Volume of $\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})$

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Abstract

We show that suitably normalized,

$$\mathrm{vol}(\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})) = \zeta(2)\zeta(3)\cdots\zeta(n),$$

where $\mathrm{SL}(n)$ is the $n \times n$ special linear group. Integration over the quotient space is carried out via the groups, using basic technique of topological groups and Poisson summation.

Introduction

This thesis calculates the volume of the quotient space

$$\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R}) = \{\mathrm{SL}_n(\mathbb{Z})g : g \in \mathrm{SL}_n(\mathbb{R})\},$$

the analogue of the circle $\mathbb{R}/\mathbb{Z}$ for the matrix group SL(2). The quotient space is no longer a group, but the topological groups $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{Z})$ carry Haar measures, and integration over the quotient space is characterized in terms of integration over the groups. Decompositions of $\mathrm{SL}_n(\mathbb{R})$ determine its Haar measure in terms of readily calculable Haar measures on subgroups. Poisson summation simplifies the integral of a dummy function until the volume of the space emerges. We discuss integration on quotient spaces in general before considering $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$, then generalize to $\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})$.

A *topological group* G is both a group and a locally compact Hausdorff topological space, with multiplication and inversion continuous,

$$\begin{aligned} G \times G \to G, &\quad g \times h \to gh \\ G \to G, &\quad g \to g^{-1}. \end{aligned}$$

We further stipulate that a topological group have a countable basis; for a topological group, a *basis* B is a collection of open sets such that every open set in G can be written as a union of elements of B.

A topological group G *acts* on a topological space X when there is a continuous map

$$G \times X \to X, \quad g \times x \to gx$$

satisfying

$$\begin{aligned} 1_G x = x &\quad \forall x \in X \\ (g_1 g_2)x = g_1(g_2 x) &\quad \forall g_1, g_2 \in G, x \in X. \end{aligned}$$

A measure μ on a group G is *left-invariant* when

$$\mathrm{d}\mu(hg) = \mathrm{d}\mu(g) \quad \forall h \in G$$

and similarly is *right-invariant* when

$$\mathrm{d}\mu(gh) = \mathrm{d}\mu(g) \quad \forall h \in G.$$

A *left Haar measure* is a measure that is left-invariant, positive, regular, and Borel, and similarly for a *right Haar measure*. Every topological group carries a left and right Haar measure, each unique up to scaling. Let μ be the left Haar measure on G. The *modular function*

$$\Delta : G \to \mathbb{R}$$

is characterized by the condition

$$\mathrm{d}\mu(gh) = \Delta(h) \cdot \mathrm{d}\mu(g).$$

The modular function of a group relates its left and right Haar measures. When they are equal, the group is *unimodular*.

The *special linear group of degree n* for a suitable ring A, denoted $\mathrm{SL}_n(A)$, is the set of n-by-n matrices with determinant 1, the operation being matrix multiplication. Of specific interest will be the *special linear group of 2-by-2 real matrices*

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 1, \quad a, b, c, d \in \mathbb{R} \right\}$$

and a discrete subgroup, the *special linear group of 2-by-2 integer matrices*,

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}) : a, b, c, d \in \mathbb{Z} \right\}.$$

Another subgroup of $\mathrm{SL}_2(\mathbb{R})$ we will be interested in is the *special orthogonal 2-group*,

$$\mathrm{SO}(2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = I \right\}.$$

Chapter 1

General Results on Topological Groups

For the pending volume calculations, we will relate integration over quotient spaces of topological groups to integration over the groups themselves. Decomposing the groups into more manageable subgroups, describing the Haar measures on the subgroups, and describing how they fit together will characterize the integral.

Lemma 1.0.1 (Hausdorff quotient groups)**.** *For a topological group G with a subgroup H, the quotient G/H is Hausdorff if and only if H is closed.*

Proof. Suppose G/H is Hausdorff and let $q : G \to G/H$ be the quotient map $q(g) = gH$. Thus for $g \notin H$, $q(g) \neq q(1)$ and so by Hausdorffness there are disjoint opens U_g and V containing $q(g)$ and $q(1)$ respectively. By continuity of q, $q^{-1}(U_g)$ and $q^{-1}(V)$ are disjoint and contain g and H respectively. Hence $q^{-1}(U_g)$ is a neighborhood of g that does not meet H for each $g \notin H$, so H^c is a union of opens and is again open, meaning H is closed.

Now suppose H is closed. Given $x \notin H$, by local compactness of G, take a neighborhood U of 1 with compact closure $\overline{U}$ so that Ux is a neighborhood of x with compact closure $\overline{U}x$. For each $y \in \overline{U}x \cap H$, since $x \notin H$, $y \neq x$. By Hausdorffness, take neighborhoods U_y of 1 and V_y of y so that $V_y \cap U_y x = \emptyset$. Since $\overline{U}x$ is compact and H is closed, $\overline{U}x \cap H$ is again compact, so there is a finite list of points $y_1, ..., y_n$ in $\overline{U}x \cap H$ such that the V_{y_i}'s cover $\overline{U}x \cap H$. Each U_{y_i} is open and so the finite intersection $W_o = \cap_i U_{y_i} x$ is open and does not meet H. Noting that $W_o x^{-1}$ is a neighborhood of 1, let V_o be an open neighborhood of 1 such that[1] $V_o^2 \subset W_o x^{-1}$ and take $V = V_o \cap V_o^{-1}$. We want $Vx \cap VH = \emptyset$. To see this, suppose $\exists y \in Vx \cap VH$, so that for some $v \in V$, $y \in Vx \cap vH$. Hence

$$v^{-1}y \in v^{-1}Vx \cap H \subset V^2 x \cap H \subset V_o^2 x \cap H \subset (W_o x^{-1})x \cap H = W_o \cap H = \emptyset.$$

