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Exploring the way rational expressions trigger the use of “mental” brackets by primary school students

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Abstract
When a number sentence includes more than one operation, students are taught to follow the rules for the order of operations to get the correct result. In this context, brackets are used to determine the operations that should be calculated first. However, it seems that the written format of an arithmetical expression has an impact on the way students evaluate this expression. It also seems that a connection exists between this way of evaluation and an understanding of structure. Both issues are examined in this paper. A number of arithmetical expressions in a rational form were given to primary school students from Greece and Sweden. The collected findings strengthen our hypothesis that this rational form of the arithmetical expressions was of critical importance for the students’ decision on how to evaluate these expressions. They temporarily put aside their knowledge about the rules for the order of operations. Instead, the way they evaluated the expressions indicates an implicit use of what we call in this paper “mental” brackets. It is very likely that the use of these “mental” brackets is closely connected with students’ structure sense.

Keywords Order of operations · Brackets · Evaluation of arithmetical expressions · Mental brackets

1 Introduction

There is a variety of forms of brackets that are frequently used in mathematics, such as parentheses ( ), square brackets [ ], braces { }, and angle brackets <>. In arithmetic

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expressions, all of them are used to denote some form of grouping, indicating where a group starts and ends, and all of them serve the same purpose: When one has to evaluate an expression that contains a bracketed sub-expression, the content in the sub-expression takes precedence over its surrounding. In this sense, since they determine which operations should be calculated first, brackets constitute an important component of the set of rules we use to evaluate arithmetical expressions. So, it is reasonable for students to relate brackets to the rules for the order of operations, given that in school brackets are introduced and used mainly in this specific context. In arithmetic, brackets can be used in two different ways. The first way is related to the order of operations, as we already mentioned, and is translated by the students as an instruction to do something first. For example, in the expression $12 \div (4 + 2)$, brackets are a signal to "do first" the operation $4 + 2$. Although the ways of perceiving brackets cannot be simply divided into a dichotomy of procedural or conceptual understanding, this "doing" can be considered as the procedural aspect of brackets. The second way highlights the role of brackets as the structural element of an expression which determines relations that exist between the various parts of the expression. For example, in the equation $\frac{6+4}{5+3} = (6+4) \div (5+3)$, brackets serve as the elements that preserve the structure of the initial rational expression and make clear the relation between the numerator and the denominator of the fraction. Although the brackets can indicate that the additions are to be done first, this way of seeing brackets as structural elements can be considered as the conceptual aspect of brackets.

In relation to these two aspects, previous studies reveal students' poor procedural knowledge and also a limited understanding of structure (Kieran, 1989; Linchevski & Livneh, 1999). Here, a structural understanding relating to arithmetic expressions refers to the student's ability to correctly parse the expression and identify the relations between the components of the expression as well as between the components and the whole.

In this landscape, our study aims to examine the possible connection between the way students evaluate rational expressions and their understanding of the expressions' underlying structure.

2 Theoretical aspects and research literature

The paper returns to an area of research that was studied extensively in the 1980s and 1990s, aiming to explore this area in a way that was less developed during these earlier years. This explains why much of the literature cited is from this period. Therefore, past research on the notion of structure sense and the role of brackets on assisting (or not) students in seeing structure in numerical and algebraic expressions is examined.

2.1 Structure sense: a necessary mathematical construct

Our theoretical framework is built upon the idea of "structure sense," a term introduced by Linchevski and Livneh (1999) and refined later by Hoch and Dreyfus (2004, 2005). In this framework, emphasis is given to structural knowledge as the ability to identify all the equivalent forms of an expression (Kieran, 1988) and discriminate between forms relevant to the task and all the others (Linchevski & Vinner, 1990). Linchevski and Livneh (1999) investigate the thesis that students' difficulties with algebraic structure reflect the difficulties these students already have with numerical structures.

They identify that three difficulties with the mathematical structure that had been observed in algebraic contexts also exist in purely numerical ones, and are widespread: (1) the detachment of a term from the indicated operation (e.g., some students consider $4+n-2+5$ equivalent to $4+n-7$); (2) misunderstanding of the order of operations (e.g., addition and subtraction are at the same level, so one can choose what to do first as is more convenient); and (3) jumping off with the posterior operation (e.g., in the equation $115-n+9=61$, some students focus on the subtraction sign following 115 and ignore the operation in front of 9 and they end up solving the equation $106-n=61$). The term "structure sense" was refined and developed later by Hoch and Dreyfus (2004, 2005) in the context of high school algebra. They see it as a collection of abilities, separate from manipulative ability, which enables students to make better use of previously learned algebraic techniques. The students display structure sense if they can deal with a compound term as a single entity, recognize equivalence to familiar structures, and choose appropriate manipulations to make the best use of the structure. Linchevski and Livneh (1999) suggest that students must be exposed to the structure of algebraic expressions in a way that lets them develop structure sense. More precisely, they have to be able to use equivalent structures of an expression flexibly. In this sense, the students' use of brackets that is necessary to preserve the structure of a mathematical expression can reveal the presence of structure sense. Writing a rational expression horizontally could be an example of the use of an equivalent structure.

