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The Sierpinski smoothie: blending area and perimeter

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Abstract
This study furthers the theory of conceptual blending as a useful tool for revealing the structure and process of student reasoning in relation to the Sierpinski triangle (ST). We use conceptual blending to investigate students' reasoning, revealing how students engage with the ST and coordinate their understandings of its area and perimeter. Our analysis of ten individual interviews with mathematics education masters' student documents diverse ways in which students reason about this situation through the constituent processes of blending: composition, completion, and elaboration. This reveals that students who share basic understandings of the area and perimeter of the ST recruit idiosyncratic ideas to engage with and resolve the paradox of a figure with infinite perimeter and zero area.

Keywords Conceptual blending · Fractal · Infinite processes · Paradox · Student thinking

> It's still hard for me to wrap my mind around the Sierpinski triangle, and that there's infinite perimeter and no area. It makes sense to me individually, but both together at

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once, I'm still, it's still mind-boggling. – Carmen, graduate mathematics education student

In this report, we present an investigation of student thinking which unpacks how their notions of geometry, limits, physical processes, and mathematics lead them to encounter the paradoxical situation of the Sierpinski triangle (ST) which has zero area and infinite perimeter. To do so, we further conceptual blending as an analytic tool to access and explore students' coordination of two simultaneous infinite iterative geometric processes. While conceptual blending has been used to unpack the concepts of actual and potential infinity (Núñez, 2005), our exploration of the constituent elements of blending (composition, completion, and elaboration) enriches the potential of this theory for future research.

In a chaos and fractal course for mathematics education masters' students, the ST was introduced through an in-class activity, wherein students were given recursive instructions (see Fig. 1) for creating the ST and asked to find the area and perimeter of the resulting figure. As suggested by the introductory quote, this was a non-trivial task. The ST is a fractal and (as encountered in this course) is the result of an infinite iterative process that begins with an equilateral triangle. At each step of the process, the area of the figure shrinks by a factor of 3/4 and the perimeter grows by a factor of 3/2. Thus, the ST has an infinite perimeter and zero area.

The ST is a rich context in which to explore students' thinking. The familiar concepts of equilateral triangle, area, and perimeter are central to the ST, making it a mathematical object accessible to a wide range of students while at the same time offering opportunities for sophisticated discussions of infinite processes, self-similarity, and dimension. The class discussion of student work on the ST further revealed the complexity of student thinking about the ST and what happens "at the end." In particular, students recognized the difficulties of thinking about the area and perimeter of the ST, in Carmen's words, "both together at once."

To investigate student reasoning about the ST, we conducted individual interviews about three weeks after the in-class investigation of the ST. Using interview data and the ideas of conceptual blending, we address our research question: How can conceptual blending be used to reveal students' individual idiosyncratic ways of coordinating the area and perimeter of the ST?

1 Background literature

Our use of the ST positions our work within a long line of research investigating student thinking about limits, infinity, and paradox. Students' proclivity to work with limits by constructing iterative processes and to apply characteristics of finite processes to infinite ones have been well documented (see reviews in Larsen, Marrongelle, Bressoud, & Graham, 2017; Rasmussen & Wawro, 2017). The cognitive dissonance brought on by paradoxical situations

a) Sketch an equilateral triangle of side a = 16 cm (*16 has been chosen for convenience*).
b) Connect the midpoints of the triangle's sides, so as to generate four congruent triangles of side a/2.
c) "Take away" the triangle in the middle (*you may cut it out or simply color it in*).
d) You are now left with three equilateral triangles. For each one of them, repeat (b), (c), and (d).

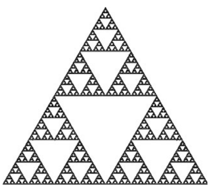

Fig. 1 Recursive instructions given in class for creating the ST, and the figure which occurs after six iterations

has been leveraged to perturb and reveal the understandings of (and difficulties with) infinity and limit. Particularly relevant are studies using infinite iterative tasks (e.g., Dubinsky, Weller, McDonald, & Brown, 2005; Ely, 2011; Mamolo & Zazkis, 2008; Radu & Weber, 2011; Wijeratne & Zazkis, 2015). The combination of physical steps (e.g., drawing a triangle, putting balls in an urn) with something physically impossible to complete is a situation not easily resolved, even by students with extensive mathematics training. The research in this area has revealed distinctions between actual and potential infinity conceptions, the projection of finite patterns onto the completed state, conceptions of limits as unreachable, and an urge to preserve consistency with the physical world. The only reference in the mathematics education literature to paradoxical situations in fractals that we are aware of is the work by Sacristán (2001). Her focus was on the role of a computer microworld in facilitating the coordination of visual and numerical representations to support resolution of the paradoxical situation of the Koch curve.

While our study is informed by existing research in similar mathematical contexts, our purpose is distinct. We take a non-deficit approach to highlighting nuanced and individualized differences in students' coordination of their understandings of the area and perimeter of the ST, and how that coordination leads to distinct understandings. Though many of these students described underlying principles and processes similarly, the application of conceptual blending theory provides insight into which ideas students leverage to make sense of a complex task. This illumination of individuality in strategies for understanding the ST speaks to the richness of students' thinking, the variation present in a single classroom, and the power of the conceptual blending approach to expose these.