This contradiction forces the intersection $Vx \cap VH$ to be empty as desired, and so for each $h \in H$, $Vxh \cap VHh = Vxh \cap VH = \emptyset$. Taking the union over such h, $VxH \cap VH = \emptyset$. With all of this in place, given y and $z \in G$ such that $yH \neq zH$, let $x = y^{-1}z$ and choose V as described above for x. Let $W = yVY^{-1}$ and see that

$$Vy^{-1}zH \cap VH = \emptyset \Longrightarrow yVy^{-1}zH \cap yVH = \emptyset \Longrightarrow WzH \cap WyH = \emptyset.$$

[1]See appendix A.

Since W is open in G and contains 1, $q(WzH)$ and $q(WyH)$ are open, disjoint, and contain $q(zH)$ and $q(yH)$ respectively. Hence given distinct cosets in G/H there are distinct open neighborhoods and so G/H is Hausdorff. □

1.1 Integration on G/H

Integration on the quotient requires a Fubini-like characterization in terms of integration on the groups forming the quotient. Expressing functions on the quotient as averages of functions on the groups gives a well-defined characterization of this iterated integral.

Let G be a topological group with H a closed subgroup, and let G/H denote the quotient space. Note that by Lemma 1.0.1, the quotient G/H is Hausdorff. Further suppose that the restriction of the modular function of G to H is exactly the modular function of H. Integration for functions $F \in C_c(G/H)$, i.e., compactly supported continuous functions on the quotient space, is characterized by the condition

$$\int_G f(g)\,\mathrm{d}g = \int_{G/H}\left[\int_H f(gh)\,\mathrm{d}h\right]\mathrm{d}\bar{g}, \quad f \in C_c(G).$$

The inner integral depends only on the equivalence class $\bar{g} \in G/H$ rather than on specific choice of g. Therefore, for a function $f \in C_c(G)$ that "winds up" over H to F,

$$F(\bar{g}) = \int_H f(gh)\,\mathrm{d}h, \quad \bar{g} \in G/H,$$

then the desired integral $\displaystyle\int_{G/H} F$ is $\displaystyle\int_G f$. Integration over a quotient space is thus

$$\int_{G/H} F(\bar{g})\,\mathrm{d}\bar{g} = \int_G f(g)\,\mathrm{d}g, \quad \text{(as above, this } f \text{ winds up to } F\text{)}.$$

Lemma 1.1.1 (Surjectivity Lemma). *Let G be a topological group with H a closed subgroup. Given $F \in C_c(G/H)$, there exists $f \in C_c(G)$ such that*

$$\int_H f(gh)\,\mathrm{d}h = F(\bar{g}), \quad \bar{g} \in G/H.$$

Proof. We first show that there exists a compact subset $C \subset G$ whose projection into G/H is exactly $\mathrm{spt}(F)$; that is, $q(C) = \mathrm{spt}(F)$ where q is the natural projection map $q : G \longrightarrow G/H$. To show that such a set exists, we take a neighborhood U of 1 in G with compact closure and so, since a quotient map is open, $q(U)$ is open in G/H and for any $g \in G$, $g \cdot q(U)$ is open in G/H with compact closure. The union $\bigcup_g (q(gU))$ covers G/H and hence covers the compact subset $\mathrm{spt}(F)$. Because $\mathrm{spt}(F)$ is compact, some finite subcover $\{q(g_iU)\}$ also covers it. Thus

$$C = \left(\bigcup_i g_i\overline{U}\right) \cap q^{-1}(\mathrm{spt}(F))$$

is the compact subset of G whose projection is exactly spt(F).

The open cover $\{cU : c \in C\}$ has a finite subcover O with compact closure $\overline{O}$. By Urysohn's Lemma, there exists a continuous function $\varphi \in C_c^o(G)$ that is identically 1 on C, nonnegative on G, with support in $\overline{O}$. Now let $\bar{\varphi}$ be φ symmetrized over H, an H-invariant function on G and also on G/H,

$$\bar{\varphi}(\bar{g}) = \int_H \varphi(gh)\,\mathrm{d}h.$$

Viewed as a function on G, $\bar{\varphi}$ is positive on C, making it positive on spt(F) when viewed as a function on G/H. Since spt(F) is compact, $\bar{\varphi}$ takes some positive minimum value m. Therefore it is well-defined to specify a function

$$f = \varphi \cdot (F \circ q)/\bar{\varphi}$$

on $\mathrm{spt}(\varphi) \cap q^{-1}(\mathrm{spt}(F))$ which can be extended continuously by 0 on the rest of G[2] to give a continuous function on G with the desired property

$$\int_H f(gh)\,\mathrm{d}h = \int_H \varphi(gh)\,\mathrm{d}h \cdot F(\bar{g})/\bar{\varphi}(\bar{g})\,\mathrm{d}h = F(\bar{g}).$$

□

1.2 Measure on *G*

With integration defined over G and H instead of G/H, evaluation requires Haar measures on G and H. The Haar measure on these groups are determined by the Haar measure of their subgroup decompositions as direct and semidirect products. As such, studying how the Haar measures of subgroups relate to the Haar measure of the whole group will determine the measure for integration.

