SPECTRAL RIGIDITY OF HURWITZ SURFACES

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Let $X_1 = \Gamma_1\backslash\mathfrak{H} \to X_0 = \Gamma_0\backslash\mathfrak{H}$ be a finite galois cover of compact Riemann (orbi) surfaces with $G = \Gamma_1\backslash\Gamma_0 = \mathrm{Gal}(X_1 \to X_0)$. Define for each surface the *Selberg zeta function*

$$\zeta_{X_i}(s) = \prod_{[\gamma]\in[\Gamma_i]_p} \prod_{k=0}^{\infty} (1 - e^{-(s+k)\ell\gamma}),$$

where $[\Gamma_i]_p$ denotes the set of conjugacy classes of primitive hyperbolic elements of Γ_i, and $\ell\gamma$ is the translation length of a representative of such a class. There is an analogue of the Artin-Takagi factorization [4]

$$\zeta_{X_1} = \prod_{\rho\in\hat{G}} L_{\Gamma_0}(s,\rho)^{\dim\rho}$$

where we define the *Artin–Selberg L function* attached to a finite dimensional unitary representation ρ of $G = \Gamma_1\backslash\Gamma_0$

$$L_{\Gamma_0}(s,\rho) = \prod_{\gamma\in[\Gamma_0]_p} \prod_{k=0}^{\infty} \det(1 - \rho(\mathrm{frob}(\gamma))e^{-(s+k)\ell(\gamma)}).$$

where $\mathrm{frob}(\gamma) \in G$ is the frobenius class associated to γ (see [3])

Now suppose we have a diamond of fuchsian groups, Γ_0 and Γ_1 the bottom and top respectively, and Λ_1 and Λ_2 the sides. We suppose Γ_1 is normal in Γ_0 and both Λ_i, and set $G = \Gamma_1\backslash\Gamma_0$ and $H_i = \Gamma_1\backslash\Lambda_i$. Denote by $X_i = \Gamma_i\backslash\mathfrak{H}$ and $Y_i = \Lambda_i\backslash\mathfrak{H} = H_i\backslash X_1$ the corresponding diamond of covers.

Claim 1. Suppose Γ_0 has the property that the collection $\{L_{\Gamma_0}(s,\rho) : \rho \in \hat{G}\}$ is multiplicatively independent. Then the following are equivalent

- Y_1 and Y_2 are isospectral
- (G, H_1, H_2) forms a Gassmann triple.

Proof. Write

$$\begin{aligned}\zeta_{Y_i}(s) &= L_{\Lambda_i}(s, 1_{\Lambda_i}) \\ &= L_{\Gamma_0}(s, \mathrm{Ind}_{\Lambda_i}^{\Gamma_0} 1_{\Lambda_i}) \\ &= \prod_{\rho\in G} L_{\Gamma_0}(s,\rho)^{\dim \hom_{\Lambda_i}(1_{\Lambda_i}, \rho|_{\Lambda_i})}.\end{aligned}$$

If Y_1 and Y_2 are isospectral, then we have an equality

$$\prod_{\rho\in G} L_{\Gamma_0}(s,\rho)^{\dim \hom_{\Lambda_1}(1_{\Lambda_1}, \rho|_{\Lambda_1})} = \prod_{\rho\in G} L_{\Gamma_0}(s,\rho)^{\dim \hom_{\Lambda_2}(1_{\Lambda_2}, \rho|_{\Lambda_2})}.$$

By multiplicative independence, we may equate exponents

$$\dim \hom_{\Lambda_1}(1_{\Lambda_1}, \rho|_{\Lambda_1}) = \dim \hom_{\Lambda_2}(1_{\Lambda_2}, \rho|_{\Lambda_2})$$

which is equivalent to Gassmann equivalence for (G, H_1, H_2). □

0.0.0.1 Multiplicative independence of L functions

In this section, K is a numberfield, L is galois over $\mathbb{Q}$ and $K \leq L$. We denote their respective rings of integers O_K and O_L. For $H = L$ or K, the norm map N_H is a multiplicative homomorphism from the monoid of nonzero ideals of O_H to $\mathbb{Z}$, which takes the value $N_H(\mathfrak{p}) = |O_H/\mathfrak{p}O_H|$ on prime ideals of O_H. The prime normset $\mathcal{N}_H$ of H is the multiset of values of N_H on prime ideals of O_H. For each integer q let $\mathcal{N}_H(q) = N_H^{-1}(q)$ be the collection of primes of O_H with norm q, and $m_H(q) = |\mathcal{N}_H(q)|$. We note that $m_H(q)$ is nonzero only for q a power of a prime.

