Platonic paradigm in which there are simple, unquestionable criteria for distinguishing right from wrong and true from false" (p. 237).

Van Bendegem's chapter, along with many others in *PMET*, suggests that no matter what mathematicians may "feel", a philosophical analysis of their practices reveals a much more complex picture. Van Bendegem points to a heterogeneity of mathematics, and Catarina Dutilh Novaes (also in Part 2) writes about the "epistemic diversity observed among mathematicians" (p. 93), which appears to be unnoticed by Steen. Mathematics educators seeking to introduce learners to mathematics and mathematical practices cannot be content with a simple reliance on the ontology, epistemology, and methodology of a Platonic paradigm. Mathematicians who may be inclined to dismiss mathematics education as a serious scientific field of research would be well advised to read Van Bendegem's chapter, amongst others, in *PMET*. However, they should heed Van Bendegem's warning: "one of the major consequences of the complexity … is that … I find that I have to work in different 'registers' when reading different texts in the field" (p. 47). Nevertheless, there does appear to be some common ground shared by Van Bendegem, the contributors to the 1998 ICMI study, and Steen, as evident in the following observation by Van Bendegem: "fundamental tensions at play … two major tasks for the future … develop a greater coherence in the field and … keep the conversation going with other philosophers of mathematics" (p. 48).

Only a very small proportion of students at school will go on to higher education to study mathematics as their major discipline, and few of those who do will become professional (research) mathematicians. In higher education, many more students follow mathematics courses as a so called "service subject" in engineering, natural sciences, economics, etc. *PMET* does not explicitly consider the mathematical needs of students following different vocational trajectories, but the chapters do provide a point from which one may begin to reflect on these diverse needs and routes. For example, in her chapter, "A Dialogical Conception of Explanation in Mathematical Proofs", Catarina Dutilh Novaes considers the role of explanation in proof, and how alternative proofs can be more or less explanatory, just as they can be more or less formal (cf. Ording, 2019). It seems to me that a learner's experience of and engagement with proof can vary in explanatory power and formality, as well as in following and understanding a proof versus constructing a proof according to the intended educational outcomes. Dutilh Novaes explains:

> What the phenomenon of different levels of granularity suggests when it comes to the explanatoriness of proofs is that, for a proof to be explanatory for its intended audience, the right level of granularity must be adopted. If a proof is to be explanatory in the sense of making 'something that is initially puzzling less puzzling; an explanation reduces mystery' (Colyvan, 2012, p. 76), mystery reduction is inherently tied to the agent *to whom* something should become less puzzling. (p. 92)

4 Concerning mathematics about/in *education*

Part 3 of *PMET* is organised around a proposal for critical mathematics education with the lead chapter "Students' foregrounds and politics of meaning in mathematics education" by Ole Skovsmose, a founding proponent of critical mathematics education. Skovsmose presents his arguments based on the notion of foreground, which he explains thus: "A basic idea of a foreground-interpretation of meaning is that students' experiences of meaning first of all emerge when they recognise that their learning actions can be directed towards features of their foregrounds" (p. 120). Skovsmose then proceeds "to discuss different socio-political formattings of foregrounds, and indicate how they structure experiences of meaning. I will talk about polarised foregrounds, destroyed foregrounds, pointed foreground[s], and multiplied foregrounds" (p. 121).

The educational focus of Part 3 is interesting because it reveals an evolution in the *political* direction of mathematics education. The publication of *The Philosophy of Mathematics Education* (Ernest, 1991) led to the promotion of a problem posing pedagogy. Sixteen years later, the book *Philosophical Dimensions in Mathematics Education* (Van Bendegem & François, 2007) promoted a pedagogy focused on ethnomathematics and culture. In *PMET*, the focus has moved to critical mathematics and includes a critique of a pedagogy framed around culture and ethnomathematics by Brendan Larvor and Karen François, "The concept of culture in critical mathematics education". Despite the divergence between Van Bendegem and François (2007) and *PMET*, there remains agreement on the importance of ethics in a mathematical education — especially the chapter by Margaret Walshaw, "Epistemological questions about school mathematics", and also the call for an ethics-oriented philosophy of mathematics education by Paul Ernest, "The ethics of mathematics: Is mathematics harmful?" The latter will be discussed further below. Another call for a philosophical inquiry component in the mathematics curriculum is made by Nadia Stoyanova Kennedy in Part 5: "Towards a wider perspective: Opening a philosophical space in the mathematics curriculum".