Proposition 1.2.1 (Some general unimodular groups)**.** *A group G is unimodular if any of the following properties hold:*

- *G is abelian.*
- *G is compact.*
- *G is discrete.*
- *G is generated by commutators.*

Proof. Let G be an abelian group with left Haar measure μ. Then

$$\mathrm{d}\mu(hg) = \mathrm{d}\mu(gh) \quad \forall g, h \in G.$$

Let G be a compact group. Because Δ is a continuous group homomorphism, the continuous image $\Delta(G)$ is compact in $\mathbb{R}_{>0}$. The only compact subgroup of $\mathbb{R}_{>0}$ is $\{1\}$, so Δ is trivial.

[2]Since the value of $\bar{\varphi}$ is bounded below by m and the value of F goes continuously to 0 as $\bar{g}$ approaches the boundary of spt(f), the function is continuous because the numerator shrink continuously to 0 without the denominator approaching 0, so the values are bounded.

Let G be a discrete group with left Haar measure μ. Because the counting measure is the only measure on a discrete group, μ is the counting measure, so every point has measure $\mathrm{d}\mu(g) = 1$.

Let G be a group generated by commutators and consider the action of Δ on commutators:

$$\begin{aligned}\Delta(aba^{-1}b^{-1}) &= \Delta(a)\Delta(b)\Delta(a)^{-1}\Delta(b)^{-1} \quad &&\text{(by properties of homomorphisms, i.e. } \Delta)\\ &= \Delta(a)\Delta(a)^{-1}\Delta(b)\Delta(b)^{-1} &&\text{(by properties of abelian groups, i.e. } \mathbb{R})\\ &= 1.\end{aligned}$$

Thus Δ is trivial on commutators, and so on G. □

Lemma 1.2.2 (Left Haar of inverse)**.** *The left Haar measure on G satisfies*

$$\mathrm{d}(g^{-1}) = \Delta(g)^{-1}\,\mathrm{d}g.$$

Proof. In general $\mathrm{d}(gh) = \Delta_G(h)\,\mathrm{d}g$. Compute that the measure $\mathrm{d}\nu(g) = \Delta_G(g)^{-1}\,\mathrm{d}g$ is right invariant,

$$\mathrm{d}\nu(gh) = \Delta_G(gh)^{-1}\,\mathrm{d}(gh) = \Delta_G(g)^{-1}\Delta_G(h)^{-1}\Delta_G(h)\,\mathrm{d}g = \Delta_G(g)^{-1}\,\mathrm{d}g = \mathrm{d}\nu(g).$$

So is the measure $\mathrm{d}\lambda(g) = \mathrm{d}(g^{-1})$,

$$\mathrm{d}\lambda(gh) = \mathrm{d}((gh)^{-1}) = \mathrm{d}(h^{-1}g^{-1}) = \mathrm{d}(g^{-1}) = \mathrm{d}\lambda(g).$$

By uniqueness of right Haar measure, the proof is complete. □

Introduce α, the averaging map

$$\alpha : C_c^o(G) \to C_c^o(G/H), \qquad \alpha(f)(g) = \int_H f(gh)\,\mathrm{d}h \quad \text{(left Haar measure d}h\text{ on }H)$$

and note that this is simply an explicit formula for the wind-up we have previously used and proved to be surjective. We will view this both as a function on G and G/H.

Proposition 1.2.3 (Inherited measure on a quotient space)**.** *Let G be a topological group with H a closed subgroup, with G acting from the left. The quotient G/H has a left-G-invariant (positive regular Borel) measure exactly when*

$$\Delta_{G|H} = \Delta_H.$$

If such a measure exists it is unique up to scalar multiples and can be normalized as follows. For left Haar measure dh on H and left Haar measure dg on G there is a unique invariant measure $d\bar{g}$ on G/H such that for $f \in C_c^o(G)$ there is an integration formula

$$\int_G f(g)\,\mathrm{d}g = \int_{G/H}\left(\int_H f(\bar{g}h)\,\mathrm{d}h\right)\mathrm{d}\bar{g}.$$

Proof. We will first show that if there is such a measure exists on the quotient space, the modular functions behave as stated. Then we will show, granting the Haar measures on G and H, that if the modular functions agree as above, the Haar measure on the quotient space exists such that the integration formula stated above is well-defined.

Suppose there is a left-G-invariant measure on G/H. For $f \in C_c^o(G)$, the map

$$I : C_c^o(G) \to \mathbb{C}, \qquad I(f) = \int_{G/H} \alpha(f)(\bar{g})\,\mathrm{d}\bar{g} = \int_{G/H}\int_H f(\bar{g}h)\,\mathrm{d}h\,\mathrm{d}\bar{g}$$

is a left-G-invariant functional and hence, invoking the uniqueness of right- and left-G-invariant functionals,[3] I is a constant multiple of the Haar integral $f \to \int_G f(g)\,\mathrm{d}g$. Define the left action of G on $C_c^o(G)$ as

$$(\gamma \cdot f)(g) = f(g\gamma^{-1}), \quad \gamma \in G.$$

We observe that I is left-G-invariant by noting that $\gamma^{-1} \in G$ acting on H permutes H:

$$I(\gamma \cdot f) = \int_{G/H}\int_H f(\bar{g}h\gamma^{-1})\,\mathrm{d}h\,\mathrm{d}\bar{g} = \int_{G/H}\int_H f(\bar{g}\tilde{h})\,\mathrm{d}\tilde{h}\,\mathrm{d}\bar{g} = I(f).$$