2 Conceptual blending theory

We use conceptual blending theory (Fauconnier & Turner, 2002) as a theoretical and methodological tool for analyzing students' coordination of two infinite iterative processes related to the ST, one increasing (perimeter) and one decreasing (area). Blending is based on the notion of mental spaces, which are "small conceptual packets constructed as we think and talk, for the purposes of local understanding and action" (p. 40) that "organize the processes that take place behind the scenes as we think and talk" (p. 51) and are composed of multiple elements. Conceptual blending is defined as the conceptual integration of two or more mental spaces to produce a new, *blended*, mental space. An important feature of this new blended space is that it can develop an emergent structure that is not explicit in either of the input mental spaces. This emergent structure is generated by the three constituent processes of blending: composition, completion, and elaboration.

Composition is the selective projection of elements from input spaces into a common space. During composition, distinct elements may be projected on top of each other or *fused*, and common elements may be projected separately. The composition process develops a new space, with the potential for structure not available in either input space. *Completion* is the process of recruiting familiar frames to the blended space, along with their entailments. That is, an individual recognizes certain aspects of a blended space as parts of a familiar frame and brings in additional knowledge, scripts, assumptions, etc., to complete the frame and provide structure for the blended space. These frames support the process of *elaboration*, which is sometimes called *running the blend*. Elaboration is the process that leads to the emergence of something new within the blended space, using the tools of the completion process and the elements that compose the blend.

Lakoff and Núñez (2000) suggest that many of the foundational ideas in mathematics (e.g., number) are conceptual metaphors which blend physical input spaces to construct abstract ones. In particular, Núñez (2005) explores the notion of *actual infinity* as the result of a conceptual blend which coordinates the finite and the unending. Alexander (2011) discusses the presence of conceptual blending in the formal structure of mathematics and suggests the actualization of blends is critical to the evolution of mathematics as a discipline. However, this theory has not been extensively used in empirical studies of student thinking.

The few examples of empirical studies in mathematics education that make use of this theory include the work of Edwards (2009) and Yoon, Thomas, and Dreyfus (2011), who analyzed how people invest their real gesture space with mathematical meaning; the use of grounded blends and physical space separates that work from what we present here. Gerson and Walter (2008) used conceptual blending to look at the emergence of calculus concepts for individuals during small group work, but did not leverage the constituent elements of the blending process as we do. Zandieh, Roh, and Knapp (2014) leverage the processes of composition, completion, and elaboration in an empirical study of students' thinking, using small groups as the unit of analysis. The theory was productive for their investigation, but our work with single students echoes the personal and individual nature of mental spaces described in the original theory (Alexander, 2011; Fauconnier & Turner, 2002; Lakoff & Núñez, 2000). Thus, we situate our work as part of a small movement to leverage conceptual blending as an analytic tool in empirical mathematics education research and our analysis indicates that conceptual blending can identify subtle variation in students' thinking. In particular, the frames which individuals recruit to complete their blended spaces can lead to distinct elaborations, even when blended spaces are composed from input spaces in a similar fashion. Moreover, in contrast to much of the prior literature on student thinking of infinity or infinite iterative processes, our use of conceptual blending offers a non-deficit approach that emphasizes the productive ways in which students' reason about complex mathematical ideas.

3 Methods

3.1 Setting and participants

The study took place in a graduate level mathematics course of 11 students taught by one of the authors. Ten students participated in the interviews, all of whom were instructors and/or tutors of secondary and/or tertiary mathematics, pursuing a master's degree in mathematics education. Students sat in four groups throughout the course: (1) Carmen, Joy, and Jackie (Jackie did not participate in interviews); (2) Shani, Soo, and Kay; (3) Mia, Kevin, and Elise; and (4) Sam and Curtis. During class time, students regularly worked on mathematical tasks in their groups and then discussed their thinking with the whole class. Data was collected as part of a larger study considering relationships between individual and collective mathematical thinking, which included video-recordings of each class session, individual task-based interviews conducted at the middle and end of the semester, and copies of all student work.

3.2 Methods for data collection

The focus of the analysis in this paper are students' responses to the following question from the mid-semester interview: "In class, we discussed the Sierpinski Triangle. How do you think about what happens to the perimeter and the area of the ST as the number of iterations tends to infinity?" This question was accompanied by a printout of the ST (as seen in Fig. 1), with follow-up prompts to tell us how confident they were (and why) and then what they thought about the following claim of a fictitious student, "Fred":

> The computation shows that the perimeter goes to infinity because the perimeter is given by $3\times(^3/_2)^n$ which increases to infinity as n tends to infinity. But, the perimeter can't really be infinitely long, because there is nothing left to draw a perimeter around, since the area goes to zero.