The artin L function attached to a pair $(\sigma, L/\mathbb{Q})$ where $L/\mathbb{Q}$ is galois, and σ is an irreducible representation of $\mathrm{Gal}(L/\mathbb{Q})$ is defined by the Euler product for $\mathrm{Re}(s) \gg 0$:

$$L(s, \sigma, L/\mathbb{Q}) := \prod_{p \text{ prime in } \mathbb{Z}} \det(1 - \mathrm{frob}(p) N_L(p)^{-s})^{-1}.$$

By computing its logarithm,

$$\begin{aligned} \log L(s, \sigma, L/\mathbb{Q}) &= -\sum_p \log\det(1 - \sigma(\mathrm{frob}(p)) N_L(p)^{-s}) \\ &= -\sum_p \mathrm{Tr}\log(1 - \sigma(\mathrm{frob}(p)) N_L(p)^{-s}) \\ &= -\sum_p \sum_m \frac{\mathrm{Tr}(\sigma(\mathrm{frob}(p)^m)) N_L(p)^{-ms}}{m}, \end{aligned}$$

we see that $L(s, \sigma, L/\mathbb{Q})$ is actually determined by the character χ_σ of the representation σ. For two irreducible representations ρ and σ, one observes $\log L(s, \sigma \bigoplus \rho, L/\mathbb{Q}) = \log L(s, \sigma, L/\mathbb{Q}) + \log L(s, \rho, L/\mathbb{Q})$, from which $L(s, \rho \bigoplus \sigma, L/\mathbb{Q}) = L(s, \sigma, L/\mathbb{Q}) L(s, \rho, L/\mathbb{Q})$ follows. Thus, we may extend our definition to allow virtual characters. That is, χ is a rational linear combination $\sum a_\rho \chi_\rho$ of irreducible characters χ_ρ of $\mathrm{Gal}(L/\mathbb{Q})$. Explicitly, for such a virtual character we have

$$L(s, \sigma, \chi) = \prod_\rho L(s, \rho, L/\mathbb{Q})^{a_\rho}.$$

The proof of the following theorem, due to Artin, will serve as the protoype for ours.

Theorem 1. Let $L/\mathbb{Q}$ be a galois extension, and χ a virtual character of $\mathrm{Gal}(L/\mathbb{Q})$. Then $L(s, \chi, L/\mathbb{Q})$ identically if and only if $\chi = 0$.

Proof. Su □

Under the GNT analogy, the norm map N_H on prime ideals corresponds to the length function ℓ_X on primitive geodesics on a Riemann surface X. The normset corresponds to the *primitive length spectrum* $\mathcal{L}_X$ of X. This is the multiset of lengths of primitive closed geodesics on X. The analogue of $\mathcal{N}_H(q)$ for $q \in \mathbb{Z}$ are $\mathcal{L}_X(l) = \ell^{-1}(l)$ the collection of primitive geodesics with length $l \in \mathbb{R}$ and $m_X(l) = |\mathcal{L}_X(l)|$.

All multiplicity in the prime normset $\mathcal{N}_K$ is controlled by symmetries of K: the primes of K norm q become conjugate in the Galois extension $L/\mathbb{Q}$ of containing K. In fact, the Galois group $\mathrm{Gal}(L/\mathbb{Q})$

acts transitively on each $\mathcal{N}_K(q)$. In particular, $\mathcal{N}_L(q)$ is uniformly bounded by the order of the finite group $\mathrm{Gal}(L/\mathbb{Q})$.

By contrast, the multiplicity in the primitive length spectrum cannot be purely the result of its finite group of symmetries: for every Riemann surface, multiplicity in the primitive length spectrum is unbounded (Randol).

Suppose K is a totally real number field, and $\Gamma \leq \mathrm{PSL}(2,K)$. Any real embedding $\eta : K \to \mathbb{R}$ identifies Γ with a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. We suppose that η is chosen so that $\eta(\Gamma)$ is a uniform lattice in $\mathrm{PSL}(2,\mathbb{R})$ and will henceforth suppress η from notation.

Let Λ be the normalizer of Γ in $\mathrm{PGL}(2,\mathbb{R})$.

