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A q-Analogue of Mahler Expansions, I

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We examine a q-analogue of Mahler expansions for continuous functions in p-adic analysis, replacing binomial coefficient polynomials $\binom{x}{n}$ with a q-analogue $\binom{x}{n}_q$ for a p-adic variable q with $|q-1|_p<1$. Mahler expansions are recovered at $q=1$ and we consider the p-adic q-Gamma function $\Gamma_{p,q}$ of Koblitz relative to its q-Mahler expansion. © 2000 Academic Press

Key Words: q-analogue; p-adic functions; Mahler expansions.

1. INTRODUCTION

Let $\mathbf{Z}_p$ be the p-adic integers, $\mathbf{Q}_p$ the p-adic rationals, and K a field extension of $\mathbf{Q}_p$ which is complete with respect to a nonarchimedean absolute value $|\cdot|_p$, normalized by $|p|_p=1/p$.

About 40 years ago, Mahler introduced in [18] an expansion for continuous functions from $\mathbf{Z}_p$ to K using special polynomials. Specifically, he observed that the nth binomial coefficient polynomial

$$\binom{x}{n}=\frac{x(x-1)\cdot\ \cdots\ \cdot(x-n+1)}{n!}$$

sends $\mathbf{Z}_p$ to $\mathbf{Z}_p$ (it sends $\mathbf{Z}$ to $\mathbf{Z}\subset\mathbf{Z}_p$, then use continuity), so $|\binom{x}{n}|_p\leqslant 1$ for all $x\in\mathbf{Z}_p$. Therefore for any sequence $c_n\in K$ with $\lim_{n\to\infty} c_n=0$, the series

$$f(x)=\sum_{n\geqslant 0} c_n\binom{x}{n}$$

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defines a continuous function $\mathbf{Z}_p \to K$. Mahler proved every continuous function from $\mathbf{Z}_p$ to K arises uniquely in this way, with

$$c_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(k), \qquad \sup_{x \in \mathbf{Z}_p} |f(x)|_p = \max_{n \geqslant 0} |c_n|_p .$$

The c_n are called the *Mahler coefficients* of f and the series $\sum c_n \binom{x}{n}$ is called the *Mahler expansion* of f.

In this paper a q-analogue of the Mahler expansion is studied, where q is a p-adic variable.

To set up the framework for our ideas, first we recall the philosophy of q-analogues over $\mathbf{R}$ and $\mathbf{C}$. For a complex number q other than 1, define the q-analogue of a positive integer n to be

$$(n)_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}.$$

As $q \to 1$, $(n)_q \to n$, and this is the hallmark of a q-analogue: the limit as $q \to 1$ recovers the classical object. There are q-analogues of most functions in classical analysis [9]. For example, the geometric series

$$(1-z)^{-a} = \sum_{n \geqslant 0} \frac{a(a+1) \cdot \cdots \cdot (a+n-1)}{n!} z^n$$

for $|z| < 1$ and $a \in \mathbf{C}$ has the q-analogue

$$1 + \frac{q^a - 1}{q - 1} z + \frac{(q^a - 1)(q^{a+1} - 1)}{(q-1)(q^2 - 1)} z^2 + \cdots = \prod_{n \geqslant 0} \frac{1 - q^{a+n} z}{1 - q^n z},$$

where the infinite product converges for $|q| < 1$. The analytic treatment of q-series in $\mathbf{C}$ usually assumes $|q| < 1$ or $0 < q < 1$. However, many results make sense in a formal way, allowing q to be viewed as an indeterminate. The study of q-analogues has connections with a number of areas of mathematics, such as partitions, modular functions, and quantum groups.

The Mahler expansion in p-adic analysis uses binomial coefficient polynomials $\binom{x}{n}$, $x \in \mathbf{Z}_p$. For $q \in K$ with $|q - 1|_p < 1$ (the p-adic substitute for the condition $|q| < 1$ in $\mathbf{C}$), we will use q-analogues $\binom{x}{n}_q$. These are exponential functions of $x \in \mathbf{Z}_p$ if q is not a root of unity, and are locally polynomials in x if q is a root of unity. In particular, $\binom{x}{n}_1 = \binom{x}{n}$. The q-analogue of Mahler's theorem is

THEOREM. *For a complete extension field* $K/\mathbf{Q}_p$ *and* $q \in K$ *with* $|q-1|_p < 1$, *every continuous function* $f\colon \mathbf{Z}_p \to K$ *has a unique expansion*

$$f(x) = \sum_{n \geqslant 0} c_{n,q} \binom{x}{n}_q,$$

where $c_{n,q} \in K$ *and* $c_{n,q} \to 0$ *as* $n \to \infty$. *Furthermore,*

$$c_{n,q} = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{(n-k)(n-k-1)/2} f(k),$$

$$\sup_{x \in \mathbf{Z}_p} |f(x)|_p = \max_{n \geqslant 0} |c_{n,q}|_p.$$

About 20 years ago, van Hamme [23] proved the p-adic analogue of a result of F. H. Jackson on real q-series, thereby giving explicit polynomial approximations for continuous functions on certain compact-open subsets V_q of $\mathbf{Z}_p$. The subset and the approximating polynomials depend on a parameter $q \in \mathbf{Z}_p^\times$ which can not be a root of unity. A. Verdoodt has continued this work. The point of view of van Hamme and Verdoodt is largely compatible with the one presented in Section 3 after a change of variables, although our approach, unlike theirs, permits a passage to the limit as $q \to 1$ to recover Mahler's theorem at $q = 1$.

The structure of the paper is as follows. In Section 2 we review some properties of q-analogues, where q will be treated mostly as an indeterminate. In Section 3 we let q be a p-adic variable and discuss the q-analogue of Mahler's theorem. Four proofs are given, having individual advantages. Because this paper may be of interest to people who work in p-adic analysis but not in q-series, and vice versa, we give extra details in Sections 2 and 3 for results that are well known to those familiar with one of these areas but not the other. In Section 4 we discuss properties of q-Mahler expansions. One aspect which is not apparent in the classical case $q = 1$ is the role of the p-adic logarithm in classifying differentiability in terms of q-Mahler expansions. In Section 5 we discuss the q-Mahler expansion of the p-adic q-Gamma function of Koblitz.

Here is a brief list of notation.

$\mathbf{N}$ is the set of natural numbers $\{0, 1, 2, ...\}$.

$\mathbf{Z}_p$ is the ring of p-adic integers.

$\mathbf{Q}_p$ is the field of p-adic numbers.

ζ denotes a root of unity.

Φ_n is the nth cyclotomic polynomial.

For a function f on $\mathbf{Z}_p$, $(E^y f)(x) = f(x+y)$ is the shift by y. In particular, $(Ef)(x) = f(x+1)$.

Let $(K, |\cdot|)$ be a complete extension field of $\mathbf{Q}_p$ with $|p| = 1/p$. The set of continuous functions from $\mathbf{Z}_p$ to K will be denoted $C(\mathbf{Z}_p, K)$ and topologized by the sup-norm $|f|_{\sup} := \sup_{x \in \mathbf{Z}_p} |f(x)|$. (We only consider p-adic absolute values, so we write $|\cdot|$ rather than $|\cdot|_p$.)

A function $\mathbf{Z}_p \to K$ is called *analytic* if it is given by a single power series that converges on $\mathbf{Z}_p$. It is called *locally analytic* if it is locally expressible by a power series around each point of $\mathbf{Z}_p$.

2. A REVIEW OF q-FORMALISM

Here we recall the features of q-analogues that are needed for our purposes, generally insofar as q can be treated as an indeterminate. Some remarks will be made about specializing q, especially at roots of unity. The focus will be on properties of q-binomial coefficients and q-difference operators.

For an integer n and an indeterminate q, the q-analogue of n is

$$(n)_q := \frac{q^n - 1}{q - 1}.$$

For example, $(0)_q = 0$, $(1)_q = 1$, $(2)_q = 1 + q$, $(-1)_q = -1/q$.

When $n \geqslant 1$, $(n)_q = 1 + q + \cdots + q^{n-1}$ is a polynomial in $\mathbf{Z}[q]$.

For any integers m and n,

$$(-n)_q = -\frac{1}{q^n}(n)_q, \qquad (n)_{1/q} = \frac{1}{q^{n-1}}(n)_q, \qquad (mn)_q = (m)_q (n)_{q^m}. \tag{2.1}$$

Specializing $q = 1$, $(n)_q$ becomes n.

The q-factorials are

$$(n)_q! := \begin{cases} 1, & n = 0; \\ (n)_q (n-1)_q \cdot \cdots \cdot (1)_q, & n \geqslant 1. \end{cases}$$

For example, $(1)_q! = 1$, $(2)_q! = 1 + q$, $(3)_q! = 1 + 2q + 2q^2 + q^3$, and

$$(n)_{1/q}! = \frac{1}{q^{n(n-1)/2}}(n)_q!. \tag{2.2}$$

The q-binomial coefficient for nonnegative integers m and n with $m \geqslant n$ is

$$\binom{m}{n}_q := \frac{(m)_q!}{(n)_q!\,(m-n)_q!} = \frac{(m)_q(m-1)_q \cdots (m-n+1)_q}{(n)_q!} = \frac{(q^m-1)(q^{m-1}-1)\cdots(q^{m-n+1}-1)}{(q^n-1)(q^{n-1}-1)\cdots(q-1)}.$$

We use the second or third expression to extend the definition of $\binom{m}{n}_q$ to any integer m. These functions go back to Gauss [10, p. 16], so they are also called Gaussian coefficients.

The first few q-binomial coefficients are

$$\binom{m}{0}_q = 1, \qquad \binom{m}{1}_q = (m)_q = \frac{q^m-1}{q-1}, \qquad \binom{m}{2}_q = \frac{(q^m-1)(q^{m-1}-1)}{(q^2-1)(q-1)}.$$

For $m \geqslant n$, $\binom{m}{n}_q = \binom{m}{m-n}_q$, and (as a rational function in q) $\binom{m}{n}_q = 0$ precisely when $0 \leqslant m < n$. The q-binomial coefficient may vanish in other cases numerically, e.g., $\binom{4}{2}_q = (1+q^2)(1+q+q^2)$, so $\binom{4}{2}_i = 0$.

The following result is essentially due to Gauss [10, p. 17].

THEOREM 2.1. *For fixed integers $m \geqslant n \geqslant 0$, $\binom{m}{n}_q \in \mathbf{Z}[q]$ with degree $n(m-n)$.*

Proof. The degree follows from the definition, once we know $\binom{m}{n}_q$ is a polynomial in q.

We give Gauss' proof that $\binom{m}{n}_q \in \mathbf{Z}[q]$ and then an alternate proof that seems to be new.

The Pascal's triangle recursion for binomial coefficients generalizes (for all m in $\mathbf{Z}$) to

$$\binom{m}{n}_q = \binom{m-1}{n-1}_q + q^n \binom{m-1}{n}_q = q^{m-n}\binom{m-1}{n-1}_q + \binom{m-1}{n}_q \tag{2.3}$$

(when $m \geqslant n$, replace n by $m-n$ to obtain either recursion from the other), and iterating the second recursion gives

$$\binom{m+n+1}{n+1}_q = q^m \binom{m+n}{n}_q + \binom{m+n}{n+1}_q = \sum_{k=0}^{m} q^k \binom{k+n}{n}_q.$$

So $\binom{m}{n}_q \in \mathbf{Z}[q]$ by induction on n (and actually all the coefficients are non-negative).

As an alternate proof, the irreducible factors of the rational function $\binom{m}{n}_q$ are cyclotomic polynomials. The multiplicity of the jth cyclotomic polynomial $\Phi_j(q)$ as a factor of $(n)_q!$ is $[n/j]$, so its multiplicity as a factor of $\binom{m}{n}_q$ is $[m/j]-[n/j]-[(m-n)/j]$, which is 0 or 1. This shows for $m \geqslant n$ not only that $\binom{m}{n}_q$ is a polynomial in q, but that its irreducible factors are all simple factors and $\Phi_j(q)$ is a factor precisely when the units' digit of m in base j is less than the units' digit of n in base j. I thank Ira Gessel for a simplification to the original form of this alternate proof. ▮

Further identities for all $m \in \mathbf{Z}$ (and $k \geqslant j \geqslant 0$) are

$$\binom{m}{n}_q = \frac{(m)_q}{(n)_q}\binom{m-1}{n-1}_q, \qquad \binom{m}{n}_{1/q} = \frac{1}{q^{n(m-n)}}\binom{m}{n}_q,$$
$$\binom{m}{k}_q\binom{k}{j}_q = \binom{m}{j}_q\binom{m-j}{k-j}_q, \tag{2.4}$$

$$\binom{-m}{n}_q = (-1)^n q^{-n(n-1)/2-mn}\binom{m+n-1}{n}_q$$
$$= (-1)^n q^{-n(n+1)/2}\binom{m+n-1}{n}_{1/q}. \tag{2.5}$$

For example, $\binom{-1}{n}_q = (-1)^n q^{-n(n+1)/2}$. By (2.5), for $m>0$ $\binom{-m}{n}_q$ is a polynomial in $1/q$ with degree $n(n-1)/2+mn$ whose coefficients are nonzero integers with sign $(-1)^n$.

The next result is a q-analogue of the binomial theorem, the *q-binomial theorem*. It goes back to Cauchy [4, p. 46, Eq. 18].

THEOREM 2.2. *For $m \geqslant 1$,*

$$(1+T)(1+qT)\cdots(1+q^{m-1}T) = \prod_{i=0}^{m-1}(1+q^iT) = \sum_{k=0}^{m}\binom{m}{k}_q q^{k(k-1)/2}T^k.$$

Equivalently, for commuting variables X and Y,

$$(X+Y)(X+qY)\cdots(X+q^{m-1}Y) = \prod_{i=0}^{m-1}(X+q^iY)$$
$$= \sum_{k=0}^{m}\binom{m}{k}_q q^{k(k-1)/2}X^{m-k}Y^k.$$

Proof. Following Cauchy [4, p. 51], let $h(T)=\prod_{i=0}^{m-1}(1+q^iT)=\sum_{k=0}^{m} a_kT^k$. Then $(1+T)\,h(qT)=h(T)(1+q^mT)$. Equating coefficients of equal powers of T,

$$a_k=\frac{q^m-q^{k-1}}{q^k-1}\,a_{k-1}=\frac{q^{m-k+1}-1}{q^k-1}\,q^{k-1}a_{k-1},$$

so $a_k=\binom{m}{k}_q q^{k(k-1)/2}$. ∎

In particular,

$$(X-1)(X-q)\cdots(X-q^{m-1})=\sum_{k=0}^{m}\binom{m}{k}_q(-1)^k\,q^{k(k-1)/2}X^{m-k}. \tag{2.6}$$

Actually, the idea of replacing T by qT to express q-products as q-series goes back to Euler [7, Chap. XVI, Sects. 306, 307].