This interview task was designed based on classroom discussion of the ST. In class, students seemed to agree that the area went to zero but were unsure of what happened to the perimeter. They publicly considered the possibilities that it went to infinity, converged to some value, or did not exist because there was nothing left for a perimeter to go around. We included sequence notation for the perimeter in the hope of foregrounding the paradoxical situation by helping students see that the perimeter diverges. The interview was structured so that we would first gain insight into their own reasoning about the area and perimeter of the ST, followed by an opportunity for them to respond to Fred's claim. All interviews were conducted by the same member of the research team, with one of the other researchers present to video-record and ask occasional follow-up questions. Each interview lasted roughly an hour, 5–20 min of which were spent on the ST segment depending on the clarity, verbosity, and assuredness of the students' responses. Interview questions were in the same order for each student, in the order of presentation of material during the course, with the ST item the fourth of five task-based items.

Fred's claim is based on an argument heard in class, presented first by Carmen, amended to include a (correct) algebraic expression. This ensured that the argument did not feel contrived to the students, and in fact several of them noted that some students in class struggled with this same scenario. So, students had previously seen the ST and considered, to some extent, the same paradoxical situation we brought up in Fred's argument. This means that when we consider students' thinking in the interviews, we gained access to a semi-retrospective account of their original thinking. As conceptual blending is not a linear process, and in fact mental spaces coexist for extended periods of time, this gave us a better chance of seeing fully blended spaces, but reduced our ability to access the completion process or identify failed blends along the way.

3.3 Methods for analysis

The transcripts and student work produced during the interviews were coded and analyzed in two rounds. The first round consisted of identifying the elements of each student's mental spaces, extending and expanding our previous work (Apkarian, Rasmussen, Tabach, & Dreyfus, 2018; Tabach, Apkarian, Dreyfus, & Rasmussen, 2017). The second round was a close analysis of the interviews to identify the blending processes which affected students' encounter with the ST.

To identify a student's input space for area (similarly for perimeter), we first marked which of their utterances were about the area. Next, we categorized these utterances into sets of ideas about the area of the ST—including the process by which it is created and the resulting product. In the spirit of grounded theory (Strauss & Corbin, 1998), these ideas were coded and compared iteratively until a coherent set of idea codes emerged. The interviews were divided into two groups and analyzed by different members of the research team. These analyses were then swapped, compared, and vetted. The multiple, iterative rounds of discussion among the research team members provided many occasions to share and defend interpretations, thereby minimizing individual bias and keeping interpretations grounded in the data (Jordan & Henderson, 1995).

We then investigated students' blending by identifying each of the three processes: composition, elaboration, and completion. To see how a student's blend was composed, we identified which elements of the students' input spaces were brought up as they considered the coordination of area and perimeter. We identified the ways students elaborated their blended spaces by identifying ideas which were not in the input spaces, but emerged as they worked to coordinate those ideas. For completion, we identified the frames (and entailed tools) students used (other than input space elements) in order to make their elaborations. Interpretation of completion and elaboration was done first as a group, with all four authors debating each point, then a more detailed pass was made by two members of the team in close comparison with the transcripts, and these analyses were then discussed again among the four authors until agreement was reached.

4 Results

We begin our report of results with a discussion of the various elements of students' input spaces for area and perimeter, the ways in which students made sense of area and perimeter of the ST independently of each other. This analysis was needed in order to determine what ideas each student might compose into a blended space, complete with another frame, and elaborate on when coordinating the two to reason about the ST. We follow this report with a detailed conceptual blending analysis of four students' work on the ST interview task. These four were selected to illustrate the power of conceptual blending to reveal nuances in student thinking, as these four composed a common blended space (which differed from others') but each completed and elaborated that space in a distinct way.

4.1 Students' ways of reasoning about area and perimeter

During the in-class discussions about the ST, there appeared to be widespread agreement that the area would go to zero but less agreement that the perimeter would diverge to infinity. We expected similar claims in the interviews and were surprised to find that only six of the ten students claimed that the area of the ST goes to zero. Soo indicated that area shrinks unendingly but was adamant it would never actually reach zero; Shani and Kay said that the area converges to something nonzero; and Kevin said only that it converged, but he had not worked out what it converged to. On the other hand, all ten students stated that the perimeter tends to infinity.

Among students' justifications for these conclusions, we identified seven qualitatively distinct mental space elements for area and seven more for perimeter. All ten students cited an infinite decreasing process related to the area of the ST, and an infinite increasing process related to its perimeter (e.g., Elise: "perimeter is forever increasing"). Nine students, all but Curtis, explicitly used the justification that area is removed at each step (e.g., Kay: "we're taking away area with each iteration") and that perimeter is added at each step (e.g., Joy: "each iteration you're creating more triangles and so you're creating, you're adding to the perimeter").

Shani and Kay, the two students of the same group who concluded that the area tended to something non-zero, were the only two students who referred to change in the rate of change with regard to area, noting that that amount of area being removed at each step is smaller than that removed at the previous step. Elise and Carmen referred to change in the rate of change for perimeter, noting that *more* is added at each step.