For an ideal $Q \triangleleft O_K$, the reduction map $O_K \to O_K/QO_K$ induces a reduction map $\mathrm{PGL}(2,O_K) \to \mathrm{PGL}(2,O_K/QO_K)$. Restricting to Γ gives rise to a reduction map $\pi_Q : \Gamma \to \mathrm{PGL}(2,O_K/QO_K)$. The kernel $\Gamma(Q)$ of π_Q is *the principal congruence subgroup of* Γ *of level* $\mathbb{Q}$. Let $S = S(\Gamma,\eta,\iota)$ be the collection of prime ideals Q in O_K such that the map π_Q is surjective. Thus, for each $Q \in S$, reduction mod Q induces an isomorphism $\Gamma/\Gamma(Q) \approx \mathrm{PGL}(2,\mathbb{F}_q)$ where $\mathbb{F}_q$ is the residue field of O_K mod Q. For $Q \in S$, we set $G_Q = \Gamma/\Gamma(Q)$. For a conjugacy class $\mathfrak{p} = \gamma^\Gamma$ of Γ, define $\mathrm{frob}(\mathfrak{p}) = [\pi_Q(\iota(\mathfrak{p}))]_{G_Q}$, a conjugacy class in G_Q.

$$
\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \pm 1 \arrow[r] \arrow[d] & F^\times \arrow[r] \arrow[d] & (F^\times)^2 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{SL}(2,F) \arrow[r] \arrow[d] & \mathrm{GL}(2,F) \arrow[r, "\det"] \arrow[d] & F^\times \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathrm{PSL}(2,F) \arrow[r] \arrow[d] & \mathrm{PGL}(2,F) \arrow[r, "\overline{\det}"] \arrow[d] & F^\times/(F^\times)^2 \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}
$$

Claim 2. Suppose Γ is as above. Then for each prime ideal Q in S, the meromorphic functions $\{L_\Gamma(s,\sigma) : \sigma \in \hat{G_Q}\}$ are multiplicatively independent.

Proof. Suppose a_σ are rational numbers, not all 0, such that

$$\prod_{\sigma \in \hat{G_Q}} L_\Gamma(s,\sigma)^{a_\sigma} = 1$$

Then taking a logarithm gives

$$
\begin{aligned}
a_\sigma \sum_\sigma \log L_\Gamma(s,\sigma) &= \sum_\sigma \sum_{\mathfrak{p}\in[\Gamma]_p} \sum_{k=0}^\infty \sum_{m=1}^\infty -a_\sigma \frac{\mathrm{tr}(\sigma(\mathrm{frob}(\mathfrak{p})^m)}{m} e^{-(s+k)m\ell(\gamma)} \\
&= \sum_\sigma \sum_{l\in\mathcal{L}_p(\Gamma)} \sum_{\mathfrak{p}:\ell(\mathfrak{p})=l} \sum_{k=0}^\infty \sum_{m=1}^\infty -a_\sigma \frac{\chi_\sigma(\mathrm{frob}(\mathfrak{p})^m)}{m} e^{-(s+k)ml}
\end{aligned}
$$

where $\mathcal{L}_p(\Gamma)$ denotes the primitive length spectrum of Γ.

If the former is identically zero for $\mathrm{Re}(s) \gg 0$, then upon taking the inverse mellin transform and applying the uniqueness principle for (generalized) Dirichlet series, we see that for each m, k, and l we have

$$\sum_{\sigma} \sum_{\mathfrak{p}:\ell(\mathfrak{p})=l} a_\sigma \frac{\chi_\sigma \operatorname{frob}(\mathfrak{p})^m}{m} e^{-kml} = 0.$$

In particular, $k = 0$, this reads for all primitive lengths l and for all $m \geq 1$

$$\sum_{\sigma} \sum_{\mathfrak{p}:\ell(\mathfrak{p})=l} a_\sigma \chi_\sigma \operatorname{frob}(\mathfrak{p}^m) = 0. \tag{1}$$

To proceed, we will follow [1] to relate the behavior of the Frobenius map as a function of length.

To this end, we first describe the conjugacy classes in $G_Q = \mathrm{PGL}(2, \mathbb{F}_q)$ in terms of the characteristic polynomial of their lifts to $GL(2, \mathbb{F}_q)$.

Let $Z = \{\mathrm{diag}(x, x) : x \in \mathbb{F}_q^\times\}$ denote the center of $\mathrm{PGL}(2, \mathbb{F}_q)$. Pick a quadratic nonresidue D in $\mathbb{F}_q$ and a square root δ in a quadratic extension E over $\mathbb{F}_q$.