The $q^{k(k-1)/2}$ term that arises in the q-binomial theorem can be removed from explicit appearance. Define the nth q-power of a polynomial $f(T)$ to be $f^{(0;\,q)}=1$ and $f^{(n;\,q)}:=f(T)\,f(qT)\cdots f(q^{n-1}T)$ for $n\geqslant 1$. Then the q-binomial theorem becomes

$$(1+T)^{(m;\,q)}=\sum_{k=0}^{m}\binom{m}{k}_q T^{(k;\,q)}.$$

We can consider q-deformed powers of a polynomial in several variables by singling out one variable, e.g., in two variables

$$f(X,Y)^{(n;\,q)}:=f(X,Y)\,f(X,qY)\cdots f(X,q^{n-1}Y).$$

This will appear later in the case of $(X\pm Y)^{(n;\,q)}$, whose value at $X=x$, $Y=y$ will be written with abuse of notation as $(x\pm y)^{(n;\,q)}$. For example,

$$(x+0)^{(n;\,q)}=x^m, \qquad (0+y)^{(n;\,q)}=q^{n(n-1)/2}y^n, \qquad \binom{m}{k}_q=\frac{(q^m-1)^{(k;\,q)}}{(q^k-1)^{(k;\,q)}}.$$

The q-Vandermonde formula for $\binom{m_1+m_2}{k}_q$ is proven as for ordinary binomial coefficients.

Theorem 2.3. *For* $m_1, m_2\geqslant 0$, $\binom{m_1+m_2}{k}_q=\sum_{j=0}^{k}\binom{m_1}{j}_q\binom{m_2}{k-j}_q q^{j(m_2-(k-j))}$.

Note the asymmetric roles of j and $k-j$ in the exponent of q on the right side.

Proof. Compare the coefficient of T^k on both sides of

$$\prod_{i=0}^{m_1+m_2-1} (1+q^i T) = \prod_{i=0}^{m_2-1} (1+q^i T) \prod_{i=0}^{m_1-1} (1+q^i q^{m_2} T). \quad \blacksquare$$

By a specialization argument, Theorem 2.3 is true for all integers m_1 and m_2, possibly negative.

The following simple fact will be used when we let q vary p-adically.

THEOREM 2.4. *For $m, n \geqslant 0$, $\binom{m}{n}_{q_1} - \binom{m}{n}_{q_2} \in (q_1 - q_2)\,\mathbf{Z}[q_1, q_2]$.*

Proof. For all $i \geqslant 0$, $q_1^i - q_2^i \in (q_1 - q_2)\,\mathbf{Z}[q_1, q_2]$. $\blacksquare$

We now discuss the value of $\binom{m}{n}_q$ for $m \geqslant n$ when q is specialized to various numbers.

When $q = 1$, $\binom{m}{n}_1 = \binom{m}{n}$ counts the number of n element subsets of an m element set. When q is a prime power, $\binom{m}{n}_q$ counts the number of n-dimensional subspaces of an m-dimensional vector space over the field of size q. This suggests the possibility of proving identities for q-binomial coefficients by letting q run through (infinitely many) prime powers and interpreting the identity as a combinatorial statement in linear algebra over finite fields. See [11] for this approach.

We now consider the case when q is specialized to a root of unity. For ζ a root of unity of order b and $n < b$, the value of $\binom{m}{n}_\zeta$ can be computed directly from the definition, since $(n)_\zeta! \neq 0$. The next theorem reduces the evaluation of all $\binom{m}{n}_\zeta$ to the case when $n < b$.

THEOREM 2.5. *Let ζ be a root of unity of order b.*

(i) *For integers k and l, with $l \geqslant 0$, $\binom{bk}{bl}_\zeta = \binom{k}{l}$.*

(ii) *For integers k and l with $l \geqslant 0$ and $0 \leqslant r, s < b$, $\binom{bk+r}{bl+s}_\zeta = \binom{bk}{bl}_\zeta \binom{r}{s}_\zeta = \binom{k}{l}\binom{r}{s}_\zeta$.*

In particular, if $n < b$ and $m_1 \equiv m_2 \bmod b$, then $\binom{m_1}{n}_\zeta = \binom{m_2}{n}_\zeta$.

Proof. (i)

$$\binom{bk}{bl}_q = \prod_{j=0}^{bl-1} \frac{q^{bk-j}-1}{q^{bl-j}-1} = \prod_{\substack{j=0 \\ j \not\equiv 0 \bmod b}}^{bl-1} \frac{q^{bk-j}-1}{q^{bl-j}-1} \cdot \prod_{i=0}^{l-1} \frac{q^{b(k-i)}-1}{q^{b(l-i)}-1}.$$

At $q = \zeta$, the right side becomes $\prod_{i=0}^{l-1} (k-i)/(l-i) = \binom{k}{l}$.

(ii) First we show $\binom{bk+a}{bl}_\zeta = \binom{bk+a-1}{bl}_\zeta$ when a is not divisible by b. Setting $m=bk+a$, $n=bl$, and $q=\zeta$ in the equation $\binom{m}{n}_q = ((m)_q/(m-n)_q)\binom{m-1}{n}_q$, we get what we want. So the theorem is true for $s=0$. For $s \geqslant 1$,

$$\binom{bk+r}{bl+s}_q = \frac{(bk+r)_q\,(bk+r-1)_q \cdots (bk+r-s+1)_q}{(bl+s)_q\,(bl+s-1)_q \cdots (bl+1)_q} \binom{bk+r-s}{bl}_q.$$

None of the terms $(bl+j)_q$ appearing in the denominator vanishes at $q=\zeta$, so we can evaluate and find

$$\binom{bk+r}{bl+s}_\zeta = \frac{(r)_\zeta\,(r-1)_\zeta \cdots (r-s+1)_\zeta}{(s)_\zeta\,(s-1)_\zeta \cdots (1)_\zeta} \binom{bk+r-s}{bl}_\zeta$$
$$= \binom{r}{s}_\zeta \binom{bk+r-s}{bl}_\zeta. \quad \blacksquare$$

COROLLARY 2.6. *Let ζ be a root of unity of order b and $n \in \mathbf{N}$. For m running through a fixed residue class mod b, $\binom{m}{n}_\zeta$ is a polynomial in m.*

Proof. By Theorem 2.5(ii), $\binom{m}{n}_\zeta$ is a polynomial in $[m/b] = (m-r)/b$ and r is fixed. $\blacksquare$

EXAMPLES.

$$\binom{19}{5}_{-1} = \binom{18+1}{4+1}_{-1} = \binom{9}{2}\binom{1}{1}_{-1} = 36,$$

$$\binom{17}{10}_i = \binom{16+1}{8+2}_i = \binom{4}{2}\binom{1}{2}_i = 0,$$

$$\binom{-5}{6}_i = \binom{-8+3}{4+2}_i = \binom{-2}{1}\binom{3}{2}_i = -2i.$$

The periodicity of $\binom{m}{n}_\zeta$ in m mod b, stated at the end of Theorem 2.5, can also be verified by computing $\binom{m+b}{n}_\zeta - \binom{m}{n}_\zeta$ with the q-Vandermonde formula.

Theorem 2.5 (and an extension to q-multinomial coefficients) can be proven by group actions [21].

For a root of unity ζ of order b, that $\binom{b}{n}_\zeta = 0$ for $1 \leqslant n \leqslant b-1$ can be seen without Theorem 2.5, since the numerator of $\binom{b}{n}_q$ vanishes at $q=\zeta$ while the denominator does not, or (using Theorem 2.2) since

$\prod_{j=0}^{b-1}(1+\zeta^j T)=1-(-T)^b$. Stated in terms of the bth cyclotomic polynomial $\Phi_b(q)$, this vanishing becomes

$$\binom{b}{n}_q \equiv 0 \qquad \bmod \Phi_b(q) \tag{2.7}$$

when $1 \leqslant n \leqslant b-1$, which is also clear from the second proof of Theorem 2.1. Specializing (2.7) at $q=1$, we recover the familiar integer congruence $\binom{p^N}{n} \equiv 0 \bmod p$ when $b=p^N$ is a power of a prime p. Since

$$\Phi_{p^N}(q)=\frac{q^{p^N}-1}{q^{p^{N-1}}-1}=(p)_{q^{p^{N-1}}},$$

when $b=p^N$ (2.7) can be written as $\binom{p^N}{n}_q \equiv 0 \bmod (p)_{q^{p^{N-1}}}$.

The q-analogue of the exponential series was introduced by Jackson [13],

$$E_q(X):=\sum_{n \geqslant 0} \frac{X^n}{(n)_q!}.$$

(In the literature, the notation $E_q(X)$ may denote a slightly different series.) Jackson's q-version of $e^{x+y}=e^x e^y$ comes from (2.2) and the q-binomial theorem,

$$\begin{aligned} E_q(X)\, E_{1/q}(Y) &= \sum_{n \geqslant 0} \frac{(X+Y)(X+qY)\cdots(X+q^{n-1}Y)}{(n)_q!} \\ &= \sum_{n \geqslant 0} \frac{(X+Y)^{(n;\, q)}}{(n)_q!}. \end{aligned} \tag{2.8}$$

In particular,

$$E_q(X)^{-1}=E_{1/q}(-X)=\sum_{n \geqslant 0}(-1)^n q^{n(n-1)/2} \frac{X^n}{(n)_q!}. \tag{2.9}$$

We now discuss q-difference operators. Powers Δ^n of the difference operator Δ, where $(\Delta h)(x)=h(x+1)-h(x)$ (here and in the rest of this section, x is an integer variable), play a role in Mahler expansions which will be taken over in the q-analogue by a sequence of operators Δ_q^n first introduced by Jackson [14, p. 256; 15, p. 145].

The powers of Δ behave nicely on binomial coefficients, namely

$$\Delta^m \binom{x}{n} = \begin{cases} \binom{x}{n-m}, & \text{if} \quad m \leqslant n; \\ 0, & \text{if} \quad m > n. \end{cases}$$

The q-analogue of powers of Δ arise naturally by considering differences of q-binomial coefficients.

First, note that in analogy with $\Delta\binom{x}{n} = \binom{x}{n-1}$,

$$\Delta \binom{x}{n}_q = q^{x+1-n} \binom{x}{n-1}_q.$$

Then, guided by the equation $\Delta^2\binom{x}{n} = \Delta\binom{x+1}{n} - \Delta\binom{x}{n} = \binom{x}{n-2}$, we compute

$$\Delta \binom{x+1}{n}_q = q^{x+2-n} \binom{x+1}{n-1}_q,$$

so we're naturally led to calculate not $\Delta\binom{x+1}{n}_q - \Delta\binom{x}{n}_q$ but

$$\begin{aligned} \Delta \binom{x+1}{n}_q - q\Delta \binom{x}{n}_q &= q^{x+2-n} \left(\binom{x+1}{n-1}_q - \binom{x}{n-1}_q \right) \\ &= q^{2(x+2-n)} \binom{x}{n-2}_q. \end{aligned}$$

Let $(Eh)(x) = h(x+1)$ be the shift operator, so we've computed

$$(E-I) \binom{x}{n}_q = q^{x+1-n} \binom{x}{n-1}_q,$$

$$(E-I)(E-q) \binom{x}{n}_q = q^{2(x+2-n)} \binom{x}{n-2}_q.$$

Of course $n \geqslant 1$ and $n \geqslant 2$ for these respective equations.

Experience with q-deformed products as in the q-binomial theorem now makes the following definition natural: $\Delta_q^n := (E-I)^{(n;\, q)} = \Delta^{(n;\, q)}$. In full, this says

$$\Delta_q^n := \begin{cases} I, & n = 0; \\ (E-I)(E-q) \cdots (E-q^{n-1}), & n \geqslant 1, \end{cases}$$

so

$$\Delta_q^m \binom{x}{n}_q = \begin{cases} q^{m(x+m-n)} \binom{x}{n-m}_q, & \text{if} \quad m \leqslant n; \\ 0, & \text{if} \quad m > n. \end{cases} \tag{2.10}$$

In particular, $\Delta_q^m \binom{x}{n}_q |_{x=0} = \delta_{mn}$. The appearance of a function of x on the right side of (2.10), outside the q-binomial coefficient, can be removed by using an alternate q-difference operator,

$$(\mathfrak{D}_q^m f)(x) := q^{-mx} (\Delta_q^m f)(x).$$

Then

$$\mathfrak{D}_q^m \binom{x}{n}_q = \begin{cases} q^{-m(n-m)} \binom{x}{n-m}_q, & \text{if} \quad m \leqslant n; \\ 0, & \text{if} \quad m > n. \end{cases}$$

By (2.6),

$$(\Delta_q^n f)(x) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{k(k-1)/2} f(x+n-k). \tag{2.11}$$

The shift E commutes with multiplication by q, so Δ_q^n and $\Delta_q^{n'}$ commute, but $\Delta_q^n \Delta_q^{n'} \neq \Delta_q^{n+n'}$. To give a formula for $\Delta_q^{n+n'}$ in terms of Δ_q^n and $\Delta_q^{n'}$,

$$\begin{aligned} \Delta_q^{n+n'} &= (E - q^{n+n'-1}) \cdot \quad \cdots \quad \cdot (E - q^{n'}) \, \Delta_q^{n'} \\ &= \sum_{k=0}^{n} \binom{n}{k}_q q^{k(k-1)/2} (-q^{n'})^k \, E^{n-k} \Delta_q^{n'} \end{aligned}$$

by the q-binomial theorem, so

$$\begin{aligned} (\Delta_q^{n+n'} f)(x) &= \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{k(k-1)/2} q^{n'k} (\Delta_q^{n'} f)(x+n-k) \\ &= q^{n'(n+x)} (\Delta_q^n g)(x), \end{aligned}$$

where $g(x) = q^{-n'x}(\Delta_q^{n'} f)(x)$. This can be written more conveniently in terms of the $\mathfrak{D}_q^m$,

$$\mathfrak{D}_q^{n+n'} = q^{nn'} \mathfrak{D}_q^n \mathfrak{D}_q^{n'}. \tag{2.12}$$

For $n \in \mathbf{Z}$, let $\mathscr{U}_n(x) = q^{nx}$ (this depends on q), so $\Delta_q^n = \mathscr{U}_n \mathfrak{D}_q^n$ and $E^k \mathscr{U}_n = q^{kn} \mathscr{U}_n E^k$. When $q = 1$ the need for $\mathscr{U}_n$ is not apparent. The notation $\mathscr{U}_n$ comes from a similar function U_n used by Verdoodt [25]. Her paper will be discussed in Section 4.

The effort to directly relate $\Delta_q^n \Delta_q^{n'}$ with $\Delta_q^{n+n'}$ led to a concise multiplicative relation (2.12) among the $\mathfrak{D}_q$'s rather than among the Δ_q's. We now use (2.12) to give a formula for $\Delta_q^n \Delta_q^{n'}$ as a linear combination of various $\Delta_q^{n''}$, so the q-difference operators are a basis of the algebra they generate (they have no linear relations by (2.10)).

