1. Selberg zeta formalism

For a closed Riemannian manifold M of negative curvature and a unitary representation $\rho : \pi_1(M, x) = \Gamma \to \mathrm{GL}(V)$, one can form the **Selberg zeta function**: for s a complex variable in a suitable right-halfplane

$$Z_\Gamma(s, \rho) = \prod_{\gamma \in \Gamma_{pcc}} \prod_{k \geq 0} \det \left(\mathrm{id}_V - \rho(\gamma) e^{-(s+k)\ell(\gamma)} \right),$$

where the product is over the set Γ_{pcc} of primitive conjugacy classes γ in Γ, and $\ell(\gamma)$ is the Riemannian length of the corresponding geodesic.

Claim 1. The following assertions are purely formal:

- Knowledge of $Z_\Gamma(s, \rho)$ amounts to the knowledge of the following data set: for each length $\ell \in \mathbb{R}_{\geq 0}$, the value of the sum
$$\sum_{\gamma \in \Gamma_{pcc}, \ell(\gamma^k) = \ell} \mathrm{tr}(\rho(\gamma))/k.$$
In particular, if $\rho = 1$ is trivial then knowledge of $Z_\Gamma(s, 1)$ amounts to knowing the length spectrum of Γ.
- **Functoriality under direct sums:** $Z_\Gamma s, \rho$ is multiplicative w/r/t direct sum of representations:
$$Z_\Gamma(s, \rho \oplus \sigma) = Z_\Gamma(s, \rho) Z_\Gamma(s, \sigma).$$
- $Z_\Gamma(s, \rho)$ depends only on the isomorphism class of ρ, so is consequently dependent only on its character $\mathrm{tr} \circ \rho$.
- **A consequence of the preceeding two observations**: While $Z_\Gamma(s, \rho)$ is initially only defined for representations ρ, we can extend its definition to any conjugacy invariant function on Γ which arises as a finite linear sum of characters of representations.
- **Functoriality under finite covers:** Suppose $N \to M$ is a finite cover and $\pi_1(N, y) = \Lambda \leq \Gamma = \pi_1(M, x)$. For any representation $\sigma : \Lambda \to \mathrm{GL}(V)$ on the cover, one can realize $Z_\Lambda(s, \sigma)$ as a Selberg zeta function for the base M via induction:
$$Z_\Lambda(s, \sigma) = Z_\Gamma(s, \mathrm{ind}_\Lambda^\Gamma \sigma).$$

In order to directly relate $Z_\Gamma(s, \rho)$ to the Laplace spectrum of M, it suffices to take M to be locally symmetric (and negatively curved). In this setting if we let $\tilde{M}$ denote the universal cover, and let Γ act on $\tilde{M}$ by isometries so that $M \approx \Gamma / \tilde{M}$, then we can define the space

$$L^2(M, \rho) = \{ f \in L^2(M, V) : f(gz) = \rho(g) f(z) \}$$

and consider the spectral problem for $\Delta_\rho = \Delta \otimes \mathrm{id}_V$ acting on it. The following is *almost* purely formal (modulo understanding the Selberg trace formula)

Claim 2. Up to a (topological) fudge factor:

$$Z_\Gamma(s, \rho) = \det \left(\Delta_\rho - s(s-1) \right).$$

In particular, away from the zeroes of the fudge factor (the so-called trivial zeroes), the zeroes of $Z_\Gamma(s, \rho)$ occur precisely at the eigenvalues of Δ_ρ acting on $L^2(M, \rho)$, taken with multiplicty.

1.1. **A converse to Sunada: formulation.** Suppose one has a diamond of Riemannian covers:

$$\begin{tikzcd} & M \arrow[dl] \arrow[dr] & \\ N_1 \arrow[dr] & & N_2 \arrow[dl] \\ & M_o & \end{tikzcd}$$

with corresponding (flipped) configuration of fundamental groups:

$$\begin{tikzcd} & \Gamma_o = \pi_1(M_o) & \\ \Lambda_1 = \pi_1(N_1) \arrow[ur, hook] & & \Lambda_2 = \pi_1(N_2) \arrow[ul, hook'] \\ & \Gamma = \pi_1(M) \arrow[ul, hook'] \arrow[ur, hook] & \end{tikzcd}$$

Question 1. Suppose N_1 and N_2 are isospectral. Are Λ_1 and Λ_2 almost-conjugate in Γ_o? That is: if N_1 and N_2 are isospectral and live in a diamond, is that diamond a sunada diamond?

Applying the Selberg zeta formalism to this question, it is equivalent to ask:

Question 2. Suppose $Z_{\Gamma_o}(s, \mathrm{ind}_{\Lambda_1}^{\Gamma_o} 1) = Z_{\Gamma_o}(s, \mathrm{ind}_{\Lambda_2}^{\Gamma_o} 1)$. Can one conclude that, as Γ_o representations, $\mathrm{ind}_{\Lambda_1}^{\Gamma_o} 1 = \mathrm{ind}_{\Lambda_2}^{\Gamma_o} 1$?

Applying the first bullet in claim 1, this amounts to asking:

Question 3. Suppose that, for all ℓ in $\mathbb{R}_{\geq 0}$, one has

$$\sum_{\gamma \in \Gamma_{opcc}, \ell(\gamma^k) = \ell} \chi_{\Lambda_1}^{\Gamma_o}(\gamma)/k = \sum_{\gamma \in \Gamma_{opcc}, \ell(\gamma^k) = \ell} \chi_{\Lambda_1}^{\Gamma_o}(\gamma)/k,$$

(where $\chi_{\Lambda_i}^{\Gamma_o}$ is the trace of the induced representation) must then $\chi_{\Lambda_1}^{\Gamma_o} = \chi_{\Lambda_1}^{\Gamma_o}$?