Now we observe how the averaging map α behaves under right translation by γ:

$$\alpha(\gamma \cdot f)(g) = \int_H f(gh\gamma^{-1})\,\mathrm{d}h = \Delta_H(\gamma)\int_H f(gh)\,\mathrm{d}h$$

upon replacing h by $h\gamma$. Then

$$\int_G f(g)\,\mathrm{d}g = \int_{G/H} \alpha(f)(g)\,\mathrm{d}\bar{g} = \Delta_H(\gamma)^{-1}\int_{G/H} \alpha(\gamma \cdot f)(g)\,\mathrm{d}\bar{g} = \Delta_H(\gamma)^{-1}\int_G f(g\gamma^{-1})\,\mathrm{d}g.$$

Replacing g by $g\gamma$ yields

$$\int_G f(g)\,\mathrm{d}g = \Delta_H(\gamma)^{-1}\Delta_G(\gamma)\int_G f(g)\,\mathrm{d}g,$$

so choosing a function f whose integral is not 0 shows that the modular function of H is the restriction of the modular function of G to H.

Now suppose G and H have Haar measures such that[4] $\Delta_{G|H} = \Delta_H$, and define integration on $C_c^o(G/H)$ by

$$\int_{G/H} \alpha f(\bar{g})\,\mathrm{d}\bar{g} = \int_G f(g)\,\mathrm{d}g,$$

since α from $C_c^o(G)$ to $C_c^o(G/H)$ is a surjection by Lemma 1.1.1. To show that this integral is well-defined, it suffices to show that when the average $\alpha(f)$ is the zero function, the integral

[3] A writeup by Paul Garrett can be found at
`http://www.math.umn.edu/~garrett/m/v/uniq_of_distns.pdf`

[4] In the case we are concerned with, G and H are unimodular, making the agreement trivial.

of f is 0. To that end, introduce an auxiliary function f_o. We will move the average from f to f_o and then choose f_o suitably to give the result,

$$\begin{aligned}
0 = \alpha(f)(g) &= \int_G f_o(g)\alpha(f)(g)\,\mathrm{d}g \quad \text{(since we are integrating 0)}\\
&= \int_G f_o(g)\left(\int_H f(gh)\,\mathrm{d}h\right)\mathrm{d}g \quad \text{(by definition of } \alpha)\\
&= \int_G\int_H f_o(g)f(gh)\,\mathrm{d}h\,\mathrm{d}g \quad \text{(passing } f_o(g) \text{ through the } H \text{ integral)}\\
&= \int_H\int_G f_o(g)f(gh)\,\mathrm{d}g\,\mathrm{d}h \quad \text{(by Fubini on compactly-supported's)}\\
&= \int_H \Delta_G(h)^{-1}\int_G f_o(gh^{-1})f(g)\,\mathrm{d}g\,\mathrm{d}h \quad \text{(replacing } g \text{ by } gh^{-1})\\
&= \int_H \Delta_G(h)\int_G f_o(gh)f(g)\,\mathrm{d}g\,\mathrm{d}(h^{-1}) \quad \text{(replacing } h \text{ by } h^{-1})\\
&= \int_H \Delta_G(h)\Delta_H(h)^{-1}\int_G f_o(gh)f(g)\,\mathrm{d}g\,\mathrm{d}h \quad \text{(by Lemma 1.2.2).}
\end{aligned}$$

The hypothesis $\Delta_{G|H} = \Delta_H$ lets us finish shifting the average from f to f_o,

$$0 = \int_H\int_G f_o(gh)f(g)\,\mathrm{d}g\,\mathrm{d}h = \int_G \alpha(f_o)(g)f(g)\,\mathrm{d}g.$$

By surjectivity of α, choose f_o so that αf_o is identically 1 on $\mathrm{supp}(f)$, giving

$$0 = \int_G f(g)\,\mathrm{d}g.$$

This completes the proof. □

Proposition 1.2.4 (Inherited product measure on some groups). *For a unimodular topological $G = G_1G_2$ with $G_1 \cap G_2$ compact, the Haar measure on G is the product of the left Haar measure on G_1 and right Haar measure on G_2.*

Proof. Let μ_1 be the left Haar measure on G_1 and μ_2 be the right Haar measure on G_2. We use a variant description of the product group $G_1 \times G_2 = \left\{(g_1, g_2^{-1})\right\}$ equipped with multiplication $(\tilde{g}_1, \tilde{g}_2^{-1}) \cdot (g_1, g_2^{-1}) = (\tilde{g}_1 g_1, g_2^{-1}\tilde{g}_2^{-1})$. $G_1 \times G_2$ is left-invariant under $\mu_1 \times \mu_2$ because

$$\mathrm{d}(\mu_1 \times \mu_2)((\tilde{g}_1, \tilde{g}_2^{-1}) \cdot (g_1, g_2)) = \mathrm{d}(\mu_1 \times \mu_2)((\tilde{g}_1 g_1, g_2^{-1}\tilde{g}_2^{-1})) = \mathrm{d}(\mu_1 \times \mu_2)(g_1, g_2)$$

and hence this product measure is the left Haar measure on $G_1 \times G_2$. Because $G_1 \cap G_2$ is compact, Proposition 1.2.1 says that its modular function and the restriction of the modular function of $G_1 \times G_2$ to $G_1 \cap G_2$ are both trivial. Thus the two modular functions agree on the restriction, so Proposition 1.2.3 says that $\mu_1 \times \mu_2$ is the Haar measure on $(G_1 \times G_2)/(G_1 \cap G_2)$.