THEOREM 2.6. *For $m, n \geqslant 0$,*

$$\begin{aligned} \Delta_q^m \Delta_q^n &= \sum_{j=0}^{m} \binom{m}{j}_q (q^n - 1)(q^n - q) \cdots (q^n - q^{m-j-1})\, \Delta_q^{n+j} \\ &= \sum_{j=0}^{m} \binom{m}{j}_q (q^n - 1)^{(m-j;\, q)}\, \Delta_q^{n+j} \\ &= \sum_{i+j=m+n} \binom{m}{i}_q \binom{n}{i}_q (q^i - 1)^{(i;\, q)}\, \Delta_q^j. \end{aligned}$$

Proof. By the q-binomial theorem,

$$\Delta_q^m \Delta_q^n = \sum_{k=0}^{m} \binom{m}{k}_q (-1)^{m-k} q^{(m-k)(m-k-1)/2} E^k \Delta_q^n.$$

To get a formula for $E^k \Delta_q^n$, we use the following identity: for all $k \geqslant 0$,

$$a^k = \sum_{i=0}^{k} \binom{k}{i}_q (a-1)(a-q) \cdots (a - q^{i-1}) = \sum_{i=0}^{k} \binom{k}{i}_q (a-1)^{(i;\, q)}.$$

This is dual to (2.6), or arises naturally from consideration of q-Mahler expansions in Section 3 (i.e., from the q-difference calculus), so we won't stop to motivate it here. Setting $a = E$,

$$E^k = \sum_{i=0}^{k} \binom{k}{i}_q \Delta_q^i. \tag{2.13}$$

Thus

$$\begin{aligned} E^k \Delta_q^n &= E^k \mathscr{U}_n \mathfrak{D}_q^n \\ &= q^{kn} \mathscr{U}_n E^k \mathfrak{D}_q^n \\ &= q^{kn} \mathscr{U}_n \sum_{i=0}^{k} \binom{k}{i}_q \mathscr{U}_i \mathfrak{D}_q^i \mathfrak{D}_q^n \qquad \text{by (2.13)} \\ &= \sum_{i=0}^{k} \binom{k}{i}_q q^{n(k-i)} \mathscr{U}_{n+i} \mathfrak{D}_q^{n+i} \qquad \text{by (2.12)} \\ &= \sum_{i=0}^{k} \binom{k}{i}_q q^{n(k-i)} \Delta_q^{n+i}, \end{aligned}$$

so

$$\begin{aligned} \Delta_q^m \Delta_q^n &= \sum_{k=0}^{m} \sum_{i=0}^{k} (-1)^{m-k} q^{(m-k)(m-k-1)/2} q^{n(k-i)} \binom{m}{k}_q \binom{k}{i}_q \Delta_q^{n+i} \\ &= \sum_{i=0}^{m} \sum_{k=0}^{m-i} (-1)^{m-i-k} q^{(m-i-k)(m-i-k-1)/2} q^{nk} \binom{m-i}{k}_q \binom{m}{i}_q \Delta_q^{n+i} \\ &= \sum_{i=0}^{m} \binom{m}{i}_q (q^n-1)^{(m-i;\, q)} \Delta_q^{n+i} \qquad \text{by (2.6)} \\ &= \sum_{i=0}^{m} \binom{m}{i}_q \binom{n}{i}_q (q^i-1)^{(i;\, q)} \Delta_q^{m+n-i}. \quad \blacksquare \end{aligned}$$

EXAMPLE. $\Delta_q^2 \Delta_q^n = (q^n-1)(q^n-q)\, \Delta_q^n + (q^n-1)(q+1)\, \Delta_q^{n+1} + \Delta_q^{n+2}$.

The case $m=1$ of Theorem 2.6 is essentially the recursive definition $\Delta_q^{n+1} = (E-q^n)\, \Delta_q^n$.

Once the formula in Theorem 2.6 is found, it can also be proven by induction on m, without using noncommuting operators E^k and $\mathscr{U}_n$, as the polynomial identity

$$\begin{aligned} (X-1)^{(m;\, q)} (X-1)^{(n;\, q)} &= \sum_{i=0}^{m} \binom{m}{i}_q (q^n-1)^{(m-i;\, q)} (X-1)^{(n+i;\, q)} \\ &= \sum_{i=0}^{m} \binom{m}{i}_q (q^n-1)^{(m-i;\, q)} (X-1)^{(n;\, q)} (X-q^n)^{(i;\, q)}. \end{aligned}$$

Dividing by $(X-1)^{(n;\, q)}$, we get an identity which is a special case of the generalized q-binomial theorem [11, p. 252].

The q-analogue of the formula

$$\Delta^n(fg) = \sum_{k=0}^{n} \binom{n}{k} (\Delta^k f)(\Delta^{n-k} E^k g)$$

is

$$\Delta_q^n(fg) = \sum_{k=0}^{n} \binom{n}{k}_q (\Delta_q^k f)(\Delta_q^{n-k} E^k g). \tag{2.14}$$

In the inductive verification of this, use (for $r \leqslant n$)

$$(E - q^n)(FG) = (E - q^r)\, F \cdot EG + q^r F \cdot (E - q^{n-r})\, G$$

with $F = \Delta_q^k f$, $G = \Delta_q^{n-k} E^k g$, and $r = k$.

3. p-ADIC FEATURES OF q-FORMALISM

In Section 2, the emphasis was on q as an indeterminate. Here it will be on q as a p-adic variable, i.e., as an element of a complete valued field K containing $\mathbf{Q}_p$. (We do not assume $q \in \mathbf{Q}_p$.) As we will have no use for the archimedean absolute value function, the absolute value on K will be denoted simply as $|\cdot|$, and ord is the corresponding additive valuation: $|z| = (1/p)^{\operatorname{ord}(z)}$. The valuation ring $\{z \in K : |z| \leqslant 1\}$ will be denoted $\mathcal{O}_K$, with maximal ideal $\mathfrak{m}_K$. We normalize the absolute value so $|p| = 1/p$.

For the benefit of readers outside of number theory, we recall some facts about power functions and roots of unity in p-adic fields.

Lemma 3.1. (i) *The roots of unity in K which reduce to* 1 *in the residue field $\mathcal{O}_K/\mathfrak{m}_K$ are exactly the pth power roots of unity in K.*

(ii) *If ζ is a root of unity of order $p^N > 1$, then*

$$|\zeta - 1| = (1/p)^{1/p^{N-1}(p-1)} \geqslant (1/p)^{1/(p-1)}.$$

The roots of unity in K are a discrete set.

(iii) *For $q \in K$, the sequence $\{1, q, q^2, q^3, ...\}$ can be extended to a continuous function q^x for $x \in \mathbf{Z}_p$ if and only if $|q-1| < 1$, in which case*

$$q^x = \sum_{n \geqslant 0} (q-1)^n \binom{x}{n}, \qquad |q^x - 1| \leqslant |q-1| < 1.$$

(iv) *If $|q-1| < 1$, then $q^x = 1$ for $x \neq 0$ if and only if q is a root of unity of order p^N and $x \in p^N \mathbf{Z}_p$.*

Proof. (i) The residue field $\mathcal{O}_K/\mathfrak{m}_K$ has characteristic p. Since $X^a - 1$ has distinct roots in characteristic p when a is prime to p, a root of unity

ζ in K of order ap^b with $a>1$ and $(a, p)=1$ has $\zeta^{p^b} \not\equiv 1 \bmod \mathfrak{m}_K$, so $\zeta \not\equiv 1 \bmod \mathfrak{m}_K$. Since the only pth power root of unity in characteristic p is 1, if $\zeta^{p^N}=1$ in K, then in the residue field of K we have $\zeta^{p^N} \equiv 1 \bmod \mathfrak{m}_K$, so $\zeta \equiv 1 \bmod \mathfrak{m}_K$.

(ii) We have

$$\prod_{\substack{i=1 \\ (p, i)=1}}^{p^N} (1-\zeta^i) = \Phi_{p^N}(1) = p,$$

so

$$p = (1-\zeta)^{p^{N-1}(p-1)} \prod_{\substack{i=1 \\ (p, i)=1}}^{p^N} \frac{1-\zeta^i}{1-\zeta},$$

and for i prime to p, the ratio $(1-\zeta^i)/(1-\zeta) = 1+\zeta+\cdots+\zeta^{i-1}$ is congruent in the residue field of K to $i \not\equiv 0 \bmod \mathfrak{m}_K$, so this ratio has absolute value 1, hence $1-\zeta$ has the indicated size.

For two distinct roots of unity ζ and ζ' in K, either $\zeta \not\equiv \zeta' \bmod \mathfrak{m}_K$, so $|\zeta-\zeta'|=1$, or $\zeta/\zeta' \equiv 1 \bmod \mathfrak{m}_K$, and then $|\zeta-\zeta'| = |\zeta/\zeta'-1| \geqslant (1/p)^{1/(p-1)}$, so the roots of unity in K are a (bounded) discrete set.

(iii) For "if," we have for any $m \in \mathbf{N}$ that

$$q^m = (1+q-1)^m = \sum_{n=0}^{m} (q-1)^n \binom{m}{n}.$$

Since $(q-1)^n \to 0$, the continuous function

$$q^x = \sum_{n \geqslant 0} (q-1)^n \binom{x}{n}$$

on $\mathbf{Z}_p$ is the p-adic interpolation of $\{q^m\}_{m \geqslant 0}$. For "only if," $q^{p^N} \to q^0 = 1$ as $N \to \infty$, so $|q|=1$ and as in (i) we conclude $|q-1|<1$.

(iv) Let $x = p^n u$ with u a unit in $\mathbf{Z}_p$. Then $q^{p^n u}=1$ if and only if $q^{p^n}=1$, by taking the $(1/u)$th power. ∎

Applying (iii) to q-analogues, $(m)_q = (q^m-1)/(q-1)$ for $m \in \mathbf{Z}$ extends to a continuous function $(x)_q$ for $x \in \mathbf{Z}_p$ if and only if $|q-1|<1$, in which case the extension to $\mathbf{Z}_p$ is

$$(x)_q = \begin{cases} \dfrac{q^x-1}{q-1}, & \text{if } q \neq 1; \\ x, & \text{if } q=1, \end{cases}$$

and by (iii), $(x)_q \equiv x \bmod \mathfrak{m}_K$. In particular, if $x \in \mathbf{Z}_p^\times$, then $(x)_q \in \mathcal{O}_K^\times$.

For $q \neq 1$, $(x)_q$ is a nonvanishing function unless, by (iv), q is a nontrivial root of unity of order p^N, in which case $(x)_q = (j)_q$ where $x \equiv j \bmod p^N$ and $0 \leqslant j \leqslant p^N - 1$.

We now define the q-analogue of binomial coefficient functions.

For $|q-1| < 1$, $\binom{m}{n}_q$ has a continuous extension from $m \in \mathbf{Z}$ to $x \in \mathbf{Z}_p$, given by

$$\binom{x}{n}_q = \frac{(x)_q\,(x-1)_q \cdots (x-n+1)_q}{(n)_q!}$$
$$= \frac{(q^x-1)(q^{x-1}-1)\cdots(q^{x-n+1}-1)}{(q^n-1)(q^{n-1}-1)\cdots(q-1)},$$

provided $(n)_q! \neq 0$, i.e., q is not a nontrivial pth power root of unity of order $\leqslant n$.

If $|q-1| < 1$ and q is a root of unity of order p^N, Corollary 2.6 implies $\binom{x}{n}_q$ is a polynomial function of x on cosets of $p^N \mathbf{Z}_p$. For $x = p^N y + r$ and $n = p^N l + s$ where $0 \leqslant r, s < p^N$, Theorem 2.5(ii) extends by continuity to

$$\binom{x}{n}_q = \binom{y}{l}\binom{r}{s}_q. \tag{3.1}$$

For example, if $p = 2$, then

$$\binom{x}{2l}_{-1} = \begin{cases} \binom{x/2}{l}, & \text{if } \; x \equiv 0 \bmod 2; \\ \binom{(x-1)/2}{l}, & \text{if } \; x \equiv 1 \bmod 2; \end{cases}$$

$$\binom{x}{2l+1}_{-1} = \begin{cases} 0, & \text{if } \; x \equiv 0 \bmod 2; \\ \binom{(x-1)/2}{l}, & \text{if } \; x \equiv 1 \bmod 2. \end{cases}$$

So $\binom{x}{n}_q$ is an exponential function of x (a polynomial in q^x) if q is not a root of unity and is locally a polynomial in x if q is a root of unity.

By Theorem 2.1, $|\binom{x}{n}_q| \leqslant 1$ for all $x \in \mathbf{Z}_p$, with equality if $x = n$.

The difference operators Δ_q^n and $\mathfrak{D}_q^n$ make sense on functions of a p-adic integer variable x, and Eqs. (2.10) and (2.11) remain true when x is any p-adic integer.

By continuity, Theorems 2.3 and 2.4 become

THEOREM 3.1. *If* $|q-1|<1$, *then for all* $x, y \in \mathbf{Z}_p$, $\binom{x+y}{k}_q = \sum_{j=0}^{k} \binom{x}{j}_q \binom{y}{k-j}_q q^{j(y-(k-j))}$.

THEOREM 3.2. *If* $x \in \mathbf{Z}_p$ *and* $|q_1-1|<1$, $|q_2-1|<1$, *then* $|\binom{x}{n}_{q_1} - \binom{x}{n}_{q_2}| \leqslant |q_1-q_2|$.

So $\binom{x}{n}_q = \lim_{q' \to q} \binom{x}{n}_{q'}$. In particular, formulas involving q-binomial coefficients when q is a root of unity can be computed first at non-roots of unity and then pass to a limit.

For example, let $1 \leqslant k \leqslant p^r$ with $k = p^j k'$ and k' prime to p. For $|q-1|<1$ with q not a root of unity,

$$\binom{p^r}{k}_q = \frac{(p^r)_q}{(k)_q} \binom{p^r-1}{k-1}_q = (p^{r-j})_{q^{p^j}} \frac{1}{(k')_{q^{p^j}}} \binom{p^r-1}{k-1}_q.$$

In $\mathcal{O}_K/\mathfrak{m}_K$, $\binom{p^r-1}{k-1}_q \equiv \binom{p^r-1}{k-1} \equiv (-1)^{k-1}$ and $(k')_{q^{p^j}} \equiv k' \not\equiv 0$, so

$$\left|\binom{p^r}{k}_q\right| = |(p^{r-j})_{q^{p^j}}|. \tag{3.2}$$

By continuity in q, (3.2) is also true when q is a root of unity. Alternatively, (3.1) could be used instead for a direct calculation when q is a root of unity.

We now discuss the q-analogue of Mahler expansions.

THEOREM 3.3 (q-Mahler Theorem). *For* $q \in K$ *with* $|q-1|<1$, *every continuous function* $f\colon \mathbf{Z}_p \to K$ *has a unique representation in the form*

$$f(x) = \sum_{n \geqslant 0} c_{n,q} \binom{x}{n}_q,$$

where $c_{n,q} \in K$ *and* $\lim_{n\to\infty} c_{n,q} = 0$. *A formula for* $c_{n,q}$ *is*

$$\begin{aligned} c_{n,q} &= (\Delta_q^n f)(0) \\ &= \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{k(k-1)/2} f(n-k) \\ &= \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n-k} q^{(n-k)(n-k-1)/2} f(k). \end{aligned}$$

We will give four proofs of Theorem 3.3 below.