2.2 Brackets and structure sense

Hoch and Dreyfus (2004), working with Grade 11 students in the context of high school algebra, found that the presence or absence of brackets affects the students' use of structure sense. More specifically, they found that (a) the lack of brackets prevents students from recognizing like terms and (b) the presence of brackets, by giving a clue of where to look and by focusing the students' attention, influences students' structure sense positively. For example, the students were able to see, without making calculations, that the expression $5+(2-3)+5-(2-3)$ equals 10, presumably because the brackets helped them see the equal terms and identify the structure $5+x+5-x$. This example corroborates that brackets enclose parts of an expression that are considered as a whole, which is important for obtaining structure sense. As Marchini and Papadopoulos (2011) noticed in a study with 2nd and 3rd graders, the use of unnecessary (they call them "useless") brackets helped students see their content as a whole and served two purposes: (a) they work as an "external" element of the structure that shapes the form of the expression and (b) they are a means to highlight the structure of the expressions, thus leading the students to successfully evaluate these expressions. Hoch and Dreyfus (2004) found that many students added useless brackets since it helped them see the structure of the expression. However, there are instances where useless brackets could become an impediment for learning the order of operations (Gunnarsson, Sönnerhed, & Hernell, 2016).

2.3 Students' errors with brackets and structure sense

Despite the importance of recognizing the role of brackets in the structure of arithmetic expressions, students seem to face difficulties in comprehending this. Linchevski and Livneh (1999) attribute this to the students' focus on the numbers rather than on the structure or the operations. The expression $a \pm b \times c$ requires detaching the middle number "b" from the preceding addition/subtraction. Linchevski and Livneh (1999) show that students often fail on this and therefore suggest the use of brackets to resolve this issue by including the multiplication within them, i.e., $a \pm (b \times c)$.

Therefore, sometimes the students ignore brackets, thus violating the priority of the involved operations, exhibiting a difficulty to understand the structural role of brackets. In Kieran's (1979) study, the students were asked to initially write $9 - 3$, and then had to multiply this by 8. The students did not feel the need to use brackets for the first operation. They suggested that it has to be $9 - 3 \times 8$ since subtraction was the operation that was given initially and therefore should be performed first. Similarly, Booth (1988) noticed students' incomplete understanding of the role of brackets, attributing this to the students' incomplete understanding of structure. The students argued that the final result of the evaluation of an expression is independent from both the order of operations and the presence of brackets. For example, some students claimed that to find the area of a rectangle with side lengths "p" and "$a + m$" the expression should be $p \times a + m$. It is the context to which the written expression relates that will determine the order of computations regardless of how the expression is written.

Blando, Kelly, Schneider, and Sleeman (1989), studying errors of Grade 7 students that result from their misconceptions about operations, grouping symbols, and the order of operations, noticed that many students tended to ignore brackets in mathematical expressions. For example, some students evaluated the expression $8 - (2 + 4)$ as $8 - 2 + 4 = 6 + 4 = 10$, hence completely ignoring the presence of brackets. In this case, the students instead worked from left to right starting with the subtraction.

In other cases, some students do not consider brackets as a compound entity consisting of an ordered pair of symbols that have certain content. Hewitt (2005) asking students from Grades 7 to 11 to translate formal algebraic equations into word statements and vice versa, found that the students showed little awareness of the notational conventions of the mathematical role that brackets play in expressions. In the same spirit, Linchevski and Herscovics (1994), working with 6th graders, found that students can ignore the mathematical structure as well as the intended meaning of the expressions and are limited to a more static view of the brackets considering the expression $926 - 167 - 167$ different from the expression $926 - (167 + 167)$. Finally, Bannerjee and Subramaniam (2005), examining students' structure sense, found that brackets can serve as a means to substitute the rules for the order of operations. The students evaluated the content of the brackets working from left to right. Hence, these students were able to follow the rules for the order of operations and correctly evaluate an expression such as $3 \times 6 + 3 \times 5$, but when brackets were inserted in the expression resulting in the form $3 \times (6 + 3 \times 5)$, the students instead performed the operations within the brackets from left to right ignoring the order of operation, i.e., $3 \times (6 + 3 \times 5) = 3 \times (9 \times 5) = 3 \times 45 = 135$.

Some studies attribute these students' difficulties about the role of brackets to both the way the teaching of brackets is approached in the classroom and the emphasis given to their use for the order of operations (Gunnarsson & Karlsson, 2015; Hewitt, 2005; Wu, 2007). Teaching the concept of brackets must present them as an ordered pair where each bracket symbol has a unique counterpart (Gunnarsson & Karlsson, 2015) and needs to be more than blindly following the relevant rules for the order of operations (Wu, 2007). Moreover, the use of mnemonics such as PEMDAS (Parentheses, Exponents, Multiplication, Division, Addition, and Subtraction) as an effective way to teach the order of operations rather seems to facilitate misunderstandings about the grouping symbols (Glidden, 2008).

