SPECTRAL RIGIDITY OF HURWITZ SURFACES

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For an element $\gamma \in \mathrm{PSL}(2,\mathbb{R})$, there is a unique lift $\tilde{\gamma} \in \mathrm{SL}(2,\mathbb{R})$ such that $\mathrm{Tr}(\tilde{\gamma}) \geq 0$. Define $\mathrm{Tr}(\gamma) := \mathrm{Tr}(\tilde{\gamma})$. We say that an element $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ is hyperbolic if $\mathrm{Tr}(\gamma) > 2$. In this case, there is a unique real number $N(\gamma) > 1$ such that $\tilde{\gamma}$ is conjugate in $\mathrm{SL}(2,\mathbb{R})$ to $\left(\begin{smallmatrix} N(\gamma) & 0 \\ 0 & N(\gamma)^{-1} \end{smallmatrix}\right)$. We call $N(\gamma)$ the *norm* of γ. The norm and trace are related by the equation $\mathrm{Tr}(\gamma) = N(\gamma) + N(\gamma)^{-1}$. In particular, they uniquely determine one another.

Let $\Gamma \leq \mathrm{PSL}(2,\mathbb{R})$ be a cofinite Fuchsian group. We say that a hyperbolic element γ is primitive if it is not a proper power. Let $[\Gamma]_{\mathrm{prim}}$ denote the collection of conjugacy classes of primitive hyperbolic elements of Γ. Since trace and norm are both conjugation invariant, we may extend their definition to $[\Gamma]_{\mathrm{prim}}$.

$$Z_\gamma(s,\rho) := \prod_{[\gamma]\in[\Gamma]_p} \prod_{k=0}^{\infty} \det(1 - \rho(\gamma)N(\gamma)^{-(s+k)})$$

0.0.0.1 Multiplicative independence

Set $\Lambda_\Gamma(s,\rho) = \log Z_\Gamma(s,\rho)$. Compute for $\mathrm{Re}(s) \gg 0$,

$$\begin{aligned}
\Lambda_\Gamma(s,\rho) &= \sum_{[\gamma]\in[\Gamma]_p} \sum_{k=0}^{\infty} \log\det(1 - \rho(\gamma)N(\gamma)^{-(s+k)}) \\
&= \sum_{[\gamma]\in[\Gamma]_p} \sum_{k=0}^{\infty} \mathrm{Tr}\log(1 - \rho(\gamma)N(\gamma)^{-(s+k)}) \\
&= -\sum_{[\gamma]\in[\Gamma]_p} \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \frac{\chi_\rho(\gamma^m)}{m} N(\gamma)^{-m(s+k)},
\end{aligned}$$

since $\log(1-A) = -\sum_{m=1}^{\infty} \frac{A^m}{m}$ for any matrix A with sufficiently small entries. In the last line, $\chi_\rho(\gamma) = \mathrm{Tr}(\rho(\gamma))$ is the character of ρ.

The expression in the last line of the previous display is sensible for χ_ρ for any class function on Γ. For χ a class function on Γ, we take this as a definition of $\Lambda_\Gamma(s,\chi)$ and thereby $Z_\Gamma(s,\chi)$. For a class function χ, suppose there are finitely many nonzero rational numbers a_ρ, for $\rho \in \hat{\Gamma}$ such that $\chi = \sum_{\rho\in\hat{\Gamma}} a_\rho \chi_\rho$. We call such a χ a *virtual character*, and write $\chi \in \mathrm{Vchar}(\Gamma)$. For such a χ,

$$\Lambda_\Gamma(s,\chi) = \sum_{\rho\in\hat{\Gamma}} a_\rho \Lambda_\Gamma(s,\rho)$$

and

$$Z_\Gamma(s,\chi) = \prod_{\rho\in\hat{\Gamma}} Z_\Gamma(s,\rho)^{a_\rho}$$

Suppose there is some multiplicative dependency among the functions $Z_\Gamma(s,\rho_1),...,Z_\Gamma(s,\rho_n)$, i.e. there are rational numbers a_{ρ_i} so that $\prod_{i=1}^n Z_\Gamma(s,\rho_i)^{a_{\rho_i}} = 1$. Then this may be expressed concisely as $Z_\Gamma(s,\chi) = 1$, where $\chi = \sum_{i=1}^n a_{\rho_i}\chi_{\rho_i}$.