In Theorem 3.3, we call $c_{n,q}$ the nth q-Mahler coefficient of f and $\sum c_{n,q}\binom{x}{n}_q$ the q-Mahler expansion of f. The terms "Mahler coefficient" and

"Mahler expansion" will refer to the case $q=1$. The formula for $c_{n,q}$ in Theorem 3.3 will be called the q-Mahler Inversion Formula.

The formula for $c_{n,q}$ follows from computing $(\Delta_q^n f)(0)$ using (2.10). Replacing f by $(E^y f)(x)=f(x+y)$, we have $\lim_{n\to\infty}(\Delta_q^n f)(y)=0$ for all $y\in\mathbf{Z}_p$. Like the case $q=1$, this limit turns out to be uniform in y, and in fact there is some uniformity in q as well (which is not apparent by looking only at the case $q=1$). Such uniformities will arise from two of the proofs of Theorem 3.3.

EXAMPLE. For $|a-1|<1$ and $|q-1|<1$,

$$a^x=\sum_{n\geqslant 0}(a-1)(a-q)\cdots(a-q^{n-1})\binom{x}{n}_q=\sum_{n\geqslant 0}(a-1)^{(n;q)}\binom{x}{n}_q. \tag{3.3}$$

EXAMPLE. Using the q-binomial theorem, the sequence $(1+t)^{(m;q)}$ extends continuously from $m\in\mathbf{N}$ to $x\in\mathbf{Z}_p$ if and only if $|t|<1$, when

$$(1+t)^{(x;q)}=\sum_{n\geqslant 0}q^{n(n-1)/2}t^n\binom{x}{n}_q.$$

This could also be proven in a style similar to that of Lemma 3.1(iii).

For any $x, y\in\mathbf{Z}_p$, $(1+t)^{(x+y;q)}=(1+t)^{(x;q)}(1+q^x t)^{(y;q)}$. Setting $y=-x$ yields

$$((1+t)^{(x;q)})^{-1}=(1+q^x t)^{(-x;q)}.$$

For example, computing $(1+q^m t)^{(-m;q)}$ in two ways for $m\geqslant 1$, we have

$$\begin{aligned}\frac{1}{(1+t)(1+qt)\cdots(1+q^{m-1}t)}&=\sum_{n\geqslant 0}q^{n(n-1)/2}(q^m t)^n\binom{-m}{n}_q\\&=\sum_{n\geqslant 0}\binom{m+n-1}{n}_q(-t)^n,\end{aligned}$$

which is due to Cauchy [4, Eq. 19, p. 46] as an identity over the complex numbers.

Warning. For $|a-1|<1$, writing $a=1+t$, it seems reasonable to define $a^{(x;q)}=(1+t)^{(x;q)}$ in the sense of the above example. However, although $|a^{(m;q)}-1|<1$ and $(1+T)^{(mn;q)}=((1+T)^{(m;q)})^{(n;q^m)}$ (which implies (2.1) by looking at the coefficient of T), it is false that $a^{(mn;q)}=(a^{(m;q)})^{(n;q^m)}$, even when $m=n=2$. A correct way to state the q-version of $(1+T)^{mn}=((1+T)^m)^n$ so that it is valid to specialize the variable is

$$(1+T)^{(mn;\,q)}=(1+T)^{(n;\,q^m)}(1+qT)^{(n;\,q^m)}\cdots(1+q^{m-1}T)^{(n;\,q^m)}.$$

Our first proof of Theorem 3.3 will deduce the result from the known case $q=1$. Recall that a countable set of vectors $\{e_n\}_{n\geqslant 0}$ in a K-Banach space $(V, \|\cdot\|)$ (we assume the norm on V is nonarchimedean: $\|v+w\|\leqslant \max(\|v\|, \|w\|)$) is called an *orthonormal basis* if every $v\in V$ has a unique representation in the form $v=\sum c_n e_n$ where $c_n\to 0$ and $\|v\|=\max|c_n|$. Mahler's theorem says the functions $\binom{x}{n}$ are an orthonormal basis of $C(\mathbf{Z}_p, K)$, topologized by the sup-norm.

The following standard lemma shows that a small perturbation of an orthonormal basis is still an orthonormal basis. The ideas in the proof are taken from [3, Proposition 2, Sect. 1.1.4, Proposition 4, Sect. 2.7.2].

Lemma 3.2. *Let K be a complete nonarchimedean nontrivially valued field and V be a K-Banach space with an orthonormal basis $\{e_n\}_{n\geqslant 0}$. If $e'_n\in V$ with $\sup_{n\geqslant 0}\|e_n-e'_n\|<1$, then $\{e'_n\}$ is an orthonormal basis of V.*

Proof. Step 1. $\|\sum_{n=0}^N c_n e'_n\|=\max_{0\leqslant n\leqslant N}|c_n|$.
Let $\varepsilon=\sup_{n\geqslant 0}\|e_n-e'_n\|<1$. Writing

$$\sum_{n=0}^N c_n e'_n=\sum_{n=0}^N c_n(e'_n-e_n)+\sum_{n=0}^N c_n e_n,$$

the first sum has size at most $\varepsilon\max|c_n|$.

Step 2. The K-linear span ($=$ finite linear combinations) of the e'_n is dense in V.

Let W be this span. For $v\in V$, let $v=\sum_{n\geqslant 0}c_n e_n$. Choose N so $|c_n|\leqslant\varepsilon\,\|v\|$ for $n\geqslant N+1$. Then

$$v-\sum_{n=0}^N c_n e'_n=\sum_{n=0}^N c_n(e_n-e'_n)+\sum_{n\geqslant N+1}c_n e_n$$

has norm $\leqslant\varepsilon\,\|v\|$. Assume W is not dense, so there is $v\in V$ such that $a=\inf_{w\in W}\|v-w\|>0$. Since $a/\varepsilon>a$, there is $w\in W$ such that $0<\|v-w\|<a/\varepsilon$. From above, there is $w'\in W$ such that

$$\|v-w-w'\|\leqslant\varepsilon\,\|v-w\|<a,$$

a contradiction.

Step 3. $\{e'_n\}$ is an orthonormal basis.

By Step 1, it suffices to show for each $v \in V$ that $v = \sum c_n e'_n$ for some sequence $c_n \to 0$ in K.

Choose $w_1 \in W$ such that $\|v - w_1\| \leqslant 1/2$. Choose $w_2 \in W$ such that $\|v - w_1 - w_2\| \leqslant 1/4$. Continuing, choose $w_m \in W$ such that $\|v - w_1 - \cdots - w_m\| \leqslant 1/2^m$. Then $\|w_m\| \to 0$ and $v = \sum w_m$. Writing $w_m = \sum_n b_{m,n} e'_n$, we have $b_{m,n} = 0$ for n large and $|b_{m,n}| \leqslant \|w_m\|$ by Step 1. Thus

$$v = \sum_m \left(\sum_n b_{m,n} e'_n \right) = \sum_n \left(\sum_m b_{m,n} \right) e'_n,$$

where the interchange of the double sum is justified by [12, Lemma 4.1.3]. ∎

Here is a first proof of Theorem 3.3.

Proof. By Mahler's theorem, $\{\binom{x}{n}\}_{n \geqslant 0}$ is an orthonormal basis of $C(\mathbf{Z}_p, K)$. For all $n \geqslant 0$, Theorem 3.2 implies

$$\left| \binom{x}{n}_q - \binom{x}{n} \right|_{\text{sup}} \leqslant |q-1| < 1.$$

Therefore we are done by Lemma 3.2. ∎

This proof of Theorem 3.3 is succinct, but depends on already having the result in the case $q = 1$. The same argument would deduce the result for all q with $|q-1| < 1$ if we had it for any one such q.

By a similar idea, since $\{\binom{x}{m}\binom{y}{n}\}$ is an orthonormal basis of $C(\mathbf{Z}_p \times \mathbf{Z}_p, K)$, topologized by the sup-norm, so is $\{\binom{x}{m}_{q_1}\binom{y}{n}_{q_2}\}$ for fixed $q_1, q_2 \in K$ with $|q_1 - 1|, |q_2 - 1| < 1$. There is a similar extension to $C(\mathbf{Z}_p^r, K)$ for any $r \geqslant 1$.

Since $(\Delta_q^n f)(x) = \Delta_q^n(E^x f)(0)$, by the q-Mahler theorem we have $\lim_{n \to \infty} (\Delta_q^n f)(x) = 0$ for each $x \in \mathbf{Z}_p$. However, this limit is actually uniform in x. To see this we give a second proof of the q-Mahler theorem, one which will not assume Mahler's theorem already. It will show directly that $\lim_{n \to \infty} \Delta_q^n f = 0$ in $C(\mathbf{Z}_p, K)$.

First we record a lemma. It gives some properties of the size of $(x)_q$. Extending (2.1) from $\mathbf{Z}$ to $\mathbf{Z}_p$, if $|q-1| < 1$ then $(xy)_q = (x)_q (y)_{q^x}$ for $x, y \in \mathbf{Z}_p$. In particular, for $n \in \mathbf{N}$ and $u \in \mathbf{Z}_p^\times$,

$$(p^n u)_q = (p^n)_q \, (u)_{q^{p^n}}. \tag{3.4}$$

LEMMA 3.3. *Let* $|q-1|<1$.

(i) *If* $x=p^n u$ *with* $u\in \mathbf{Z}_p^\times$, $|(x)_q|=|(p^n)_q|$.

(ii) $|(p^n)_q|\leqslant \prod_{i=0}^{n-1}\max(|q^{p^i}-1|, 1/p)\leqslant \max(|q-1|, 1/p)^n<1$.

(iii) *If* $|q-1|<(1/p)^{1/(p-1)}$, *then* $|(x)_q|=|x|$ *for all* $x\in \mathbf{Z}_p$.

Proof. (i) Use (3.4), recalling $(u)_{q^{p^n}}\equiv u\not\equiv 0 \bmod \mathfrak{m}_K$.

(ii) By (2.1),

$$(p^n)_q=(p)_q(p)_{q^p}\cdot \;\cdots\; \cdot (p)_{q^{p^{n-1}}}, \tag{3.5}$$

so it suffices to show for $|q-1|<1$ that $|(p)_q|\leqslant \max(|q-1|, 1/p)$. In $\mathcal{O}_K/(q-1, p)$,

$$(p)_q=\Phi_p(q)\equiv (q-1)^{p-1}\equiv 0.$$

(iii) By (i), we only need to show the result for $x=p^n$. Moreover, by (3.5) and $|q^{p^i}-1|\leqslant |q-1|<(1/p)^{1/(p-1)}$, it suffices to show the result for $x=p$. Since

$$(p)_q=\frac{q^p-1}{q-1}=\sum_{k=1}^{p}\binom{p}{k}(q-1)^{k-1}$$

and each term in the sum except the one for $k=1$ has size less than $1/p$, we're done. ∎

As a consequence of (i) and (ii), we have

$$|(x)_q-(y)_q|=|(x-y)_q|\leqslant \max(|q-1|, 1/p)^{\operatorname{ord}(x-y)},$$

which can be rewritten as $|q^x-q^y|\leqslant |q-1|\ \max(|q-1|, 1/p)^{\operatorname{ord}(x-y)}$, in which form it appears in [22, Theorem 32.4].

From (ii), (3.2) can be weakened to

$$\left|\binom{p^r}{k}_q\right|\leqslant \max(|q-1|, 1/p)^{r-j}, \tag{3.6}$$

where we recall $1\leqslant k\leqslant p^r$, $j=\operatorname{ord}(k)$.

We now give a second proof of Theorem 3.3. The idea is taken from the proof of Mahler's theorem in [22, Exercise 52.E].

Proof. Since $|\Delta_q^{n+1} f|_{\sup} \leqslant |\Delta_q^n f|_{\sup}$, it suffices to show $\lim_{r\to\infty} \Delta_q^{p^r} f = 0$. We have

$$\begin{aligned}(\Delta_q^{p^r} f)(x) &= \sum_{k=0}^{p^r} \binom{p^r}{k}_q (-1)^{p^r-k} q^{(p^r-k)(p^r-k-1)/2} f(k+x) \\ &= \sum_{k=0}^{p^r} \binom{p^r}{k}_q (-1)^{p^r-k} q^{(p^r-k)(p^r-k-1)/2} (f(k+x)-f(x)).\end{aligned}$$

The $k=0$ term vanishes, so by (3.6)

$$|\Delta_q^{p^r} f|_{\sup} \leqslant \max_{i+j=r} \max(|q-1|, 1/p)^i \rho_j(f),$$

where $\rho_j(f) = \sup_{|x-y|\leqslant 1/p^j} |f(x)-f(y)|$. The terms indexed by i and j are both uniformly bounded above, and each tends to zero for large values of the index. ∎

Not only does this show $\lim_{n\to\infty} (\Delta_q^n f)(x) = 0$ uniformly in x, but also (for fixed $\delta \in (0, 1)$) uniformly in q for $|q-1| \leqslant \delta < 1$.

For the third proof of the q-Mahler theorem, we extend a periodicity property of ordinary binomial coefficients to q-binomial coefficients: for any $N \geqslant 1$ and all $n < p^N$,

$$a \equiv b \bmod p^N \Rightarrow \binom{a}{n} \equiv \binom{b}{n} \bmod p.$$

For q-binomial coefficients, the same result is true provided N is taken large enough depending on q.

LEMMA 3.4. *Let* $|q-1| < 1$. *For* N *large, depending on* q, *if* $x \equiv y \bmod p^N \mathbf{Z}_p$ *and* $n < p^N$ *then*

$$\left|\binom{x}{n}_q - \binom{y}{n}_q\right| \leqslant \frac{1}{p}.$$

More precisely, this is true if $1/(p^{N-1}(p-1)) < \operatorname{ord}(q-1)$.

Proof. By Theorem 2.5,

$$m_1 \equiv m_2 \bmod p^N \Rightarrow \binom{m_1}{n}_q - \binom{m_2}{n}_q \in \Phi_{p^N}(q)\, \mathbf{Z}[q].$$

So by continuity,

$$\left|\binom{x}{n}_q - \binom{y}{n}_q\right| \leqslant |\Phi_{p^N}(q)| = |(p)_{q^{p^{N-1}}}|.$$

For N large, $|q^{p^{N-1}} - 1| < (1/p)^{1/(p-1)}$, so $(p)_{q^{p^{N-1}}}$ has size $|p| = 1/p$ by Lemma 3.3(iii).