Curtis was the only student to express multiplicative reasoning in regard to the area of the ST, noting that the area decreases by a factor of 3/4 at every step, while others focused on removal of area, suggesting additive reasoning. Curtis and Carmen both described the change in the perimeter as increasing by a factor of 3/2 at each step. For both area and perimeter, Curtis wrote algebraic limits using this multiplicative relation. Curtis was the only student to express a conception of area and perimeter as each composed of congruent components, not just as numeric sequences. This way of reasoning coincides with him being the only student to describe the area, and one of only two who described perimeter, multiplicatively. Kevin and Mia (groupmates) both indicated that "$r > 1$" indicated divergence and "$r < 1$" indicated convergence, with Kevin explicitly motivated by conceptualizing perimeter and area as geometric series. Mia conceived of the area of the ST as computed from "leftover" triangles and the perimeter as computed from "removed" triangles during the recursive creation process explored in class.

In light of students' informal ways of reasoning, the parallelism between area and perimeter ideas is interesting. Each element of reasoning about area had a corresponding element of reasoning about perimeter. Some of these ideas were present in many students' reasoning, while others were rare. Our focus in this paper is on the variation in students' informal reasoning and coordination of area and perimeter of the ST, and not their "correctness," but we must note that in many cases students presented incomplete justification for their claims.

4.2 Students' blending of area and perimeter

We now turn to the focus of our paper, the leveraging of conceptual blending to unpack students' individualized coordination of area and perimeter processes in their investigation of the ST. In doing so, we note the composition of blends from students' projection of input space elements, discuss their completions, and identify the resulting elaborations.

One element appears in each student's blended space which did not appear in the area/perimeter section: *infinite creation process*. This element is a result of fusion, wherein two input space elements (here, infinite increasing and infinite decreasing) are projected onto one element. As students were introduced to the ST as something created through an iterative, recursive process affecting both area and perimeter, the students are re-fusing elements which they previously separated.

We present four students' blends in some detail, highlighting how composition, completion, and elaboration reveal idiosyncrasies in students' thinking about the ST despite their having been in the same class and discussing the topic with their peers. These four students were selected because the composition of their blended spaces is the same, and no others composed the same space. The framings which they used to complete those blended spaces varied in ways which impacted their elaboration of the blended space and hence highlight the power of conceptual blending to reveal nuances in student thinking.

4.3 Joy's blending process

We first detail the analysis of Joy's blending process regarding area and perimeter of the ST. This includes exposition of the ideas she raised regarding area and perimeter separately, as described in the previous section, as well as how her blended space was composed, completed, and elaborated following her consideration of Fred's argument. We start with her response to the first question about the ST:

> So, for the perimeter, I think it goes towards infinity because each iteration you're creating more triangles and so you're creating, you're adding to the perimeter. For the area, I think it tends towards zero because you keep taking - so all that white area are spaces and you don't count [them] and so you keep taking out more and more of the area as time goes on.

The content of Joy's first sentence is associated with her input space for perimeter and the second with her input space for area. We coded her first sentence as containing the ideas *perimeter tends to infinity* and *perimeter added at each step*; the second was coded with *area tends to zero*, *infinite decreasing process*, and *area removed at each step*. Joy indicated that she was fairly confident in her response, and when asked to elaborate said:

> For perimeter I'm more confident because you keep drawing more and more of the triangles and you keep adding on to it. For the area, it's harder to visualize because I'm taking things away [from] something that exists there. So it's hard to think of something as having no area.

Again, Joy clearly distinguishes her ideas about perimeter from those about area. Thus, these ideas are considered part of the input spaces related to perimeter and area. In her first sentence, we coded her description of an unending additive process as *infinite increasing process*; we interpreted the second sentence as Joy revisiting the ideas of *area tends to zero* and *infinite decreasing process*. After Joy read Fred's argument, she said of the latter part of his claim:

> I disagree because we thought about it in terms of like fencing. So if you keep putting in more fencing, so eventually, in a sense, it's all fence. Like there's, if you keep putting in more and more, you're drawing around it and you're creating more and more spaces, but the spaces in our cases, a lot of them we don't count their space, but there is still a perimeter associated with it.

Here we see evidence of Joy's blending process. In this quote, Joy talks about area and perimeter together, linking the adding of perimeter to the creation of new regions, though their area does not get counted. We interpret her statements as a fusion of *infinite increasing process* and *infinite decreasing process* into *infinite creation process*. She maintains (here and in other passages) that the perimeter of the ST tends to infinity and the area of the ST goes to zero, projecting both into a common space.

This segment also references the fence metaphor, which Joy recruited as a frame to complete her blended space in order to think about the whole of the ST. This frame came with several entailments, used in her elaboration of the blended space. One is that fences remain, even if the space they enclose is no longer there, as evidenced by the final sentence of the above segment. Another entailment is that fences not only have length but may also take up space, and this possibly contributes to Joy's statement that "eventually, in a sense, it's all fence." This frame brings with it entailments from the physical world, which Joy apparently recognized. Her final statement highlighted her understanding of this juxtaposition of an infinite mathematical process with the physical world as the source of what Fred encountered as a paradox:

> I can see where it gets funny for Fred because he doesn't think there can be a perimeter around something that has no area, but it's hard because this is not a real object. This is an idea that if you zoomed into infinity, you can still keep zooming in, but in real life you can't do that, so I think that's where the disconnect is.