Let $\Gamma' \leq \Gamma$ be a finite index normal subgroup, and set $G = \Gamma'\backslash\Gamma$. For each class function χ of G, let $\tilde{\chi}$ be its inflation to Γ. Now suppose χ is a virtual character of G such that $Z_\Gamma(s,\tilde{\chi}) = 1$. This is equivalent to

$$\Lambda_\Gamma(s,\tilde{\chi}) = -\sum_{[\gamma]\in[\Gamma]_p}\sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\frac{\tilde{\chi}(\gamma^m)}{m}N(\gamma)^{-m(s+k)} = 0$$

Recall (find ref) the uniqueness principle for generalized Dirichlet series: let $\nu_1,\nu_2,...$ be a sequence of distinct positive real numbers and $a(\nu_1),a(\nu_2),...$ a sequence of complex numbers such that the series

$$\sum_{i=1}^{\infty} a(\nu_i)\nu_i^{-s}$$

converges absolutely to an analytic function $f(s)$ for $\mathrm{Re}\, s \gg 0$. Then $f(s) = 0$ identically if and only if $a(\nu_i) = 0$ for all ν_i.

In order to apply the uniqueness principle for (generalized) Dirichlet series, we must collect summands according to their base. To this end, let $\mathcal{N}_\Gamma$ denote the *primitive norm spectrum of* Γ. That is, $\mathcal{N}_\Gamma$ is the (multi-)set of values taken by the norm map N at primitive hyperbolic conjugacy classes in Γ. For every real number ν, let $\mathcal{N}_\Gamma(\nu)$ denote the collection of classes $[\gamma] \in [\Gamma]_p$ with $N(\gamma) = \nu$. By definition, $\mathcal{N}_\Gamma(\nu)$ is empty unless $\nu \in \mathcal{N}_\Gamma$. Now we have

$$\Lambda_\Gamma(s,\tilde{\chi}) = -\sum_{\nu\in\mathcal{N}_\Gamma}\sum_{[\gamma]\in\mathcal{N}_\Gamma(\nu)}\sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\frac{\tilde{\chi}(\gamma^m)}{m}\nu^{-m(s+k)} = 0.$$

The sum over k is a geometric series with base ν^{-m}, so

$$\Lambda_\Gamma(s,\tilde{\chi}) = -\sum_{\nu\in\mathcal{N}_\Gamma}\sum_{[\gamma]\in\mathcal{N}_\Gamma(\nu)}\sum_{m=1}^{\infty}\frac{\tilde{\chi}(\gamma^m)}{m(1-\nu^{-m})}\nu^{-ms} = 0$$

The set consisting of all powers of a fixed primitive norm and the set consisting of a fixed power of all primitive norms may intersect nontrivially. Indeed, by McReynolds-Lafont, arithmetic non-compact hyperbolic surfaces admit arbitrarily long arithmetic progressions in their length spectrum.

For each primitive norm ν, and each integer $m > 0$, let $A_m(\nu)$ denote the set of primitive hyperbolic conjugacy classes $[\gamma]$ in Γ, such that $N(\gamma^m) = \nu$. For a fixed $\nu \in \mathcal{N}_\Gamma$, only finitely many $A_m(\nu)$ are nonempty. Let $m(\nu)$ be the largest m such that $A_m(\nu)$ is nonempty. Then we have a partition $\mathcal{N}_\Gamma(\nu) = A_1(\nu) \sqcup ... \sqcup A_{m(\nu)}(\nu)$. Thus, for each $\nu \in \mathcal{N}_\Gamma$, the coefficient $b(\nu,\chi)$ of ν^{-s} in $\Lambda_\Gamma(s,\chi)$ is

$$b(\nu,\chi) := -\sum_{m=1}^{m(\nu)}\frac{1}{m(1-\nu^{-m})}\sum_{[\gamma]\in A_m(\nu)}\tilde{\chi}(\gamma^m).$$

which we conclude is zero for every primitive length ν. This does not immediately imply that the class function $\tilde{\chi}$ is zero.