Let's be more precise about how large N has to be. For any N,

$$\Phi_{p^N}(q) = \prod_{\substack{\zeta^{p^N}=1 \\ \zeta^{p^{N-1}} \neq 1}} (q-\zeta).$$

There are $p^{N-1}(p-1)$ terms in the product. When $1/p^{N-1}(p-1) < \operatorname{ord}(q-1)$, then $|q-1| < |\zeta - 1| = (1/p)^{1/p^{N-1}(p-1)}$ for all such ζ by Lemma 3.1(ii), so all the terms have the same size and therefore

$$|\Phi_{p^N}(q)| = \frac{1}{p}. \quad \blacksquare$$

If we work modulo $(q-1, p)$, then for $x \equiv y \bmod p^N$ and $n < p^N$, $\binom{x}{n}_q \equiv \binom{x}{n} \equiv \binom{y}{n} \equiv \binom{y}{n}_q$, so without needing N to be large, we have $|\binom{x}{n}_q - \binom{y}{n}_q| \leqslant \max(|q-1|, 1/p)$.

Now we give a third proof of Theorem 3.3. Like the second, it does not require prior knowledge at $q = 1$. It is based on the proof in [17, pp. 99–100].

Proof. Let

$$L\colon \{(c_n)_{n \geqslant 0} \colon c_n \in K,\, c_n \to 0\} \to C(\mathbf{Z}_p, K)$$

by $(c_n) \mapsto \sum_{n \geqslant 0} c_n \binom{x}{n}_q$. This is K-linear and continuous, where the domain and range are both topologized by the appropriate sup-norm. We want to show L is onto. By scaling it suffices to show the restriction $L\colon B \to C(\mathbf{Z}_p, \mathcal{O}_K)$ is onto, where

$$B = \{(c_n) \colon |c_n| \leqslant 1,\, c_n \to 0\}.$$

By completeness of B and continuity of L, it is enough to show that for any $f \in C(\mathbf{Z}_p, \mathcal{O}_K)$, there is some $s \in B$ such that $|f - L(s)| \leqslant |p|$. (Then apply the result to $g = (f - L(s))/p$ to get $s' \in B$ such that $|f - L(s + ps')| \leqslant |p^2|$, etc.) That is, we want to show surjectivity of the map

$$\{(c_n) \colon c_n \in \mathcal{O}_K/p,\, c_n = 0 \text{ for large } n\} \to C(\mathbf{Z}_p, \mathcal{O}_K/p)$$

given by

$$(c_n) \mapsto \sum_{n \geqslant 0} c_n \binom{x}{n}_q \qquad \bmod p. \tag{3.7}$$

Note that the quotient topology on $\mathcal{O}_K/p$ is the discrete topology. Thus

$$C(\mathbf{Z}_p, \mathcal{O}_K/p) = \bigcup_{N \geqslant 1} \mathrm{Maps}(\mathbf{Z}_p/p^N\mathbf{Z}_p, \mathcal{O}_K/p). \tag{3.8}$$

The union in (3.8) can be taken over just large integers. Lemma 3.4 suggests that at least for large N (depending on q), $f \in C(\mathbf{Z}_p, \mathcal{O}_K/p)$ factors through $\mathbf{Z}_p/p^N\mathbf{Z}_p$ when its nth q-Mahler coefficient vanishes for $n \geqslant p^N$, thus suggesting the more precise surjectivity of

$$\{(c_n)_{n=0}^{p^N-1} : c_n \in \mathcal{O}_K/p\} \to \mathrm{Maps}(\mathbf{Z}_p/p^N\mathbf{Z}_p, \mathcal{O}_K/p) \tag{3.9}$$

given by (3.10) with the sum over $0 \leqslant n \leqslant p^N - 1$. (Note that by Lemma 3.4, $\binom{x}{n}_q$ mod p is well-defined on $\mathbf{Z}_p/p^N\mathbf{Z}_p$ for N large and $n < p^N$.) The surjectivity (even bijectivity) of (3.9) follows from the argument that q-Mahler coefficients are unique. ∎

We could have worked in $\mathcal{O}_K/(q-1, p)$ and not needed to use only large N at the end of the proof.

Here's a fourth proof of Theorem 3.3, which like the second will yield some uniformity statements in q.

Proof. Define the numbers $c_n = c_{n,q}$ as in the statement of Theorem 3.3, so

$$f(m) = \sum_{n \geqslant 0} c_n \binom{m}{n}_q$$

for all nonnegative integers m. We thus only need to show that $|c_n| \to 0$. To do this we adapt Bojanic's argument in [2].

Bojanic's proof uses two different formulas for $(\Delta^n f)(m)$. First,

$$(\Delta^n f)(m) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(k+m).$$

Writing $(\Delta^n f)(m) = (\Delta^n E^m f)(0)$, we also have

$$E^m = (I + \Delta)^m = \sum_{j=0}^{m} \binom{m}{j} \Delta^j \Rightarrow (\Delta^n f)(m) = \sum_{j=0}^{m} \binom{m}{j} (\Delta^{n+j} f)(0).$$

For the q-analogue of these, (2.11) gives

$$(\Delta_q^n f)(m)=\sum_{k=0}^{n}\binom{n}{k}_q(-1)^{n-k}\,q^{(n-k)(n-k-1)/2}f(k+m),$$

while the equation $E^m=\sum_{j=0}^{m}\binom{m}{j}_q\,q^{n(m-j)}(E-q^n)^{(j;\,q)}$ gives

$$\begin{aligned}(\Delta_q^n f)(m)&=(\Delta_q^n E^m f)(0)\\&=\sum_{j=0}^{m}\binom{m}{j}_q\;q^{n(m-j)}(\Delta_q^{n+j}f)(0).\end{aligned}$$

Equating these formulas for $(\Delta_q^n f)(m)$ and isolating the $j=m$ term,

$$\begin{aligned}c_{n+m}=&\sum_{k=0}^{n}\binom{n}{k}_q(-1)^{n-k}\,q^{(n-k)(n-k-1)/2}f(k+m)\\&-\sum_{j=0}^{m-1}\binom{m}{j}_q\;q^{(m-j)\,n}c_{n+j}.\end{aligned}$$

With this formula we show $|c_n|\to 0$.

The $j=0$ term is $q^{mn}c_n=q^{mn}\sum_{k=0}^{n}\binom{n}{k}_q(-1)^{n-k}\,q^{(n-k)(n-k-1)/2}f(k)$, so

$$\begin{aligned}c_{n+m}=&\sum_{k=0}^{n}\binom{n}{k}_q(-1)^{n-k}\,q^{(n-k)(n-k-1)/2}(f(k+m)-q^{mn}f(k))\\&-\sum_{j=1}^{m-1}\binom{m}{j}_q\;q^{(m-j)\,n}c_{n+j}.\end{aligned}$$

Scaling, we may assume $|f(x)|\leqslant 1$ for all $x\in\mathbf{Z}_p$, so $|c_n|\leqslant 1$ for all n.

Let $m=p^r$, for r to be determined. Then

$$|c_{n+p^r}|\leqslant\max_{\substack{0\leqslant k\leqslant n\\1\leqslant j\leqslant p^r-1}}\left\{|f(k+p^r)-q^{p^rn}f(k)|,\;\left|\binom{p^r}{j}_q c_{n+j}\right|\right\}.$$

For such j, $|\binom{p^r}{j}_q|\leqslant|\Phi_{p^r}(q)|$ by (2.7).

Choose $\varepsilon>0$. For large r, depending on f,

$$|x-y|\leqslant\frac{1}{p^r}\Rightarrow|f(x)-f(y)|\leqslant\varepsilon.$$

Thus $f(k+p^r)-q^{p^rn}f(k)=f(k+p^r)-f(k)+f(k)(1-q^{p^rn})$, where the first term has size at most ε, while the second is at most $|q-1|\cdot\max(|q-1|,1/p)^r$, which is $\leqslant\varepsilon$ for r large (depending on q).

By the proof of Lemma 3.4, $|\Phi_{p^r}(q)| = 1/p$ for all large r, depending on q. So there is a large r such that for all $n \geqslant 0$,

$$\begin{aligned} |c_{n+p^r}| &\leqslant \max_{1 \leqslant j \leqslant p^r - 1} (\varepsilon, (1/p)\, |c_{n+j}|) \\ &\leqslant \max(\varepsilon, 1/p). \end{aligned}$$

Thus $|c_n| \leqslant \max(\varepsilon, 1/p)$ for $n \geqslant p^r$. Replacing n by $n + p^r$ gives, for all $n \geqslant 0$,

$$\begin{aligned} |c_{n+2p^r}| &\leqslant \max_{1 \leqslant j \leqslant p^r - 1} (\varepsilon, (1/p)\, |c_{n+p^r+j}|) \\ &\leqslant \max(\varepsilon, 1/p^2). \end{aligned}$$

So

$$|c_n| \leqslant \max(\varepsilon, 1/p^2)$$

for $n \geqslant 2p^r$. Repeating this $s-1$ more times gives

$$|c_n| \leqslant \max(\varepsilon, 1/p^s)$$

for $n \geqslant sp^r$. Choosing s so large that $1/p^s \leqslant \varepsilon$ we have $|c_n| \leqslant \varepsilon$ if $n \geqslant sp^r$. ▮

Since the functions $E^x f$ are equicontinuous, this proof shows $\lim_{n \to \infty} \Delta_q^n f = 0$ uniformly in q for $|q-1| \leqslant \delta < 1$.

For the reader who knows about q-derivatives, a second way to obtain the two formulas for $(\Delta_q^n f)(m)$ in the proof above is to make the proposed equality of these two expressions a universal polynomial identity, and to establish it by q-differentiating the equation

$$\sum_{k \geqslant 0} f(k) \frac{X^k}{(k)_q!} = E_q(X) \sum_{n \geqslant 0} c_n \frac{X^n}{(n)_q!}$$

m times, dividing by $E_q(X)$, and then equating coefficients of X^n.

Although the q-Mahler expansion is treated above for a single function $f \in C(\mathbf{Z}_p, K)$, we will look in Section 5 at an example of a family of functions $f_q \in C(\mathbf{Z}_p, K)$ that depends continuously on q and consider the expansion of f_q relative to the q-Mahler basis.

4. PROPERTIES OF q-MAHLER EXPANSIONS

We now go through properties of q-Mahler expansions that are analogous to properties of Mahler expansions. *Throughout this section*, $|q-1| < 1$.

First, note that $\{q \in K : |q-1| < 1\}$ is a multiplicative group, unlike the parameter set that arises for q-series over $\mathbf{C}$, the open unit disk. So we can also consider $1/q$-Mahler expansions.

THEOREM 4.1. *Let* $|q-1|<1$, $f \in C(\mathbf{Z}_p, K)$ *with* q-*Mahler coefficients* $c_{n,q}$. *Then*

(i) $\sup_{x \in \mathbf{Z}_p} |f(x)| = \max_{n \geqslant 0} |c_{n,q}|$.

(ii) $f(x+1) = \sum_{n \geqslant 0} (q^n c_{n,q} + c_{n+1,q}) \binom{x}{n}_q$.

(iii) $f(x+y) = \sum_{n \geqslant 0} (\Delta_q^n f)(y) \binom{x}{n}_q$.

(iv) $f(-x) = \sum_{n \geqslant 0} c_{n,q} (-1)^n q^{-n(n+1)/2} \binom{x+n-1}{n}_{1/q}$.

(v) $(x)_q f(x) = \sum_{n \geqslant 1} (n)_q (c_{n,q} + q^{n-1} c_{n-1,q}) \binom{x}{n}_q$.

Proof. Part (i) follows from the q-Mahler Inversion Formula, or from the first proof of Theorem 3.3.

Part (ii) is a special case of part (iii) or can be done on its own. For part (iii), note $f(x+y) = (E^y f)(x)$ and the nth q-Mahler coefficient of $E^y f$ is $(\Delta_q^n(E^y f))(0) = (\Delta_q^n f)(y)$.

Part (iii) can also be proven by using the q-Vandermonde formula and an interchange of a double sum, which is Mahler's original method at $q=1$.

For part (iv), use (2.5). Note that the expansion given in (iv) is related to a $1/q$-Mahler expansion, which can be explicitly computed using Theorem 3.1.

For part (v), use $(n)_q \binom{x}{n}_q = (x)_q \binom{x-1}{n-1}_q$. ∎

In light of (iii), $(\Delta_q^n f)(y)$ should be called the nth q-Mahler coefficient of f at y.

As with Mahler expansions, a function $\mathbf{Z}_p \to K$ with a pointwise representation as $\sum c_{n,q} \binom{x}{n}_q$ must be continuous, since $c_{n,q} \to 0$ by looking at $x = -1$.

Let's see how the difference operators act on q-Mahler expansions. For $q=1$,

$$\Delta^m \left(\sum_{j \geqslant 0} c_j \binom{x}{j} \right) = \sum_{j \geqslant 0} c_{m+j} \binom{x}{j},$$

but for general q, (2.10) implies

$$\Delta_q^m \left(\sum_{j \geqslant 0} c_{j,q} \binom{x}{j}_q \right) = \sum_{j \geqslant 0} c_{m+j,q} q^{m(x-j)} \binom{x}{j}_q, \tag{4.1}$$

which is not a q-Mahler expansion, because of the term q^{mx}. So using the operator $(\mathfrak{D}_q^m f)(x) = q^{-mx}(\Delta_q^m f)(x)$, we can write this instead as

$$\mathfrak{D}_q^m \left(\sum_{j \geqslant 0} c_{j,q} \binom{x}{j}_q \right) = \sum_{j \geqslant 0} c_{m+j,q} q^{-mj} \binom{x}{j}_q.$$

The formula in part (v) of Theorem 4.1 can be extended to $\binom{x}{m}_q f(x)$, computing the nth q-Mahler coefficient by (2.14) and (4.1) for $n \geqslant m$,

$$\begin{aligned} \Delta_q^n \left(\binom{x}{m}_q f(x) \right)(0) &= \sum_{k=0}^{n} \binom{n}{k}_q \left(\Delta_q^k \binom{x}{m}_q \right)(0)(\Delta_q^{n-k} f)(k) \\ &= \binom{n}{m}_q (\Delta_q^{n-m} f)(m) \\ &= \binom{n}{m}_q \sum_{k=0}^{m} q^{(n-m)k} \binom{m}{k}_q c_{n-k,q}. \end{aligned}$$

We now discuss the relation between differentiability and q-Mahler expansions. When $q=1$, Mahler shows in [18, Theorem 3; 19] that $f \in C(\mathbf{Z}_p, K)$ is differentiable at y if and only if $\lim_{m \to \infty} (\Delta^m f)(y)/m = 0$ and then

$$f'(y) = \sum_{m \geqslant 1} \frac{(\Delta^m f)(y)}{m} (-1)^{m-1}. \tag{4.2}$$

The extension of this result to general $|q-1|<1$ involves the p-adic logarithm, whose properties we will summarize for the convenience of readers outside of number theory. These readers should notice in particular part (iv) below, which says the p-adic logarithm is locally an isometry.

LEMMA 4.1. (i) *The series* $\log_p(1+z) = \sum_{n \geqslant 1} (-1)^{n-1} z^n/n$ *converges at* $z \in K$ *if and only if* $|z|<1$.