The following table describes the conjugacy classes of $\mathrm{PGL}(2, \mathbb{F}_q)$:

Name	Representative	Constraints	Redundancy	Size	Number
I	$i := \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$	none	none	1	1
N	$n := \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$	none	none	$q^2 - 1$	1
$H(x)$	$h(x) := \left[\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right]$	$x \in \mathbb{F}_q^\times \setminus \pm 1$	$H(x) = H(x^{-1})$	$q(q+1)$	$(q-3)/2$
R	$r := \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$	none	none	$q(q+1)/2$	1
$E(x, y)$	$e(x, y) := \left[\begin{smallmatrix} x & yD \\ y & x \end{smallmatrix}\right]$	$x^2 - Dy^2 = 1$ and $x \neq 0$	$E(x, y) = E(x, -y)$	$q(q-1)/2$	$q(q-1)$
$E(\delta)$	$e(0, 1) := \left[\begin{smallmatrix} 0 & D \\ 1 & 0 \end{smallmatrix}\right]$	none	none	1	$q(q-1)/2$

Here, the specified representative is unique up to the stated redundancy.

For a primitive hyperbolic conjugacy class $\mathfrak{p}$ in Γ, we let $f_{\mathfrak{p}}(x) = x^2 - \mathrm{Tr}(\gamma)x + 1$ denote its characteristic polynomial, and $\widetilde{f_{\mathfrak{p}}}(x) = f(x) \mod I$ the characteristic polynomial of $\operatorname{frob}(\mathfrak{p})$ as an endomorphism of $O_K/IO_K \times O_K/IO_K = F_q \times F_q$.

Fix an element $D \in F_q^\times$ nonsquare. We break the conjugacy classes of $G_Q = \mathrm{SL}_2(q)$ into four disjoint families: say

- g is I-central if it is $\pm\,\mathrm{id}$. Equivalently, its characteristic poly has repeated root and g is diagonal.
- g is I-parabolic if it is conjugate to one of
$$c_p^{\pm}(1) := \pm \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right], c_p^{\pm}(D) := \left[\begin{smallmatrix} 1 & D \\ 0 & 1 \end{smallmatrix}\right]$$
Equivalently, its characteristic poly has a repeated root and is not diagonal.
- g is I-hyperbolic if it is conjugate to
$$c_h(x) := \left[\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right]$$
for some $x \in F_q^\times \setminus \pm 1$. Equivalently, its characteristic poly has distinct roots in $F_q^\times$. Note that $c_h(x)$ and $c_h(x^-1)$ are conjugate via $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$.
- g is I-elliptic if it is conjugate to
$$c_e(x, y) := \left[\begin{smallmatrix} x & y \\ -yD & x \end{smallmatrix}\right]$$

for $x, y \in F_q^\times$ satisfying $x^2 - Dy^2 = 1$. Equivalently, its characteristic poly has no roots in F_q. Note that $c_e(x, y)$ and $c_e(x, -y)$ are conjugate via $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$

We say a primitive hyperbolic conjugacy class $\mathfrak{p}$ of Γ has a given I-type according to that of $\mathrm{frob}(\mathfrak{p})$.

For such a class $\mathfrak{p}$, the length of the corresponding primitive closed geodesic is $\ell(\mathfrak{p}) = 2\,\mathrm{arccosh}(\mathrm{Tr}(\mathfrak{p})/2)$. The content of the following proposition (from [1]) is that, for a non I-central primitive hyperbolic conjugacy class $\mathfrak{p}$ of Γ, the I-type of $\mathfrak{p}$ is completely determined by its length.

Proposition 1. Suppose I is an odd prime lying over the rational prime p. Suppose that the characteristic polynomial $f_\mathfrak{p}$ of a prime $\mathfrak{p}$ of Γ has no root in K. Let $\varepsilon_\mathfrak{p}$ be its root with $i(\varepsilon_\mathfrak{p}) > 1$. Let $c(\varepsilon_\mathfrak{p})$ denote the conductor of the order $O_K[\varepsilon_\mathfrak{p}]$ in $O_{K(\varepsilon_\mathfrak{p})}$, the integers in the quadratic extension $K(\varepsilon)$ of K. Further, suppose that p does not divide $c(\varepsilon_\mathfrak{p})$, and that $\mathrm{frob}(\mathfrak{p})$ is not I-central. Then