2.4 Mental brackets

In this study, our interest lies in the use of “mental” brackets when rational expressions are written horizontally. “Mental” brackets were introduced by Linchevski and Livneh (1999) and this is the only reference we found in the literature. In their study, some of the students, evaluating the expression $926 - 167 + 167 = ?$, put “mental” brackets around $167 + 167$. It seems that students imagined these brackets (not physically present) and view the equation as $926 - (167 + 167)$. Additionally, a considerable percentage of their students put “mental” brackets around the multiplicative terms in the expression $24 \div 3 \times 2$ in contradiction to the order of operations. So, in this case they preferred to multiply first 3 by 2 and then divided 24 by 6. Hence, Linchevski and Livneh (1999) could understand students’ behavior through introducing the concept of “mental” brackets.

We apply this concept to study a different but adjacent topic. The influence of the written form of an expression upon the way this expression is evaluated is a topic that so far has been less explored, but where research is needed.

So, in this setting, our research questions are:

> How does the rational form of an arithmetical expression guide the way students write and evaluate the expression in horizontal form?
> Is the interpretation of the written form related to the structural understanding of these expressions?

3 The setting of the study

The study took place in Sweden and Greece simultaneously. The participants (11–12 years old) were 112 Grade 6 students from Greece and 123 Grade 5 students from Sweden. All the students had been taught the rules for the order of operations. The different grades were chosen on the basis of the age of the participating students. In Sweden, students enter Grade 1 of primary school when they are 7 years old, whereas in Greece the corresponding age is 6 years old. So, all participants were of the same age and they had been taught the rules for the order of operations. In the Swedish curriculum, there is no specific grade where brackets are taught, but traditionally they are introduced along with the rules for the order of operations. In Greece, the official introduction of the use of brackets takes place in Grade 6 when the order of operations is discussed. Therefore, the students did not have another formal teaching experience on brackets and their role and use. Additionally, the students’ prior experience had included work with numerical fractions and their equivalent horizontal form in terms of a division operation.

The instrument of this study was a collection of groups of tasks with arithmetic expressions designed to reveal how the students understand the role and use of brackets while evaluating arithmetic expressions. The instrument was designed in Greek, Swedish, and English. The latter version was used for corroboration of the equivalence of words and phrases. A pilot study was conducted, and its findings

You know that $\frac{8}{4}$ can be written on a single line as $8 \div 4$. First, re-write each fraction (without calculating) below on a single line, and then evaluate each arithmetic expression.
Example: $\frac{8}{4} = 8 \div 4 = 2$.

1) $\frac{12}{4} + 2 =$ ________________ = ________

2) $\frac{12}{4+2} =$ ________________ = ________

3) $\frac{8+12}{3+2} =$ ________________ = ________

4) $\frac{20}{\frac{4}{2}} =$ ________________ = ________

5) $\frac{12+2\cdot 3}{3} =$ ________________ = ________

Fig. 1 The group of arithmetic tasks in rational form delivered to students

helped redesign and finalize the instrument which at the end included five groups of activities.

For the purpose of this paper, we examine the results from one group of activities. All the activities in this group invite students to initially rewrite a rational (fractional) expression in horizontal form and then to evaluate the horizontal expression (Fig. 1). The aim of this group was to shed light on whether there is a connection between the format of the written expression and the way the students evaluate it.

The first two items are the same when written horizontally without brackets, i.e., $12 \div 4 + 2$, and therefore, the presence of brackets (for the second item) is necessary to preserve the initial structure. So, in the first case brackets ensure that the division precedes the addition, and therefore, they can be put around the initial fraction $(12 \div 4) + 2$. However, this is not necessary if one follows the rules for the order of operations. In the second case, the role of the brackets is to keep the denominator of the fraction together, in order to preserve the initial structure, $12 \div (4 + 2)$.

The third item demands two pairs of brackets to keep the numerator and the denominator together, thus preserving the structure, i.e., $(8 + 12) \div (3 + 2)$. Otherwise, the division $12 \div 3$ must be done first and the result will be wrong.

The next item when transferred horizontally shows a sequence of divisions. Without brackets, we have to work from left to right according to the rules for the order of operations, but this violates the form of the initial expression that wants the denominator to be kept as a whole, i.e., $20 \div (4 \div 2)$.

Finally, the last item combines both the structural role of brackets and the rules for the order of operations since brackets are necessary to keep the numerator as a whole, but then, the rules are necessary to evaluate the content of the brackets, $(12 + 2 \times 3) \div 3$.

The process was the same in both countries: no time limit and the same instructions.

The students' worksheets constituted our data and the analysis took place on two levels: qualitative and quantitative. The qualitative part was based on content analysis (Mayring, 2014), and the aim was twofold: initially to check the correctness of the result and later to organize the students' answers into categories according to the solution strategies used for the evaluation of the expressions. Each activity was examined separately, and the data were post-coded independently by the two authors. The coding results were compared, codes were clarified, and some data were recoded until agreement.

