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Tensorial Properties of Multiple View Constraints

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We define and derive some properties of the different multiple view tensors. The multiple view geometry is described using a four-dimensional linear manifold in $\mathbb{R}^{3m}$, where m denotes the number of images. The Grassman co-ordinates of this manifold build up the components of the different multiple view tensors. All relations between these Grassman co-ordinates can be expressed using the quadratic p-relations. From this formalism it is evident that the multiple view geometry is described by four different kinds of projective invariants; the epipoles, the fundamental matrices, the trifocal tensors and the quadrifocal tensors. We derive all constraint equations on these tensors that can be used to estimate them from corresponding points and/or lines in the images as well as all transfer equations that can be used to transfer features seen in some images to another image.

As an application of this formalism we show how a representation of the multiple view geometry can be calculated from different combinations of multiple view tensors and how some tensors can be extracted from others. We also give necessary and sufficient conditions for the tensor components, i.e. the constraints they have to obey in order to build up a correct tensor, as well as for arbitrary combinations of tensors. Finally, the tensorial rank of the different multiple view tensors are considered and calculated. Copyright © 2000 John Wiley & Sons, Ltd.

1. Introduction

There has been an intensive research on multiple view geometry over the last few years. Several different approaches to treat the case of many images, taken by uncalibrated cameras, of a 3D object have been proposed in the literature. These approaches range from using camera matrices using multilinear forms to using multiple view tensors. In this paper we will concentrate on the tensorial description and show its relation to other descriptions.

1.1. *Baseline*

One of the first approaches to multiple view geometry is to use the camera matrices directly, as in [15], where canonic representations for the multiple view geometry are given. The disadvantage of the use of camera matrices is that one has to choose a co-ordinate system in 3D in order to write down the camera matrices. This fixes

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some degrees of freedom that are natural in the uncalibrated case, which we argue is less intuitive than leaving these degrees of freedom unfixed.

The multilinear forms have been treated by many researchers. The bilinear form, expressed by the fundamental matrix, has been known for a long time. The trilinear forms were originally discovered by Spetsakis and Aloimonis in [19], in the calibrated case, and rediscovered by Shashua, in [16, 18] in the uncalibrated case. Faugeras and Mourrain, in [1, 2, also available in (3)] use Grassman–Cayley algebra to derive relation between different multilinear forms. Heyden uses a reduced form of the multilinear constraints in [11]. In [12, 13] the algebraic relations between the different multilinear constraints have been explored in detail. One drawback of the multilinear forms is that one has to be careful about what can be estimated from image measurements and what can be calculated from the multilinear forms. These two things are often, unintentionally, mixed up.

The tensorial description was introduced, in the case of three images, by Hartley, in [8, 10] and in a general projective framework by Triggs, in [20]. Hartley has also put all the different tensors in a general framework in [9], where the constraint equations on the tensor components obtained from corresponding points/lines in the images are also derived. Reduced forms of the multiple view tensors, together with necessary and sufficient conditions for the tensor components, have been introduced by Heyden, in [11]. Necessary and sufficient conditions for the ordinary trifocal tensor components have been found by Faugeras and Papadopoulo, in [5, also available in [4]; 6]. These conditions are needed when estimating the tensor components using non-linear methods. The advantages of using the multiple view tensors are that no co-ordinate system has to be fixed and that the difference between the entities that can be estimated from image measurements and those estimated from the tensor components is more evident.

The novelty of the present paper is mainly the general framework introduced by viewing the tensor components as Grassman co-ordinates of a certain linear subspace. A number of new results has also been obtained. The necessary and sufficient conditions for the quadrifocal tensor are to our knowledge new as well as the tensorial rank of the quadrifocal tensor. The complete derivation of all constraint equations and transfer equations are also new although the equations *per se* have been used before. Also the derivation of the number of independent constraints on the trifocal and quadrifocal tensors are new. When preparing this paper we came to know that these results have been obtained in [9], but we believe that our derivations are simpler. Finally, the explicit relations between the different tensor components that allows us to calculate one tensor from another are new, although some relations for the trifocal tensor can be found in [5].

1.2. *Background on computer vision*

Computer vision deals with the problem of extracting information about the three-dimensional world from a number of its two-dimensional perspective images. Different tasks could be to recognize some pre-defined objects or build up a three-dimensional map of the surrounding world. Multiple view geometry deals with the problem of investigating the geometric situation encountered when trying to

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reconstruct the three-dimensional world from a sequence of images taken from different locations and at the same time recovering the unknown motion of the camera. A camera is usually modelled by the equation

$$\lambda \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \gamma f & sf & x_0 \\ 0 & f & y_0 \\ 0 & 0 & 1 \end{bmatrix} [R \mid -Rt] \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \Leftrightarrow \lambda \mathbf{x} = K[R \mid -Rt]\mathbf{X} = P\mathbf{X}. \tag{1}$$

Here $\mathbf{X} = [X \;\; Y \;\; Z \;\; 1]^{\mathrm{T}}$ denotes object co-ordinates in the extended form and $\mathbf{x} = [x \;\; y \;\; 1]^{\mathrm{T}}$ denotes extended image co-ordinates. The scale factor λ accounts for perspective effects and (R, t) represents a rigid transformation of the object, i.e. R denotes a 3×3 rotation matrix and t a 3×1 translation vector. Finally, the parameters in K represents intrinsic properties of the image formation system: f represents the focal length, the aspect ratio γ and the skew s model non-quadratic light sensitive elements and (x_0, y_0) is called the principal point and is interpreted as the orthogonal projection of the focal point onto the image plane. The parameters in R and t are called extrinsic parameters and the parameters in K are called the intrinsic parameters. Observe that there are 6 extrinsic and 5 intrinsic parameters, totally 11, the same number as in an arbitrary 3×4 matrix defined up to a scale factor. Observe that the centre of projection, also called the focal point, C, is obtained as the nullspace to P, i.e. $PC = 0$. Furthermore, the line, called the optical ray, defined by the focal point C and an image point $\mathbf{x}$ is given by $C + tP^{+}\mathbf{x}$, where P^{+} denotes the More–Penrose pseudoinverse of P, since $P(C + tP^{+}\mathbf{x}) = PC + tPP^{+}\mathbf{x} = 0 + t\mathbf{x} = \lambda\mathbf{x}$, with $\lambda = t$. Similarly, from a line in the image, represented by $\mathbf{1} = [a\, b\, c]^{\mathrm{T}}$,† a plane, in 3D containing this line and the focal point can be obtained by the formula $\mathbf{N} = P^{\mathrm{T}}\mathbf{1}$, where $\mathbf{N} = [n_1 n_2 n_3 n_4]^{\mathrm{T}}$ represents the plane. In fact, given any point $\mathbf{x}$ on the line, i.e. $\mathbf{x}^{\mathrm{T}}\mathbf{1} = 0$, we have $0 = \mathbf{x}^{\mathrm{T}}\mathbf{1} = \mathbf{X}^{\mathrm{T}}P^{\mathrm{T}}\mathbf{1}$, which in turn implies that $P\mathbf{X}$ is a point on the line $\mathbf{1}$. Observe also that when a change of co-ordinate system is made in the image according to $\mathbf{x} = S\hat{\mathbf{x}}$, where S denotes a non-singular 3×3 matrix, the camera matrix changes to $P = S\hat{P}$. Note also that $\mathbf{1}$ representing a line changes according to $\hat{\mathbf{1}} = S^{\mathrm{T}}\mathbf{1}$, since $\mathbf{1}^{\mathrm{T}}\mathbf{x} = \hat{\mathbf{1}}^{\mathrm{T}}\hat{\mathbf{x}} = 0$, when $\mathbf{x}$ is on the line $\mathbf{1}$.

If the intrinsic parameters are known, the camera is said to be calibrated. On the other hand, if both the extrinsic and intrinsic parameters are unknown, we say that the camera is uncalibrated. If the extrinsic as well as the intrinsic parameters are unknown and allowed to vary between the different imaging instants, $i = 1, \ldots, m$, (1) can compactly be written

$$\lambda_i \mathbf{x}_i = P_i \mathbf{X} \tag{2}$$

and the image sequence is said to be uncalibrated. Since there is an unknown scale factor involved in (2) it is natural to use projective spaces for representing object points and image points. Using homogeneous co-ordinates in the images and in the object, now represented by $\mathbf{x}_i = [x_i \;\; y_i \;\; z_i]^{\mathrm{T}}$ and $\mathbf{X} = [X \;\; Y \;\; Z \;\; W]^{\mathrm{T}}$,

† $\mathbf{x}$ belongs to the line represented by $\mathbf{1}$ iff $\mathbf{1}^{\mathrm{T}}\,\mathbf{x} = 0$.

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(2) can be written

$$\mathbf{x}_i \sim P_i\mathbf{X}, \tag{3}$$

where $\sim$ means equality up to scale. This switch to homogeneous co-ordinates will be made when convenient.

The purpose of most reconstruction techniques, based on multiple view tensors, is to reconstruct the scene from a number of images in a series of linear steps. Therefore these algorithms are called linear reconstruction algorithms. A common theme for these linear reconstruction algorithms is that they are based on the following procedure.

1. Obtain point correspondences between the images.
2. Estimate multilinear constraints from these correspondences.
3. Reconstruct the object from the multilinear constraints.

The first step is based on low-level vision operations, e.g. corner detection and tracking. In the second step one has to choose a subset of all available multilinear constraints, when the number of images is large, and estimate the different tensor components using these constraints. The third step depends on the subset of estimated tensor components. Thus an interesting question is what subset of multiple view tensors should be chosen. To answer this question one has to know the relation between different tensor components (and also between the components of one fixed tensor). This is the problem we address.

1.3. *Outline of paper*

In this paper we will present a common framework for the definition and manipulation of multiple view tensors. The definitions and properties of the multiple view tensors, including constraint equations and transfer equations, will be given in section 2. The bifocal tensor will be treated in section 2.1, the trifocal tensor in section 2.2, the quadrifocal tensor in section 2.3 and, finally, the monofocal tensor in section 2.4. In section 3, we will show how the camera matrices relates to the multiple view tensors and introduce a normal form for the representation of the multiple view geometry. The intrinsic properties of the tensors will be studied in section 4. The two view case will be studied in detail in section 4.1, the three view case in section 4.2 and the four view case in section 4.3. Here we show how to calculate lower order tensors from the quadrifocal, trifocal or bifocal tensors and how to obtain a representative of the multiple view geometry. In section 4.4 the N view case will be commented upon, before the conclusions will be given in section 5.

2. The multiple view tensors

In this section the multiple view tensors will be defined in a generic way. We start with the standard camera equation, for an uncalibrated pinhole camera:

$$\lambda\mathbf{x} = P\mathbf{X}, \tag{4}$$

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where $\mathbf{X} = [X \;\; Y \;\; Z \;\; 1]^{\mathrm{T}}$ denote extended object co-ordinates, $\mathbf{x} = [x \;\; y \;\; 1]^{\mathrm{T}}$ denote extended image co-ordinates, P denotes the camera matrix and λ a scale-factor.

Assume that we have m images of n points taken by a camera, modelled as in (4). Then we can write

$$\lambda_{i,j}\mathbf{x}_{i,j} = P_i\mathbf{X}_j, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n. \tag{5}$$

Consider one object point $\mathbf{X}_1$ and its images $\mathbf{x}_{i,1}$. Observing that (5) is linear in $\mathbf{X}_1$ and $\lambda_{i,1}$, it can be written, using the notation $\mathbf{X} = \mathbf{X}_1$, $\mathbf{x}_i = \mathbf{x}_{i,1}$ and $\lambda_i = \lambda_{i,1}$,

$$\underbrace{\begin{bmatrix} P_1 & \mathbf{x}_1 & 0 & 0 & \cdots & 0 \\ P_2 & 0 & \mathbf{x}_2 & 0 & \cdots & 0 \\ P_3 & 0 & 0 & \mathbf{x}_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ P_m & 0 & 0 & 0 & \cdots & \mathbf{x}_m \end{bmatrix}}_{M} \begin{bmatrix} \mathbf{X} \\ -\lambda_1 \\ -\lambda_2 \\ -\lambda_3 \\ \vdots \\ -\lambda_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{6}$$

Since M has a non-trivial right nullspace

$$\operatorname{rank} M < m + 4 \tag{7}$$

holds, i.e. all $(m+4) \times (m+4)$ minors of M vanish. Using Laplace expansions, see [7], of these minors it is seen that they can be written as sums of products of determinants of four rows taken from the first four columns of M and products of image co-ordinates. Note that minors not containing at least one row from each camera matrix block is identically zero, since then one column contains only zeros. Furthermore, when only one of the three rows corresponding to one camera matrix is contained in the minor, the image co-ordinate corresponding to this row could be factored out. Since M has $m + 4$ columns and $3m$ rows and at least one row from each block has to be present in a non-trivial minor, at most four camera matrices can be involved. This observation leads us to study the special case of four images.