- $\mathrm{frob}(\mathfrak{p})$ is I-parabolic if and only if I ramifies in $K(\varepsilon_\mathfrak{p})$ if and only if $\mathrm{Tr}(\mathfrak{p}) = \pm 2 \mod I$ and $\mathrm{frob}(\mathfrak{p})$ is not diagonal.
- $\mathrm{frob}(\mathfrak{p})$ is I-hyperbolic if and only if I is split in $K(\varepsilon_\mathfrak{p})$ if and only if $\mathrm{Tr}(\mathfrak{p}) \neq \pm 2$ and $\mathrm{Tr}(\mathfrak{p})^2 - 4$ is a square $\mod I$.
- $\mathrm{frob}(\mathfrak{p})$ is I-elliptic if and only if I is inert in $K(\varepsilon_\mathfrak{p})$ if and only if $\mathrm{Tr}(\mathfrak{p})^2 - 4$ is not a square mod I.

Furthermore, if $\mathfrak{p}$ is I-hyperbolic or I-elliptic, then $\mathrm{frob}(()\mathfrak{p})$ is completely determined by $\mathrm{Tr}(\mathfrak{p})$:

- If $\mathrm{Tr}(\mathfrak{p})^2 - 4$ is a nonzero square mod I, then $\mathrm{frob}(\mathfrak{p}) = [c_h(x_\mathfrak{p})]_{G_Q}$ where
$$x_p = \frac{-\mathrm{Tr}(\mathfrak{p}) + \sqrt{\mathrm{Tr}(\mathfrak{p})^2 - 4}}{2} \mod I$$
- If $\mathrm{Tr}(\mathfrak{p})^2 - 4$ is not a square mod I, then $\mathrm{frob}(\mathfrak{p}) = [c_e(x_\mathfrak{p}, y_\mathfrak{p})]_{G_Q}$ where $x_\mathfrak{p} = \mathrm{Tr}\,\mathfrak{p}/2$ and $y_\mathfrak{p}$ is either of the two solutions to $y^2 = \frac{\mathrm{Tr}(x_\mathfrak{p})^2 - 4}{4D} \mod I$.

We are now ready to return to the sum in 1. Since the irreducible characters are linearly independent as class functions on G_Q, there is a $g \in G_Q$ such that $\sum_\sigma a_\sigma \chi_\sigma(g) \neq 0$. Suppose g is I-hyperbolic, conjugate to some $c_h(x)$ for $x \in F_q^\times \setminus \pm 1$. By Chebotarev's density theorem, there is a set of primitive hyperbolic conjugacy classes $\mathfrak{p}$ of positive natural density such that $\mathrm{frob}(p) = [c_h(x)]_{G_Q}$. Pick one such $\mathfrak{p}$, and evaluate the sum 1 with $l = \ell(\mathfrak{p})$ and $m = 1$. Then, by 1 we have

$$m(l) \sum_\sigma a_\sigma \chi_\sigma(c_h(x_\mathfrak{p})) = 0$$

where $m(l)$ is the multiplicity of the length l. Thus, $\sum_\sigma a_\sigma \chi_\sigma$ is supported away from the I-hyperbolic conjugacy classes.

An identical argument shows that $\sum_\sigma a_\sigma \chi_\sigma$ is supported away from I-elliptic classes.

Now suppose that $\sum_\sigma a_\sigma \chi_\sigma$ is supported on the conjugacy class $[c_p^+(1)]$. By Chebotarev's density theorem, there is a set of primitive hyperbolic conjugacy classes $\mathfrak{p}$ of positive natural density such that $\mathrm{frob}(\mathfrak{p}) = [c_p(1)]_{G_Q}$. Pick one such $\mathfrak{p}$ of length l_o. The set of primitive hyperbolic conjugacy classes of this length l_o consists of I-central classes $C(l_o)$ and I-parabolic classes $P(l_o)$. We further partition the parabolic classes according to their image under the frobenius map, $P(l_o) = P_1(l_o) \cup P_D(l_o)$, where $P_1(l_o)$ consists of those which map to $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $P_D(l_o)$ are those which map to $\left[\begin{smallmatrix} 1 & D \\ 0 & 1 \end{smallmatrix}\right]$. Now evaluating 1 at $l = l_o$ and $m = 1$, we have

$$\sum_{\mathfrak{p}:\ell(\mathfrak{p})=l_o} \sum_{\sigma} a_\sigma \chi_\sigma(\mathrm{frob}(\mathfrak{p})) = \sum_{\sigma} a_\sigma \left(|C(l_o)| \chi_\sigma(1) + |P_1(l_o)| \chi_\sigma([\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}]) + |P_D(l_o)| \chi_\sigma(\left[\begin{smallmatrix} 1 & D \\ 0 & 1 \end{smallmatrix}\right]) \right) = 0$$