$G_1 \times G_2$ acts on G,

$$G_1 \times G_2 \times G \to G, \qquad (g_1, g_2^{-1}) \cdot g \mapsto g_1 g g_2^{-1}.$$

This is a left action,

$$(\tilde{g}_1, \tilde{g}_2^{-1}) \cdot ((g_1, g_2^{-1}) \cdot g) = \tilde{g}_1(g_1 g g_2^{-1})\tilde{g}_2^{-1} = (\tilde{g}_1 g_1) g (\tilde{g}_2 g_2)^{-1} = ((\tilde{g}_1, \tilde{g}_2^{-1}) \cdot (g_1, g_2^{-1})) \cdot g.$$

The action is transitive because any $g \in G$ takes the form $g_1 \cdot g_2^{-1}$ and hence

$$(g_1, g_2^{-1}) \cdot 1_G = g_1 g_2^{-1}.$$

Identifying $G_1 \cap G_2$ with its embedded image $\left\{(g, g^{-1})\right\} \mapsto G_1 \times G_2$, the intersection

$$\left\{g_1 \times g_2 : g_1 1 g_2^{-1} = 1\right\} = G_1 \cap G_2$$

is exactly the isotropy of 1_G for the action of $G_1 \times G_2$ on G and so we have an isomorphism of Hausdorff spaces[5],

$$(G_1 \times G_2)/(G_1 \cap G_2) \approx G.$$

Hence G inherits the product measure as a left-$(G_1 \times G_2)$-invariant measure. However, the Haar measure on G is also left-$(G_1 \times G_2)$-invariant by unimodularity of G, since

$$\mathrm{d}\mu((g_1, g_2^{-1}) \cdot g) = \mathrm{d}\mu(g_1 g g_2^{-1}) = \mathrm{d}\mu(g).$$

Thus the inherited product measure agrees with the Haar measure on G, and so the Haar measure on G is the product measure $\mu_1 \times \mu_2$ by uniqueness of Haar. □

While unimodular groups inherit product measure even when not proper products, some subgroup decompositions in this calculation will require Haar measure on a semidirect product. Given a semidirect product $P = NM$ with both N and M unimodular groups, the group structure determines the Haar measure on P as a modified product of the Haar measures on N and M.

Let P be a group with normal subgroup N and complementary subgroup M, meaning

$$P = NM, \quad N \triangleleft P, \quad N \cap M = 1_P,$$

and define the map

$$\sigma : M \longrightarrow \mathrm{Aut}(N), \quad m \longmapsto (\sigma_m : n \mapsto mnm^{-1}).$$

Thus the group operation of P is

$$nm \cdot \tilde{n}\tilde{m} = n \cdot m\tilde{n}m^{-1} \cdot m\tilde{m} = n\sigma_m(\tilde{n})m\tilde{m}.$$

[5] See appendix B.

This describes the semidirect product $P = N \times_\sigma M$. To express the Haar measure on P in terms of the product measure $\mu_N \times \mu_M$ of the respective subgroups' left Haar measures, note that

$$\mathrm{d}(\mu_N \times \mu_M)(\tilde{n}\tilde{m} \cdot nm) = \mathrm{d}(\mu_N \times \mu_M)(\tilde{n}\sigma_{\tilde{m}}(n)\tilde{m}m) = \mathrm{d}(\mu_N \times \mu_M)(\sigma_{\tilde{m}}(n) \cdot m).$$

Thus a left invariant measure as a modification of the product measure satisfies

$$\mathrm{d}\mu_{N\times M}(\sigma_{\tilde{m}}(n) \cdot m) = \phi(\tilde{m})\,\mathrm{d}(\sigma_{\tilde{m}}(n))\,\mathrm{d}m = \mathrm{d}n\,\mathrm{d}m = \mathrm{d}\mu_{N\times M}(n \cdot m),$$

and hence

$$\mathrm{d}n = \phi(\tilde{m})\,\mathrm{d}(\sigma_{\tilde{m}}(n)).$$

Thus Haar measure on the semidirect product is product measure with a factor correction for conjugation of the normal subgroup by the complementary subgroup. The Haar measure on a semidirect product then requires only describing the measures of the subgroups and studying explicitly how conjugation interacts with the left Haar measure on N.

Chapter 2

$\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$

With integration over quotient spaces in hand, we specialize to $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{Z})$, the base case for $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{Z})$. We first confirm that results from chapter 1 apply to these groups, then calculate the volume of the quotient space $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$. A related classical result is well known, and we explain how the results are compatible.

2.1 Applying general ideas to $\mathrm{SL}_2(k)$

Establishing unimodularity of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{Z})$ makes the results of chapter 1 hold, and so describing the Haar measure on G will provide the Haar measure on the quotient. With the measure in hand, integration is a series of understood steps.