Consider the first four images and call the first four camera matrices A, B, C and D. The equations in (5) for one object point $\mathbf{X}_1$ can be written, now using the notation $\mathbf{X} = \mathbf{X}_1$, $\mathbf{x}_A = \mathbf{x}_{1,1}$, etc, and $\lambda_A = \lambda_{1,1}$, etc.,

$$\underbrace{\begin{bmatrix} A & \mathbf{x}_A & 0 & 0 & 0 \\ B & 0 & \mathbf{x}_B & 0 & 0 \\ C & 0 & 0 & \mathbf{x}_C & 0 \\ D & 0 & 0 & 0 & \mathbf{x}_D \end{bmatrix}}_{M} \begin{bmatrix} \mathbf{X} \\ -\lambda_A \\ -\lambda_B \\ -\lambda_C \\ -\lambda_D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{8}$$

In the same way as before we conclude that

$$\operatorname{rank} M_4 < 8 \tag{9}$$

and using Laplace expansions of suitable 8×8 minors from M_4 we observe that there are essentially three different types of such minors. As commented upon above, one

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row has to be selected from each camera matrix block and then the remaining four rows can be selected from two, three or four different blocks.

2.1. *The bifocal tensor*

The first type of minors from M_4 involves all three rows from one camera matrix and all three rows from another, e.g.

$$\det \begin{bmatrix} A^1 & x_A^1 & 0 \\ A^2 & x_A^2 & 0 \\ A^3 & x_A^3 & 0 \\ B^1 & 0 & x_B^1 \\ B^2 & 0 & x_B^2 \\ B^3 & 0 & x_B^3 \end{bmatrix} = 0, \tag{10}$$

where A^i denotes the ith row of A and the notation $\mathbf{x}_A = [x_A^1 \;\; x_A^2 \;\; x_A^3]$ has been used. To be precise the actual 8×8 minor from M_4 is a product of one image co-ordinate in image 3, one image co-ordinate in image 4 and the determinant in (10), but since these image co-ordinates can be factored out they are omitted. Expanding the determinant in (10) gives

$$\frac{1}{4}\,\varepsilon_{ii'i''}\varepsilon_{jj'j''} \det \begin{bmatrix} A^{i'} \\ A^{i''} \\ B^{j'} \\ B^{j''} \end{bmatrix} x_A^i x_B^j = 0, \tag{11}$$

where ε_{ijk} denote the permutation symbol (i.e. $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$, $\varepsilon_{321} = \varepsilon_{213} = \varepsilon_{132} = -1$ and $\varepsilon_{ijk} = 0$ if two indices are equal) and Einsteins' summation convention has been used. Note that the left-hand side of (11) is a bilinear form in the image coordinates of the two images.

Definition 2.1. *The constraint in* (11) *will be called the* bilinear constraint *or the* epipolar constraint.

The coefficients of the bilinear forms are the components of the bifocal tensor defined as follows.

Definition 2.2. *Let P_I^i denote the* ith *row of the camera matrix P_I for image I. Then the tensor*

$$_{IJ}F_{ij} = \frac{1}{4}\,e_{ii'i''}\varepsilon_{jj'j''} \det \begin{bmatrix} P_I^{i'} \\ P_I^{i''} \\ P_J^{j'} \\ P_J^{j''} \end{bmatrix}, \tag{12}$$

where Einsteins' summation convention has been used, is called the bifocal tensor *corresponding to views I and J.*

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Using the bifocal tensor, the bilinear constraints in (11) can be written

$$ {}_{IJ}F_{ij}\mathbf{x}_I^i\mathbf{x}_J^j = 0. \tag{13}$$

Proposition 2.1. *Corresponding points in two images fulfill the bilinear constraints in* (13).

When the notation in (8) has been used, the bifocal tensor will be denoted by ${}_{AB}F$, ${}_{CB}F$, etc. Note that from Definition 2.2 it follows that ${}_{IJ}F_{ij} = {}_{JI}F_{ji}$.

Remark. The bilinear constraint in (11) can be written as

$$\mathbf{x}_A^{\mathrm{T}}\mathbf{F}\mathbf{x}_B = 0,$$

where *the fundamental matrix* F is defined according to $\mathbf{F}_{(i,j)} = {}_{IJ}F_{ij}$ ($\mathbf{F}_{(i,j)}$ denotes the element with row-index i and column-index j of $\mathbf{F}$).

In fact, ${}_{IJ}F_{ij}$ can be viewed as a tensor of degree 2, which is covariant in both indices, which alludes to how the tensor components change when making co-ordinate changes in the images. Choose new co-ordinate systems in image I and J, by changing co-ordinates from $\mathbf{x}_I$ to $\hat{\mathbf{x}}_I$ according to $\mathbf{x}_I = S\hat{\mathbf{x}}_I$, i.e. $\mathbf{x}_I^i = s_{i'}^i\hat{\mathbf{x}}_I^{i'}$, where $s_{i'}^i = S_{(i,i')}$, and similarly for image J using the U instead of S. Then the camera matrices change to $A = S\hat{A}$, i.e. $A_j^i = s_{i'}^i\hat{A}_j^{i'}$ and similarly for B using U instead of S. It follows from (12) (or from (13) that the tensor ${}_{IJ}\hat{F}_{i'j'}$ describing the relations between the new co-ordinates in image I and the new co-ordinates in image J can be written

$${}_{IJ}\hat{F}_{i'j'} = s_{i'}^i u_{j'}^j {}_{IJ}F_{ij}.$$

The tensor can also be used to transfer a point in one image, say the point $\mathbf{x}_I$ in image I, to the so-called *epipolar line*, $\mathbf{1}^J$ in the second image according to the following theorem.

Theorem 2.1. *Let* $\mathbf{1}^J$ *denote the line defined as the projection in camera J of the line defined by a point*, $\mathbf{x}_I$, *in camera I and the focal point*, C_I, *of camera I, i.e. the projection of the optical ray corresponding to* $\mathbf{x}_I$. *Then the line* $\mathbf{1}^J$ *is given by the transfer formula*

$$\mathbf{1}_j^J = {}_{IJ}F_{ij}\mathbf{x}_I^i. \tag{14}$$

Proof. Given a point, $\mathbf{x}_I$, in image I. Wherever the corresponding object point $\mathbf{X}$ is located on the optical ray the corresponding image point in image J fulfils the bilinear constraint in (13). This bilinear constraint can be interpreted as

$${}_{IJ}F_{ij}\mathbf{x}_I^i\mathbf{x}_J^j = ({}_{IJ}F_{ij}\mathbf{x}_I^i)\mathbf{x}_J^j = 0,$$

i.e. as $\mathbf{x}_I$ lying on the line $\mathbf{1}^J$ defined in (14). □

The transfer equation in (14) tells us that given a point, $\mathbf{x}_I$ in image I, the corresponding point, $\mathbf{x}_J$, in image J is constrained to lie on the epipolar line $\mathbf{1}^J$, i.e. $\mathbf{1}_j^J\mathbf{x}_J^j = 0$, which of course gives (13).

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Table 1. Bilinear constraint and transfer

Constraint	Equation
Two points	${}_{IJ}F_{ij}\mathbf{x}_I^i\mathbf{x}_J^j = 0$
Transfer	Equation
Point to line	$\mathbf{1}_j^J = {}_{IJ}F_{ij}\mathbf{x}_I^i$

Observe also that all epipolar lines intersect in one point, called *the epipole*, which is the projection of the focal point of camera I in camera J. We conclude this subsection with the well-known theorem:

Theorem 2.2. *The bifocal tensor can in general be linearly calculated from at least* 8 *corresponding points in two images.*

Proof. Each point correspondence gives one homogeneous linear constraint on the tensor components according to (11), implying that 8 corresponding points is sufficient to calculate the tensor components up to an unknown global scale, if the constraints are linearly independent (in the tensor components). □

We summarize the constraints and the transfer equations for the bifocal tensor in Table 1.

2.2. *The trifocal tensor*

The second type of minors from M_4 involves three rows from one camera matrix, two rows from another camera matrix and two rows from a third camera matrix, e.g.

$$\det\begin{bmatrix} A^1 & x_A^1 & 0 & 0 \\ A^2 & x_A^2 & 0 & 0 \\ A^3 & x_A^3 & 0 & 0 \\ B^j & 0 & x_B^j & 0 \\ B^{j'} & 0 & x_B^{j'} & 0 \\ C^k & 0 & 0 & x_C^k \\ C^{k'} & 0 & 0 & x_C^{k'} \end{bmatrix} = 0. \tag{15}$$

Expanding the determinant in (15) gives

$$\frac{1}{2}\,\varepsilon_{ii'i''}\varepsilon_{jj'j''}\varepsilon_{kk'k''}\det\begin{bmatrix} A^{i'} \\ A^{i''} \\ B^j \\ C^k \end{bmatrix} x_A^i x_B^{j'} x_C^{k'} = 0_{j''k''}, \tag{16}$$

where j'' and k'' denote free indices corresponding to the omitted row from the second camera matrix block and the third camera matrix block, respectively. Note that the left-hand side of (16) is a trilinear form in the image coordinates in the three images.

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Definition 2.3. *The constraints in* (16) *will be called the* trilinear constraint.

The coefficients of the trilinear forms are the components of the trifocal tensor defined as follows.

Definition 2.4. *Using the notations in Definition* 2.2, *the tensor*

$$ {}^{JK}_{I}T^{jk}_{i} = \frac{1}{2}\,\varepsilon_{ii'i''}\det\begin{bmatrix} P^{i'}_{I} \\ P^{i''}_{I} \\ P^{j}_{J} \\ P^{k}_{K} \end{bmatrix} \tag{17} $$

is called the trifocal tensor *corresponding to views I, J and K.*

Note that the ordering of the views is important, since two rows are chosen from the first camera matrix and one from each of the other in the determinant in (17). Using the trifocal tensor, the trilinear constraint in (16) can be written

$$ {}^{JK}_{I}T^{jk}_{i}\mathbf{x}^{i}_{I}\varepsilon_{jj'j''}\mathbf{x}^{j'}_{J}\varepsilon_{kk'k''}\mathbf{x}^{k'}_{K} = 0_{j''k''}. \tag{18} $$

Proposition 2.2. *Corresponding points in three images fulfil the trilinear constraint in* (18).

When the notation in (8) has been used, the trifocal tensors will be denoted by ${}^{BC}_{A}T$, ${}^{AD}_{C}T$, etc. Note that from Definition 2.4 it follows that ${}^{BC}_{A}T^{jk}_{i} = -\,{}^{CB}_{A}T^{kj}_{i}$. Unfortunately, the relations between ${}^{BC}_{A}T^{jk}_{i}$ and ${}^{AC}_{B}T^{jk}_{i}$ are not so easy to express as this stage, hence, we will come back to this issue later.

The numbers ${}^{JK}_{I}T^{jk}_{i}$ can be viewed as a tensor of degree 3, that are covariant in one index and contravariant in the other indices. This means that changes of coordinates in image I, image J and image K using matrices S, U and V, respectively, change the components of the tensor, from ${}^{JK}_{I}T^{jk}_{i}$ to ${}^{JK}_{I}\hat{T}^{jk}_{i}$, according to

$$ u^{j}_{j'}v^{k}_{k'}\,{}^{JK}_{I}\hat{T}^{j'k'}_{i'} = s^{i}_{i'}\,{}^{JK}_{I}T^{jk}_{i}. \tag{19} $$

This follows directly from (17), or more easily, from the transfer equation (20) that will be obtained shortly. Observe that the tensor components changes differently when co-ordinate changes are made in image I (contraction of T^{jk}_{i} with $s^{i}_{i'}$) than when co-ordinate changes are made in image J or K (contraction of $\hat{T}^{j'k'}_{i'}$ with $u^{j}_{j'}$ and $v^{k}_{k'}$).

The tensor can also be used to transfer two lines in image J and image K, $\mathbf{1}^{J}$ and $\mathbf{1}^{K}$ to the corresponding line in image I, $\mathbf{1}^{I}$, see also [5, 9, 21], according to the following theorem.