Is the move from problem posing, through ethnomathematics to critical mathematics, evidence of an evolution (as suggested above)—that is, a progressive development in the field? Or, are the differences merely evidence of changing zeitgeist or fashion? The temptation is to conclude the latter, due to the fact that the authors in this collection are likely to be self-selecting. I do not believe the contents of *PMET* indicate that problem-based, inquiry-based, or other mathematically oriented pedagogies are dead or dying. Rather, as noted above, a rather narrow group of countries is represented by the authors, and I believe this is also reflected in the pedagogical discussions.

I was very pleased to read Paul Ernest's acknowledgement in Part 1:

> Let me start by saying that progressive child-centred pedagogies conducted well are very likely more successful than traditional teacher-centred pedagogies done

> badly, both in terms of student learning outcomes and student satisfaction. But likewise, traditional teacher-centred pedagogies conducted well very likely have better outcomes than progressive child-centred pedagogies done badly. The key factor is not one of technique, or even ideology, but of effective application of teacher knowledge and teaching skill. (p. 24)

Of course, this leaves open questions such as: What can be agreed about desirable outcomes? How can these be measured? What is counted as effectiveness? And even, how can pedagogies be evaluated to be conducted "well" or "badly"? If "we", presumed mathematics education experts, make demands on teachers to reframe their pedagogy, as can happen in developmental research projects, I believe it is possible to undermine competence and their success as teachers. In his chapter, "The struggle is pedagogical: Learning to teach critical mathematics", Eric "Rico" Gutstein points directly at the problem of learning a new pedagogy. Mathematics educators need to be cautious in seeking changes in pedagogy. Let us seek to do things well, not necessarily differently. This is not to say that I believe mathematics educators should not be continually reflecting on the relevance of the curriculum. For example, Richard Barwell's chapter "Some thoughts on a mathematics education for environmental sustainability" makes a lot of sense to me, as an educator and emphasises the important contribution that mathematics has to make in the education of the next generation. Barwell argues that to be informed and socially responsible citizens, it is necessary for people to have more than mathematical knowledge (the goal of mathematical and statistical literacy), and more than technological knowledge (about how technology facilitates and harnesses the power of mathematics). In this chapter, Barwell focuses on a major global issue of our time, environmental sustainability, and asserts the importance of mathematics education that will develop learners' critical awareness of the uncertainties present in "post-normal" science[2], and statistical and mathematical models. "Participation in the wider peer community of science … requires mathematically literate citizens; in particular, future citizens need an education that includes a critical understanding of the role of mathematics in understanding and creating current environmental problems" (p. 150).

In "The ethics of mathematics: Is mathematics harmful?" Paul Ernest focuses attention on a *possible* ethical dimension of mathematics. I write possible in italics because this is my insertion. For Ernest, there are at least three strands to an ethical dimension of *mathematics*: mathematics can lead to styles of thinking that can shape and control society, mathematics can be used for beneficial as well as harmful purposes, and mathematics can harm an individual's self-esteem and self-efficacy. For myself, I could go along with the claims if Ernest used *mathematics education* (or mathematics applications) rather than mathematics. My issue probably lies in a divergence between the ontologies of mathematics that Ernest and I espouse. However, I cannot help but wonder about Ernest's use/understanding of the *ontology* of mathematics. He writes, for example, "I have critiqued the idea that mathematics is an

[2] Barwell explains "post-normal science", "Post-normal science is a way of responding to problems with particular features: high levels of uncertainty, urgency, high stakes and the interrelation of facts and values. Such problems feature contradictions and our collective response needs to involve an extended peer community. Normal science with its standardised procedures, ways of defining problems and separation of facts from values is insufficient" (p. 154).

untrammeled *force* for good" (p. 205, my italics). This appears to ascribe to mathematics an intrinsic dynamic; can this be? Later he writes "I also claim that mathematics has a bad *face*" (p. 205, my italics). Exactly what is meant by the faces of mathematics? However, on the following page Ernest appears to contradict himself; he declares: "The force of my critique is not directed at mathematics itself, but at the social institutions of mathematics, including training in mathematics, and the false social images of mathematics that they can legitimate and project" (p. 206). He also declares "mathematics is not intrinsically bad or harmful" (p. 206). So what does he mean when stating that mathematics "is a force" and "has a face"? I want to respond that the force and face are outcomes of the uses and applications of mathematics and especially mathematics education (that is education of, in, and mediated by mathematics). I urge readers to engage with Ernest's chapter for themselves, and not to rely on my interpretation. I hope, however, the reader will agree with Ernest's conclusion, irrespective of nuances in the argument, as other contributors to Part 3 and I do. Mathematical instruction at all stages should include an element of philosophy that enables learners to take a critical stance towards the applications of mathematics and the unwanted possible consequences of (aspects of) their own mathematical education.