The first summand on the right hand side vanishes, by evaluating 1 at any length and with $m = |G_Q|$. Using the notation for characters of $\mathrm{PSL}_2(q)$ in [2], what remains is

$$\sum_{\sigma} a_\sigma \left(A\chi_\sigma([\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}]) + B\chi_\sigma(\left[\begin{smallmatrix} 1 & D \\ 0 & 1 \end{smallmatrix}\right]) \right) = (A + B)(a_1 - \sum_{\varphi} a_{X_\varphi} + \sum_{\alpha} a_{W_\alpha}) \tag{2}$$

$$+ A \left(a_{W^+} S_q^+(1) + a_{W^-} S_q^-(1) \right) \tag{3}$$

$$+ B \left(a_{W^+} S_q^+(D) + a_{W^-} S_q^-(D) \right) = 0. \tag{4}$$

where $S_q^\pm(x) = \frac{1}{2}(1 \pm \tau(x)\sqrt{q})$ if $q = 1 \mod 4$ and $S_q^\pm(x) = \frac{1}{2}(-1 \pm \tau(x)\sqrt{-q})$ if $q = 3 \mod 4$, and τ is the nontrivial quadratic character of $F_q^\times$. Thus

$$A \left(a_{W^+} S_q^+(1) + a_{W^-} S_q^-(1) \right) = \frac{A}{2}(a_{W^+} + a_{W^-}) + \frac{A\sqrt{\pm q}}{2}(a_{W^+} - a_{W^-})$$
$$B \left(a_{W^+} S_q^+(D) + a_{W^-} S_q^-(D) \right) = \frac{B}{2}(a_{W^+} + a_{W^-}) - \frac{B\sqrt{\pm q}}{2}(a_{W^+} - a_{W^-})$$

Now, since the a_σ are rational, the sum in 2 is zero only if either $a_{W^+} = a_{W^-}$ or $A = B$.

Suppose $A = B$.

By the chebotarev's density theorem of [3], for each $x \in F_q^\times \setminus \pm 1$, there are infinitely many primes $\mathfrak{p}$ with $\mathrm{frob}(\mathfrak{p}) = c_h(x)$. Evaluating the above sum at each such, then citing linear independence of characters, we find $a_\sigma \neq 0$ if and only if χ_σ is supported away from the I-hyperbolic conjugacy classes.

Similarly, if $l = 2\cosh^{-1}(\mathrm{Tr}(\gamma)/2)$ where $t^2 - 4$ is not a square mod I, then the inner sum in 1 consists entirely of primes $\mathfrak{p}$ with $\mathrm{frob}(\mathfrak{p}) = c_e(x_t) := \left[\begin{smallmatrix} x_t & y_t \\ Dy_t & x_t \end{smallmatrix}\right]$. Applying chebotarev for each $x_t + y_t\delta$ with $x_t^2 - Dy_t^2$, evaluating 1, and citing linear independence of characters, we find that $a_\sigma \neq 0$ if and only if χ_σ is supported away from the I-elliptic conjugacy classes.

To conclude the proof, one looks at a character table for $\mathrm{SL}_2(q)$ and observes that any irreducible character which is supported away from the I-hyperbolic classes has support containing the I-elliptic classes, and vice versa. Thus, a_σ is zero for all $\sigma \in \hat{G}$, a contradiction. We conclude that $\{L_\Gamma(s, \sigma) : \sigma \in \hat{G}\}$ is a multiplicatively independent set. □

Thus, for $\ell \in \mathcal{L}_{I-h}(\Gamma)$ or $\mathcal{L}_{I-h}$ we may unambiguously define $\mathrm{frob}(\ell)$ as the frobenius class attained by all $\mathfrak{p}$ with length ℓ. We note that the Chebotarev density theorem of [3] implies that for any conjugacy class C of G, there is a set of positive natural density in the set of primes $\mathfrak{p}$ of Γ such that $\mathrm{frob}(p) = C$. Thus, for any I-hyperbolic or I-elliptic conjugacy class, C the set of ℓ in the primitive length spectrum with $\mathrm{frob}(\ell) = C$ has positive natural density.

We are now prepared to prove multiplicative independence for the Artin-Selberg L-functions for congruence covers of Γ.

References

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