Theorem 2.3. *Let* $\mathbf{1}^{J}$ *and* $\mathbf{1}^{K}$ *denote lines in images J and K, respectively. Introduce the plane* Π_J *defined by the image line* $\mathbf{1}^{J}$ *and the focal point* C_J *and similarly the plane* Π_K. *These planes intersect in a line* $\mathbf{L}_{JK}$. *Then the projection of this line in image I,* $\mathbf{1}^{I}$, *is given by the transfer formula*

$$ \mathbf{1}^{I}_{i} = {}^{JK}_{I}T^{jk}_{i}\mathbf{1}^{J}_{j}\mathbf{1}^{K}_{k}. \tag{20} $$

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Proof. Consider a fixed point $\mathbf{X}$ on the line $\mathbf{L}_{jk}$ and the projection of this point in the three images, i.e.

$$\lambda_I \mathbf{x}_I = P_I \mathbf{X}, \quad \lambda_J \mathbf{x}_J = P_J \mathbf{X}, \quad \lambda_K \mathbf{x}_K = P_K \mathbf{X}. \tag{21}$$

From the last two equations in (21) and the fact that the points $\mathbf{x}_J$ and $\mathbf{x}_K$ are coincident with the lines $\mathbf{1}^J$ and $\mathbf{1}^K$ respectively, it follows that

$$(\mathbf{1}^J)^{\mathrm{T}} P_J \mathbf{X} = 0, \quad (\mathbf{1}^K)^T P_K \mathbf{X} = 0. \tag{22}$$

Note that the first equation in (21) and the two equations in (22) are linear in $\mathbf{X}$ and λ_I and can be written in matrix form as

$$\begin{bmatrix} P_I & \mathbf{x}_I \\ (\mathbf{1}^J)^T P_J & 0 \\ (\mathbf{1}^K)^{\mathrm{T}} P_K & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ -\lambda_I \end{bmatrix} = \mathbf{0}. \tag{23}$$

Thus the leftmost matrix in (23) is singular and expanding the determinant by the last column gives

$$\frac{1}{2} \mathbf{x}_I^i \varepsilon_{ii'i''} \det \begin{bmatrix} P_I^{i'} \\ P_I^{i''} \\ (\mathbf{1}^J)^{\mathrm{T}} P_J \\ (\mathbf{1}^K)^{\mathrm{T}} P_K \end{bmatrix} = \frac{1}{2} \mathbf{x}_I^i \mathbf{1}_j^J \mathbf{1}_k^K \varepsilon_{ii'i''} \det \begin{bmatrix} P_I^{i'} \\ P_I^{i''} \\ P_J^j \\ P_K^k \end{bmatrix} = \frac{1}{2} \mathbf{x}_I^i \mathbf{1}_j^J \mathbf{1}_k^K T_i^{jk} = 0. \tag{24}$$

Since this equation is valid for every point $\mathbf{x}_I$ which is a projection of the line $\mathbf{L}_{JK}$, i.e. for every point on the line $\mathbf{1}^I$, the expression $T_i^{jk} \mathbf{1}_j^J \mathbf{1}_k^K$ is a representative of this line giving the transfer equation in (20). □

In addition we have the following transfer equations, see also [21].

Theorem 2.4. *Given a point*, $\mathbf{x}_I$, *in image* I, *and a line*, $\mathbf{1}^J$, *in image* J. *Construct the corresponding optical ray in camera* I *and the corresponding plane* Π_K *in camera* K (*defined by the focal point and the image line*). *This line and plane intersect in a point* $\mathbf{X}$ *and the projection of this point in image* K *is given by the transfer equation*

$$\mathbf{x}_K^k = {}^{JK}_{I}T_i^{jk} \mathbf{x}_I^i \mathbf{1}_j^J. \tag{25}$$

Similarly, *with obvious interpretation*, *the transfer formula*

$$\mathbf{x}_J^j = {}^{JK}_{I}T_i^{jk} \mathbf{x}_I^i \mathbf{1}_k^K \tag{26}$$

holds.

Proof. Consider the point $\mathbf{X}$, its projection in image I and the fact that its projection in image J is coincident with the line $\mathbf{1}^J$,

$$\lambda_I \mathbf{x}_I = P_I \mathbf{X}, \quad (\mathbf{1}^J)^{\mathrm{T}} P_J \mathbf{X} = 0. \tag{27}$$

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Write these equations (linear in $\mathbf{X}$) as

$$\begin{bmatrix} P_I \\ (\mathbf{1}^J)^{\mathrm{T}} P_J \end{bmatrix} \mathbf{X} = \begin{bmatrix} \mathbf{x}_I^i \\ 0 \end{bmatrix} \Rightarrow \mathbf{X} = \begin{bmatrix} P_I \\ (\mathbf{1}^J)^{\mathrm{T}} P_J \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x}_I^i \\ 0 \end{bmatrix}. \tag{28}$$

Now the projection of $\mathbf{X}$ in image K is given by

$$x_K^k \sim P_K^k \begin{bmatrix} P_I \\ (\mathbf{1}^J)^{\mathrm{T}} P_J \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x}_I^i \\ 0 \end{bmatrix} \sim \varepsilon_{ii'i''} \det \begin{bmatrix} P_I^{i'} \\ P_I^{i''} \\ (\mathbf{1}^J)^{\mathrm{T}} P_J \\ P_K^k \end{bmatrix} \mathbf{x}_I^i = \varepsilon_{ii'i''} \det \begin{bmatrix} P_I^{i'} \\ P_I^{i''} \\ P_J^j \\ P_K^k \end{bmatrix} \mathbf{x}_I^i \mathbf{1}_j^J$$

$$= {}_I^{JK}T_i^{jk} \mathbf{x}_I^i \mathbf{1}_j^J,$$

where $\sim$ means equality up to scale and Cramers' rule has been used. The transfer equation in (26) follows by the symmetry of ${}_I^{JK}T_i^{jk}$. □

The trilinear constraint in (16) can be used to estimate the trifocal tensor from point correspondences. Another constraint follows immediately from (20) by contraction with $\mathbf{x}_I^i$ on both sides, is actually obtained in (24), giving

$${}_I^{JK}T_i^{jk} \mathbf{x}_I^i \mathbf{1}_j^J \mathbf{1}_k^K = 0. \tag{29}$$

This constraint can be used if a point in image I and lines in images J and K are corresponding, meaning that the corresponding optical ray and planes intersect at one point. If instead of a line in image J or K a corresponding point is available, new constraints can be obtained by using all lines coincident with the point. Thus, in (29) $\mathbf{1}_j^J$ can be replaced by $\varepsilon_{jj'j''} \mathbf{x}_J^{j'}$, representing lines through $\mathbf{x}_J$,‡ see also [21], and similarly for $\mathbf{1}_k^K$, giving

$${}_I^{JK}T_i^{jk} \mathbf{x}_I^i \varepsilon_{jj'j''} \mathbf{x}_{j'}^J \mathbf{1}_k^K = 0_{j''}, \quad {}_I^{JK}T_i^{jk} \mathbf{x}_I^i \mathbf{1}_{j'}^J \varepsilon_{kk'k''} \mathbf{x}_K^{k'} = 0_{k''}. \tag{30}$$

Finally, a constraint involving only lines can be obtained from (20), by taking the cross product with $\mathbf{1}^I$, i.e. by contracting with $\varepsilon^{ii'i''} \mathbf{1}_{i'}^I$, giving

$${}_I^{JK}T_i^{jk} \varepsilon^{ii'i''} \mathbf{1}_{i'}^I \mathbf{1}_j^J \mathbf{1}_k^K = 0^{i''}. \tag{31}$$

Proposition 2.3. *Corresponding lines in three image fulfil the trilinear constraint in* (31).

From (16) it follows that each corresponding point in three images gives 9 linear constraint on the trifocal tensor components. However, these constraints are not linearly independent. The number of linearly independent constraints obtained for each point correspondence is given in the following theorem, see also [9].

Theorem 2.5. *There are in general* 4 *linearly independent* (*in the tensor components*) *constraints among the* 9 *trilinear constraints* (16).

‡When $j'' = 1$, $\varepsilon_{jj'j''} \mathbf{x}_J^{j'} = \varepsilon_{jj'1} \mathbf{x}_J^{j'}$ represents a horizontal line through $\mathbf{x}_J$, when $j'' = 2$ a vertical line and when $j'' = 3$ a diagonal line.

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Proof. Start with the 9×7-matrix

$$M = \begin{bmatrix} P_1 & \mathbf{x}_1 & 0 & 0 \\ P_2 & 0 & \mathbf{x}_2 & 0 \\ P_3 & 0 & 0 & \mathbf{x}_3 \end{bmatrix}. \tag{32}$$

The 9 minors obtained by removing one row corresponding to the second image (i.e. rows 3–6) and one row corresponding to the third image (i.e. rows 7–9) give the 9 trilinear constraints. Consider now the 9×8 matrix

$$M_e = \begin{bmatrix} P_1 & \mathbf{x}_1 & 0 & 0 & 0 \\ P_2 & 0 & \mathbf{x}_2 & 0 & \mathbf{x}_2 \\ P_3 & 0 & 0 & \mathbf{x}_3 & 0 \end{bmatrix}. \tag{33}$$

obtained by duplicating the 6th column to M in (32). Expanding the 3 minors of M_e in (33) obtained by removing one of the rows corresponding to the second image (i.e. rows 4–6) gives 3 linear constraints on the previously obtained linear constraints, called-second order linear constraints. The same thing can be done by duplicating the 7th column of M, giving in total 6 second-order linear constraints. On the other hand, every linear dependency among the trilinear constraints can be written in matrix form as in (33) with the coefficients (independent of P_1, P_2 and P_3) of the vanishing linear combination in the last column. Thus, we have obtained all second-order linear constraints.

However, the process does not stop here, there are even linear constraints between these 6 second-order linear constraints, which can be obtained from the determinant of the 9×9 matrix

$$\begin{bmatrix} P_1 & \mathbf{x}_1 & 0 & 0 & 0 & 0 \\ P_2 & 0 & \mathbf{x}_2 & 0 & \mathbf{x}_2 & 0 \\ P_3 & 0 & 0 & \mathbf{x}_3 & 0 & \mathbf{x}_3 \end{bmatrix}, \tag{34}$$

giving one linear constraint on these 6 constraints, called third-order linear constraints. Again, using the same argument as above, it is evident that all third-order linear constraints have been obtained. To sum up, the number of linearly independent constraints on the tensor components from the last equation in (30) is

$$9 - 6 + 1 = \binom{0}{2} 3^2 - \binom{2}{1} 3^1 + \binom{2}{2} 3^0 = (1-3)^2 = 4,$$

which concludes the proof. □

We immediately get the following important corollary.

Corollary 2.1. *The trifocal tensor can in general be linearly calculated from at least 7 corresponding points in 3 images.*

Proof. The total number of linearly independent constraints are $7 \times 4 = 28$ and there are 27 (homogeneous) components of the trifocal tensor. □

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Again, in general, means that the 4 linearly independent constraints obtained from each point are all together linearly independent.

Correspondences consisting of one point (in image I) and two lines (in image J and K) give one linearly independent constraint on the tensor components from (29). Finally, correspondences consisting of two points and one line (in image J or K) give two linearly independent constraints on the tensor components. This can be seen by writing the constraints in (30) as the three minors of

$$\begin{bmatrix} P_I & \mathbf{x}_I & 0 \\ P_J & 0 & \mathbf{x}_J \\ (\mathbf{1}^K)^{\mathrm{T}} P_K & 0 & 0 \end{bmatrix}$$

obtained by deleting one row from the second block of camera matrices. (This matrix can also be obtained by looking at the projection of a point $\mathbf{X}$ onto $\mathbf{x}_I$ in image I, $\mathbf{x}_J$ in image J and on the line $\mathbf{1}^K$ in image K.) Then one second-order linear constraint can be obtained by duplicating the last column. In the same way it can be shown that three corresponding lines also give two linearly independent constraints on the tensor components, from (31).§ This is also obvious from (20), consisting of three equations, where the representations of the lines can only be obtained up to an unknown scale. We summarize the constraints and the transfer equations for the trifocal tensor in Table 2.

2.3. *The quadrifocal tensor*

The third type of minors from M_4 involves two rows from each of the four different camera matrices, e.g.

$$\det \begin{bmatrix} A^i & x_A^i & 0 & 0 & 0 \\ A^{i'} & x_A^{i'} & 0 & 0 & 0 \\ B^j & 0 & x_B^j & 0 & 0 \\ B^{j'} & 0 & x_B^{j'} & 0 & 0 \\ C^k & 0 & 0 & x_C^k & 0 \\ C^{k'} & 0 & 0 & x_C^{k'} & 0 \\ D^l & 0 & 0 & 0 & x_D^l \\ D^{l'} & 0 & 0 & 0 & x_D^{l'} \end{bmatrix} = 0. \tag{35}$$

Expanding the determinant in (35) gives

$$\varepsilon_{ii'i''}\varepsilon_{jj'j''}\varepsilon_{kk'k''}\varepsilon_{ll'l''} \det \begin{bmatrix} A^i \\ B^j \\ C^k \\ D^l \end{bmatrix} x_A^{i'} x_B^{j'} x_C^{k'} x_D^{l'} = 0_{i''j''k''l''}, \tag{36}$$

§Use the leftmost matrix in (23) and duplicate the last column to obtain the second-order linear constraint.