The quantitative part is limited to the frequencies of the answers that belong in each solution strategy. The decision to collect data from the two countries was because there is a collaboration between the two researchers. Although the study includes frequency data (of students' responses) from Greece and Sweden separately, the focus is not on elements that refer to a comparison between these two countries.

4 Results

In this section, we present the four strategies that lead to correct answers. The section starts with an overview of the distribution of the students' answers in both countries for each item according to the strategy of evaluation employed. Then, each strategy is presented separately by giving its description and some representative examples. The section ends with the distribution of the students' incorrect answers according to the origin of their errors.

The analysis is based on two levels: correctness and solution strategy. This resulted in nine categories of answers. The first group includes the four strategies that correspond to correct answers. A brief description of them can be found in the work of Papadopoulos and Gunnarsson (2018). The second group includes the wrong answers which were further organized according to the origin of these answers. In addition, in our coding, we found unanswered items as well as answers to items that were not codable.

Table 1 presents the distribution of the students' correct answers across the four strategies that correspond to correct solutions. Since wrong or blank answers are not included in this table, the sums in each column do not add up to 123 (Swe) and 112 (Gre).

Table 1 Distribution of the students' answers according to the employed strategy of evaluation

	Item 1		Item 2		Item 3		Item 4		Item 5	
	$\frac{12}{4}+2$		$\frac{12}{4+2}$		$\frac{8+12}{3+2}$		$\frac{20}{\frac{4}{2}}$		$\frac{12+2\cdot 3}{3}$	
	Swe	Gre	Swe	Gre	Swe	Gre	Swe	Gre	Swe	Gre
C-brackets	0	3	2	5	3	7	3	10	1	6
C-no brackets	62	52	33	82	32	76	28	54	31	56
C-2nd step	15	8	73	6	73	9	59	15	69	12
C-fraction operation	0	9	0	0	0	0	0	2	0	0
Total	77	72	108	93	108	92	90	81	101	74

4.1 Strategies that correspond to correct answers

In this part of the section, these categories are described in detail. The strategies that correspond to correct answers (indicated by a "C" below) are as follows: (i) using brackets (C-brackets), (ii) not using brackets (C-no brackets), (iii) avoiding brackets by making necessary calculations in the second step before the horizontal rewriting (C-2nd step), and (iv) using knowledge of operations of fractions (C-fraction operations). Examples from the students' answers are presented and the qualitative difference between the categories is discussed.

4.1.1 The "C-brackets" strategy

Answers that fall into this category are using brackets in a correct way to preserve the structure of the rational expression. It is worth mentioning here that the first item of this group (see Fig. 1) can be written horizontally without brackets. The rules for the order of operations ensure the correct result. This can be seen in the example of a student's work in Fig. 2. The student made use of brackets for all the items except for the first one.

This decision of the student seems to be an indication that he realized that any pair of brackets around the division $12 \div 4$ would be "useless." The rules for the order of operation are sufficient for preserving the structure of the given expression without using brackets. Only three Greek students used unnecessary brackets for the first item.

For the remaining four items, the use of brackets is necessary to get the correct result. However, as can be seen in Table 1, the number of students who were able to reach the correct answer using the necessary brackets in the horizontal form (C-brackets row) was very small. A total of 31 answers for the Greek students were identified which means an average (over all items) of 6.2 students. The relevant

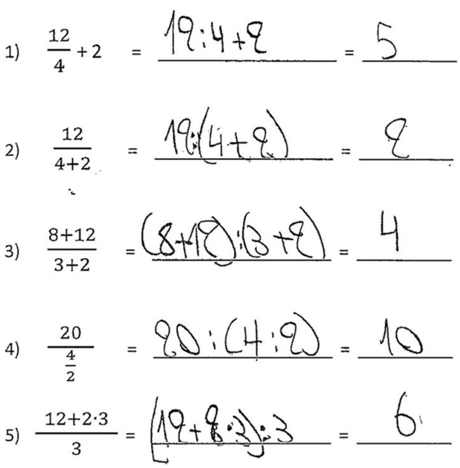


Fig. 2 An example of answers that use brackets in a correct way (C-brackets category)

numbers for the Swedish students were 9 answers, an average of 1.8 Swedish students.

4.1.2 The "C-no brackets" strategy

What makes this strategy especially interesting is that the students managed to get the correct arithmetic result following a seemingly mistaken way of evaluation since they did not use the required brackets. Indeed, if one considers the horizontal expression and the final result, one very easily notices that from the mathematical point of view this is an incorrect process since it violates the order of operations. However, the students manage to get the correct result (Fig. 3).