Indeed, for each fixed ν, as we vary χ among the virtual characters, the function $\lambda_\nu : \chi \mapsto b(\nu,\chi)$ is actually a linear functional on the vectorspace $\mathrm{Vchar}(\Gamma'\backslash\Gamma)$. In order to conclude that $\chi = 0$, we must demonstrate that the (infinite!) set of functionals $\{\lambda_\nu : \nu \in \mathcal{N}_\Gamma\}$ spans the (finite dimensional!) dual space to $\mathrm{Vchar}(\Gamma'\backslash\Gamma)$.

0.0.0.2 Conjugacy classes in $\mathrm{GL}(2,q)$ and $\mathrm{PGL}(2,q)$

Let q be an odd prime power and set $G = \mathrm{GL}(2,q)$ and $\bar{G} = \mathrm{PGL}(2,q) = G/Z$ where Z is the center of G. For an element $g \in G$, define the *characteristic polynomial* $p_g := x^2 - \mathrm{tr}(g)x + \det(g) \in F_q[x]$. Among non central conjugacy classes, the characteristic polynomial of an element completely determines its class:

Lemma 1. *Suppose g and h are non central. Then g and h are conjugate in G if and only if $p_g = p_h$*

Proof. This follows from the theory of Jordan normal forms. □

Conjugation in $\bar{G}$ can be described in terms of conjugation in G:

Lemma 2. *Elements gZ and hZ in $\bar{G}$ are conjugate if and only if g is conjugate to λh in G for some $\lambda \in Z$.*

For an element gZ of $\bar{G}$, let $\bar{p}_{gZ}$ denote the collection of characteristic polynomials of lifts of gZ to G. That is, $\bar{p}_{gZ} = \{p_{\lambda g} : \lambda \in Z\}$. Combining the preceding lemmas, we obtain a characterization of nonidentity conjugacy classes in $\bar{G}$:

Lemma 3. *Nonidentity elements gZ and hZ are conjugate in $\bar{G}$ if and only if $\bar{p}_{gZ} = \bar{p}_{hZ}$.*

0.0.0.3 Multiplicative independence for principal congruence covers of semi-arithmetic surfaces

Now suppose Γ is a subgroup of $\mathrm{PGL}(2, \mathrm{O}_K)$ where K is a totally real number field, and O_K is its ring of integers. We further suppose that under some embedding $K \to \mathbb{R}$, the image of Γ is a lattice in $\mathrm{PGL}(2,\mathbb{R})$. For any prime ideal $\mathfrak{p}$ of O_K, the reduction mod $\mathfrak{p}$ map $\mathrm{O}_K \to \mathbb{F}_\mathfrak{p} := \mathrm{O}_K / \mathfrak{p}\, \mathrm{O}_K$ induces a map $\mathrm{PGL}(2, O_K) \to G(\mathfrak{p}) := \mathrm{PGL}(2, \mathbb{F}_\mathfrak{p})$. Let $\pi_\mathfrak{p}$ be its restriction to Γ. For each $\mathfrak{p}$, define the *principal congruence subgroup* $\Gamma(\mathfrak{p}) := \ker \pi_\mathfrak{p}$ *of level* $\mathfrak{p}$. Let S be the set of primes $\mathfrak{p}$ such that the map $\pi_\mathfrak{p}$ is surjective. Thus, when $\mathfrak{p} \in S$, $\pi_\mathfrak{p}$ induces an isomorphism of $\Gamma(\mathfrak{p})\backslash\Gamma$ with $G(\mathfrak{p}) = \mathrm{PGL}(2, O_K/\mathfrak{p}O_K)$. If H is one of the groups $\Gamma, \Gamma(\mathfrak{p})$ or $G(\mathfrak{p})$, let $\tilde{H}$ denote its lift to $\mathrm{GL}(2, \mathbb{F}_\mathfrak{p})$.