In her elaboration, or running of the blend, Joy encounters and recognizes the paradox of a region with infinite perimeter and zero area. However, her fencing metaphor and recognition of that as a metaphor led her to dismiss the paradox on the grounds that the ST is not a "real object" in the physical world.

From this analysis and interpretation of Joy's interview about the ST, the diagram in Fig. 2 was created. The work reported in the first results section resulted in the language used to describe mental space elements in the input and blended spaces; recruited frames and elaborations arise from close inspection of each student. Similar diagrams were created for all ten students during our analysis.

4.4 Elise's blending process

We next present our conceptual blending analysis of Elise's thinking, with comparable detail to that of Joy. Elise's initial response to the question of what happens to the area and perimeter of the ST was that "the area goes to zero," and "I think that the perimeter goes to infinity," from which we identified the mental elements of *area tends to zero* and *perimeter tends to infinity*. In a further explanation of how she thinks about the area, Elise says:

> I do my first iteration. I take out this huge chunk of area. And then I do the next iteration and I take out a big chunk of area […] And then at some point I'm taking out tiny chunks, so it doesn't seem like that matters. […] if I'm going to infinity I'm going to eventually take out everything so I don't think there is anything left.

This segment is focused exclusively on the area of the ST and was coded as referencing the mental space elements *area tends to zero*, *area removed at each step*, and *infinite decreasing process*. This makes her area input space identical to Joy's. While she hints that the amount of area "taken out" changes throughout the process, she does not quantify that change nor does it appear to factor into her final conclusion. Elise also expanded her initial thoughts about the perimeter to explain her reasoning, saying "I'm thinking if you go to infinity that you are adding side lengths infinitely so you're just like forever adding length to your perimeter, so I feel like your perimeter is forever increasing." This quote provides evidence for the mental space elements *perimeter added at each step* and *infinite increasing process* as part of Elise's perimeter input space. Elise then paused to compute the change in perimeter at each step, and reiterated "I think it's going to infinity. I'm pretty confident now. Because […] every time after

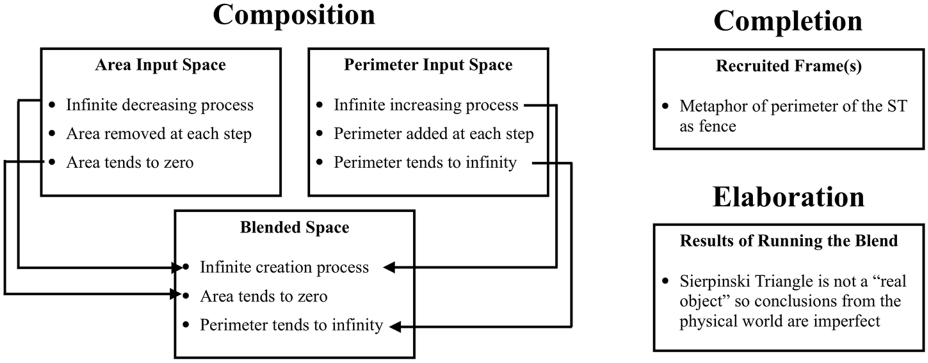

Fig. 2 Diagram of Joy's blending process, including composition of a blended space from input spaces for area and perimeter, frames recruited to complete the space, and the results of her elaboration

the first iteration I'm adding more perimeter than I added before." This marks the inclusion of *adding more perimeter at each step*, an element of her perimeter input space that was not part of Joy's—though it did not figure into her blended space.

Elise does not initially remark on the fact that this figure has infinite perimeter *and* zero area. Upon reading Fred's argument, she recalls that "we had a discussion about this in class. Like, do you have to still have the area?" It is here, after Fred, that we gain insight into her blended space which includes both area and perimeter of the ST. She explains:

> But I guess I'm thinking of our perimeter as like, like I guess I think at the end of this I have this skeleton, so I have no area, nothing is left inside, but I think I still have the framework. And so, I think the perimeter still exists.

As Elise talks about coordinating area and perimeter of the ST, it becomes apparent that her blended space is composed identically to Joy's, with the projection of *infinite creation process*, *perimeter tending to infinity*, and *area tending to zero* from her input spaces. She did not again reference her calculations of the rate at which the perimeter changes. She completes this space with a distinct metaphor of a skeleton. This skeleton metaphor is not entirely dissimilar from Joy's fencing metaphor, but it comes with somewhat different entailments. While Joy spoke of fencing "filling in" the original outline of a triangle, Elise says that "the area goes to zero. I don't think it matters though, because I think you still have the skeleton." Thus, the skeleton metaphor brings with it entailments of bones remaining when flesh has gone, mapping perimeter to bones and area to flesh. She elaborated her blend, saying, "I'm thinking of our perimeter as like, like I guess I think at the end of this I have this skeleton, so I have no area, nothing is left inside but I think I still have the framework." Note that Elise mentions "at the end" in her elaboration, perhaps hinting that she sees the ST as an abstract object at the end of a generating process. Though she also recognizes the confusion brought on by an object with infinite perimeter and zero area, she resolves this using her skeleton framing of the blend—the area (flesh) may be gone, but the perimeter (skeleton) must remain.