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Table 2. Trilinear constraints and transfers, including the number of linearly independent constraints

Constraint	Equation	No. of. lin. ind. constr.
Three points	${}_{IJ}T_i^{jk}\mathbf{x}_I^i\varepsilon_{jj'j''}\mathbf{x}_J^{j'}\varepsilon_{kk'k''}\mathbf{x}_K^{k'} = 0_{j''k''}$	4
Two points, one line	${}^{JK}_{I}T_i^{jk}\mathbf{x}_I^i\varepsilon_{jj'j''}\mathbf{x}_{j'}^J\mathbf{1}_k^K = 0_{j''}$	2
Two points, one line	${}^{JK}_{I}T_i^{jk}\mathbf{x}_I^i\mathbf{1}_{j'}^J\varepsilon_{kk'k''}\mathbf{x}_K^{k'} = 0_{k''}$	2
One point, two lines	${}^{JK}_{I}T_i^{jk}\mathbf{x}_I^i\mathbf{1}_j^J\mathbf{1}_k^K = 0$	1
Three lines	${}^{JK}_{I}T_i^{jk}\varepsilon^{ii'i''}\mathbf{1}_{i'}^I\mathbf{1}_j^J\mathbf{1}_k^K = 0^{i''}$	2
Transfer	**Equation**	
Two lines to a line	$\mathbf{1}_i^I = {}^{JK}_{I}T_i^{jk}\mathbf{1}_j^J\mathbf{1}_k^K$	
Point and line to a point	$\mathbf{x}_K^k = {}^{JK}_{I}T_i^{jk}\mathbf{x}_I^i\mathbf{1}_j^J$	
Point and line to a point	$\mathbf{x}_J^j = {}^{JK}_{I}T_i^{jk}\mathbf{x}_I^i\mathbf{1}_k^K$	

where i'', j'', k'' and l'' denote free indices corresponding to the omitted row from each camera matrix block. Note that the left-hand side of (36) is a quadrilinear form in the image co-ordinates in the four images.

Definition 2.5. *The constraint in* (36) *will be called the* quadrilinear constraint.

The coefficients of the quadrilinear forms are the components of the quadrifocal tensor defined as follows.

Definition 2.6. *Using the notations in Definition* 2.2, *the tensor*

$$
{}^{IJKL}Q^{ijkl} = \det \begin{bmatrix} P_I^i \\ P_J^i \\ P_K^k \\ P_L^l \end{bmatrix}, \tag{37}
$$

is called the quadrifocal tensor *corresponding to views* I, J, K *and* L.

Using the quadrifocal tensor, the quadrilinear constraint in (36) can be written

$$
{}^{IJKL}Q^{ijkl}\varepsilon_{ii'i''}\mathbf{x}_I^{i'}\varepsilon_{jj'j''}\mathbf{x}_J^{j'}\varepsilon_{kk'k''}\mathbf{x}_K^{k'}\varepsilon_{ll'l''}\mathbf{x}_L^{l'} = 0_{i''j''k''l''}. \tag{38}
$$

Proposition 2.4. *Corresponding points in four images fulfil the quadrilinear constraint in* (38).

When the notation in (8) has been used, the quadrifocal tensors will be denoted by ${}^{ABCD}Q$, ${}^{CBAD}Q$, etc. Note that from Definition 2.6 it follows that ${}^{ABCD}Q^{ijkl} = -{}^{BACD}Q^{jikl}$, and similarly for other permutations of the images.

The numbers ${}^{IJKL}Q^{ijkl}$ can be viewed as a tensor of degree 4, that are contravariant in all indices. This means that making changes of co-ordinates, for example in image 1 using the matrix S, changes the components of the tensor according to

$$S_{i'}^{iIJKL}\hat{Q}^{i'jkl} = {}^{IJKL}Q^{ijkl}, \tag{39}$$

which follows directly from (37). The tensor can be used to transfer three lines, say in images J, K and L, $\mathbf{l}^J$, $\mathbf{l}^K$ and $\mathbf{l}^L$ to the corresponding point in image I, $\mathbf{x}_I$ according to the following theorem.

Theorem 2.6. *Let $\mathbf{l}^J$, $\mathbf{l}^k$ and $\mathbf{l}^L$ denote lines in images J, K and L, respectively. Introduce the plane Π_J defined by the image line $\mathbf{l}^J$ and the focal point C_J and similarly the planes Π_K and Π_L. These planes intersect at a point $\mathbf{X}$. Then the projection of this point in image I, $\mathbf{x}_I$, is given by the transfer formula*

$$\mathbf{x}_I^i = {}^{IJKL}Q^{ijkl}\mathbf{l}_j^J\mathbf{l}_k^K\mathbf{l}_l^L. \tag{40}$$

Proof. Consider the projection of $\mathbf{X}$ in the three images, i.e.

$$\lambda_I\mathbf{x}_I = P_I\mathbf{X}, \quad \lambda_J\mathbf{x}_J = P_J\mathbf{X}, \quad \lambda_K\mathbf{x}_K = P_K\mathbf{X}, \quad \lambda_L\mathbf{x}_L = P_L\mathbf{X}. \tag{41}$$

From the fact that the points $\mathbf{x}_J$, $\mathbf{x}_K$ and $\mathbf{x}_L$ are coincident with the lines $\mathbf{l}^J$, $\mathbf{l}^K$ and $\mathbf{l}^L$, respectively, it follows that

$$(\mathbf{l}^J)^{\mathrm{T}}P_J\mathbf{X} = 0, \quad (\mathbf{l}^K)^{\mathrm{T}}P_K\mathbf{X} = 0, \quad (\mathbf{l}^L)^{\mathrm{T}}P_L\mathbf{X} = 0, \tag{42}$$

which can be written as

$$\begin{bmatrix} (\mathbf{l}^I)^{\mathrm{T}}P_I \\ (\mathbf{l}^J)^{\mathrm{T}}P_J \\ (\mathbf{l}^K)^{\mathrm{T}}P_K \end{bmatrix} \mathbf{X} = 0 \Rightarrow \mathbf{X} = \mathcal{N}\begin{bmatrix} (\mathbf{l}^I)^{\mathrm{T}}P_I \\ (\mathbf{l}^J)^{\mathrm{T}}P_J \\ (\mathbf{l}^K)^{\mathrm{T}}P_K \end{bmatrix}, \tag{43}$$

where $\mathcal{N}$ denotes the nullspace. Now the projection of $\mathbf{X}$ in image I is given by

$$x_I^i \sim P_I^i\mathcal{N}\begin{bmatrix} (\mathbf{l}^I)^{\mathrm{T}}P_I \\ (\mathbf{l}^J)^{\mathrm{T}}P_J \\ (\mathbf{l}^K)^{\mathrm{T}}P_K \end{bmatrix} \sim \det\begin{bmatrix} P_I^i \\ (\mathbf{l}^J)^{\mathrm{T}}P_J \\ (\mathbf{l}^K)^{\mathrm{T}}P_K \\ (\mathbf{l}^L)^{\mathrm{T}}P_L \end{bmatrix} = \det\begin{bmatrix} P_I^i \\ P_J^i \\ P_K^k \\ P_L^l \end{bmatrix} \mathbf{l}_j^J\mathbf{l}_k^K\mathbf{l}_l^L = Q^{ijkl}\mathbf{l}_j^J\mathbf{l}_k^k\mathbf{l}_l^L.$$

□

There are of course three other similar transfer equations obtained by a permutation of the images.

The quadrilinear constraint in (36) can be used to estimate the quadrifocal tensor from point correspondences. Another constraint follows immediately from (40) by contraction with $\mathbf{l}_i^I$ on both sides, giving

$${}^{IJKL}Q^{ijkl}\mathbf{l}_I^i\mathbf{l}_j^J\mathbf{l}_k^K\mathbf{l}_l^L = 0. \tag{44}$$

Proposition 2.5. *Corresponding lines in four images fulfil the quadrilinear constraint in* (44).

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Remark. The constraint in (44) can easily be derived by considering a point $\mathbf{X}$ that projects to image points lying on the different lines, giving

$$\begin{bmatrix} (\mathbf{1}^I)^{\mathrm{T}} P_I \\ (\mathbf{1}^J)^{\mathrm{T}} P_J \\ (\mathbf{1}^K)^{\mathrm{T}} P_K \\ (\mathbf{1}^L)^{\mathrm{T}} P_L \end{bmatrix} \mathbf{X} = 0.$$

Expanding the determinant of the leftmost matrix gives directly (44).

Similarly to the incidence relations for the trifocal tensor each occurrence of a line in (44) can be replaced by a point, using the permutation symbol, e.g.

$${}^{IJKL}Q^{ijkl}\mathbf{1}^i_I \varepsilon_{jj'j''}\mathbf{x}^J_{j'}\varepsilon_{kk'k''}\mathbf{x}^K_{k'}\varepsilon_{ll'l''}\mathbf{x}^L_{l'} = 0_{j''k''l''}. \tag{45}$$

$${}^{IJKL}Q^{ijkl}\mathbf{1}^i_I \mathbf{1}^j_J \varepsilon_{kk'k''}\mathbf{x}^K_{k'}\varepsilon_{ll'l''}\mathbf{x}^L_{l'} = 0_{k''l''}. \tag{46}$$

$${}^{IJKL}Q^{ijkl}\mathbf{1}^i_I \mathbf{1}^j_J \mathbf{1}^k_K \varepsilon_{ll'l''}\mathbf{x}^L_{l'} = 0_{l''}. \tag{47}$$

These constraints can be used for different combinations of corresponding points and lines in the images.

From (38) it follows that each corresponding point in four images gives 27 linear constraint on the quadrifocal tensor components. However, these constraints are not linearly independent. The number of linearly independent constraints obtained for each point correspondence is given in the following theorem, see also [9].

Theorem 2.7. *There are in general* 16 *linearly independent* (*in the tensor components*) *constraints among the* 27 *quadrilinear constraints in* (36).

Proof. Start with the 12×8-matrix

$$M = \begin{bmatrix} P_1 & \mathbf{x}_1 & 0 & 0 & 0 \\ P_2 & 0 & \mathbf{x}_2 & 0 & 0 \\ P_3 & 0 & 0 & \mathbf{x}_3 & 0 \\ P_4 & 0 & 0 & 0 & \mathbf{x}_4 \end{bmatrix}. \tag{48}$$

The 9 minors obtained by removing one row corresponding to each one of the four images give the previously mentioned linear constraints. Consider the 12×9 matrix

$$M = \begin{bmatrix} P_1 & \mathbf{x}_1 & 0 & 0 & 0 & \mathbf{x}_1 \\ P_2 & 0 & \mathbf{x}_2 & 0 & 0 & 0 \\ P_3 & 0 & 0 & \mathbf{x}_3 & 0 & 0 \\ P_4 & 0 & 0 & 0 & \mathbf{x}_4 & 0 \end{bmatrix}, \tag{49}$$

obtained by duplicating the 5th column to M in (48). Expanding the 27 minors of M_e in (49) obtained by removing one of the rows corresponding to each one of the second, third and fourth image gives 27 second-order linear constraints. The same thing can be done by duplicating the 6th, 7th and 8th column of M, giving in total 108 second-order linear constraints. On the other hand, every linear dependency among the quadrilinear constraints can be written in matrix form as in (49) with the coefficients of the vanishing linear combination in the last column. Thus, we have obtained all second-order linear constraints.

Continuing in the same way, by expanding the 9 minors of the 12×10 matrix

$$\begin{bmatrix} P_1 & \mathbf{x}_1 & 0 & 0 & 0 & \mathbf{x}_1 & 0 \\ P_2 & 0 & \mathbf{x}_2 & 0 & 0 & 0 & \mathbf{x}_2 \\ P_3 & 0 & 0 & \mathbf{x}_3 & 0 & 0 & 0 \\ P_4 & 0 & 0 & 0 & \mathbf{x}_4 & 0 & 0 \end{bmatrix} \tag{50}$$

obtained by removing one row corresponding to each one of the second and third image, we obtain 9 third-order linear constraints. By duplicating the other combinations of two different columns picked from columns 5–8 gives in total 54 third-order linear constraints. Again, using the same argument as above, it is evident that all third-order linear constraints have been obtained.

By duplicating all 4 combinations of 3 different columns picked from columns 5–8 and expanding minors obtained by removing one row from the image without a duplicated column gives in total 12 fourth-order linear constraints. Finally, the determinant of the 12×12 matrix obtained by duplicating all of the columns 5–8 gives 1 fifth-order linear constraint. To sum up, the number of linearly independent constraints obtained on the tensor components in (44) is

$$\begin{aligned} 81 - 108 + 54 - 12 + 1 &= \binom{4}{0} 3^4 - \binom{4}{1} 3^3 + \binom{4}{2} 3^2 - \binom{4}{3} 3^1 + \binom{4}{4} 3^0 \\ &= (1-3)^4 = 16. \end{aligned} \qquad \square$$

Thus (44) gives 16 linearly independent constraints for one point alone. However, it turns out that there exists a linear dependency between the 32 constraints obtained for two different corresponding points according to the following lemma.