Theorem 1. *Suppose $\gamma \in \Gamma - \Gamma(\mathfrak{p})$ is hyperbolic and that the characteristic polynomial p_γ is irreducible over K. If $\lambda \in \Gamma$ is such that $N(\gamma) = N(\lambda)$, then $\pi_\mathfrak{p}(\gamma)$ and $\pi_\mathfrak{p}(\lambda)$ are conjugate in $G(\mathfrak{p})$.*

Proof. It suffices to show that $N(\gamma)$ completely determines the characteristic polynomial p_γ. To this end, observe that the characteristic polynomial p_γ of γ is also the minimal polynomial for the unit N_γ in the quadratic extension $K_\gamma := K(N(\gamma))$ of K. Let σ denote the nontrivial Galois automorphism of K_γ/K. Then $p_\gamma = (t - N(\gamma))(t - N(\gamma)^\sigma)$, as claimed. Specifically, the constant term $\det(\gamma)$ of p_γ is the Galois norm $N(\gamma)N(\gamma)^\sigma$ of the quadratic unit $N(\gamma)$. □

Corollary 1. *Suppose χ is the inflation of a class function from $G(\mathfrak{p})$ to Γ. Then χ is constant along each set $\mathcal{N}_\Gamma(\nu)$ such that $\nu \neq 1 \mod \mathfrak{p}$. If $\nu = 1 \mod \mathfrak{p}$, and $\mathcal{N}_\Gamma(\nu) \cap \Gamma(\mathfrak{p}) = \emptyset$, then χ is constant on $\mathcal{N}_\Gamma(\nu)$.*

Theorem 2. *Suppose Γ is as above, and $\mathfrak{p} \in S$. Let χ be a class function on Γ inflated from one on $G(\mathfrak{p})$. Then $\Lambda_\Gamma(s, \chi) = 0$ if and only if $\chi = 0$.*

Proof. If $\Lambda_\Gamma(s, \chi) = 0$, then for all $\nu \in \mathcal{N}_\Gamma$, we have $b(\nu, \chi) = 0$.

First, suppose $\nu \neq 1 \mod \mathfrak{p}$. Then along the set $\mathcal{N}_\Gamma(\nu)$ of primitive hyperbolic $\gamma \in \Gamma$ with norm ν, the function χ takes a common value which we denote $\chi(\nu)$. Let $a_m(\nu)$ denote the cardinality of $A_m(\nu)$, the set of primitive hyperbolic $\gamma \in \Gamma$ such that $N(\gamma^m) = \nu$. Then

$$b(\nu, \chi) = -\sum_{m=1}^{m(\nu)} \frac{a_m(\nu)}{m(1-\nu^{-m})}\chi(\nu) = 0$$

To conclude that $\chi(\nu) = 0$, it suffices to demonstrate the existence of a primitive hyperbolic $\gamma \in \Gamma$ such that $N(\gamma) = \nu \mod \mathfrak{p}$. This follows from the Cheboratev density theorem of (cite Sarnak):

Theorem 3. *Let C be a conjugacy class in $G(\mathfrak{p})$ and $F(T, C)$ denote the collection of primitive hyperbolic γ in Γ with $N(\gamma) < T$ such that $\pi_\mathfrak{p}(\gamma) \in C$. Then as $T \to \infty$,*

$$F(T, C) = \frac{|C|}{|G(\mathfrak{p})|}\frac{T}{\log T} + o(1).$$

Thus, for any primitive hyperbolic $\gamma \in \Gamma$ with $N(\gamma) \neq 1 \mod \mathfrak{p}$, we have $\chi(\gamma) = 0$.

Now suppose $\nu = 1 \mod \mathfrak{p}$. Then if $\gamma \in \mathcal{N}_\Gamma(\nu)$, either $\pi_\mathfrak{p}(\gamma) \in I$ the identity conjugacy class, or $\pi_\mathfrak{p}(\gamma) \in P$ the unique parabolic conjugacy class in $G(\mathfrak{p})$. Note that $|I| = 1$, and $|P| = q^2 - 1$, where $q = |O_K/\mathfrak{p}O_K|$. By Sarnak's version of Chebotarev's density theorem, there are infinitely $\nu \in \mathcal{N}_\Gamma$ with $\nu = 1 \mod \mathfrak{p}$ such that $\mathcal{N}_\Gamma(\nu) \cap \Gamma(\mathfrak{p}) = \emptyset$. For any such ν, χ is constant along $\mathcal{N}_\Gamma(\nu)$. As above, for such ν, $b(\nu, \chi) = 0$ implies $\chi(\nu) = 0$.

$\square$