4.5 Carmen's blending process

Carmen, the student quoted at the beginning of this paper, shared many aspects of her input and blended spaces with Joy and Elise. At the beginning of her interview, she clearly separated the area and perimeter of the ST and spoke at some length about each of them. About the area, she concluded "eventually the area gets to zero, but that's if you could do it infinitely many times." Despite some back-and-forth which indicated tensions between potential and actual infinity, Carmen wrote "area = 0." Her area input space included the elements *area tends to zero*, *infinite decreasing process*, and *area removed at each step*. Carmen acknowledged that perimeter "took [her] a while to get." She reviewed the creation process used in class (*infinite increasing process*), describing how at each step more triangles would be drawn (*perimeter added at each step*), but was not initially certain if the perimeter would reach some finite limit or increase to infinity as the new triangles are smaller. In contrast to Elise, she did not calculate those values until she looked at Fred's statement, at which point *change in the rate of change* and *perimeter tends to infinity* became part of her perimeter input space—at that point still separate from area.

Carmen's blended space is composed in same way as Joy's and Elise's. The completion of her blend, however, was more complex. She recruited a calculus frame and identified "analogies to calculus or real analysis," including Riemann sums, that she saw as similar to Fred's paradox. Her appeals to calculus indicate that to Carmen, the ST paradox is like other situations that she has seen in previous mathematics courses and accepts. Upon reading Fred's arguments during the interview, Carmen stops to query whether he is implying zero perimeter or some non-zero finite length; she then eliminates each, leaving only the possibility that the perimeter is indeed infinite and Fred is wrong. During this episode, two more frames appeared. Like Joy, she brought in a fence metaphor for the perimeter and the entailment that fencing should remain. Her argument against zero perimeter came via elaborations using the fence frame, saying, "you have sort of your old triangle fences that you had before [...] we still have this fence around, that big triangle and the center, and we still have those other ones we made before." Finally, she brought the frame of self-similarity, with the entailment that "we can keep zooming in." The elaboration using this frame was that the perimeter cannot be a finite value, which she explained via contradiction, saying "I think if we could [stop] then you could say, ok it's this number," but the zooming goes on forever, "so that's kind of why it can't be a number." Carmen encountered the paradox, as seen in the quote which starts this paper, but she resolved it by running the blend of a space completed with several frames.

4.6 Curtis' blending process

Curtis' ST interview segment was the shortest of all the students', lasting just about five minutes. Upon hearing the initial question, he sketched the first few iterations of the ST as it had been introduced in class and described the relationship between the new triangles being added at each stage and the existing figure. For perimeter, he noted that "the whole thing increases by three-halves at each stage. So, […] that's three-halves to the n" and wrote $(^3/_2)^n$ on paper. He then pointed at this expression and said "$(^3/_2)^n$ gives you the perimeter at n. Since you're looking at the limit as n goes to infinity, that's gonna equal positive infinity because that number is greater than one." Curtis' statement about "increases […] at each stage" was coded as *infinite increasing process*, and his quantification of the limits as *each step increases by a factor of 3/2* and *perimeter tends to infinity (limit)*. As Curtis drew the first iteration of the ST (i.e., connected the midpoints of a triangle and scratched out the resulting center figure), he pointed to one of the three remaining triangles and said its perimeter was half that of the original large figure, and "since each of these [small triangles] is the same thing, the whole thing increases by three-halves at each stage" which we coded as *computable from congruent components*.

He then switched to the area, noting that "after one [iteration], we're gonna have two- three-fourths. Since you do the same thing with each one…" and trailed off while writing $(^3/_4)^n$ and finished "so that's zero." He further described his thinking as "I'm just confident that at each time it's $^3/_4$ the previous, so, I mean, I know that when you keep multiplying $^3/_4$ by itself you get, you get zero out." This brief description, coupled with gestures to the ST he had drawn and his work on the scratch paper, revealed the presence of the area mental space elements *infinite decreasing process*, *computable from congruent components*, *each step decreases by a factor of* $^3/_4$, and *area tends to zero (limit)*.

Curtis was the only student to write complete formulations of limits for the area and perimeter of the ST. He was also the only student to consider the ST process from a multiplicative perspective, while all others engaged with additive processes. This sets his input spaces apart from his peers (compare Figs. 2 and 3). However, he did not make use of these unique elements when composing his blended space in response to Fred, leveraging only the fact that the results of his limits were zero and infinity.