Lemma 2.1. *There are in general* 31 *linearly independent constraints on the components of the quadrifocal tensor from two corresponding points in four images. That is, there is a linear dependency between the different groups of* 16 *linearly independent constraints obtained from each point alone.*

Proof. The linear dependency can be obtained from the determinant of the matrix

$$\begin{bmatrix} P_1 & \mathbf{x}_1 & 0 & 0 & 0 & \hat{\mathbf{x}}_1 & 0 & 0 & 0 \\ P_2 & 0 & \mathbf{x}_2 & 0 & 0 & 0 & \hat{\mathbf{x}}_2 & 0 & 0 \\ P_3 & 0 & 0 & \mathbf{x}_3 & 0 & 0 & 0 & \hat{\mathbf{x}}_3 & 0 \\ P_4 & 0 & 0 & 0 & \mathbf{x}_4 & 0 & 0 & 0 & \hat{\mathbf{x}}_4 \end{bmatrix}, \tag{51}$$

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Table 3. Quadrilinear constraints and transfer, including the number of linearly independent constraints

Constraint	Equation	No. of. lin. ind. constr.
Four points	${}^{IJKL}Q^{ijkl}\varepsilon_{ii'i''}\mathbf{x}_I^{i'}\varepsilon_{jj'j''}\mathbf{x}_J^{j'}\varepsilon_{kk'k''}\mathbf{x}_K^{k'}\varepsilon_{ll'l''}\mathbf{x}_L^{l'} = 0_{i''j''k''l''}$	16
Three points, one line	${}^{IJKL}Q^{ijkl}\mathbf{1}_I^i\varepsilon_{jj'j''}\mathbf{x}_J^{j'}\varepsilon_{kk'k''}\mathbf{x}_K^{k'}\varepsilon_{ll'l''}\mathbf{x}_L^{l'} = 0_{j''k''l''}$	8
Two points, two lines	${}^{IJKL}Q^{ijkl}\mathbf{1}_I^i\mathbf{1}_J^j\varepsilon_{kk'k''}\mathbf{x}_{k'}^K\varepsilon_{ll'l''}\mathbf{x}_{l'}^L = 0_{k''l''}$	4
One point, three lines	${}^{IJKL}Q^{ijkl}\mathbf{1}_I^i\mathbf{1}_J^j\mathbf{1}_K^k\varepsilon_{ll'l''}\mathbf{x}_{l'}^L = 0_{l''}$	2
Four lines	${}^{IJKL}Q^{ijkl}\mathbf{1}_I^i\mathbf{1}_j^J\mathbf{1}_k^K\mathbf{1}_l^L = 0$	1
Transfer	**Equation**	
Three lines to a point	$\mathbf{x}_I^i = {}^{IJKL}Q^{ijkl}\mathbf{1}_j^J\mathbf{1}_k^k\mathbf{1}_l^L$	

where $\mathbf{x}_i$ denotes the first point and $\hat{\mathbf{x}}_i$ the second point. In tensor notation this relation can be written as

$$ {}^{IJKL}Q^{ijkl}\varepsilon_{ii'i''}\mathbf{x}_{i'}^I\varepsilon_{jj'j''}\mathbf{x}_{j'}^J\varepsilon_{kk'k''}\mathbf{x}_{k'}^K\varepsilon_{ll'l''}\mathbf{x}_{l'}^L\hat{\mathbf{x}}_{i''}^I\hat{\mathbf{x}}_{j''}^J\hat{\mathbf{x}}_{k''}^K\hat{\mathbf{x}}_{l''}^L = 0. \tag{52} $$

We immediately get the following important corollary.

Corollary 2.2. *From $n < 6$ corresponding points in 4 images $16n - n(n-1)/2$ linearly independent constraints on the quadrifocal tensor components can be obtained. From $n \geqslant 6$ corresponding points 80 linearly independent constraints can be obtained. Thus the components of the quadrifocal tensor can be linearly calculated from at least 6 corresponding points in 4 images.*

Proof. The corollary follows from the fact that one linear constraint is removed for each pair of corresponding points and the obvious fact that there are no further relations between triplets of corresponding points, etc. Note also that since there are 81 components of the quadrifocal tensor, there can at most be 80 linearly independent constraints. □

Similar arguments indicate that there are 8 linearly independent constraints from one line and three points using (45), 4 linearly independent constraints from two lines and two points using (46) and 2 linearly independent constraints from three lines and one point using (47). We summarize the constraints and the transfer equations for the trifocal tensor in Table 3.

2.4. *The monofocal tensor*

Looking at the three different tensors in Definitions 2.2, 2.4 and 2.6 we observe that all determinants of matrices obtained by picking four rows from the different camera matrices P_I, P_j, P_K and P_L have been used except for the case of all the three rows from one camera matrix and one row from another, e.g.

$$\begin{bmatrix} P_I^1 \\ P_I^2 \\ P_I^3 \\ P_J^j \end{bmatrix}. \tag{53}$$

The reason for this is that there are no minors of M_4 where coefficients of this type appear. Note that the minors of the same type as in (53) (that is three rows from one camera matrix and one from another) cannot be estimated directly from image measurements in the same way as the bifocal, trifocal and quadrifocal tensors can be estimated. For the sake of completeness we make the following definition:

Definition 2.7. *The tensor*

$$_{IJ}e^j = \det \begin{bmatrix} P_I^1 \\ P_I^2 \\ P_I^3 \\ P_J^j \end{bmatrix} \tag{54}$$

is called the monofocal tensor *corresponding to views I and J.*

Observe that $_{IJ}e^j$ is a contravariant tensor of degree 1, which again alludes to the fact that a change of co-ordinates in image J, using the matrix S, changes the tensor components according to

$$_{IJ}\hat{e}^{j'} = s_j^{j'} \, _{IJ}e^j,$$

which follows from (54). Note that (53) can be interpreted as the jth component of the epipole from camera I in image J according to the following proposition.

Proposition 2.6. *The projection of the focal point of camera I, C_I in camera J is given by $(_{IJ}e^1, {}_{IJ}e^2, {}_{IJ}e^3)$.*

Proof. The fact that C_I is the focal point of camera I can be written as

$$P_I C_I = 0 \Rightarrow C_I \in \mathcal{N}(P_i).$$

Consider now co-ordinate number j of the projection of C_I in camera J:

$$P_J^j C_I = P_J^j \mathcal{N}(P_I) = P_J^j \mathcal{N} \begin{bmatrix} P_I^1 \\ P_I^2 \\ P_I^3 \end{bmatrix} \sim \det \begin{bmatrix} P_I^1 \\ P_I^2 \\ P_I^3 \\ P_J^j \end{bmatrix} = {}_{IJ}e^j. \qquad \square$$

We conclude this section with a formal definition.

Definition 2.8. *The* epipole *of camera I in camera J is the projection of the focal point of camera I in camera J, $_{IJ}e^j$.*

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3. Representations of the multiple view geometry

We now turn to the question of how to represent the multiple view geometry. The following proposition is fundamental.

Proposition 3.1. *Given an uncalibrated image sequence. Then it is only possible to reconstruct the object up to an unknown projective transformation from corresponding points in the images. The camera matrices can also be only recovered up to an unknown projective transformation.*

Proof. Given camera matrices P_i and object co-ordinates $\mathbf{X}_j$ that fulfills (5), i.e.

$$\lambda_{i,j}\mathbf{x}_{i,j} = P_i\mathbf{X}_j, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n.$$

Then also P_iH^{-1} and HX fulfils (5), for any non-singular 4×4 matrix H. This ambiguity in representation corresponds to arbitrary projective transformations of the object. □

One way to represent the multiple view geometry is by taking the first four columns of M in (6), containing all camera matrices, i.e.

$$\mathbf{P} = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_m \end{bmatrix}. \tag{55}$$

The previously mentioned ambiguity corresponds in this representation to the fact that $\mathbf{P}$ and $\mathbf{P}H$ represents the same viewing geometry. Observe that all tensor components are minors taken from $\mathbf{P}$ and replacing $\mathbf{P}$ by $\mathbf{P}H$ changes only the overall scale of all tensor components by the factor $\det H$. Another ambiguity in representation in the scale for every camera matrix P_i, which can be chosen arbitrarily. This can be seen from (5) by changing P_i to $\mu_i P_i$ and λ_i to $\mu_i\lambda_i$ and still obtaining a solution.

Proposition 3.2. *Let $\mathscr{G}$ denote the group of all projective transformations in* 3*D, represented by* 4×4 *matrices defined up to scale. Then the multiple view geometry for m views can be uniquely represented as an equivalence class of m* 3×4 *matrices defined up to scale under the action of right multiplication with elements from the group $\mathscr{G}$.*

From the proposition it follows that the minimal number of parameters needed to represent the multiple view geometry for m views is $11m - 15$.

Note that the tensor components, i.e. the minors of $\mathbf{P}$ in (55) are Grassman co-ordinates of the linear subspace in R^{3m} spanned by the columns of $\mathbf{P}$. This indicates another way to look at the multiple view geometry, that is as this subspace, represented by its Grassman co-ordinates. However, the ambiguity corresponding to rescaling the camera matrices has not been taken into account. Another problem is that the different tensors can only be estimated up to an unknown scalar factor from image

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correspondences. Furthermore, the relative scale between the tensors cannot be determined either. One natural way to deal with this fact is to use a slight generalization of the Grassman co-ordinates obtained as follows. We divide the set of Grassman co-ordinates into different subsets, each corresponding to one of the multiple view tensors. Then we define each subset of Grassman co-ordinates up to an unknown scale and use these modified Grassman co-ordinates as a representation.

Definition 3.1. *The set of multiple view tensor components, defined up to an unknown scale for each tensor, will be called* modified Grassman co-ordinates.

Proposition 3.3. *The multiple view geometry for m views can be uniquely represented by the modified Grassman co-ordinates, i.e. by the different multiple view tensors each one defined up to scale.*

This representation is of course not minimal. Firstly, the individual tensors are only defined up to scale, but the scale between all different tensors cannot be chosen arbitrarily when they are considered as Grassman co-ordinates. For instance, the scale between ${}_{AB}F$, ${}_{AB}e$ and ${}_{BA}e$ cannot be chosen arbitrarily, which will be seen in the next section. Secondly, there exist dependencies between the different tensor components. For the ordinary Grassman coordinates, all such dependencies can be written using the so-called quadratic p-relations, see [14]. These can be written

$$\begin{aligned} &p_{i_1 i_2 i_3 i_4} p_{j_1 j_2 j_3 j_4} - p_{j_1 i_2 i_3 i_4} p_{i_1 j_2 j_3 j_4} + p_{j_2 i_2 i_3 i_4} p_{j_1 i_1 j_3 j_4} - p_{j_3 i_2 i_3 i_4} p_{j_1 j_2 i_1 j_4} \\ &\quad + p_{j_4 i_2 i_3 i_4} p_{j_1 j_2 j_3 i_1} = 0, \end{aligned} \tag{56}$$

where p_{ijkl} denotes the determinant of the matrix obtained from rows i, j, k and l from $\mathbf{P}$ in (55). One example is, using the notation $[A^2 A^3 B^1 B^2]$ for the determinant of the matrix formed by the rows A^2, A^3, B^1 and B^2,

$$\begin{aligned} &[A^2 A^3 B^1 B^2][A^1 A^2 A^3 B^3] - [A^1 A^2 A^3 B^2][B^1 A^2 A^3 B^3] \\ &\quad + [A^1 A^2 A^3 B^1][B^2 A^2 A^3 B^3] \\ &= {}_{AB}F_{13} \cdot {}_{AB}e^3 + {}_{AB}F_{12} \cdot {}_{AB}e^2 + {}_{AB}F_{11} \cdot {}_{AB}e^1 = 0, \end{aligned} \tag{57}$$

i.e. the well-known fact that the epipole in image 2 can be obtained as the left nullspace to the fundamental matrix between images 1 and 2.

A necessary and sufficient condition for numbers p_{ijkl} to be Grassman co-ordinates of a subspace is that they fulfill all quadratic p-relations. Thus given a subset of multiple view tensors, the relation between the components of this subset of tensors can be obtained by considering the ideal of all quadratic p-relations and then calculating the elimination ideal, where the tensor components outside the subset have been eliminated. Another application of the quadratic p-relations is to calculate one tensor from other tensor/tensors and to calculate a representation of the multiple view geometry in the form of the camera matrices. These two applications will be described in the next sections.

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From the representation $\mathbf{P}$ in (55) and the action of the group $\mathscr{G}$ of all projective transformations in 3D on $\mathbf{P}$, a unique representative of camera matrices can be obtained by setting $P_1 = [I \mid 0]$, P_2^1 to $[0\,0\,0\,1]$ and fixing the scale of the other camera matrices arbitrarily. According to the definition of the multiple view tensors, this normal from can be written, in the case of 4 images, as

$$\mathbf{P}_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ {}_{AB}F_{13} & {}_{AB}F_{23} & {}_{AB}F_{33} & {}_{AB}e^2 \\ -\,{}_{AB}F_{12} & -\,{}_{AB}F_{22} & -\,{}_{AB}F_{32} & {}_{AB}e^3 \\ {}^{BC}_{A}T^{11}_{1} & {}^{BC}_{A}T^{11}_{2} & {}^{BC}_{A}T^{11}_{3} & {}_{AC}e^1 \\ {}^{BC}_{A}T^{12}_{1} & {}^{BC}_{A}T^{12}_{2} & {}^{BC}_{A}T^{12}_{3} & {}_{AC}e^2 \\ {}^{BC}_{A}T^{13}_{1} & {}^{BC}_{A}T^{13}_{2} & {}^{BC}_{A}T^{13}_{3} & {}_{AC}e^3 \\ {}^{BD}_{A}T^{11}_{1} & {}^{BD}_{A}T^{11}_{2} & {}^{BD}_{A}T^{11}_{3} & {}_{AD}e^1 \\ {}^{BD}_{A}T^{12}_{1} & {}^{BD}_{A}T^{12}_{2} & {}^{BD}_{A}T^{12}_{3} & {}_{AD}e^2 \\ {}^{BD}_{A}T^{13}_{1} & {}^{BD}_{A}T^{13}_{2} & {}^{BD}_{A}T^{13}_{3} & {}_{AD}e^3 \end{bmatrix}. \tag{58}$$

Definition 3.2. *The representation of the multiple view geometry in* (58) *will be called a* normal form. *Observe that the mono-focal tensor* ${}_{AB}e^i$ *has been normalized such that* ${}_{AB}e^1 = 1$.