5 A philosophy of mathematics education

Part 4 focuses on philosophy of mathematics education as a research field. The common ground in this part appears to be communication, language, and discourse. Out of interest, I searched for the number of times different philosophers were cited in the book—I am not sure if any meaning can be read into this because the count included the references at the end of each chapter. The count may be indicative of "influence" in the field as represented by the chapters of *PMET*. The largest number of citations was to Plato (97), followed by Foucault (72), Wittgenstein (67), Frege (50), and Freire (50). This seems to reflect the focus of Part 2 on (critiquing) a Platonist paradigm, Part 3 on critical mathematics, and Part 4 on language and discourse. Other philosophers with over 20 references each include Kant (42), Lakatos (36), Hilbert (28), Descartes (26), and Aristotle and Husserl (24). The point I want to make, again, is that *PMET* may be a very useful resource, but it should not be considered comprehensive. The title of the volume is quite assertive *The* philosophy of mathematics education *today*, to be accurate, and it is composed of a collection of chapters that present a self-selection of philosophies that underpin a section of the field of current inquiries in mathematics education. The philosophies underpinning some very important components of current research in mathematics education, such as socio-cultural, cognitive, and anthropological theories, are not considered.

Part 2 opens with a chapter by Anna Sfard, "On the need for theory of mathematics learning and the promise of 'commognition'". Sfard starts by empathising with the plight of PhD fellows who look for a suitable theory for their research in mathematics education and explains why theory is important for her research: "Theory is the ground to stand on while trying to move and the signpost to follow while looking for a direction. Without it, I feel like walking on a thin ice in the middle of night" (p. 220). The chapter continues with an outline of the rationale for Sfard's development of a commognitive theory of mathematics learning.

Following Sfard's chapter, Ladislav Kvasz writes "On the role of language in mathematics education", in which he sets out a case for the study of the language of mathematics. Kvasz uses an argument based on a historical consideration of the development of mathematics to propose a linguistic framework of mathematics that bridges the divide between objective and subjective knowledge of mathematics, and which supports epistemological analysis. Uwe Schurmann also adopts an historical perspective in his chapter "The separation of mathematics from reality in scientific and educational discourse". This will certainly interest those who seek an explanation for the development of the philosophy of mathematics, and to a large extent, it complements Otte's chapter in Part 2, discussed above. The final chapter in Part 4, "Mathematics education actualized in the cyberspace: A philosophical essay" by Maria Aparecida Viggiani Bicudo considers the "virtual world" created as one engages with mathematics digitally. Throughout the chapter, I repeatedly reflected on how the experience of mathematics mediated by a computer screen differed from reading, reflectively, a printed text; I would have welcomed some consideration of this in the chapter. As I engaged with the discussion about ontology and epistemology, even "modes of dialogue present in the humans-with-computer" (p. 263), I was able to insert the phase "and humans-with-printed text" into Bicudo's argument without difficulty. I would also have liked to read about the relation between the philosophical arguments articulated and the instrumental approach (Trouche, 2005). However, on this latter point, I recognise that I am expecting too much of Bicudo's chapter, but I do hope that those researching the mediation of mathematics by digital technology will engage with Bicudo's argument.

6 Philosophy of mathematics education where it counts—in the classroom

The final section of *PMET*, Part 5, focuses on the philosophy of/in teaching, learning, and doing mathematics. Based on a count of chapters, it is the largest part of the book, composed of eight chapters. I am sure that Ernest, as the editor of *PMET*, had his reasons for selecting the chapters for inclusion in this section, but I think some of the chapters have more in common with those in other parts. For example, I have already drawn attention to the call by Nadia Stoyanova Kennedy for philosophical elements to be incorporated in the school mathematics curriculum, alongside Paul Ernest's claim for an ethical stance, discussed in Part 3. Another contender to be included in a different Part, I believe, is John Mason's chapter "Making distinctions: A phenomenological exploration in mathematics education", which opens Part 5. Mason sets out his arguments for the development of awareness, based on the work of Caleb Gattegno, and Ference Marton's theory of variation. I have always been a bit destabilised by Mason's work because he appeals to theoretical/philosophical/linguistic notions and argumentation that seem rather tangential to the mainstream. This is not a criticism; rather, his appeal to sources with which I am not familiar is part of the rich stimulation I experience from his work. I think Mason's chapter could well have been placed within Part 4; it would usefully have broadened its scope and supported an appreciation of his approach within the mainstream of theories endorsed by researchers in mathematics education.

