A BRIEF ATTMEPT AT REFORMULATING THE SETUP IN TERMS OF ALGEBRAIC GROUPS

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1. Arithmetic of indefinite quaternion algebras (following shimura)

1.1. **Indefinite quaternion algebras.** Let F be a totally real field, with finite degree t over $\mathbb{Q}$. A quaternion algebra D over F is a central simple algebra over F such that $|D : F| = 4$. Write R for the ring of integers of F. For a prime ideal $\mathfrak{p}$ of R, we let $R_\mathfrak{p}$, $F_\mathfrak{p}$, and $D_\mathfrak{p}$ denote the $\mathfrak{p}$-adic completions of R, F, and D respectively. We enumerate the t infinite places of F and denote them by $\mathfrak{p}_{\infty,1}, \cdots \mathfrak{p}_{\infty,t}$. For $i \leq t$ we write D^i for the completion of D at $\mathfrak{p}_{\infty,i}$ so that

$$D \otimes_\mathbb{Q} \mathbb{R} \approx D^1 \times \cdots \times D^t$$

There are exactly two quaternion algebras over $\mathbb{R}$: the split algebra $M_2(\mathbb{R})$, and the nonsplit algebra $\mathcal{H}$. After reindexing, we may assume that for $i \leq r$, we have $D^i \approx M_2(\mathbb{R})$, and for $r < i \leq t$, we have $D^i \approx \mathcal{H}$. We say that D is **indefinite** if $r > 0$, and henceforth will assume that this is the case.

For an element $a \in D$, we write its image in D^i by a^i. Thus, for each $i \leq r$ we have $a^i \in M_2(\mathbb{R})$ and $a^i \in \mathcal{H}$ for $r < i \leq t$. Note that the restrictions of the maps $(\cdot)^i$ to the central copy of F in D yeild all of the embeddings of F into $\mathbb{R}$. We write F^i for that image.

The algebras D (resp $D_\mathfrak{p}, D^i$) are each equipped with an involution $a \mapsto a'$ characterized by the condition that $F[a] \approx F[x]/((X-a)(X-a'))$ (resp. $F_\mathfrak{p}[a], \mathbb{R}[a]$). Set, for each $a \in D$ (resp, in $D_\mathfrak{p}, D^i$)

$$N(a) = aa' \quad \mathrm{tr}(a) = a + a'.$$

For those $i \leq r$, under the identification $D^i \approx M_2(\mathbb{R})$, the maps N and tr coincide with det and tr of matrices.

Letting $N_{F/\mathbb{Q}}$ and $\mathrm{tr}_{F/\mathbb{Q}}$ denote the absolute norm and trace maps on F, we define absolute maps for $a \in D$:

$$N_{D/\mathbb{Q}}(a) = N_{F/\mathbb{Q}}(N(a)) \quad \mathrm{tr}_{D/\mathbb{Q}}(a) = \mathrm{tr}_{F/\mathbb{Q}}(\mathrm{tr}(a))$$

1.2. **Ideal theory in D.** An R (resp. $R_\mathfrak{p}$) lattice in D (resp. $D_\mathfrak{p}$) is a finitely generated R-module (resp. $R_\mathfrak{p}$-module) M in D (resp. $D_\mathfrak{p}$) such that $FM = D$ (resp. $F_\mathfrak{p}M = D_\mathfrak{p}$).

Date: June 3, 2023.

1.2.1. *consider introducing the set $\mathcal{L}(D)$ of lattices in D, as well as its local counterparts.* A subring of D containg R is an **order** if it is also an R lattice. An order is maximal if its not properly contained in any other order. Maximal orders exist, and any order is contained in a maximal one.

For an order $\mathfrak{o}$, a lattice M in D is a right (resp. left) $\mathfrak{o}$-ideal if $M\mathfrak{o} \subset M$ (resp. $\mathfrak{o}M \subset M$). We say M is a two-sided $\mathfrak{o}$ ideal if it is both a left and a right $\mathfrak{o}$-ideal.

1.3. **The local theory, split case:** In this section F is a finite extension of $\mathbb{Q}_p$ with ring of integers R, π a uniformizer, ord the normalized valuation. For an element $y \in \mathrm{GL}(2, F)$ write $\bar{y}$ for its image in $\mathrm{PGL}(2, F)$. Say y (or $\bar{y}$) is <u>even</u> if $\mathrm{ord}(\det(y))$ is so, and odd otherwise. Write $F \times F$ as rwo vectors, and let $M(2, F)$ act on it from the right.

A maximal R order $\mathcal{O}$ of $M(2, F)$ takes the form

$$\mathcal{O} = \mathrm{End}_R(\Lambda) = \{x \in M(2, F) : \Lambda x \subset \lambda\}$$

for some R lattice $\Lambda \subset F \times F$, uniquely determined by $\mathcal{O}$ up to homothety. Conversely, for any such lattice Λ, the ring $\mathrm{End}_R(\Lambda)$ is a maximal order.

Given two maximal orders $\mathcal{O}_1$ and $\mathcal{O}_2$, pick a lattice $\Lambda_1 = Rf + Rg$ so that $\mathcal{O}_1 = \mathrm{End}_R(\Lambda_1)$. Then there is a lattice Λ_2 such that $\mathcal{O}_2 = \mathrm{End}_R(\Lambda_2)$ and $\Lambda_2 = Rf + \pi^n Rg \leq \Lambda_1$. The integer $n = d(\mathcal{O}_1, \mathcal{O}_2)$ is uniquely determined by $\mathcal{O}_1$ and $\mathcal{O}_2$.