Observe that the scale for the block containing ${}_{AB}F$ can be fixed independently of the scale of the block containing ${}_{AB}e$, giving $5 + 2 = 7$ parameters for the first two views and $7 + 11(n-2)$ parameters for $n > 2$ views. It follows that the camera matrices are determined from the fundamental matrix between views 1 and 2, the trifocal tensors between views 1, 2 and $i \geqslant 3$ and the epipoles in image 1 from image $i \geqslant 2$, when the scales between ${}_{AB}F$ and ${}_{AB}e$ and between ${}^{2I}_{1}T$ and ${}_{1I}e$ have been determined. These scales can easily be obtained by calculated suitable minors of $\mathbf{P}_n$, e.g. the expression for ${}^{BC}_{A}T^{21}_{1}$ gives one linear constraint on these scales. We conclude this section with the observation that minimal parameterizations of the multiple view geometry can easily be obtained from (58).

4. Intrinsic properties of the multiple view tensors

In this section we will study the different multiple view tensors in more detail. Especially, their intrinsic properties and the relations between components of different tensors. Of particular interest in applications is the possibility to calculate one tensor from another (or several others).

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4.1. *Two views*

In this section we will study the case of two images in detail. The only tensors appearing here are the bifocal tensors ${}_{AB}F$ and ${}_{BA}F$ and the two monofocal tensors ${}_{AB}e$ and ${}_{BA}e$. The relations between them can be obtained by exploring the quadratic p-relations. Putting $i_1 = 1, i_2 = 2, i_3 = 3, i_4 = 6, j_1 = 2, j_2 = 3, j_3 = 4$ and $j_4 = 5$ gives the quadratic p-relation

$$\begin{aligned}&[A^1A^2A^3B^3][A^2A^3B^1B^2] - [B^1A^2A^3B^3][A^1A^2A^3B^2]\\&\qquad + [B^2A^2A^3B^3][A^1A^2A^3B^1]\\&\quad = {}_{AB}e^3 \cdot {}_{AB}F_{13} + {}_{AB}F_{12} \cdot {}_{AB}e^2 + {}_{AB}F_{11} \cdot {}_{AB}e^1 = 0.\end{aligned} \tag{59}$$

Similar quadratic p-relations give

$${}_{AB}F_{ij} \cdot {}_{AB}e^i = 0_j, \quad {}_{AB}F_{ij} \cdot {}_{BA}e^j = 0_i \tag{60}$$

and similarly for ${}_{BA}F$. From (60) it follows that if we use the notation ${}_{AB}F_{\cdot\cdot}$ for the matrix with components ${}_{AB}F_{ij}$, then this matrix is singular, its right nullspace is ${}_{BA}e$ and its left nullspace is ${}_{AB}e$, written out

$${}_{AB}e^{\mathrm{T}}\,{}_{AB}F_{\cdot\cdot} = 0, \quad {}_{AB}F_{\cdot\cdot}\,{}_{BA}e = 0. \tag{61}$$

This means that ${}_{BA}e$ and ${}_{AB}e$ can be found up to proportionality by considering suitable minors of ${}_{AB}F_{\cdot\cdot}$. In fact, putting $i_1 = 1$, $i_2 = 1$, $i_3 = 3$, $i_4 = 4$, $j_1 = 2$, $j_2 = 4$, $j_3 = 5$ and $j_4 = 6$ gives the quadratic p-relation

$$\begin{aligned}&- [A^2A^1A^3B^1][A^1B^1B^2B^3] - [B^2A^1A^3B^1][A^1A^2B^1B^3]\\&\quad + [A^1A^3B^1B^3][A^1A^2B^1B^2] = 0.\end{aligned} \tag{62}$$

Similar quadratic p-relations gives (writing $F = {}_{AB}F$, $e = {}_{AB}e$ and $\bar{e} = {}_{BA}e$)

$$\begin{aligned}&F_{22}F_{33} - F_{23}F_{32} = e^1\bar{e}^1, \quad F_{23}F_{31} - F_{21}F_{33} = e^1\bar{e}^2,\\&F_{21}F_{32} - F_{22}F_{31} = e^1\bar{e}^3,\\&F_{13}F_{32} - F_{12}F_{33} = e^2\bar{e}^1, \quad F_{11}F_{33} - F_{13}F_{31} = e^2\bar{e}^2,\\&F_{12}F_{31} - F_{11}F_{32} = e^2\bar{e}^3,\\&F_{12}F_{23} - F_{13}F_{22} = e^3\bar{e}^1, \quad F_{13}F_{21} - F_{11}F_{23} = e^3\bar{e}^2,\\&F_{11}F_{22} - F_{12}F_{21} = e^3\bar{e}^3\end{aligned} \tag{63}$$

giving the relative scale between F, e and $\bar{e}$. Let adj(F) denote the adjoint matrix to ${}_{AB}F_{\cdot\cdot}$. Then the equations in (63) can be interpreted as

$$\mathrm{adj}({}_{AB}F_{\cdot\cdot}) = {}_{AB}e \cdot {}_{BA}e^{\mathrm{T}}. \tag{64}$$

Thus the epipoles can be calculated directly from the fundamental matrix, with consistent scale factors, using (64). We state this important result as a theorem.

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Table 4. Relations used for the two-view case

Quadratic p-relations	Tensor notation	Matrix notation
	${}_{AB}F_{ij} \cdot {}_{AB}e^{i} = 0_j$	${}_{AB}e^{T}\, e_{AB}F_{\bullet\bullet} = 0$
	${}_{AB}F_{ij} \cdot {}_{BA}e^{j} = 0_i$	${}_{AB}F_{\bullet\bullet\,BA}e = 0$
	$\frac{1}{2}\,\varepsilon^{ii'i''}\varepsilon^{jj'j''}{}_{AB}F_{i'j'\,AB}F_{i''j''}F_{i''j''} = {}_{AB}e^{i}{}_{BA}e^{j}$	$\mathrm{adj}({}_{AB}F_{\bullet\bullet}) = {}_{AB}e^{\mathrm{T}} \cdot {}_{BA}e$
Constraint	**Tensor notation**	**Matrix notation**
	$\varepsilon_{jj'j''\,AB}F_{1j\,AB}F_{2j'\,AB}F_{3j''} = 0$	$\det({}_{AB}F_{\bullet\bullet}) = 0$

Theorem 4.1. *The monofocal tensors (epipoles) can be calculated from the bifocal tensor (fundamental matrix) from* (60) *or* (61), *i.e. as the left and right nullspace to the fundamental matrix. Consistent scales can be obtained from* (63) *or* (64).

Corollary 4.1. *The bifocal tensor encodes the bifocal geometry in the sense that a unique representative (up to projective transformations) of the bifocal geometry can be obtained from the bifocal tensor.*

From noisy data the epipoles can easily be estimated from the singular value decomposition of adj(F). Moreover, this means that a representative for the multiple view geometry in the form of (58) can be written out directly. From the previous equations we immediately get the following well-known theorem.

Theorem 4.2. *The numbers ${}_{AB}F_{ij}$ constitute the components of a bifocal tensor if and only if ${}_{AB}F_{\bullet\bullet}$ is a singular matrix.*

Proof. The necessity of the conditions follows from (60). The sufficiency follows from the fact that given such numbers, call them ${}_{AB}F_{ij}$, with $\det({}_{AB}F_{\bullet\bullet}) = 0$ we can calculate ${}_{AB}e$ from (60) and obtain a representative of the bifocal geometry in the form of (58). It is then easy to check that when ${}_{AB}F_{ij}$ is calculated from this normal form using Definition 2.2, the original numbers ${}_{AB}F_{ij}$ are obtained. □

Observe that in non-degenerate cases rank ${}_{AB}F_{\bullet\bullet} = 2$. We summarize the equations used for the two-view case in Table 4.

4.2. *Three views*

In this subsection we will study the case of three views in detail. The tensors appearing here are the six trifocal tensors ${}^{BC}_{A}T$, ${}^{CB}_{A}T$, ${}^{AC}_{B}T$, etc., the six bifocal tensors ${}_{BA}F$, ${}_{BA}F$, ${}_{AC}F$, etc., and the six monofocal tensors ${}_{AB}e$, ${}_{BA}e$, ${}_{AC}e$, etc. Taking the previously mentioned symmetries into account, we have essentially 3 trifocal tensors, 3 bifocal tensors and 6 monofocal tensors. The relations between them can be obtained by exploring the quadratic p-relations. We will show how these quadratic p-relations can be used to calculate one trifocal tensor from another, how the bifocal

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tensors can be calculated from the trifocal tensors and finally how a representative of the trifocal geometry can be obtained from only one trifocal tensor.

Studying quadratic p-relations of the type

$$\begin{aligned}&[B^1A^1A^2B^3][A^1A^2B^2C^1]-[B^2A^1A^2B^3][A^1A^2B^1C^1]\\&\quad+[C^1A^1A^2B^3][A^1A^2B^1B^2]\\&=-{}_{AB}F_{32}\cdot{}^{BC}_{A}T^{21}_{3}-{}_{AB}F_{31}\cdot{}^{BC}_{A}T^{11}_{3}-{}^{BC}_{A}T^{31}_{3}\cdot{}_{AB}F_{33}=0\end{aligned}\tag{65}$$

gives the following relations, writing $F={}_{AB}F$, $\bar{F}={}_{AC}F$ and $T={}^{BC}_{A}T$:

$$\begin{aligned}&T^{j1}_{i}\cdot F_{1j}=0_i,\quad T^{j2}_{i}\cdot F_{2j}=0_i,\quad T^{j3}_{i}\cdot F_{3j}=0_i,\\&T^{1k}_{i}\cdot\bar{F}_{1k}=0_j,\quad T^{2k}_{i}\cdot\bar{F}_{2k}=0_j,\quad T^{3k}_{i}\cdot\bar{F}_{3k}=0_j,\end{aligned}\tag{66}$$

implying that $F_{i\bullet}$ and $\bar{F}_{i\bullet}$ can be obtained as the left and right nullspace, respectively, to the matrix $T^{\bullet\bullet}_{i}$. However, the scale between $F_{i\bullet}$ for different i cannot be determined in this way. Since the monofocal tensor ${}_{AB}e$, according to (60), can be obtained from $F_{i\bullet}$, we can also calculate the monofocal tensors ${}_{AB}e$ and ${}_{AC}e$ easily from ${}^{BC}_{A}T$. The simplest way to obtain the scale between the different $F_{i\bullet}$, as well as the other bifocal tensors and monofocal tensors, is to calculate the other trifocal tensors from ${}^{BC}_{A}T$. For this calculation study the quadratic p-relation

$$\begin{aligned}&[A^1A^3B^2C^1][A^2B^1B^2C^1]-[A^2A^3B^2C^1][A^1B^1B^2C^1]\\&\quad+[B^1A^3B^2C^1][A^1A^2B^2C^1]\\&=-{}^{BC}_{A}T^{21}_{2}\cdot{}^{AC}_{B}T^{21}_{3}-{}^{BC}_{A}T^{21}_{1}\cdot{}^{AC}_{B}T^{11}_{3}-{}^{AC}_{B}T^{31}_{3}\cdot{}^{BC}_{A}T^{21}_{3}=0.\end{aligned}\tag{67}$$

Similar quadratic p-relations give, using the notation $T={}^{BC}_{T}T$ and $U={}^{AC}_{B}T$,

$$\begin{aligned}&T^{1K}_{i}\cdot U^{iK}_{3}=0,\quad T^{2K}_{i}\cdot U^{iK}_{3}=0,\quad T^{1K}_{i}\cdot U^{iK}_{2}=0,\quad T^{3K}_{i}\cdot U^{iK}_{2}=0,\\&T^{2K}_{i}\cdot U^{iK}_{1}=0,\quad T^{3K}_{i}\cdot U^{iK}_{1}=0,\end{aligned}\tag{68}$$

where $K=1,2,3$ is fixed and summation is done over i only. Thus (68) implies that $U^{\bullet k}_{j}$ can be obtained up to an unknown scale from T as the cross products of $T^{j'k}_{\bullet}$ and $T^{j''k}_{\bullet}$, where j, j' and j'' are different. A correct scale between different $U^{\bullet k}_{j}$ can be obtained from the quadratic p-relations:

$$T^{1K}_{\bullet}\times T^{2K}_{\bullet}={}_{AC}e^{K}U^{\bullet K}_{3},\quad T^{1K}_{\bullet}\times T^{3K}_{\bullet}={}_{AC}e^{K}U^{\bullet K}_{2},\quad T^{2K}_{\bullet}\times T^{3K}_{\bullet}={}_{AC}e^{K}U^{\bullet K}_{1},\tag{69}$$

where $a\times b$ denotes the cross-product of the vectors a and b. Since ${}_{AC}e^{K}$ can be calculated from T as described above, the scale between different $U^{\bullet K}_{j}$ can be recovered. Thus all trifocal tensors can be calculated from ${}^{BC}_{A}T$.