Proposition 2.1.1 (Unimodularity of $\mathrm{SL}_2(k)$)**.** *Let k be an algebraically closed field with at least four elements. Then $\mathrm{SL}_2(k)$ is unimodular.*

Proof. We show that $\mathrm{SL}_2(k)$ is generated by commutators. We denote commutators $[g,h] = ghg^{-1}h^{-1}$, and note that

$$\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}^{-1}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & (a^2-1)b \\ 0 & 1 \end{bmatrix}.$$

Hence if $a^2 \neq 1$ for some[1] $a \in k^\times$, every element $\left[\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right]$ is a commutator. Replacing $\left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]$ with $\left[\begin{smallmatrix} 1 & 0 \\ b & 1 \end{smallmatrix}\right]$, every element $\left[\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right]$ is a commutator under the same condition. Now we observe through a string of commutator multiplications that

$$\begin{aligned}
\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & b \\ c & 1+cb \end{bmatrix} \\
\begin{bmatrix} 1 & b \\ c & 1+cb \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{-c}{1+cb} & 1 \end{bmatrix} &= \begin{bmatrix} \frac{1}{1+cb} & b \\ 0 & 1+cb \end{bmatrix} \\
\begin{bmatrix} \frac{1}{1+cb} & b \\ 0 & 1+cb \end{bmatrix}\begin{bmatrix} 1 & -b(1+cb) \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} \frac{1}{1+cb} & 0 \\ 0 & 1+cb \end{bmatrix}.
\end{aligned}$$

[1] Such a exist in every field with more than four elements, and hence the restriction on the size of k.

Choosing $c = 1$, $b = a^{-1} - 1$ for $a \in k^\times$, we now have $\left[\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right]$ as a commutator. $\mathrm{SL}_2(k)$ can be expressed as a disjoin union $\mathrm{SL}_2(k) = \{[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}] \sqcup [\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}], (c \neq 0)\}$ followed by the Bruhat decomposition

$$\mathrm{SL}_2(k) = P^+ \sqcup P^+ wN = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \sqcup \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} w \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \qquad (\text{with } w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}).$$

To establish this decomposition, given $A = [\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in \mathrm{SL}_2(k)$ with $c \neq 0$ we note that since the parabolic subgroup $[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}]$ and the unipotent radical $[\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}]$ are closed under inverses, the decomposition is established by a short sequence of matrix multiplications demonstrating the existence of matrices

$$\begin{bmatrix} c & -a \\ 0 & \frac{1}{c} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & -\frac{d}{c} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

It is clear from this that every element of $\mathrm{SL}_2(k)$ is in P^+ or is in P^+wN, so if the sets are disjoint we are done. This holds if there is no matrix $[\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}] \in P^+wN$, so

$$\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} b & -a \\ a^{-1} & 0 \end{bmatrix} \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} b & bc - a \\ a^{-1} & a^{-1}c \end{bmatrix}$$

shows that we cannot produce a matrix with $a^{-1} = 0$, so the sets are disjoint. Thus we see that every component of the decomposition is generated by commutators, so if the Weyl element w is also, then all of G is. To that end, letting $c = 1$:

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 1 & 1+b \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1+b \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1+b \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = w.$$

Because G is generated by commutators, G is unimodular by Proposition 1.2.1. □

In the case $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, both G and Γ are unimodular and so by Proposition 1.2.3 the Haar measure on $\Gamma\backslash G$ is exactly the Haar measure on G. The Haar measure on G is described through decompositions of the group. We first consider the action of G on the complex upper half-plane

$$\mathfrak{H} = \{x + iy : y > 0\}.$$

An element $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$ acts on $\mathfrak{H}$ by fractional linear transformations $g(z) = \frac{az+b}{cz+d}$. The isotropy of i in G is exactly the special orthogonal group $\mathrm{SO}(2)$. For a given g taking i to some $x + iy$, note also that for $p = [\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}]\left[\begin{smallmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{smallmatrix}\right] \in G$, $p(i) = x + iy$. Hence $p^{-1}g = k \in \mathrm{SO}(2)$, so every g can be expressed $g = pk$. This naturally suggests the Iwasawa decomposition

$$G = P^+K \quad (\text{to be explained momentarily}),$$

which will combine with Proposition 1.2.4 to describe the the Haar measure on G explicitly. Let P^+ denote the positive parabolic subgroup and K the special orthogonal group,

$$P^+ = \left\{\begin{bmatrix} t & * \\ 0 & t^{-1} \end{bmatrix} : t > 0\right\}$$
$$K = \mathrm{SO}(2) = \{g \in G : g^\mathsf{T} g = 1_2\}.$$

As seen above, $G = P^+K$ because[2] for any $g \in G$, there are $p \in P^+$ and $k \in K$ such that $g = pk$.

Since $P^+ \cap K = \{1_G\}$ is compact, by Proposition 1.2.4 the Haar measure on $G = P^+K$ is the product of the left Haar measure on P^+ and the right Haar measure on K. K is compact and hence unimodular, so the Haar measure on K is θ-measure normalized to $\mu_K(K) = 2\pi$, and the measure on G is determined by the measure on P^+. The parabolic group P^+ does not inherit unimodularity from G, but is a semidirect product, so its Haar measure is described by the measures of its subgroups and how conjugation among those subgroups affects elements.

The earlier expression $g = pk$ with $p = \left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{smallmatrix}\right]$ suggests the further Levi–Maltsev decompososition

$$P^+ = NM^+$$

of the unipotent radical and Levi component subgroups,

$$N = \left\{\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}\right\}$$
$$M^+ = \left\{\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} : t > 0\right\}.$$

Note that N is a normal subgroup of P^+ and M^+ is the complementary subgroup.