As can be seen in Fig. 3, the students' first step was to write the total expression without brackets horizontally, and for items 1 and 2, the student wrote the same horizontal expression twice. However, the two results are different (i.e., 5 and 2). It is evident that item 1 was evaluated as $3+2$ whereas item 2 as $12 \div 6$. The student considered the part $12 \div 4$ as a whole in the first case and the part $4+2$ as a whole in the second. This would be in alignment with the expected correct approach if brackets were physically present. However, brackets are not physically present and perhaps the correct result is obtained with the aid of imagined "mental" brackets as it will be discussed later. This is strengthened by noticing in Fig. 3 that the spacing of the written work is different between the first and the second items. In the horizontal form there is extra spacing around the rewritten fraction so as to set it off from the rest of the expression (Landy & Goldstone, 2007, 2010). In this way, the student

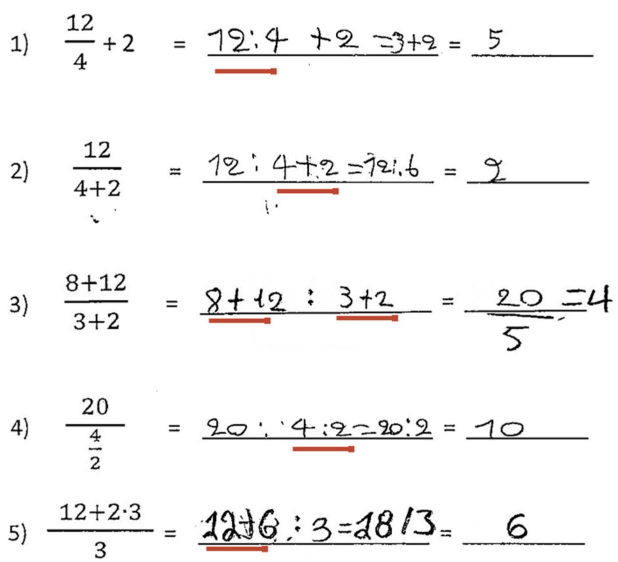


Fig. 3 An example of correct answers that did not use brackets (C-no brackets category)—The lines below some parts of the expressions have been inserted by the authors

indicated to himself which operation(s) was/were to be carried out first, thus feeling comfortable to move between these two unbracketed identical expressions that give different results. Both results (5 and 2) are in complete alignment with the structure of the initial rational expressions.

In a similar way, the evaluation of item 3 in Fig. 3 should result in $8 + 12 \div 3 + 2 = 8 + 4 + 2 = 14$. However, again, the student's second step in the solution process implies two separate sums, $8 + 12$ and $3 + 2$, which correspond to the numerator and denominator of the initial fraction respectively. Then, what remains is the division $20 \div 5 = 4$.

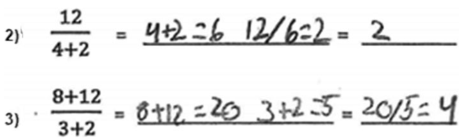
The fourth item is a compound fraction that when rewritten horizontally includes operations of the same priority. So, the evaluation should be $20 \div 4 \div 2 = 5 \div 2 = 2.5$. Perhaps in this case, it is not easy to say whether this answer is the result of the proper application of the rules for the order of operations or just a tendency to calculate left-to-right because in this example these two coincide. However, the second step of the student's evaluation is $20 \div 2$ which means that the denominator has been calculated first and thereby despite the absence of brackets the structure of the original expression is preserved.

The 5th item is the most cognitively demanding since its numerator cannot be evaluated without the correct use of the precedence rules. In its horizontal form without the use of brackets, the expression should be evaluated as $12 + 2 \times 3 \div 3 = 12 + 6 \div 3 = 12 + 2 = 14$. However, again a seemingly unorthodox way of evaluation leads to the correct answer. In his second step, the student makes the multiplication 2×3 and now the expression becomes $12 + 6 \div 3$. Then, an extra step has been added that again violates the precedence rules. The student decided to make the addition instead of the division that should be calculated first according to the rules. Therefore, this step, $18 \div 3$, leads to the correct answer 6.

4.1.3 The "C-2nd step" strategy

Answers were also found that were correct but not in complete alignment with the instructions of the task. The instruction was to write the rational expression in its horizontal form. However, a large number of students preferred to calculate separately the operations on each term of the fraction instead of writing the expression horizontally (see Fig. 4). In this case, there is no need for brackets. In total, 289 such answers were given by the Swedish students. The corresponding numbers for the Greek students were 50 answers. What seems odd here is the noticeable difference between Sweden and Greece in this respect. These answers are not arithmetically incorrect. On the contrary, they follow the rules for the order of operations. However,

Fig. 4 An example of correct answers that worked on each term of the fraction separately



the students simply do not follow the instruction of the task. A possible explanation might be the influence of the rational form of the expression, as will be discussed in the next section.

For item 2, it is evident that the student calculated initially the sum $4+2$ (denominator) and then the horizontal form was limited to the division $12 \div 6$. For item 3, the student made the operations on both the numerator and denominator of the fraction (20 and 5 respectively) separately before the division $20 \div 5 = 4$. By doing the calculation on each term separately, the need for brackets is eliminated and the request to write a horizontal expression does not pose a dilemma for the students.