(ii) *If* $|u_1 - 1|, |u_2 - 1| < 1$, *then* $\log_p(u_1 u_2) = \log_p(u_1) + \log_p(u_2)$.

(iii) *For* $|q-1|<1$, $\lim_{x \to 0} ((q^x - 1)/x) = \log_p q$.

(iv) *If* $|u-v| < (1/p)^{1/(p-1)}$, *then* $|\log_p u - \log_p v| = |u-v|$.

(v) $\log_p u = 0$ *if and only if* u *is a* p*th power root of unity in* K.

(vi) *If* $|\zeta - 1| < 1$ *and* $\zeta^m = 1$, *then* $\lim_{q \to \zeta} ((\log_p q)/(q^m - 1)) = 1/m$.

Proof. (i) $|z|^n \leqslant |z^n/n| \leqslant n\,|z|^n$.

(ii) See [12, Proposition 4.5.3].

(iii) For $x \neq 0$, $(q^x - 1)/x = \sum_{n \geqslant 1} ((q-1)^n/n)\binom{x-1}{n-1}$ and $(q-1)^n/n \to 0$ by (i).

(iv) By (ii) we may take $v=1$. The first term of the series for $\log_p u$ is $u-1$. For $u \neq 1$, all the remaining terms have size less than $|u-1|$ since for $n \geqslant 2$, the unique minimum of $|n|^{1/(n-1)} = (1/p)^{\mathrm{ord}(n)/(n-1)}$ occurs at $n=p$.

(v) For any integer r, $\log_p u = 0$ if and only if $\log_p (u^{p^r}) = 0$. For r large, $|u^{p^r} - 1| < (1/p)^{1/(p-1)}$, and by (iv) the only z with $|z-1| < (1/p)^{1/(p-1)}$ and $\log_p z = 0$ is $z = 1$.

(vi) $(\log_p q)/(q^m - 1) = \log_p (q/\zeta)/((q/\zeta)^m - 1)$ and $\lim_{u \to 1} (\log_p u)/(u^m - 1) = 1/m$ since $\lim_{u \to 1} (\log_p u)/(u-1) = 1$ from the definition of $\log_p u$. ∎

Lemma 4.2. *Let $g\colon \mathbf{Z}_p \to K$ be continuous on $\mathbf{Z}_p - \{-1\}$, with $g(x) = \sum_{n \geqslant 0} c_n \binom{x}{n}_q$ for $x \neq -1$. Then g is continuous at -1 if and only if $c_n \to 0$, in which case $g(-1) = \sum_{n \geqslant 0} c_n \binom{-1}{n}_q$.*

Proof. The "if" direction is clear. For "only if," continuity of g at -1 is the same as continuity of g on $\mathbf{Z}_p$, by our hypothesis. Letting x run through the nonnegative integers, we see by the q-Mahler Inversion Formula that c_n is the nth q-Mahler coefficient of g, so we're done by Theorem 3.3. ∎

Here is the test for differentiability with q-Mahler expansions. Compare with formulas for the derivative in (4.2).

Theorem 4.2. *Let $f \in C(\mathbf{Z}_p, K)$.*

(i) *When q is not a (nontrivial) root of unity, f is differentiable at $x \in \mathbf{Z}_p$ if and only if $\lim_{m \to \infty} (\Delta_q^m f)(x)/(m)_q = 0$, in which case*

$$f'(x) = \frac{\log_p q}{q-1} \sum_{m \geqslant 1} \frac{(\Delta_q^m f)(x)}{(m)_q} (-1)^{m-1} q^{-m(m-1)/2}.$$

(ii) *When q is a root of unity of order p^N ($N \geqslant 0$), f is differentiable at $x \in \mathbf{Z}_p$ if and only if $\lim_{l \to \infty} (\Delta_q^{p^N l} f)(x)/p^N l = 0$, in which case*

$$f'(x) = \sum_{l \geqslant 1} \frac{(\Delta_q^{p^N l} f)(x)}{p^N l} (-1)^{l-1}.$$

Proof. (i) For $h \neq 0$, $f(x+h) = f(x) + \sum_{m \geqslant 1} (\Delta_q^m f)(x) \binom{h}{m}_q$ by Theorem 4.1. Therefore

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \sum_{m \geqslant 1} (\Delta_q^m f)(x) \frac{1}{h} \binom{h}{m}_q \\ &= \frac{(h)_q}{h} \sum_{m \geqslant 1} \frac{(\Delta_q^m f)(x)}{(m)_q} \binom{h-1}{m-1}_q. \end{aligned} \tag{4.4}$$

Since $(h)_q/h=(q^h-1)/(h(q-1))$ is continuous at all $h\in\mathbf{Z}_p-\{0\}$ and its limit as $h\to 0$ is $(\log_p q)/(q-1)\neq 0$ (even if $q=1$), the function $(h)_q/h$ is continuous and nowhere vanishing. So by Lemma 4.2 (with $h-1$ as the variable), $f'(x)$ exists if and only if $(\Delta_q^m f)(x)/(m)_q\to 0$ and then $f'(x)$ has the indicated form.

(ii) We consider only suitably small h, say $h=p^N z$ for $z\in\mathbf{Z}_p$. For $z\neq 0$,

$$\frac{f(x+p^N z)-f(x)}{p^N z}=\sum_{m\geqslant 1}(\Delta_q^m f)(x)\,\frac{1}{p^N z}\binom{p^N z}{m}_q,$$

and by (3.1),

$$\binom{p^N z}{m}_q=\begin{cases}\dbinom{z}{m/p^N}, & \text{if } p^N\mid m;\\ 0, & \text{if } p^N\nmid m,\end{cases}$$

so

$$\begin{aligned}\frac{f(x+p^N z)-f(x)}{p^N z}&=\sum_{l\geqslant 1}(\Delta_q^{p^N l} f)(x)\,\frac{1}{p^N z}\binom{z}{l}\\ &=\sum_{l\geqslant 1}\frac{(\Delta_q^{p^N l} f)(x)}{p^N l}\binom{z-1}{l-1}.\end{aligned}$$

Apply Lemma 4.2 (for $q=1$) with $z-1$ as the variable. ∎

Let's unify both parts of this theorem. For q not a root of unity, $(\log_p q)/((q-1)(m)_q)=(\log_p q)/(q^m-1)$, while Lemma 4.1(vi) shows that for q a root of unity, $(\log_p q)/(q^m-1)$ equals $1/m$ when $q^m=1$ and equals 0 otherwise. Moreover, if q is a root of unity of order p^N, then $\binom{-1}{p^N l-1}_q=(-1)^{l-1}$ for $l\geqslant 1$. So for any $|q-1|<1$, a root of unity or not, f is differentiable at x if and only if $\lim_{m\to\infty}(\Delta_q^m f)(x)(\log_p q)/(q^m-1)=0$, in which case

$$f'(x)=\sum_{m\geqslant 1}(\Delta_q^m f)(x)\,\frac{\log_p q}{q^m-1}\binom{-1}{m-1}_q.$$

In particular,

$$f'(0)=\sum_{m\geqslant 1}c_{m,q}\,\frac{\log_p q}{q^m-1}\binom{-1}{m-1}_q.$$

When $f(x) = \sum c_n \binom{x}{n}$ is differentiable and f' is continuous, Mahler [18, Theorem 4] gives the Mahler expansion for f',

$$f'(x) = \sum_{n \geqslant 0} \left(\sum_{j \geqslant 1} \frac{c_{n+j}}{j} (-1)^{j-1} \right) \binom{x}{n}. \tag{4.3}$$

For the q-analogue, we use the following q-analogue of [22, Proposition 47.4],

$$p^k \leqslant n < p^{k+1} \Rightarrow \left| \binom{x}{n} - \binom{y}{n} \right| \leqslant p^k \, |x - y|.$$

LEMMA 4.3. *Let $n \geqslant 1$, $p^k \leqslant n < p^{k+1}$.*

(i) *When q is not a (nontrivial) root of unity,*

$$\begin{aligned} \left| \binom{x}{n}_q - \binom{y}{n}_q \right| &\leqslant \frac{1}{|(p^k)_q|} \, |(x)_q - (y)_q| \\ &\leqslant \frac{1}{|(p^k)_q|} \max(|q-1|, 1/p)^{\operatorname{ord}(x-y)}. \end{aligned}$$

(ii) *When q is a root of unity of order p^N $(N \geqslant 0)$ and $x \equiv y \bmod p^N$,*

$$\left| \binom{x}{n}_q - \binom{y}{n}_q \right| \leqslant p^k \, |x - y|.$$

Proof. (i) Let $x = y + z$, so by Theorem 3.1,

$$\binom{x}{n}_q - \binom{y}{n}_q = \sum_{j=1}^{n} \frac{(z)_q}{(j)_q} \binom{z-1}{j-1}_q \binom{y}{n-j}_q q^{j(y+j-n)},$$

hence

$$\left| \binom{x}{n}_q - \binom{y}{n}_q \right| \leqslant \max_{1 \leqslant j \leqslant n} \left| \frac{(z)_q}{(j)_q} \right| = \max_{m \leqslant k} \frac{1}{|(p^m)_q|} \, |(x)_q - (y)_q|.$$

(ii) The difference vanishes if $n < p^N$, so we may assume $n \geqslant p^N$, i.e., $k \geqslant N$. Let $x \equiv y \equiv r \bmod p^N$, $0 \leqslant r \leqslant p^N - 1$. Write $x = p^N x' + r$, $y = p^N y' + r$, $n = p^N l + s$, $0 \leqslant s \leqslant p^N - 1$, so $p^{k-N} \leqslant l < p^{k+1-N}$. Then $\binom{x}{n}_q - \binom{y}{n}_q = \left(\binom{x'}{l} - \binom{y'}{l}\right)\binom{r}{s}_q$, so (knowing the case $q = 1$ already)

$$\left| \binom{x}{n}_q - \binom{y}{n}_q \right| \leqslant \left| \binom{x'}{l} - \binom{y'}{l} \right| \leqslant p^{k-N} \, |x' - y'| = p^k \, |x - y|. \quad \blacksquare$$

If $|q-1|<(1/p)^{1/(p-1)}$, then part (i) reduces to $|\binom{x}{n}_q-\binom{y}{n}_q|\leqslant p^k\,|x-y|$, which (for $q\in\mathbf{Z}$) is a special case of [8, Theorem 4.5].

Here is the q-analogue of the Mahler expansion of f' when f' is continuous, extending (4.3).

THEOREM 4.3. *Let $f(x)=\sum_{n\geqslant 0} c_{n,q}\binom{x}{n}_q$ be a continuous function from $\mathbf{Z}_p$ to K with a continuous derivative. The q-Mahler expansion of f' is*

$$\begin{aligned}f'(x)&=\sum_{n\geqslant 0}\left(nc_{n,q}\log_p q+\sum_{j\geqslant 1} c_{n+j,q}\frac{\log_p q}{q^j-1}\binom{-1}{j-1}_q q^{-jn}\right)\binom{x}{n}_q\\&=\sum_{n\geqslant 0}\left(nc_{n,q}\log_p q+\sum_{j\geqslant 1} c_{n+j,q}\frac{\log_p q}{q^j-1}(-1)^{j-1}q^{-j(j-1)/2-jn}\right)\binom{x}{n}_q.\end{aligned}$$

Proof. Apply $\lim_{m\to\infty}(\Delta_q^m f)(x)(\log_p q)/(q^m-1)=0$ at $x=0,1,2,\ldots$ to see $\lim_{m\to\infty} c_{n+m,q}(\log_p q)/(q^m-1)=0$ for all $n\in\mathbf{N}$.

For $y\neq 0$,

$$\frac{f(x+y)-f(x)}{y}=\sum_{n\geqslant 0}\left(\frac{(\Delta_q^n f)(y)-c_{n,q}}{y}\right)\binom{x}{n}_q.$$

By (4.1),

$$\frac{(\Delta_q^n f)(y)-c_{n,q}}{y}=c_{n,q}\left(\frac{q^{yn}-1}{y}\right)+\sum_{j\geqslant 1} c_{n+j,q}q^{n(y-j)}\frac{1}{y}\binom{y}{j}_q.$$

How does each term behave as $y\to 0$? The first term tends to $c_{n,q}\log_p(q^n)=nc_{n,q}\log_p q$. For the other terms,

$$\begin{aligned}q^{n(y-j)}\frac{1}{y}\binom{y}{j}_q&=q^{n(y-j)}\frac{q^y-1}{y}\frac{1}{q^j-1}\binom{y-1}{j-1}_q\\&\to\frac{\log_p q}{q^j-1}\binom{-1}{j-1}_q q^{-jn}\\&=\frac{\log_p q}{q^j-1}(-1)^{j-1}q^{-j(j-1)/2-jn}.\end{aligned}$$

This calculation is valid only if $q^j\neq 1$, but the result is true if $q^j=1$ by using (3.1). So we expect

$$f'(x)=\sum_{n\geqslant 0}\left(nc_{n,q}\log_p q+\sum_{j\geqslant 1} c_{n+j,q}\frac{\log_p q}{q^j-1}(-1)^{j-1}q^{-j(j-1)/2-jn}\right)\binom{x}{n}_q. \tag{4.4}$$

However, though we know $\lim_{j\to\infty} c_{n+j,q}(\log_p q)/(q^j-1)=0$ for each n, so the putative q-Mahler coefficients of f' in (4.4) do make sense, we don't yet know

$$\lim_{n\to\infty}\sum_{j\geqslant 1} c_{n+j,q}\frac{\log_p q}{q^j-1}\binom{-1}{j-1}_q q^{-jn}=0,$$

so convergence of the infinite series over n in (4.4) is not clear. To get around this, we use the idea of Mahler from his proof of Theorem 4.3 at $q=1$, namely by the hypothesis of continuity of f' it suffices to verify (4.4) when $x=m\in\mathbf{N}$. In this case the sum over n becomes finite,

$$\frac{f(m+y)-f(m)}{y}=\sum_{n=0}^{m}\left(c_{n,q}\left(\frac{q^{yn}-1}{y}\right)+\sum_{j\geqslant 1} c_{n+j,q}q^{n(y-j)}\frac{1}{y}\binom{y}{j}_q\right)\binom{m}{n}_q.$$

The outer sum is finite, so to verify termwise evaluation of $\lim_{y\to 0}$ all we need to do is check

$$\lim_{y\to 0} c_{n+j,q}\frac{1}{y}\binom{y}{j}_q=c_{n+j,q}\frac{\log_p q}{q^j-1}\binom{-1}{j-1}_q$$

uniformly in j (but perhaps not in q or n).

Case 1. q is a root of unity of order p^N, so $\lim_{j\to\infty} c_{n+j,q}/j=0$, as j runs through multiples of p^N.

If $q^j\neq 1$, then $\binom{y}{j}_q=0$ for $|y|\leqslant 1/p^N$.