Theorem 4.3. *Given the trifocal tensor* ${}^{BC}_{A}T$. *Then it is possible to calculate the trifocal tensor* ${}^{AC}_{B}T$ *from* (68) *and* (69). *Thus all* 6 *trifocal tensors for the trifocal geometry can be calculated from one.*

We summarize the method to calculate ${}^{AC}_{B}T$ from ${}^{BC}_{A}T$ in the following algorithm.

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Algorithm 4.1. $[{}^{BC}_{A}T$ to ${}^{AC}_{B}T]$

1. Calculate ${}_{AC}F_{i\bullet}$ up to scale from (66).
2. Calculate ${}_{AC}e^{K}$ up to scale from ${}_{AC}F_{i\bullet}$ using (61).
3. Calculate ${}^{AC}_{B}T_{j}^{\bullet k}$ up to scale from (68).
4. Calculate a consistent scale between different ${}^{AC}_{B}T_{j}^{\bullet k}$ using (69).

Now, from ${}^{AC}_{B}T$, it is possible to calculate ${}_{AB}F_{\bullet j}$ up to scale from equations similar to (66). Thus when both ${}_{AB}F_{i\bullet}$ and ${}_{AB}F_{\bullet j}$ are known, up to scale, it is possible to calculate ${}_{AB}F$.

Theorem 4.4. *Given the trifocal tensor* ${}^{BC}_{A}T$. *Then it is possible to calculate the bifocal tensor* ${}_{AB}F$ *from* ${}^{BC}_{A}T$. *Thus all* 6 *bifocal tensors for the trifocal geometry can be calculated from one trifocal tensor.*

The algorithm looks as follows

Algorithm 4.2. $[{}^{BC}_{A}T$ to ${}_{AB}F]$

1. Calculate ${}_{AB}F_{i\bullet}$ up to scale from (66).
2. Calculate ${}^{AC}_{B}T$ from ${}^{BC}_{A}T$ using the previous algorithm.
3. Calculate a consistent scale between different ${}_{AB}F_{i\bullet}$ using (66) and ${}^{AC}_{B}T$.

We have thus shown how to calculate all other multiple view tensors from ${}^{BC}_{A}T$, using only polynomial equations of degree 2.

Theorem 4.5. *Given the trifocal tensor* ${}^{BC}_{A}T$. *Then it is possible to calculate the other* 5 *trifocal tensors, the* 6 *bifocal tensors as well as the* 6 *monofocal tensors.*

Moreover, a representative of the trifocal view geometry can be obtained from the trifocal tensor, ${}^{BC}_{A}T$. Starting from ${}^{BC}_{A}T$, we have shown how to calculate ${}_{AB}F$, ${}_{AB}e$ and ${}_{AC}e$, up to scale. Thus everything in (58) is known apart from the scale between ${}^{BC}_{A}T$ and ${}_{AC}e$, that has to be compatible with the other fixed scales (the ones for ${}_{AB}F$ and ${}_{AB}e$). This scale can be obtained from (58) by calculating minors from (58) corresponding to ${}^{BC}_{A}T_{i}^{jk}$, when $j \neq 1$. The expression obtained from these minors can also be derived from suitable quadratic p-relations. We thus have

Corollary 4.2. *The trifocal tensor encodes the trifocal geometry in the sense that a unique representative* (*up to projective transformations*) *of the trifocal geometry can be obtained from the trifocal tensor.*

From (66) and (60), it follows that the components of the trifocal tensor must fulfil

$$\det T_{i}^{\bullet\bullet} = 0 \quad \text{and} \quad \det[\mathcal{N}(T_{1}^{\bullet\bullet})\mathcal{N}(T_{2}^{\bullet\bullet})\mathcal{N}(T_{3}^{\bullet\bullet})] = 0. \tag{70}$$

Moreover, since T_{i}^{jk} is a third-order tensor that transforms according to (19) and that the conditions on the tensor components in (70) are generic, i.e. independent of the chosen coordinate systems in the images, it follows that (70) are valid also when T_i are

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replaced by $\lambda_i^j T_j$ for any nonsingular matrix with components λ_i^j. In fact, it turns out that it is only necessary to replace T_i by $\lambda^i T_i$ in the first equation in (70) to obtain necessary and sufficient conditions for the trifocal tensor components, according to the following theorem.

Theorem 4.6 *A necessary and sufficient condition for* 27 *numbers* T_i^{jk} *to constitute the components of a trifocal tensor is*

$$\det \lambda^i T_i^{\bullet\bullet} = 0, \quad \forall \lambda_i \quad \textit{and} \quad \det[\mathcal{N}(T_1^{\bullet\bullet})\mathcal{N}(T_2^{\bullet\bullet})\mathcal{N}(T_3^{\bullet\bullet})] = 0. \tag{71}$$

Proof. The necessity follows from the previous discussion. For the sufficiency see [5]. □

Another way to represent the trifocal geometry is to use the three different bifocal tensors, ${}_{AB}F_{ij}$, ${}_{AC}F_{ik}$ and ${}_{AB}F_{jk}$. Calculating the number of parameters in these three bifocal tensors, gives that there are $3 \times 7 = 21$ (3 tensors with 9 homogeneous elements obeying one constraint each) parameters, whereas 18 parameters is the minimal number to parametrize the trifocal geometry. However, three constraints are easily obtained from the bilinear constraint and the fact that the epipoles from one camera in the other two are in correspondence since they are both images of the focal point of that camera. This simple observation gives the three constraints

$${}_{AB}F_{ij\,CA}e^i{}_{CB}e^j = 0, \quad {}_{AC}F_{ik\,BA}e^i{}_{BC}e^k = 0, \quad {}_{BC}F_{jk\,AB}e^j{}_{AC}e^k = 0. \tag{72}$$

Theorem 4.7. *A necessary condition for* 27 *numbers to constitute the components of the* 3 *bifocal tensors* ${}_{AB}F$, ${}_{AC}F$ *and* ${}_{BC}F$ *is to fulfil the conditions in Theorem* 4.2 *and* (72).

We conjecture that these conditions are sufficient also. It is easy to see that a representative of the trifocal geometry can be obtained from these three bifocal tensors, which means that they encode the trifocal geometry, but the calculations are rather lengthy and omitted here.

Another property of the trifocal tensor is that it has rank 4, where the rank of a tensor is defined as the minimal number of terms needed in order to write the tensor as a sum of rank 1 tensors and a rank 1 tensor is an outer product of three vectors. This fact has been proven in [17], however, a simpler proof will be given here.

Theorem 4.8. *The trifocal tensor*, T, *fulfils*

$$\operatorname{rank} T = 4.$$

Proof. First observe that the rank of the trifocal tensor is a generic property, i.e. it does not depend on the chosen co-ordinate systems in the images. Thus, we may chose a representative for the multiple view geometry, using the camera matrices

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \tag{73}$$

Table 5. Relations used for the three-view case

Quadratic p-relations
${}^{BC}_{A}T_i^{jK}\,{}_{AB}F_{Kj} = 0_i$
${}^{BC}_{A}T_i^{jK}\,{}_{AB}F_{Jk} = 0_i$
${}^{BC}_{A}T_i^{J'K}\,{}^{AC}_{B}T_{I'}^{iK} = 0_i, \quad I' \neq J'$
${}^{BC}_{A}T_i^{JK}\,{}^{BC}_{A}T_{i'}^{J'K}\varepsilon^{ii'i''}\varepsilon_{JJ'J''} = {}_{AC}e^{K}\,{}^{AC}_{B}T_{J''}^{i''K}$
Constraints
$\det \lambda^i T_i^{\bullet\bullet} = 0, \quad \forall \lambda_i$
$\det[\mathcal{N}(T_1^{\bullet\bullet})\mathcal{N}(T_2^{\bullet\bullet})\mathcal{N}(T_3^{\bullet\bullet})] = 0$
${}_{AB}F_{ijCA}e^{i}{}_{CB}e^{j} = 0, \quad {}_{AC}F_{ikBA}e^{i}{}_{BC}e^{k} = 0, \quad {}_{BC}F_{jkAB}e^{j}{}_{AC}e^{k} = 0$

i.e. by choosing a 3D co-ordinate system such that the focal points of the three cameras have projective co-ordinates (0, 0, 0, 1), (0, 0, 1, 0) and (0, 1, 0, 0), respectively, and the 2D co-ordinate systems in the images have been chosen such that the remaining 3×3 part of the camera matrix becomes the identity matrix. Calculating the trifocal tensor from the camera matrices in (73) gives $T_i^{jk} = 0$ except for,

$$T_3^{11} = -1, \quad T_1^{22} = 1, \quad T_2^{32} = 1 \quad \text{and} \quad T_1^{13} = -1, \tag{74}$$

which immediately implies that rank $T \leqslant 4$. Looking at the non-zero coefficients of T in (74), it is obvious that it is not possible to write T as a sum of less than 4 rank 1 tensors. This observation concludes the proof. □

We summarize the equations used for the three-view case in Table 5.

4.3. *Four views*

We now turn to the case of four views, where the tensors appearing are the quadrifocal tensor, 12 trifocal tensors, 6 bifocal tensors and 12 monofocal tensors, when the symmetries have been taken into account. We will show how the quadratic p-relations can be used to calculate one trifocal tensor from the quadrifocal tensor and how a representative of the quadrifocal geometry can be obtained.

The quadratic p-relations of the type

$$\begin{aligned}
&[A^1B^1C^1D^1][A^2A^3B^1C^1] - [A^2B^1C^1D^1][A^1A^3B^1C^1] \\
&\quad + [A^3B^1C^1D^1][A^1A^2B^1C^1] \\
&= {}^{ABCD}Q^{1111}\cdot{}^{BC}_{A}T_1^{11} + {}^{ABCD}Q^{2111}\cdot{}^{BC}_{A}T_2^{11} + {}^{ABCD}Q^{3111}\cdot{}^{BC}_{A}T_3^{11} = 0,
\end{aligned} \tag{75}$$

implies that, writing $Q = {}^{ABCD}Q$,

$$Q^{iJKl}\cdot{}^{BC}_{A}T_i^{JK} = 0^l, \quad Q^{iJkL}\cdot{}^{BD}_{A}T_i^{JL} = 0^k, \quad Q^{ijKL}\cdot{}^{CD}_{A}T_i^{KL} = 0^j,$$

$$Q^{IjKl}\cdot{}^{AC}_{B}T_j^{IK} = 0^l, \quad Q^{IjkL}\cdot{}^{AD}_{B}T_j^{IL} = 0^k, \quad Q^{ijKL}\cdot{}^{CD}_{B}T_j^{KL} = 0^i,$$

$$Q^{IJkl}\cdot{}^{AB}_{C}T^{IJ}_{k}=0^{l},\quad Q^{IjkL}\cdot{}^{AD}_{C}T^{IL}_{k}=0^{j},\quad Q^{iJkL}\cdot{}^{BD}_{C}T^{JL}_{k}=0^{i},$$
$$Q^{IJkl}\cdot{}^{AB}_{D}T^{IJ}_{l}=0^{k},\quad Q^{IjKl}\cdot{}^{AC}_{D}T^{IK}_{l}=0^{j},\quad Q^{iJKl}\cdot{}^{BC}_{D}T^{JK}_{l}=0^{i},\tag{76}$$

where again capital letters denote indices where no summation is done. Thus ${}^{BC}_{A}T^{jk}_{\bullet}$, etc., can be calculated, up to scale, from Q. A first step towards the fixation of these scales is to consider quadratic p-relations of the type

$$\begin{aligned}&-[A^2A^1A^3B^1][A^1B^1C^1D^1]-[C^1A^1A^3B^1][A^1A^2B^1D^1]\\&\quad+[D^1A^1A^3B^1][A^1A^2B^1C^1]\\&={}_{AB}e^{1}\cdot{}^{ABCD}Q^{1111}-{}^{BC}_{A}T^{11}_{2}\cdot{}^{BD}_{A}T^{11}_{3}-{}^{BD}_{A}T^{11}_{2}\cdot{}^{BC}_{A}T^{11}_{3}=0.\end{aligned}\tag{77}$$

Similar quadratic p-relations gives the following equations, writing $T={}^{BC}_{A}T$, $U={}^{BD}_{A}T$ and $e={}_{AB}e$,

$$T^{JK}_{\bullet}\times U^{JL}_{\bullet}=e^{J}Q^{\bullet JKL},\tag{78}$$

making it possible to recover the scale between $T^{JK}_{\bullet}$ for different K by fixing $U^{JL}_{\bullet}$. What remains to be done is to calculate the scale between $T^{J\bullet}_{\bullet}$ for different J. This can be done from quadratic p-relations, similar to the ones in (78).

$${}^{BC}_{A}T^{JK}_{\bullet}\times{}^{CD}_{A}T^{KL}_{\bullet}=e^{K}Q^{\bullet JKL}.\tag{79}$$

We have thus shown how to calculate the trifocal tensor ${}^{BC}_{A}T$ from the quadrifocal tensor ${}^{ABCD}Q$.