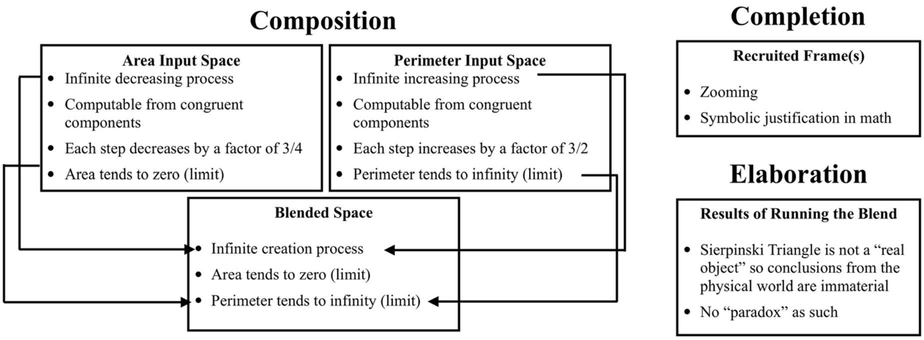

Fig. 3 Diagram of Curtis's blending process, including composition, completion, and elaboration

When considering Fred's paradox, his completion process brought in a zooming frame, saying, "we could say you could zoom in for infinitely, as much as you want, and you could get like these as tiny and tiny as you want, there's still more perimeter to draw." The second frame we saw Curtis leverage is one related to mathematics classes (e.g., calculus, analysis) where symbolic manipulations are sufficient, as seen in his confidence related to his knowledge of limits of geometric sequences. Curtis provided the same elaboration as Joy, saying "[the ST] isn't like, not physically drawing something like a perimeter. It's kind of just a concept." Curtis' blend did not lead him to encounter the paradox as his peers did, perhaps because of his confidence in the limits he calculated.

4.7 The other six students

We have presented in some detail the similarities and differences between the blended spaces of these four students. There was some variation in their mental spaces for area and perimeter, but they composed blended spaces of the same elements. Following composition, these students recruited distinct frames to complete those spaces. These frames, in turn, resulted in some distinct elaborations and interactions with the ST and the apparent paradox of a figure with no area and an infinite perimeter. The other six students composed different blended spaces, with their own completions and elaborations. We now present some highlights of these analyses.

Shani, Kay, and Soo were groupmates and shared the idea of the area as infinitely decreasing, but *not* converging to zero. Shani and Kay described area as converging to some "infinitely small" yet nonzero value, while Soo described an unending decreasing process which never reaches zero. Both conceptualizations resulted in similar blended spaces that did not include *area goes to zero*, though in other respects they were the same as those of the previously discussed students. Shani and Soo's completion processes were framed by their understanding that the process does not end, while Kay brought in a frame of perimeter with the entailment that "a perimeter encloses something." We have no evidence that any of the three proceeded to the elaboration phase of blending, and none of the three seemed to encounter the paradox.

In addition to the elements composed into their blended spaces by the first four described students, Sam projected a unique fusion of *removing area at each step* and *adding perimeter at each step*, into the *simultaneous addition of perimeter and removal of area*. Sam completed his blend by bringing in a frame about the nature of infinity and infinite iterative processes, saying about area that "at infinity it's going to zero. It's not [zero] before infinity." These elements and entailments supported his running of the blend, establishing that the infinite aspect of the process resolves the paradox, while “before infinity [Fred’s] statement will be right.”

Kevin also composed a similar blended space to that of the first four students, though he did not commit to a value for area—only that area would converge. Additionally, he explained that an infinite perimeter and finite area are only a paradoxical situation if the figure is of integer dimension. He completed the blend with a frame of non-integer dimension, and his elaboration of this space did not result in a paradox.

Mia uniquely conceptualized the ST as two figures: the triangles that remain and the triangles which are removed during the recursive creation process. This composition, with two distinct figures, did not make sense to Mia but also did not lead to a paradox. She saw these figures separately and did not make a concerted attempt to coordinate them. We saw no evidence that she moved beyond composition into completion or elaboration of a blended space.

4.8 Summary

Our analysis of students’ blending processes, especially as provoked by encountering Fred’s argument, revealed how students encounter, cope with, or resolve the paradox of coordinating infinite perimeter and zero area associated with the ST. All students *composed* a blended space from their area and perimeter input spaces following Fred’s prompt, and most of them also *completed* their blended space with additional frames, which then supported *elaboration* of the blend—leading to new implications. In two students’ interviews, we saw evidence of completion but not elaboration (Shani and Soo); only for one student (Mia) do we not have evidence of completion.

We saw one commonality across all students’ composition processes: the fusion of *infinite (increasing) process* and *infinite (decreasing) process* into a unified *infinite creation process* for the stepwise creation of the ST. This is not to say that there was a shared conception of exactly what happens at each step, only that the process is infinite. We purposefully refer to these elements as *infinite*, with all the ambiguity about potential/actual infinity it entails, because our data does not support conclusions about the nature of students’ conception of the infinite.

We found that four students did not encounter the paradox on their own, and this seems related to the composition of their blended spaces. That Mia’s blend was composed of processes based on two distinct figures (removed vs. leftover triangles) prevented her from encountering a paradox, since she did not see one figure with infinite perimeter and zero area. Shani and Kay’s input spaces for area included a non-zero limit for area, and their completions allowed them to coordinate this without experiencing a paradox. Soo did not encounter the paradox on her own, because she did not see an end where area would equal zero.