Proposition 2.1.2 (Measure on P^+)**.** *The Haar measure on P^+ is*

$$\mathrm{d}\mu_{P^+}(n_x m_t) = \frac{\mathrm{d}\mu_N(n_x)\,\mathrm{d}\mu_{M^+}(m_t)}{t^2}$$

where $\mathrm{d}\mu_{P^+}(n_x) = \mathrm{d}x$ *and* $\mathrm{d}\mu_{P^+}(m_t) = \mathrm{d}t/t$.

Proof. First, the Haar measure on N: since $N \approx (\mathbb{R}, +)$ after identifying the matrix $\left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right]$ with x, the Haar measure on N is simply the standard additive $\mathbb{R}$ Haar measure,

$$n_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \quad \mathrm{d}\mu_N(n_x) = \mathrm{d}x.$$

Next, the Haar measure on M^+: since $M^+ \approx (\mathbb{R}^+, \cdot)$ after identifying the matrix $\left[\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right]$ with t, the Haar measure on M^+ is the standard multiplicative $\mathbb{R}^+$ Haar measure,

$$m_t = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}, \quad \mathrm{d}\mu_{M^+}(m_t) = \frac{\mathrm{d}t}{t}.$$

[2]Be aware that this is not a proper product!

Because N and M^+ are abelian, they are unimodular. As noted in the conclusion of Section 1.2, we now need only study how the modified product measure on P^+ must compensate for conjugation. Hence

$$\mathrm{d}\mu_{P^+}(n_y m_u \cdot n_x m_t) = \mathrm{d}\mu_{P^+}(n_x m_t)$$

and by matrix multiplication[3] $n_y m_u n_x m_t = n_{y+u^2x} m_{ut}$. Using the Haar measure on N,

$$\mathrm{d}\mu_N(m_u n_x m_u^{-1}) = u^2\,\mathrm{d}\mu_N(nx),$$

so we have

$$\mathrm{d}\mu_{P^+}(n_y m_u \cdot n_x m_t) = \phi(ut)u^2\,\mathrm{d}\mu_N(n_x)\,\mathrm{d}\mu_{M^+}(m_t).$$

Thus ϕ must satisfy $u^2\phi(ut) = \phi(t)$, forcing $\phi(t) = t^{-2}$ and hence

$$\mathrm{d}\mu_{P^+}(n_x m_t) = \frac{\mathrm{d}\mu_{P^+}(n_x)\,\mathrm{d}\mu_{P^+}(m_t)}{t^2} = \frac{\mathrm{d}x\ \mathrm{d}t/t}{t^2}.$$

□

We have now proved that for any absolutely integrable function φ on G,

$$\int_G \varphi(g)\,\mathrm{d}g = \int_{P^+}\int_K \varphi(pk)\,\mathrm{d}k\,\mathrm{d}p.$$

2.2 Approach to the volume of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$

With integration described over quotient spaces, the calculation of $\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R}))$ reduces to integrating a test function over the quotient. A rotation-invariant function simplifies integration, and periodicizing will allow Fourier transformation and Poisson summation, making the integral a series of coordinate changes.

Choosing a rotation-invariant Schwartz function f on $\mathbb{R}^2$, we make a function F on G

$$F(g) = \sum_{v\in\mathbb{Z}^2} f(vg).$$

The vectors v here are row vectors acted on from the right by 2-by-2 matrices. By design, this function is left Γ-invariant[4]. We will evaluate $\int_{\Gamma\backslash G} F(g)\,\mathrm{d}g$ by using Poisson summation to remove unncecessary factors of the calculation.

Each positive integer l defines an orbit of Γ in $\mathbb{Z}^2$,

$$\{(c,d) : \gcd(c,d) = l\}.$$

Choosing $(0,1)$ as a reference point, we observe that

$$\mathbb{Z}^2 - \{0\} = \{l\cdot(0,1)\cdot\gamma : \gamma\in\Gamma, l > 0\}.$$

[3]Note that $n_y m_u \cdot n_x m_t = n_y m_u n_x m_u^{-1} m_u m_t$, and conjugation acts by $m_u n_x m_u^{-1} = n_{u^2x}$.

[4]$F(\gamma g) = \sum_{v\in\mathbb{Z}^2} f(v\gamma g) = \sum_{\tilde v\in\mathbb{Z}^2} f(\tilde v g) = F(g)$ because Γ permutes $\mathbb{Z}^2$ and the sum is absolutely convergent.

The stabilizer of $(0,1)$ in Γ is exactly[5]

$$\{\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix} : x \in \mathbb{Z}\} = N_{\mathbb{Z}} = P^+ \cap \Gamma.$$