4.1.4 The "C-fraction operations" strategy

This strategy is somehow qualitatively similar to the "C-2nd step" strategy, since both are not completely aligned with the task's instructions. The students who applied this strategy seem to have exploited their knowledge of the operations of fractions to get the result and thereby avoided writing down the horizontal expression of the task. No Swedish student gave answers that fit in this category and the number of Greek students who applied this strategy was very small. A total of 11 answers were collected (an average of 2.2 students).

For item 2 (Fig. 5), the student combined her knowledge of equivalent fractions and of the addition of fractions to get the correct result. For item 4, another student followed a rule about the simplification of a whole number over a fraction (a special case of a more general rule for simplifying compound fractions). So, the student rewrote the expression creating a fraction on the numerator $\left(\frac{20}{1}\right)$, and then followed the rule: *Multiply the numerator of the first fraction with the denominator of the second* ($20 \times 2 = 40$). *The result will be the numerator of the final fraction. Then multiply the denominator of the first fraction with the numerator of the second* ($1 \times 4 = 4$). *This will be the denominator of the final fraction*. This is a common practice in Greek mathematics classrooms. Many Greek students rely heavily on this rule and perhaps this can explain why the answers assigned to this category were given by Greek students only.

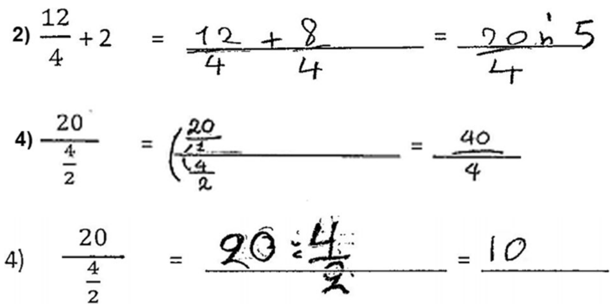


Fig. 5 An example of correct answers aligned with the "C-fraction operation" strategy

4.2 Incorrect answers and their origin

The incorrect answers are divided into three sub-groups: (i) miscalculations, (ii) left-to-right calculations, and (iii) incomplete understanding of fractions. The group of incorrect answers also includes unanswered items and answers that were not codable. Table 2 presents the distribution of these incorrect answers according to the origin of the error as well as the number of unanswered instances per item and country.

For the "miscalculate" answers, the students knew the correct way of solving, but they made mistakes while executing the operations. In Fig. 6, the student correctly wrote the rational expression horizontally and started with the division (following the order of operations). However, the addition at the final step was evaluated as 7 instead of 8.

The second errors' origin was the tendency of the students to evaluate arithmetical expressions from left to right (Blando et al., 1989; Booth, 1988; Kieran, 1979).

Both examples in Fig. 7 clearly show the application of this rule. In the first example, the student followed, in a very explicit way, the operations in the order they are written from left to right (addition and multiplication) and finally made the division.

Finally, the third case of errors was related to an incomplete understanding of the notion of fractions and the operations with fractions.

Both examples in Fig. 8 refer to item 1 that deals with the addition of a fraction with a whole number. Some students add the single term to the numerator (top example in Fig. 8). Other students add the single term to the denominator (bottom example in Fig. 8). Both examples reflect a limited understanding of the notion of fraction and addition of fractions and are in accordance with relevant research findings (Siegler, Thompson, & Schneider, 2011) concerning students' difficulties with fractions and their operations.

Table 2 Distribution of incorrect answers across their possible origin

	Item 1		Item 2		Item 3		Item 4		Item 5	
	$\frac{12}{4}+2$		$\frac{12}{4+2}$		$\frac{8+12}{3+2}$		$\frac{20}{\frac{4}{2}}$		$\frac{12+2\cdot 3}{3}$	
	Swe	Gre	Swe	Gre	Swe	Gre	Swe	Gre	Swe	Gre
Miscalculate	12	12	9	2	8	3	6	0	8	5
Left-to-right	0	0	2	5	0	0	12	2	3	18
Incomplete understanding	10	17	2	1	1	5	0	0	0	1
Unanswered	24	11	2	11	6	12	15	29	11	14
Total	46	40	15	19	15	20	33	31	22	38

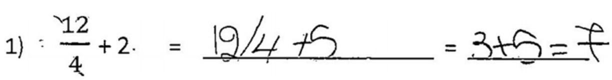


Fig. 6 Incorrect answer due to miscalculation error

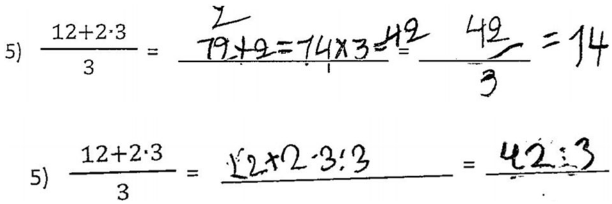


Fig. 7 Incorrect answers due to the left-to-right calculation

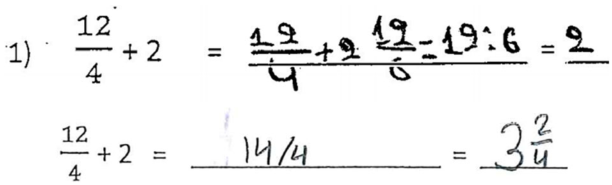


Fig. 8 Incorrect answers due to an incomplete understanding of operations with rational numbers