Of the six remaining chapters in Part 5, two are fundamentally theoretical presentations and four are illustrated with data generated from classrooms. The first theoretical presentation is "Creativity research in mathematics education simplified: Using the concept of bisociation as Ockham's razor" by Bronislaw Czarnocha, William Baker, and Olen Dias. The authors apply a framework proposed by Arthur Koestler (1964) who explains the principle of bisociation:

> Bisociation is 'a spontaneous leap of insight which connects previously unconnected matrices of experience' (Koestler, 1964 p. 45)—known also as an 'Aha!' Moment, or

> Eureka experience. A bisociative framework is the framework composed of 'two unconnected matrices of experience,' where one may find a 'hidden analogy'—the content of insight. The definition asserts that the presence of the bisociative framework is the necessary condition for the 'Aha!' Moment to occur. (p. 323).

The other theoretical presentation in Part 5 is by Regina Dorothea Möller: "Teaching of velocity in mathematics classes—Chances for philosophical ideas". This begins with a didactical analysis of the presentation of the concept of velocity in German school texts, and then moves on to consider the historical and philosophical development of the concept. A discussion of the how the philosophical aspects might be developed in class leads Möller to propose "This kind of endeavour invites the pupils to think and to experiment for their arguments and can lead to a thorough understanding …" (p. 341).

Finally, the first of four chapters drawing on data generated from classrooms is "Using rules for elaborating mathematical concepts" by Michael Meyer, who uses Wittengenstein's notion of language games: "… we can understand conceptual learning as an inferential use of words. ... understanding (students') conceptual learning processes means to understand the rules they use in order to establish meaning" (p. 306). The second, "Time for work: Finding worth-while-ness in making mathematics", by Hilary Povey, Gill Adams, and Colin Jackson, reports from an intervention in which primary age children were engaged in practical activities preparing a mathematics exhibition open to other students, their parents, and the general public. The authors outline a set of theoretical foundations for the work that is intended to give the students practical "hands-on" (literally), meaningful tasks. The third, by Walther Paravicini, Jörn Schnieder, and Ingrid Scharlau, has the rather ominous sounding (at least in English) title, "Hades—The invisible side of mathematical thinking". Here, "HADES" is an acronym for different methods of thinking: Hermeneutic, Analytical, Dialectical, Experience-based or phenomenological, and Speculative. The authors propose that these "philosophical methods of thinking and working" are developed as approaches to "reflective reading and writing strategies for mathematical texts" (p. 354). The final chapter in Part 5, and the book, is "Developing rules due to the use of family resemblances in classroom communication", by Jessica Kunsteller, in which the notion of family resemblances arises from the work of Wittgenstein, and is used to illustrate how students can develop meanings of concepts through appreciation of resemblances in different ways—phonetic, semantic, inferential, solution process, iconic, and/or symbolic.

7 Conclusion

As for any review, this is as much about the reviewer's interpretation as it is about the book, or collection of chapters under review. In the foregoing, I have drawn attention to issues, problems, questions, etc. that I have experienced as provocative. This is a very subjective and individual account, and it is unlikely that another reader will be stimulated to reflect in the same way. However, despite my critical comments about lack of cohesion and brevity of the chapters overall, I am confident that *The Philosophy of Mathematics Education Today* will stimulate reflection and reaction, and inform a broad range of readers, including experienced researchers in mathematics education and mathematics, mathematics teachers, PhD advisors, and fellows.

8 Postscript: a note about the quality of production of *The Philosophy of Mathematics Education Today*

In some of the chapters, my reading was seriously disturbed by errors that should have been corrected by careful proof reading or competent language checking. For me, with English as my first language, the errors are frustrating and irritating, but with a little bit of effort and guesswork, I managed to make sense. However, the PhD fellows who have been included in the Mathematics Education PhD programme at my university do not have the same advantage; none has English as first language, and over ten different first languages are represented. They will find this book of great value, and they will inevitably be challenged by the complexity of arguments in some chapters, but it should not be necessary for them to negotiate errors.

Another issue I have with the book is the absence of an index. I guess it would be argued that if one values an index, then it would be better to download a *searchable* eBook version. However, a good index can add a dimension to the value of a book.[3] A good index opens up themes that run throughout a book, and in a collection of independent chapters such as this, it would be a very valuable addition. More than this, when taking up a book for the first time, it will be the contents pages—and the index—that help to demonstrate that the book is worth attention. Furthermore, the possibility of only being able to initiate one's own search runs the risk of missing unexpected themes which may be introduced and illuminated through the text.

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[3] Subsequent to reading the hard copy, I did download the e-version through my university library. In particular, I wanted to get an idea of the extent to which the collection of articles referred to "established" philosophers. The search tool worked well, but I had to enter separately the names of individual philosophers, and it still required an amount of work to determine whether the citations were concentrated within a single or small number of articles. It is not the same as a well-prepared index.