If $q^j=1$, say $j=p^N j'$, then

$$\lim_{y\to 0} c_{n+j,q}\frac{1}{y}\binom{y}{j}_q=\lim_{z\to 0} c_{n+j,q}\frac{1}{j}\binom{z-1}{j'-1}.$$

We consider the difference

$$c_{n+j,q}\frac{1}{j}\binom{z-1}{j'-1}-c_{n+j,q}\frac{1}{j}\binom{-1}{j-1}_q=\frac{c_{n+j,q}}{j}\left(\binom{z-1}{j'-1}-\binom{-1}{j'-1}\right).$$

Choose a power of p, say p^r, such that $|c_{n+j,q}/j|\leqslant\delta$ for $j\geqslant p^r$ (and $p^N\mid j$). For $j<p^r$, $\binom{z-1}{j'-1}-\binom{-1}{j'-1}$ has size at most $p^{r-1}\,|z|$ by Lemma 4.3.

Therefore

$$\lim_{y\to 0} c_{n+j,q}\frac{1}{y}\binom{y}{j}_q=c_{n+j,q}\frac{\log_p q}{q^j-1}\binom{-1}{j-1}_q,$$

uniformly in j.

Case 2. q is not a root of unity.
So $\log_p q \neq 0$, hence $\lim_{j\to\infty} c_{n+j,q}/(q^j-1)=0$.
Since

$$c_{n+j,q}\,\frac{1}{y}\binom{y}{j}_q - c_{n+j,q}\,\frac{\log_p q}{q^j-1}\binom{-1}{j-1}_q$$
$$=\frac{c_{n+j,q}}{q^j-1}\left(\frac{q^y-1}{y}\binom{y-1}{j-1}_q - \log_p q\binom{-1}{j-1}_q\right)$$
$$=\frac{c_{n+j,q}}{q^j-1}\left(\frac{q^y-1}{y} - \log_p q\right)\binom{y-1}{j-1}_q$$
$$+\frac{c_{n+j,q}}{q^j-1}\log_p q\left(\binom{y-1}{j-1}_q - \binom{-1}{j-1}_q\right),$$

we need to show that

$$\lim_{y\to 0}\frac{c_{n+j,q}}{q^j-1}\left(\binom{y-1}{j-1}_q - \binom{-1}{j-1}_q\right)=0$$

uniformly in j. For $\delta>0$, choose p^r so $|c_{n+j,q}/(q^j-1)|\leqslant\delta$ for $j\geqslant p^r$. For $j<p^r$, Lemma 4.3 implies

$$\left|\binom{y-1}{j-1}_q - \binom{-1}{j-1}_q\right| \leqslant \frac{1}{|(p^{r-1})_q|}\max(|q-1|, 1/p)^{\operatorname{ord}(y)},$$

which is $\leqslant\delta$ for $\operatorname{ord}(y)$ large enough. ∎

So for f' continuous and q not a root of unity,

$$f'(x)=\frac{\log_p q}{q-1}\sum_{n\geqslant 0}\left((q-1)\,nc_{n,q}+\sum_{j\geqslant 1}\frac{c_{n+j,q}}{(j)_q}(-1)^{j-1}q^{-j(j-1)/2-jn}\right)\binom{x}{n}_q,$$

while for q a root of unity of order p^N,

$$f'(x)=\sum_{n\geqslant 0}\left(\sum_{\substack{j\geqslant 1\\ p^N\mid j}}\frac{c_{n+j,q}}{j}(-1)^{j/p^N-1}\right)\binom{x}{n}_q.$$

The Mahler expansion characterizes analyticity: $\sum c_n\binom{x}{n}$ is analytic if and only if $c_n/n!\to 0$ [22, Theorem 54.4]. For example, the function q^x is an analytic function of x if and only if $|q-1|<(1/p)^{1/(p-1)}$, in which case its mth Taylor coefficient at $x=0$ is $(\log_p q)^m/m!$. For other q, $|q^{p^r}-1|<(1/p)^{1/(p-1)}$ for r large, so $(x)_q$ is locally analytic.

To describe analyticity in terms of q-Mahler expansions, we only consider $|q-1|<(1/p)^{1/(p-1)}$, since this is the region of q where the functions $\binom{x}{n}_q$ are all analytic. For such q, $|(x)_q|=|x|$. In particular, $|n!|=|(n)_q!|$.

LEMMA 4.4. *Let $a_1, b_1, ..., a_m, b_m \in K$ with $|a_j|, |b_j| \leqslant 1$. Then*

$$|a_1 a_2 \cdots a_n - b_1 b_2 \cdots b_n| \leqslant \max |a_j - b_j|.$$

Proof. In $\mathcal{O}_K/(a_1-b_1, ..., a_n-b_n)$, $a_1 \cdots a_n \equiv b_1 \cdots b_n$. ∎

THEOREM 4.4. *For $|q-1|<(1/p)^{1/(p-1)}$, $\sum c_n\binom{x}{n}_q$ is analytic if and only if $c_n/(n)_q! \to 0$.*

Proof. As with the first proof of Theorem 3.3, we'll get the result for general q from the case $q=1$ by Lemma 3.2.

Let $A(\mathbf{Z}_p, K)=\{f(x)=\sum a_n x^n : a_n \in K, a_n \to 0\}$ be the analytic functions from $\mathbf{Z}_p$ to K. It is a K-Banach space under the norm $\|f\|=\max |a_n|$. (This norm does not generally coincide with the sup-norm over $\mathbf{Z}_p$, e.g., $\|x^p-x\|=1$, but $|x^p-x|_{\sup}=1/p$.)

Writing

$$\sum a_n x^n = \sum b_n x(x-1)\cdots(x-n+1) = \sum n!\, b_n \binom{x}{n},$$

we see $a_n - b_n \in \mathbf{Z}[[b_{n+1}, b_{n+2}, ...]]$, so $\max |a_n| = \max |b_n|$. Therefore the norm in $A(\mathbf{Z}_p, K)$ of an analytic function written as $\sum c_n\binom{x}{n}$ is $\max |c_n/n!|$. In other words, the functions $n!\binom{x}{n}=x(x-1)\cdots(x-n+1)$ are an orthonormal basis of $A(\mathbf{Z}_p, K)$.

The theorem amounts to showing the functions $(n)_q!\binom{x}{n}_q=(x)_q(x-1)_q\cdots(x-n+1)_q$ are an orthonormal basis of $A(\mathbf{Z}_p, K)$. To show this we compare these functions to $n!\binom{x}{n}$ in order to use Lemma 3.2. By Lemma 4.4, it suffices to find an $\varepsilon<1$ such that $\|(x-j)_q-(x-j)\| \leqslant \varepsilon$ for all $j \in \mathbf{N}$. Well,

$$(x-j)_q-(x-j)=\left(\frac{\log_p q}{q-1}-1\right)(x-j)+\frac{\log_p q}{q-1}\sum_{r\geqslant 2}\frac{(\log_p q)^{r-1}}{r!}(x-j)^r. \tag{4.5}$$

We want a uniform upper bound <1 on the Taylor coefficients. (The definition of the norm on $A(\mathbf{Z}_p, K)$ is based on a Taylor expansion around 0, but recentering the series at j does not affect the maximum size of the Taylor coefficients.)

The coefficient of $x-j$ on the right side of (4.5) is

$$\frac{\log_p q}{q-1} - 1 = \sum_{n \geqslant 2} \frac{(q-1)^{n-1}}{n} (-1)^{n-1}.$$

Note $|(q-1)^{n-1}/n| \leqslant |(q-1)^{n-1}/n!|$. By Lemma 4.4(iv), the coefficients of the higher powers of $x-j$ in (4.5) have size

$$\left|\frac{\log_p q}{q-1} \frac{(\log_p q)^{r-1}}{r!}\right| = \left|\frac{(q-1)^{r-1}}{r!}\right|.$$

So provided $\sup_{r \geqslant 2} |(q-1)^{r-1}/r!| < 1$, we're done. Letting $s_p(r)$ be the sum of the base p digits of r,

$$\begin{aligned}\left|\frac{(q-1)^{r-1}}{r!}\right| &= |q-1|^{r-1}\, p^{(r-s_p(r))/(p-1)} \\ &\leqslant |q-1|^{r-1}\, p^{(r-1)/(p-1)} \leqslant |q-1|\, p^{1/(p-1)}. \quad \blacksquare\end{aligned}$$

COROLLARY 4.1. *For* $|q-1| < (1/p)^{1/(p-1)}$ *and* $|t| < 1$, $(1+t)^{(x;\, q)}$ *is analytic on* $\mathbf{Z}_p$ *if and only if* $|t| < (1/p)^{1/(p-1)}$.

We now connect the work here with that of van Hamme and Verdoodt. They consider the following. Let $a, q \in \mathbf{Z}_p^\times$, perhaps $q \not\equiv 1 \bmod p$, and assume q is not a root of unity. Let V_q denote the closure of the set $\{aq^n\}_{n \geqslant 0}$ in $\mathbf{Z}_p$. It is a compact subset of $\mathbf{Z}_p$, and open since q is not a root of unity. As $q \to 1$, V_q "shrinks" to $\{a\}$. In [23], van Hamme proves every continuous function $f\colon V_q \to \mathbf{Q}_p$ has the form

$$f(x) = \sum_{n \geqslant 0} \frac{(D_q^n f)(a)}{(n)_q!} (x-a)^{(n;\, q)} \tag{4.6}$$

for $x \in V_q$, where $(D_q f)(x) := (f(qx) - f(x))/(qx - x)$ is the q-derivative, D_q^n its nth iterate. Note that the domain V_q of the function depends on q and a. Having $(n)_q!$ in the denominator of (4.12) keeps q away from roots of unity.

When $q \in 1 + p\mathbf{Z}_p$ and is not a root of unity, (4.6) is essentially a q-Mahler expansion. Indeed, in this case the elements of V_q have the form $x = aq^y$ for unique $y \in \mathbf{Z}_p$, in which case

$$\begin{aligned}\frac{(D_q^n f)(a)}{(n)_q!}(x-a)^{(n;\,q)} &= \frac{(D_q^n f)(a)}{(n)_q!}(aq^y-a)^{(n;\,q)} \\ &= (D_q^n f)(a)\cdot a^n(q-1)^n \\ &\quad\times\frac{(q^y-1)(q^y-q)\cdots(q^y-q^{n-1})}{(q^n-1)(q^{n-1}-1)\cdots(q-1)} \\ &= (D_q^n f)(a)\cdot a^n(q-1)^n\, q^{n(n-1)/2}\binom{y}{n}_q.\end{aligned}$$

This last expression has an alternate form by [23, Lemma 3],

$$(D_q^n f)(a)\cdot a^n(q-1)^n\, q^{n(n-1)/2} = \sum_{k=0}^{n}(-1)^k\, q^{k(k-1)/2}\binom{n}{k}_q f(aq^{n-k}).$$

This goes back to Jackson [14, Eq. 12].

Letting $g(y)=f(aq^y)$ be the pullback of f to a continuous function on $\mathbf{Z}_p$, van Hamme's expansion (4.6) becomes

$$g(y)=\sum_{n\geqslant 0}\left(\sum_{k=0}^{n}(-1)^k\, q^{k(k-1)/2}\binom{n}{k}_q g(n-k)\right)\binom{y}{n}_q,$$

which is the q-Mahler expansion of g. But q-Mahler expansions do allow q to be a root of unity, as well as to lie outside of $\mathbf{Q}_p$, though subject to the restriction $|q-1|<1$. In [6], a q-analogue of Mahler expansions will be described for $q\in K$, $|q|=1$, that will reduce to van Hamme's expansion when $q\in\mathbf{Z}_p^\times$ and q is not a root of unity.

In [24, Theorem 3], van Hamme gives a remainder formula for the Mahler expansion. For a complete extension field $K/\mathbf{Q}_p$ and a continuous function $f\colon \mathbf{Z}_p\to K$ with Mahler coefficients c_n,

$$f(x)=c_0+c_1\binom{x}{1}+\cdots+c_n\binom{x}{n}+\Delta^{n+1}f *' \binom{\cdot}{n}, \tag{4.7}$$

where $*'$ is a modified convolution of continuous functions that we now recall. For two continuous functions g and h from $\mathbf{Z}_p$ to K, let $g * h\colon \mathbf{Z}_p\to K$ be the p-adic interpolation to $\mathbf{Z}_p$ of the function $\mathbf{N}\to K$ given by $n\mapsto\sum_{k=0}^n g(k)\,h(n-k)$. (For a proof that this sequence interpolates, see [22, Exercises 34.E, 52.J; 24, Lemma 1].) The operation $*$ is an associative, commutative multiplication on $C(\mathbf{Z}_p, K)$ and $|g*h|_{\sup}\leqslant|g|_{\sup}\,|h|_{\sup}$. By definition, $(g*'h)(x):=(g*h)(x-1)$. Since $\Delta^{n+1}f\to 0$ in $C(\mathbf{Z}_p, K)$, (4.7) is a Mahler expansion with remainder.

Here is the q-Mahler expansion with remainder.

THEOREM 4.5. *Choose $q \in K$ with $|q-1|<1$ and $f \in C(\mathbf{Z}_p, K)$. Letting $c_{0,q}, c_{1,q}, \ldots$ be the q-Mahler coefficients of f,*

$$f(x)=c_{0,q}+c_{1,q}\binom{x}{1}_q+\cdots+c_{n,q}\binom{x}{n}_q+\Delta_q^{n+1}f *' \binom{\cdot}{n}_q,$$

Our proof below will be a translation of Verdoodt's ideas in [25], where she proves a version of this expansion with remainder for functions on the sets V_q. To simplify the comparison with [25], we write the variable in $\mathbf{Z}_p$ as y.

For $y \in \mathbf{Z}_p$, set $\mathcal{U}_n(y)=q^{ny}$, so $\mathcal{U}_0(y)=\binom{y}{0}_q$. (The functions $\mathcal{U}_n=\mathcal{U}_{n,q}$ were already used in Section 3.)

LEMMA 4.5. *For any $n \geqslant 0$, $f=f(0)\,\mathcal{U}_n+(E-q^n)\,f *' \mathcal{U}_n$.*

Proof. We evaluate the right hand side at $y=m \in \mathbf{Z}^+$,

$$\begin{aligned}((E-q^n)\,f *' \mathcal{U}_n)(m)&=\sum_{i=0}^{m-1}(f(i+1)-q^nf(i))\,q^{n(m-1-i)}\\&=\sum_{i=0}^{m-1}f(i+1)\,q^{n(m-(i+1))}-\sum_{i=0}^{m-1}f(i)\,q^{n(m-i)}\\&=f(m)-f(0)\,q^{nm}. \quad \blacksquare\end{aligned}$$

LEMMA 4.6. *For all n,*

$$\mathcal{U}_{n+1} *' \binom{\cdot}{n}_q=\binom{\cdot}{n+1}_q.$$

Proof. Using the first recursion in (2.3),

$$\begin{aligned}\binom{m}{n+1}_q&=\binom{m-1}{n}_q+q^{n+1}\binom{m-1}{n+1}_q\\&=\binom{m-1}{n}_q+q^{n+1}\binom{m-2}{n}_q+q^{2(n+1)}\binom{m-2}{n+1}_q\\&=\sum_{i=0}^{m-1}\binom{m-1-i}{n}_q\,q^{i(n+1)}\\&=\mathcal{U}_{n+1}(y) * \binom{y}{n}_q \qquad \text{at} \quad y=m-1\\&=\mathcal{U}_{n+1}(y) *' \binom{y}{n}_q \qquad \text{at} \quad y=m. \quad \blacksquare\end{aligned}$$

Now we prove Theorem 4.5.