5 Discussion

The analysis of the data suggested that the rational form in itself has an impact on the way students write and evaluate the expression in its horizontal form. At the same time, it shows a possible relation between students’ interpretation of the written form and their structure sense, as will be discussed below.

This paper examines the potential impact of the rational form of an expression on the way the students evaluate the expression when it is rewritten horizontally. Brackets should play an important role in this process since they could be used to preserve the mathematical structure in the rewriting process. Among the strategies exhibited by the students, the “C-no brackets” strategy attracts particular interest since it seems to violate the precedence rules, but the students get the correct result. More than half (56.5%) of the correct answers that were collected belong to this strategy. This strong preference to this strategy is rather *connected with the rational form* of the expressions that guided the students’ evaluation of these expressions. Given that a relevant example was included in the statement of the task, it was expected to see expressions of the form $\frac{a}{b}$ to be translated as $a \div b$. However, this translation alone is necessary but not sufficient for the correct evaluation of a rational expression. The first two items were designed intentionally to highlight that the mere horizontal writing is not sufficient if one ignores the role of brackets. They can lead to contrast between expressions that are identical in their horizontal form. If brackets are ignored, both expressions are written as $12 \div 4 + 2$. The students evaluated the second item in a way that violates the precedence rules. A pair of brackets was necessary around the denominator to preserve the structure of the second rational expression. The students did not physically write the brackets but evaluated the expression in a way that mirrors the presence of brackets—we believe that the students acted as if the brackets were there. We note that item 1 does not need the use of brackets. Only three students made use of brackets to evaluate this item. It seems that they were added to “see” the terms of the fraction better. Thus, in this case, these “useless” brackets helped students “looking” before “doing” which is a feature of structure sense (Hoch & Dreyfus, 2004; Marchini & Papadopoulos, 2011). But, for item 2, the students using the “C-no brackets” strategy used “mental’ brackets around the sum $4 + 2$. This becomes even more obvious in item 3. The students wrote $8 + 12 \div 3 + 2$ but the way they evaluate this expression using (i.e., $20 \div 5$) implies the use of “mental” brackets around the two sums (numerator and denominator of the initial rational expression), and therefore, the structure of the initial fraction is preserved. This use of “mental” brackets as a grouping mechanism was evident in the evaluation of all the items. This is in accordance with Booth (1988) who has noted the role of the context in determining the order in which the calculations should be carried out. For the study described in this manuscript, the context was one involving fractions—a numerical context that was just as powerful as a context involving real-world problems.

However, this raises the question: Is the use of the “C-no brackets” strategy a mere consequence of the influence of the expressions’ written format, thus indicating an incomplete knowledge about the precedence rules? One could claim that a solid understanding of the notion of fractions is sufficient for correctly evaluating the first four items and in this case, we do not have any implication about the students’ knowledge on the precedence rules. The rational form of the expression seems to impose the separate calculation of each term of the fraction. Keeping this in mind and given that the first four items include only one operation on each term, the correct answer is easily obtained. Such operations are a simple addition (denominator, item 2), a division (denominator, item 4), or two simple additions (both terms, item 3). However, if one is based only on the translation of $\frac{a}{b}$ to the equivalent form of $a \div b$, it makes hard to evaluate correctly item 5. The expression on the numerator demands the knowledge of the rules for the order of operations. This expression includes two operations and according to the precedence rules, multiplication must be done first. It can be seen in Table 1 that 56 Greek students and 31 Swedish ones used this strategy and found the correct result. Their success cannot be attributed to the sole impact of the expression’s written (rational) format.

To form a valid opinion on whether the students who applied the “C-no brackets” strategy were aware of the rules for the order of operations, it was considered necessary to follow their work while evaluating the first four items and view this work in combination with the answers they gave for the fifth item. The examination of the data revealed that 51 Greek students (out of 56 who evaluated item 5 correctly) used the “C-no brackets” strategy consistently for the remaining 4 items. The corresponding numbers for the Swedish students were 28 (out of 31). These students displayed structure sense. First, they were able to identify an equivalent form of the initial expression (Kieran, 1988; Linchevski & Livneh, 1999; Hoch & Dreyfus, 2005) as they shift from the rational to the horizontal form. Second, they seemed to know and apply the precedence rules (Linchevski & Livneh, 1999), especially for item 5. Third, their interpretation of the structure of the expressions was consistent (Linchevski & Livneh, 1999). This finding raises an interesting question. If these students are aware of the rules for the order of operations and prove themselves able to use these rules (as it happened in item 5), how could this unorthodox behavior of the “C-no brackets” strategy then be explained? We believe that the answer is the use of “mental” brackets.