While composition is an important part of the coordination of area and perimeter, and relates to who encountered the paradox or not, it was not enough to explain students’ different ways of reasoning. For example, the blended spaces of Joy, Elise, Curtis, and Carmen included the same three elements, but they completed these spaces with different frames or different entailments of similar frames. For each student (except Mia), we have evidence of 1–3 distinct frames being used to complete their blended spaces. In most cases, one of the frames has to do with the nature of mathematics (e.g., the nature of infinite processes). Four students also used physical frames and their entailments to coordinate area and perimeter and to make sense of that coordination.

Elaboration of the blend was the most varied of the processes we analyzed, due in part to its dependence on both the composition and completion of the blended spaces. We observe that students could arrive at similar conclusions based on different lines of reasoning (e.g., Joy and

Curtis concluding the ST is a concept, not a real object) and that students with similar starting points could bring in different frames and develop different conceptualizations. Our non-evaluative and non-deficit conceptual blending approach allowed us to see these nuances as students' lines of reasoning separated and coalesced in a non-deterministic way.

5 Limitations

The students who participated in this study are part of a select population—students who have already completed a mathematics degree and are pursuing an advanced degree in mathematics education. Their career intentions may impact the ways in which they engage with both the material (e.g., knowing that they may teach it one day) as well as the ways in which they engaged with the instructor and researchers. Another limitation of our study is that the students who participated in interviews about the ST had already seen the figure in a classroom setting, discussed it with their peers, and had ample time to think about or look up explanations of the ST on their own. Thus, some students began by trying to *recall* answers instead of puzzling through the task in the moment. We do not have access to the details of students' earliest ideas, which might provide even richer information. Conceptual blending analysis of students' composition of a blended space, completion with other frames, and elaboration of that space requires that students attempt to coordinate mental spaces. This is highly appropriate for the present study given the nature of the ST and its in-class presentation and discussion. However, we cannot be sure what, if any, contextual limitations there are to the use of this theory.

6 Discussion

As designed, the in-class ST activity was intended to be a brief interlude, a relatively simple yet interesting task that would serve as motivation for a discussion about self-similarity and later fractal dimension. However, it proved to be challenging for the students and rich for analysis. The classroom discussion prompted us to design a related interview task which provided an appropriate avenue for conceptual blending analysis—and this analysis gave evidence of the highly individualized ways in which students can coordinate ideas which appear to be shared.

Given students' similar academic backgrounds and that they worked together in class on the ST task, we expected to see many similarities in their responses. There were some similarities, but also distinctions, in how students approached the area and perimeter of the ST. In any event, the elements we identified of students' mental spaces for area and perimeter are compatible with the existing literature on students' thinking about infinite iterative processes.

As students in this class considered the ST, they attempted to coordinate their understandings of its area with its perimeter. Conceptual blending is a theory about the coordination of ideas, and so we turned to it. We accessed students' input spaces for area and perimeter through our initial line of questioning, but the presentation of Fred's argument (which resonated with students, as it was built from a classroom episode) was a rich interview task, as students' responses to Fred's argument were, in general, more extensive than their original responses. This item allowed us to see the extent to which students maintained the ideas expressed in a classroom discussion, pick out nuances of individual thinking that were not accessible in the larger group, and see how students would defend and elaborate their ideas in the face of an alternative view (Rasmussen, Apkarian, Tabach, & Dreyfus, in review).

The conceptual blending analyses we conducted on students' interviews related to the ST revealed nuances in students' coordination of ideas. Despite variation in their input spaces, the composition of most students' blended spaces included a core set of ideas. Most students had the ideas of perimeter being added and area being removed at each step, but only Sam fused these ideas and mapped them to his blended space. And while several students had blended spaces composed of the same elements, the development of those spaces showed more variation. Students completed their blends with ideas from calculus or analysis (e.g., Carmen, Curtis), fractal dimension (Kevin), and physical metaphors (e.g., Joy, Elise). These frames resulted in varied elaborations. Some related to the nature of the ST, such as "it's not a real object" (Joy, Curtis), its non-integer dimension (Kevin), or that it is only the remaining outline (Elise's skeleton, Carmen's fence); others framed the nature of the paradox itself, such Sam's statement that the paradox only exists "before" infinity.

Our articulation of the component process of conceptual blending in a mathematical context allows for nuanced analysis of students' reasoning—an empirical application of a known and respected theory that is particularly relevant for situations where students must bring together multiple ideas. Indeed, such situations are frequent in learning mathematics, and hence, the approach developed here has the potential to transcend the case of the ST.

Identifying all three processes—composition, completion, and elaboration—enables us to examine not only the main ideas students mention, but how they are used and enacted, or what leverage they give students in thinking about mathematical objects. This is in contrast to other approaches which highlight student difficulties. The analytic power of the completion process, in particular, helped us to articulate the tools by which students elaborate their blends.

This nuance among students' blending processes enables us to see how students' ideas converged and diverged from each other, separate from questions of how well they understand the constituent mathematical elements such as infinity. Our use of these processes as analytic tools allows us insight into student thinking that, on the surface, appeared as a jumble of ideas and conclusions with little connection. We further see that the completion process, that of recognizing elements and bringing in a frame and its entailments, was critical in students' ability to think deeply about the ST. Regardless of how a blended space is composed, it is completion which allows for elaboration and the formation of new understandings.

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