On a converse to Sunada

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Sunada introduced a group theoretic mechanism for producing pairs of isospectral manifolds. In this note, I formulate a converse to that construction and discuss some avenues by which one might prove them.

Gassman triples Let G be a finite group and H_1 and H_2 subgroups. We say that the triple (G, H_1, H_2) is Gassman, or that H_1 and H_2 are almost conjugate in G, if any of the following three equivalent conditions are met:

- For any G-conjugacy class $C \subset G$, the cardinality of the set $C \cap H_i$ is independent of i.
- The $\mathbb{Q}[G]$ isomorphism type of $\mathbb{Q}[G/H_i]$ is independent of i
- The $k[G]$ isomorphism type of $k[G/H_i]$ is independent of i for any field k of characteristic zero.
- For any irreducible representation (ρ, V) of G, the dimension of V^{H_i} is independent of i.

Sunada diamonds Let M_1 and M_2 be closed Riemannian manifolds, and suppose that the universal covers $\tilde{M}_i$, equipped with the pullback metric, are isometric; in this case write $\tilde{M}$ for $\tilde{M}_i$. Write $\Gamma_i = \pi_1(M_i)$, which we view as subgroups of $\mathrm{Isom}(\tilde{M})$ acting by deck transformations of the universal covering map, so that $M_i \approx \Gamma_i \backslash \tilde{M}$. We say that M_1 and M_2 are **commensurable** if they admit a common finite Riemannian cover, and **strongly commensurable** if furthermore there is a Riemannian manifold admitting both M_1 and M_2 as finite Riemannian covers. Then M_1 and M_2 are commensurable if, after a suitable conjugation, the intersection $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_i, and are strongly commensurable if each Γ_i has finite index in the group generated by the product $\Gamma_1 \Gamma_2$ in $\mathrm{Isom}(\tilde{M})$.

Suppose now that M_1 and M_2 are strongly commensurable, with common cover M_o, commonly covering M, and that the cover $M_o \to M$ is regular, with deck group G. If $\Gamma = \pi_1(M)$ and $\Gamma_o = \pi_1(M_o)$ then this assumption amounts to $\Gamma_o \triangleleft$

$$\begin{tikzcd}
B & M \arrow[dl, two heads] \arrow[dr, two heads] & \\
M_1 \arrow[dr, two heads] & & M_2 \arrow[dl, two heads] \\
& M_o &
\end{tikzcd}$$

Taking a finite cover, if necessary, we assume that the cover $M \to M_o$ is regular, and set $G = \mathrm{Gal}(M/M_o)$. Let $H_i = \mathrm{Gal}(M/M_i)$ denote the subgroups corresponding to the intermediate covers $M \twoheadrightarrow M_i$ for $i = 1, 2$. Letting G act on M by deck transformations, the diagram above is

$$\begin{tikzcd}
& M \arrow[dl, two heads] \arrow[dr, two heads] & \\
H_1\backslash M \arrow[dr, two heads] & & H_2\backslash M \arrow[dl, two heads] \\
& G\backslash M &
\end{tikzcd}$$