Theorem 4.9. *Given the quadrifocal tensor ${}^{ABCD}Q$ it is possible to calculate the trifocal tensor ${}^{BC}_{A}T$.*

The algorithm looks as follows:

Algorithm 4.3. $[{}^{ABCD}Q$ to ${}^{BC}_{A}T]$

1. Calculate ${}^{BC}_{A}T^{jk}_{\bullet}$ up to scale from (76).
2. Calculate ${}^{BD}_{A}T^{jl}_{\bullet}$ up to scale from (76).
3. Calculate a consistent scale between different ${}^{BC}_{A}T^{j\bullet}_{\bullet}$ from (78).
4. Calculate ${}^{CD}_{A}T^{kl}_{\bullet}$ up to scale from (76).
5. Calculate a consistent scale between different ${}^{BC}_{A}T^{j\bullet}_{\bullet}$ from (79).

This implies that all trifocal tensors can be calculated from the quadrifocal tensor and also all bifocal and monofocal tensor using the method presented in the previous subsections. We have thus shown how to calculate all other multiple view tensors from ${}^{ABCD}Q$, using only polynomial equations of degree 2.

Theorem 4.10. *Given the quadrifocal tensor ${}^{ABCD}Q$. Then it is possible to calculate the other* 12 *trifocal tensors, the* 6 *bifocal tensors as well as the* 12 *monofocal tensors.*

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Moreover, a representative of the quadrifocal view geometry can be obtained from the quadrifocal tensor, ${}^{ABCD}Q$. Starting from ${}^{ABCD}Q$, we have shown how to calculate ${}^{BC}_{A}T$, ${}^{BD}_{A}T$, ${}_{AB}F$, ${}_{AB}e$, ${}_{AC}e$ and ${}_{AD}e$, up to scale. Thus everything in (58) is known apart from the scale between ${}^{BC}_{A}T$ and ${}_{AC}e$ and the scale between ${}^{BD}_{A}T$ and ${}_{AD}e$. These scales have to be compatible with the other fixed scales and can be obtained from (58) by calculating minors from (58) corresponding to ${}^{ABCD}Q^{ijkl}$, for all i, j, k and l. The expressions obtained from these minors can also be derived from suitable quadratic p-relations. We thus have:

Corollary 4.3. *The quadrifocal tensor encodes the quadrifocal geometry in the sense that a unique representative* (*up to projective transformations*) *of the quadrifocal geometry can be obtained from the quadrifocal tensor.*

Looking at (76) and using the transformation properties of the quadrifocal tensor we get the following theorem.

Theorem 4.11. *A necessary condition for* 81 *numbers of* Q^{ijkl} *to constitute the components of a quadrifocal tensor is*

$$\det(\lambda^1_{ij}Q^{ij\cdot\cdot}) = 0, \quad \det(\lambda^2_{ik}Q^{i\cdot k\cdot}) = 0, \quad \det(\lambda^3_{il}Q^{i\cdot\cdot l}) = 0,$$

$$\det(\lambda^4_{jk}Q^{\cdot jk\cdot}) = 0, \quad \det(\lambda^5_{jl}Q^{\cdot j\cdot l}) = 0, \quad \det(\lambda^6_{kl}Q^{\cdot\cdot kl}) = 0, \tag{80}$$

where λ^M_{ij}, $M = 1, \ldots, 6$, *denote the components of arbitrary non-singular matrices.*

To our knowledge, it is an open question whether or not the equations in (80) are also sufficient.

Another way to represent the quadrifocal geometry is to use two different trifocal tensors, ${}^{BC}_{A}T$ and ${}^{CD}_{B}T$. Calculating the number of parameters in these two trifocal tensors, gives that there are $2 \times 18 = 36$ (2 tensors with 18 independent parameters each) parameters, whereas 29 parameters is the minimal number to parameterize the trifocal geometry. However, 7 constraints are easily obtained by calculating the bifocal tensor ${}_{BC}F$ from both ${}^{BC}_{A}T$ and ${}^{CD}_{B}T$ and writing down the equations that they are equal up to scale. That gives $8 - 1 = 7$ constraints since ${}_{BC}F$ only contains 7 independent parameters. We conjecture that this is both a necessary and sufficient condition for 54 numbers to constitute the components of two trifocal tensors ${}^{BC}_{A}T$ and ${}^{CD}_{B}T$. It is also easy to see that a representative of the quadrifocal geometry can be obtained from these two trifocal tensors, which means that they encode the quadrifocal geometry, but the calculations are rather lengthy and omitted here.

Yet, another way to represent the quadrifocal geometry is to use the five different bifocal tensors ${}_{AB}F$, ${}_{AC}F$, ${}_{BC}F$, ${}_{BD}F$ and ${}_{CD}F$. Calculating the number of parameters in these 5 bifocal tensors, gives that there are $5 \times 7 = 35$ (5 tensors with 7 independent parameters each) parameters, whereas 29 parameters is the minimal numbers to parameterize the trifocal geometry. However, 6 constraints are easily obtained from (72) for the triplet (${}_{AB}F$, ${}_{AC}F$, ${}_{BC}F$) and the triplet (${}_{BC}F$, ${}_{BD}F$, ${}_{CD}F$). We conjecture that these conditions are sufficient also. It is also easy to see that a representative of the quadrifocal geometry can be obtained from these five bifocal tensors, which means that they encode the quadrifocal geometry, but the calculations are again rather lengthy and omitted here. Observe that adding the bifocal tensor $_{AD}F$ gives no extra information, since it can be calculated from the other bifocal tensors. This observation gives immediately the necessary and sufficient conditions for 54 numbers to constitute the components of all 6 bifocal tensors, since 7 new constraints can be obtained by calculating $_{AD}F$ from the other bifocal tensors and writing down the equations that this calculated $_{AD}F$ is equal to the given $_{AD}F$ up to scale.

Another property of the quadrifocal tensor is that it has rank 9, where the rank of a fourth order tensor is defined similarly to the case of a third order tensor, described above.

Theorem 4.12. *The quadrifocal tensor, Q, fulfils*

$$\operatorname{rank} Q = 9.$$

Proof. The proof is similar to the proof of rank of the trifocal tensor. Choose co-ordinate systems according to (73) and

$$P_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{81}$$

and calculate the quadrifocal tensor. This gives $Q^{ijkl} = 0$ except for

$$\begin{aligned} &Q^{1111} = -1, \quad Q^{3221} = -1, \quad Q^{3131} = 1, \\ &Q^{1322} = -1, \quad Q^{2222} = 1, \quad Q^{2132} = -1, \\ &Q^{1313} = 1, \quad Q^{2213} = -1 \text{ and } Q^{3333} = -1, \end{aligned} \tag{82}$$

which in the same way as for the trifocal tensor implies that rank $Q = 9$. □

We summarize the equations used for the four-view case in Table 6.

4.4. *N views*

There are essentially three different ways to encode the N-view geometry using multifocal tensors; using bifocal tensors, using trifocal tensors and using quadrifocal

Table 6. Relations used for the four-view case

Quadratic p-relations
${}^{ABCD}Q^{iJKl}\ {}^{BC}_{A}T_i^{JK} = 0^l$
${}_{AB}e^J\ {}^{ABCD}Q^{iJKl} = \varepsilon^{ii'i''}\ {}^{BC}_{A}T_{i'}^{JK}\ {}^{BD}_{A}T_{i''}^{JL}$
Constraints
$\det(\lambda_{ij}Q^{ij\bullet\bullet}) = 0$

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tensors. There are of course many possible combinations of these, but they can be easily handled when the three methods above are understood. Thus we will concentrate on these three different ways of representing the N-view geometry here.

When only bifocal tensors are used for a sequence of image, it is sufficient to use the tensors ${}_{I,I+1}F$ and ${}_{I,I+2}F$, subject to the constraints in (72) for every triplet $({}_{I,I+1}F, {}_{I,I+2}F, {}_{I+1,I+2}F)$. Using this representation we have $N-1+N-2=2N-3$ bifocal tensors and $N-2$ such triplets obeying three constraints each, giving in total $7(2N-3)-3(N-2)=11N-15$ parameters describing the N-view geometry, i.e. the minimal number.

Theorem 4.13. *Necessary and sufficient conditions for* $9(2N-3)$ *numbers to constitute the components of the* $2N-3$ *bifocal tensors* ${}_{I,I+1}F$ *and* ${}_{I,I+2}F$ *are*

$$\det({}_{I,I+1}F)=0, \tag{83}$$

$$\begin{aligned} &{}_{I+2,I}e^T\,{}_{I,I+1}F_{I+2,I+1}e=0, \quad {}_{I+1,I}e^T\,{}_{I,I+2}F_{I+1,I+2}e=0,\\ &{}_{I,I+1}e^T\,{}_{I+1,I+2}F_{I,I+2}e=0. \end{aligned} \tag{84}$$

When only trifocal tensors are used it is sufficient to use the tensors ${}^{I+1,I+2}_{\quad\;\; I}T$. Using this representation we have $N-2$ trifocal tensors with 18 independent parameters each and for every consecutive pair $({}^{I+1,I+2}_{\quad\;\; I}T, {}^{I+2,I+3}_{\quad\;\; I+1}T)$ of trifocal tensors we get 7 constraints from the compatibility of ${}_{I+1,I+2}F$, that can be calculated from both. Since there are $N-3$ such constraints, we get $18(N-2)-7(N-3)=11N-15$ parameters describing the N-view geometry, i.e. the minimal number.

Theorem 4.14. *Necessary and sufficient conditions for* $27(N-2)$ *numbers to constitute the components of the* $N-2$ *trifocal tensors* ${}^{I+1,I+2}_{\quad\;\; I}T$ *are the individual condition in* (71) *and the compatibility conditions that the bifocal tensors* ${}_{I+1,I+2}F$ *are equal up to scale when calculated from* ${}^{I+1,I+2}_{\quad\;\; I}T$ *and* ${}^{I+2,I+3}_{\quad\;\; I+1}T$).

When only quadrifocal tensors are used it is sufficient to use the tensors ${}^{I,I+1,I+2,I+3}Q$. Using this representation we have $N-3$ quadrifocal tensors with 29 independent parameters each and for every consecutive pair $({}^{I,I+1,I+2,I+3}Q, {}^{I+1,I+2,I+3,I+4}Q)$ of quadrifocal tensors we get 18 constraints from the compatibility of ${}^{I+2,I+3}_{\quad\;\; I+1}T$, that can be calculated from both. Since there are $N-4$ such constraints, we get $29(N-3)-18(N-4)=11N-15$ parameters describing the N-view geometry, i.e. the minimal number.

Theorem 4.15. *Necessary and sufficient conditions for* $81(N-3)$ *numbers to constitute the components of the* $N-3$ *quadrifocal tensors* ${}^{I,I+1,I+2,I+3}Q$ *are the individual condition in* (80) *and the compatibility conditions that the trifocal tensors* ${}^{I+2,I+3}_{\quad\;\; I+1}T$ *are equal up to scale when calculated from* ${}^{I,I+1,I+2,I+3}Q$ *and* ${}^{I+1,I+2,I+3,I+4}Q$.

The general case of necessary and sufficient conditions for any collection of multifocal tensors can easily be handled using the minimal representations of the multifocal geometry described above in the following way.

Algorithm 4.4. [Necessary and sufficient conditions for an arbitrary collection of multiple view tensors]

1. Pick out a minimal set (or several minimal sets for suitable subsequences) of overlapping tensors that encodes the multifocal geometry.
2. Write down the constraints for this minimal set in the form of (72) for overlapping bifocal tensors or compatibility relations for overlapping trifocal or quadrifocal tensors.
3. Write down the constraints for the compatibility of the tensors not used in the minimal set, since these can be calculated from the minimal set.

Note that when minimal subsets has to be used, the multiple view geometry cannot be recovered for the whole sequence, only for each subsequence.

Theorem 4.16. *Necessary and sufficient conditions for a collection of numbers to constitute the components of an arbitrary collection of multifocal tensors can be obtained from Algorithm* 4.4.

5. Conclusions

In this paper we have presented a new framework for multiple view geometry and multiple view tensors, using Grassman co-ordinates and quadratic p-relations. Using this framework, relations between different tensors and constraints on tensor components can be derived. This has been exemplified by calculating bifocal tensors from a trifocal tensor and by calculating trifocal tensors from a quadrifocal tensor. Having these tools available, it is possible to calculate a representative of the multiple view geometry, in the form of the camera matrices, from different combinations of multiple view tensors. It is also possible to give necessary and sufficient conditions for all possible combinations of multiple view tensors.

It has also been shown that the rank of the trifocal tensor is 4 and that the rank of the quadrifocal tensor is 9. We have also proved that it is possible to recover the trifocal tensor from at least 7 corresponding points in 3 images and the quadrifocal tensor from at least 6 corresponding points in 4 images linearly. Using the previously mentioned methods, the camera matrices can be calculated form the tensor components and, finally, a reconstruction can be obtained.

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