Proof. Writing $g *' h|_y$ for $(g *' h)(y)$ in order to cut down on parentheses,

$$\begin{aligned} f(y) &= f(0)\,\mathscr{U}_0(y) + (E-I)\,f *' \mathscr{U}_0|_y \\ &= f(0) + \Delta f *' \mathscr{U}_0|_y \\ &= f(0) + ((\Delta f)(0)\,\mathscr{U}_1 + (E-q)\,\Delta f *' \mathscr{U}_1) *' \mathscr{U}_0|_y \qquad \text{by Lemma 4.5} \\ &= f(0) + (\Delta f)(0)(\mathscr{U}_1 *' \mathscr{U}_0)|_y + \Delta_q^2 f *' (\mathscr{U}_1 *' \mathscr{U}_0)|_y \\ &= f(0) + (\Delta f)(0)\binom{y}{1}_q + \Delta_q^2 f *' \binom{\cdot}{1}_q\bigg|_y \qquad \text{by Lemma 4.6.} \end{aligned}$$

Assuming

$$f(y) = f(0) + (\Delta_q f)(0)\binom{y}{1}_q + \cdots + (\Delta_q^n f)(0)\binom{y}{n}_q + \Delta_q^{n+1} f *' \binom{\cdot}{n}_q\bigg|_y,$$

apply Lemma 4.5 at $n+1$ with the function $\Delta_q^{n+1} f$, and then use Lemma 4.6. ∎

It is left to the reader to extend the q-Mahler expansion and some properties of it in this section to the case when K is a complete field of characteristic p or a complete commutative $\mathbf{Z}_p$-algebra.

In addition to the q-numbers and q-binomial coefficients we have used, the study of quantum groups has focused attention on the q-analogues

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + \frac{1}{q^{n-3}} + \frac{1}{q^{n-1}} = \frac{1}{q^{n-1}}\,(n)_{q^2},$$

$$[n]_q! := [n]_q\,[n-1]_q \cdots [1]_q = \frac{1}{q^{n(n-1)/2}}\,(n)_{q^2}!,$$

and

$$\begin{bmatrix} m \\ n \end{bmatrix}_q := \frac{[m]_q\,[m-1]_q \cdots [m-n+1]_q}{[n]_q!} = \frac{1}{q^{(m-n)\,n}}\binom{m}{n}_{q^2}.$$

The extra property these have is invariance when q is replaced by $1/q$.

All the properties of $\binom{m}{n}_q$ have analogues for $\left[{m \atop n}\right]_q$, such as

$$[-n]_q = -[n]_q, \qquad [n]_{1/q} = [n]_q, \qquad [mn]_q = [m]_q [n]_{q^m},$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_q \in \mathbf{Z}[q, 1/q],$$

$$\begin{bmatrix} -m \\ n \end{bmatrix}_q = (-1)^n \begin{bmatrix} m+n-1 \\ n \end{bmatrix}_q,$$

$$\begin{bmatrix} m_1+m_2 \\ k \end{bmatrix}_q = \sum_{i+j=k} \begin{bmatrix} m_1 \\ i \end{bmatrix}_q \begin{bmatrix} m_2 \\ j \end{bmatrix}_q q^{m_2 i - m_1 j}.$$

That $\left[{m \atop n}\right]_q$ is related to $\binom{m}{n}_{q^2}$ means there is a different formula for $\left[{m \atop n}\right]_\zeta$ in the case when ζ is an odd or even order root of unity.

For $|q-1|<1$, we get a continuous extension $\left[{x \atop n}\right]_q = (1/q^{n(x-n)})\binom{x}{n}_{q^2}$, and $|\left[{x \atop n}\right]_q - \binom{x}{n}| \leqslant |q-1|$, so the functions $\left[{x \atop n}\right]_q$ form an orthonormal basis of $C(\mathbf{Z}_p, K)$.

It is left to the reader to formulate all the results of this paper so far in this context. As an example of some differences, let $\mathscr{E}_q(X) = \sum X^n/[n]_q!$. Then $\mathscr{E}_{1/q}(X) = \mathscr{E}_q(X)$ and $\mathscr{E}_q(X)\,\mathscr{E}_q(Y)$ equals

$$\sum_{n \geqslant 0} \frac{1}{[n]_q!} \left(\sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q X^{n-m} Y^m \right)$$
$$= \sum_{n \geqslant 0} \frac{(X+Y/q^{n-1})(X+Y/q^{n-3}) \cdots (X+q^{n-1}Y)}{[n]_q!},$$

where powers of q in consecutive terms of the product on the right hand side differ by two.

Set

$$(X+Y)^{[n;q]} := (X+Y/q^{n-1})(X+Y/q^{n-3}) \cdots (X+q^{n-1}Y)$$
$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q X^{n-k} Y^k,$$

so $\mathscr{E}_q(X)\,\mathscr{E}_q(Y) = \sum_{n \geqslant 0} (X+Y)^{[n;q]}/[n]_q!$ and $(X+Y)^{[m+n;q]} = (X+q^n Y)^{[m;q]}\,(X+Y/q^m)^{[n;q]}$. Note $(X-X)^{[n;q]} \neq 0$ if n is even. In particular, $\mathscr{E}_q(X)\,\mathscr{E}_q(-X) \neq 1$, and there doesn't seem to be a simple formula for the coefficients of $\mathscr{E}_q(X)^{-1}$. For example,

$$\mathscr{E}_q(X)^{-1} = 1 - X + \frac{q^2 - q + 1}{q^2 + 1} X^2 - \frac{q^6 - 2q^5 + 2q^4 - q^3 + 2q^2 - 2q + 1}{(1+q^2)(1+q^2+q^4)} X^3 + \cdots,$$

and the numerator of the coefficient of X^3 is irreducible in $\mathbf{Z}[q]$.

We define polynomials $\mu_n(q)$ by $\mathscr{E}_q(X)^{-1} = \sum_{n \geqslant 0} \mu_n(q)\, X^n/[n]_q!$, using the notation μ by analogy with combinatorial inversion formulas. Then

$$f(x) = \sum_{n \geqslant 0} C_{n,q} \begin{bmatrix} x \\ n \end{bmatrix}_q \Leftrightarrow C_{n,q} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mu_{n-k}(q)\, f(k).$$

5. THE p-ADIC q-GAMMA FUNCTION

To illustrate the possibility of using q-Mahler expansions with a family of functions depending continuously on a parameter, we consider Morita's p-adic Gamma function Γ_p and its q-analogue $\Gamma_{p,q}$ as defined by Koblitz.

For a nonnegative integer n, Morita [20] defines

$$\Gamma_p(n+1) := (-1)^{n+1} \prod_{\substack{1 \leqslant j \leqslant n \\ (p,j)=1}} j = \frac{(-1)^{n+1}\, n!}{p^{[n/p]}[n/p]!}$$

for $n \geqslant 1$ and $\Gamma_p(1) = -1$. Morita's proof that Γ_p is p-adically continuous is based on congruence properties of the sequence $\{\Gamma_p(n+1)\}$. For our treatment here, it is Barsky's proof [1] of the continuity which is of primary interest. Barsky's method is based on the identity

$$\sum_{n \geqslant 0} \frac{(-1)^{n+1}\Gamma_p(n+1)}{n!} X^n = (1 + X + \cdots + X^{p-1})\, e^{X^p/p}, \tag{5.1}$$

which implies that the Mahler coefficients $\tau_p(n)$ (say) of the sequence $\Gamma_p(n+1)$ satisfy

$$\sum_{n \geqslant 0} \frac{(-1)^{n+1}\, \tau_p(n)}{n!} X^n = (1 + X + \cdots + X^{p-1})\, e^{X + X^p/p}. \tag{5.2}$$

Writing $e^{X+X^p/p}=\sum_{n\geqslant 0}(b_{p,n}/n!)\,X^n$, estimates of Dwork [17, p. 320] imply $b_{p,n}\to 0$ p-adically as $n\to\infty$, so $\tau_p(n)\to 0$ as $n\to\infty$. Therefore Γ_p extends continuously from $\mathbf{N}$ to $\mathbf{Z}_p$.

We recall Dwork's proof that $b_{p,n}\to 0$. Multiply $\exp(X+X^p/p)$ by the additional terms $\exp(X^{p^j}/p^j)$ for $j\geqslant 2$ and then remove them:

$$e^{X+X^p/p}=\exp\left(\sum_{j\geqslant 0}\frac{X^{p^j}}{p^j}\right)\prod_{j\geqslant 2}e^{-X^{p^j}/p^j}. \tag{5.3}$$

We want to show $\exp(X+X^p/p)$ is in the space of p-adic divided power series $\sum c_n X^n/n!$ where $c_n\to 0$. Such series form the Leopoldt space. It is a Banach algebra when we norm such series by $\sup|c_n|$. Since $\exp(\sum_{j\geqslant 0}X^{p^j}/p^j)$ is the Artin–Hasse series, which has $\mathbf{Z}_p$-coefficients, it is a Leopoldt series. (Any series with bounded coefficients is a Leopoldt series.) By a direct calculation, $\exp(\pm X^{p^j}/p^j)$ is a Leopoldt series and $\to 1$ in the Leopoldt norm as $j\to\infty$. So by completeness the right side of (5.3) is a Leopoldt series. Thus $b_{p,n}\to 0$.

For $|q-1|<1$, the q-analogue $\Gamma_{p,q}$ of Γ_p is defined by Koblitz [16] by

$$\begin{aligned}\Gamma_{p,q}(n+1) &:= (-1)^{n+1}\prod_{\substack{1\leqslant j\leqslant n\\(p,j)=1}}\frac{q^j-1}{q-1}\\ &= (-1)^{n+1}\prod_{\substack{1\leqslant j\leqslant n\\(p,j)=1}}(1+q+\cdots+q^{j-1})\end{aligned}$$

for $n\geqslant 1$ and $\Gamma_{p,q}(1)=-1$. For fixed q with $0<|q-1|<1$, Koblitz shows that the sequence $\Gamma_{p,q}(n+1)$ p-adically interpolates to $\mathbf{Z}_p$ by comparing $\Gamma_{p,q}$ with Γ_p, whose continuity is already known. There are alternate proofs of the interpolation for $\Gamma_{p,q}$ (cf. [5]), but we would like to have available a proof of the interpolation based on Barsky's method, proceeding as follows.

For any integer j, $(j)_{q_1}\equiv(j)_{q_2} \bmod q_1-q_2$, so $|\Gamma_{p,q_1}(n+1)-\Gamma_{p,q_2}(n+1)|\leqslant|q_1-q_2|$. Thus p-adic interpolation of $\Gamma_{p,q}(n+1)$ for general q will follow from that for a dense set of q. So we may suppose q is *not* a root of unity, making $(n)!_q$ nonzero for all n.

In this case, which we may assume we are in from now on,

$$\Gamma_{p,q}(n+1)=\frac{(-1)^{n+1}\,(n)!_q}{\prod_{k\leqslant[n/p]}(pk)_q}=\frac{(-1)^{n+1}\,(n)!_q}{(p)_q^{[n/p]}\,([n/p])!_{q^p}}.$$

Following Barsky, we consider

$$\sum_{n\geqslant 0} \frac{(-1)^{n+1}\,\Gamma_{p,q}(n+1)}{(n)!_q} X^n = \sum_{n\geqslant 0} \frac{1}{(p)_q^{[n/p]}\,([n/p])!_{q^p}} X^n$$
$$= \sum_{r=0}^{p-1} \sum_{m\geqslant 0} \frac{1}{(p)_q^m\,(m)!_{q^p}} X^{pm+r}$$
$$= (1+X+\cdots+X^{p-1}) \sum_{m\geqslant 0} \frac{(X^p/(p)_q)^m}{(m)!_{q^p}}$$
$$= (1+X+\cdots+X^{p-1})\, E_{q^p}(X^p/(p)_q).$$

Let $\tau_{p,q}(n)$ be the nth q-Mahler coefficient of the sequence $\Gamma_{p,q}(n+1)$. We want to show $\tau_{p,q}(n)\to 0$ as $n\to\infty$. Continuing with the above calculations, we obtain

$$\sum_{n\geqslant 0} \frac{(-1)^{n+1}\,\tau_{p,q}(n)}{(n)!_q} X^n = (1+X+\cdots+X^{p-1})\, E_q(-X)^{-1}\, E_{q^p}(X^p/(p)_q)$$
$$= (1+X+\cdots+X^{p-1})\, E_{1/q}(X)\, E_{q^p}(X^p/(p)_q).$$

Comparing this with (5.2) shows the q-analogue of $e^{X+X^p/p}$ is apparently

$$E_{1/q}(X)\, E_{q^p}(X^p/(p)_q) = (E_{1/q}(X)\, E_{1/q}(-X))\cdot E_q(X)\, E_{q^p}(X^p/(p)_q).$$

By the q-Mahler theorem, the existence of a p-adic interpolation for $\Gamma_{p,q}(n+1)$ is thus equivalent to the fact that, when we write

$$E_{1/q}(X)\, E_{q^p}(X^p/(p)_q) = \sum_{n\geqslant 0} b_{p,q,n} \frac{X^n}{(n)!_q},$$

the sequence $b_{p,q,n}$ tends to 0 as $n\to\infty$. This suggests looking at a q-Leopoldt space, namely the q-divided power series $\sum c_n X^n/(n)!_q$ where $c_n\to 0$. By a direct calculation for $j\geqslant 2$, $E_{q^{p^j}}(X^{p^j}/(p^j)_q)$ is a unit in the q-Leopoldt space, so carrying out a q-version of Barsky's argument comes down to checking that a q-analogue of the Artin–Hasse series,

$$E_{1/q}(X) \prod_{j\geqslant 1} E_{q^{p^j}}(X^{p^j}/(p^j)_q), \tag{5.4}$$

is in the q-Leopoldt space. (Since $E_{1/q}(X)\, E_{1/q}(-X)$ is a unit in the q-Leopoldt space, we can replace $E_{1/q}(X)$ with $E_q(X)$ in (5.4) without affecting the property of being or not being a q-Leopoldt series.)

Here we are left with a gap, as we do not see how to establish (5.4) is a q-Leopoldt series without referring to the preexisting fact that $\Gamma_{p,q}(n+1)$ interpolates. Is there a method of analyzing (5.4) without using anything about $\Gamma_{p,q}$, and ideally also not relying on the case $q=1$ first? It may be possible to carry out this task more easily when $|q-1|<(1/p)^{1/(p-1)}$, but ultimately there should be an argument valid for $|q-1|<1$.

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