More precisely, the initial rational form of the expression can, when accompanied by a solid understanding of the notion of fraction, impose the way the students perceive and evaluate the expression. Based on this understanding, the students respect the form, but they do not check the accuracy in the evaluation process. All the examples collected in the “C-no brackets” strategy reflect this impact of the written form on the way of evaluation. The seemingly incorrect way of evaluation and the violation of the precedence rules are compensated by the use of “mental” brackets (Linchevski & Livneh,1999) that serve here as “prostheses of the mind to accomplish actions as required by the contextual activities in which the individuals engage” (Radford, 2000, p. 241). This use of “mental’ brackets seems to help students perceive the expressions in the way they should be perceived while their writing does not preserve the structure of the rational expression. In this case, the horizontal expressions are apparently violating the order of operations, but the "mental" brackets that are put in these expressions made the students find the correct results of the calculations. In some cases, this interpretation seems to be strengthened by the way the students write down the horizontal expressions. A more careful examination of Fig. 3 (see signs below the terms inserted by the authors) shows a grouping of the terms that fits the proper use of brackets. Some students in this category used spacing as a way to mark the numbers that are to be paired up first. For example, for item 1, and in its horizontal form, there exists a noticeable blank space between the $12 \div 4$ part and the sign of the addition that follows, whereas for item 2, this blank space exists between the sign of the division and the $4 + 2$ part. Similar instances can be seen in the remaining items in Fig. 3. This seems to be in alignment with Landy and Goldstone (2007, 2010) who discussed the role of the spatial format on arithmetic computations describing how precedence was affected by spacing. However, the difference here is that, in their study, the researchers presented arithmetic expression with different pre-defined spacings between the terms. In our study, it was *the students themselves* who created this spacing. This could be yet another indication of the use of "mental" brackets. Despite all the items being typed and printed with equal spaces between operations and terms to avoid the impact of visual cues on students' performance (Kirshner, 1989), it was the students who used spacing in this case as a visual feature that guides the evaluation process.

An additional impact of the rational form of the expression might be the increased number of students who preferred to apply the "C-2nd step' strategy. It is clear that this strategy is aligned with a correct understanding of the notion of fraction, emphasizing the separate calculation of each term (numerator, denominator) before their division. Perhaps this strong impact made the students ignore the instruction of the task.

So, there is a kind of flexibility in the way students evaluate these rational expressions. The obligation to apply the precedence rules in their horizontal expression seems to be of minor importance when the use of "mental' brackets is sufficient to get the correct result (the first four items) even though the rewriting is not in agreement with the conventions. Our conjecture is that in some way they see the horizontal writing as obsolete and therefore do not check the alignment between the structure of the expression and the result of the calculation. However, even if they do so, they can turn to the precedence rules when the knowledge provoked by the format (fractions) is not sufficient (i.e., item 5) for evaluating the whole expression. Hence, the way the students interpret this (unorthodox) co-existence of arithmetical expressions that are seemingly evaluated in a way that violates the rules, but anyway obtain the correct answers, corroborates the idea that this has to do with the students' structural understanding. This understanding then seems to be triggered by the (rational) *form* of the expression.

The fact that this common behavior occurred among students who come from different educational systems suggests that this is not the result of some particular kind of teaching, thus increasing the chance for the generalizability of the observations. However, there are some teaching implications that could be extracted and that strengthen relevant suggestions made by Linchevski and Livneh (1999): Students should be "able to use equivalent structures of an expression flexibly and creatively" (p. 191). Teaching might include evaluation of rational expressions using alternative equivalent numerical expressions. This fosters an ability to decompose and recompose an expression and the use of "mental brackets" constitutes an efficient mental means for manipulating the recomposed expressions. Moreover, the use of such tasks in the classroom connects the formal aspect of the rules (for example, brackets first to evaluate each term of the fraction) with the numerical context (fraction, division) and consequently with the students' common sense (fraction translated as division). Using such tasks, teachers have the chance to help students see structure and use brackets for considering a sub-expression as an entity. Moreover, the analysis of these (rational and horizontal) expressions in terms of their structural properties can be the goal of classroom activities.

6 Conclusions

In this paper, the impact of the written format of an expression on the way students evaluate this expression was examined. The role of the use of brackets for both evaluating arithmetic expressions and exhibiting structure sense has been acknowledged. At the same time, the collected data give evidence that the (purposeful?) avoidance of brackets when writing arithmetic expressions does not always necessarily mean lack of structural understanding. It seems that when students write rational expressions horizontally, they often use "mental" brackets to preserve the formal structure of the expression in a way that violates the conventions but leads to the correct result. The findings of this study deepen our understanding of the way young students evaluate arithmetic expressions.

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