Thus we may establish a bijection

$$\mathbb{Z}^2 - \{0\} \longleftrightarrow \mathbb{Z}^+ \times N_{\mathbb{Z}}\backslash\Gamma$$

by

$$l \cdot (0,1)\gamma \longleftrightarrow l \times N_{\mathbb{Z}}\gamma.$$

With descriptions of each component of the integral, the calculation is now

$$\begin{aligned}\int_{\Gamma\backslash G} F(\bar{g})\,\mathrm{d}g &= \int_{\Gamma\backslash G}\sum_{v\in\mathbb{Z}^2} f(vg)\,\mathrm{d}g = \int_{\Gamma\backslash G} f(0)\,\mathrm{d}g + \int_{\Gamma\backslash G}\sum_{v\in\mathbb{Z}^2, v\neq 0} f(vg)\,\mathrm{d}g\\ &= f(0)\cdot\mathrm{vol}(\Gamma\backslash G) + \sum_{l>0}\int_{N_{\mathbb{Z}}\backslash G} f(l\cdot(0,1)g)\,\mathrm{d}g\\ &= f(0)\cdot\mathrm{vol}(\Gamma\backslash G)\ + \sum_{l>0}\int_{N_{\mathbb{Z}}\backslash P^+}\int_K f(l\cdot(0,1)pk)\,\mathrm{d}k\,\mathrm{d}p.\end{aligned}$$

Because f is rotation-invariant,

$$f(l(0,1)pk) = f(l(0,1)p).$$

Hence this term is constant in the K-integral, producing a factor of $\mathrm{vol}(K) = 2\pi$. Writing the integral now with P^+ expressed as N and M^+,

$$\begin{aligned}\int_{\Gamma\backslash G} F(\bar{g})\,\mathrm{d}g &= f(0)\cdot\mathrm{vol}(\Gamma\backslash G) + 2\pi\sum_{l>0}\int_{N_{\mathbb{Z}}\backslash P^+} f(l(0,1)p)\,\mathrm{d}p\\ &= f(0)\cdot\mathrm{vol}(\Gamma\backslash G) + 2\pi\sum_{l>0}\int_{M^+}\int_{N_{\mathbb{Z}}\backslash N} f(l(0,1)n_x m_t)\frac{\mathrm{d}n\,\mathrm{d}m}{t^2}.\end{aligned}$$

As we noted earlier, n_x fixes $(0,1)$ and thus

$$f(l(0,1)n_x m_t) = f(l(0,1)m_t).$$

Hence this term is constant in the N-integral, gaining a factor of $\mathrm{vol}(N_{\mathbb{Z}}\backslash N) = \mathrm{vol}(\mathbb{R}/\mathbb{Z}) = 1$. This simplifies the integral to

$$\int_{\Gamma\backslash G} F(\bar{g})\,\mathrm{d}g = f(0)\cdot\mathrm{vol}(\Gamma\backslash G) + 2\pi\sum_{l>0}\int_{M^+} f(l(0,1)m_t)\frac{\mathrm{d}m}{t^2}.$$

[5]The set of determinant 1, integer-entry matrices satisfying $\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}0 & 1\end{bmatrix}$ forces $c = 0, d = 1$ in order for the action to be correct, which forces $a = 1$ to preserve determinant 1, while b remains free.

At this point we simplify the term being integrated. By matrix multiplication,

$$l(0,1)m_t = (0, lt^{-1}).$$

Replacing t with $(lt)^{-1}$ and writing $\mathrm{d}m$ in terms of t reduces the integral to

$$\begin{aligned}\int_{\Gamma\backslash G} F(\bar{g})\,\mathrm{d}g &= f(0)\cdot \mathrm{vol}(\Gamma\backslash G) + 2\pi\sum_{l>0} l^{-2}\int_0^\infty f(0,t)t^2\frac{\mathrm{d}t}{t}\\ &= f(0)\cdot \mathrm{vol}(\Gamma\backslash G) + \zeta(2)2\pi\int_0^\infty f(0,t)t^2\frac{\mathrm{d}t}{t}.\end{aligned}$$

Because f is rotation invariant, we can turn the line integral into a plane integral, which we then recognize as a Fourier transform at 0,

$$\int_0^\infty f(0,t)t^2\frac{\mathrm{d}t}{t} = \int_0^\infty f(0,t)t\,\mathrm{d}t = \frac{1}{2\pi}\int_{\mathbb{R}^2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2\pi}\mathcal{F}f(0)$$

making the conclusion of the integral (after cancelling factors of 2π)

$$\int_{\Gamma\backslash G} F(\bar{g})\,\mathrm{d}g = \int_{\Gamma\backslash G}\sum_{v\in\mathbb{Z}^2} f(vg)\,\mathrm{d}g = f(0)\cdot\mathrm{vol}(\Gamma\backslash G) + \zeta(2)\mathcal{F}f(0).$$

2.3 Poisson summation

The result of section 2.2 depends on $f(0)$ and $\mathcal{F}f(0)$, which Poisson summation will eliminate. Poisson summation will show that the roles of f and $\mathcal{F}f$ are interchangeable and remove them from the calculation, leaving only the volume of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$.

This requires Poisson summation on a translated version of the test function,

$$f_g(v) = f(vg).$$

For a Schwartz function $f : \mathbb{R}^2 \to \mathbb{C}$ the Fourier transform

$$\mathcal{F}f : \mathbb{R}^2 \to \mathbb{C}, \quad \mathcal{F}f(\xi) = \int_{t\in\mathbb{R}^2} f(t)e^{-2\pi i\langle\xi,t\rangle}dt$$

gives the simple formula for Poisson summation

$$\sum_{v\in\mathbb{Z}^2} f(v) = \sum_{w\in\mathbb{Z}^2}\mathcal{F}f(w).$$

To see why this is true, we introduce the automorphization of f as a $\mathbb{Z}^2$-periodic function,

$$F : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{C}, \quad F(x) = \sum_{v\in Z^2} f(x+v).$$

This function is desirable because it matches its Fourier series. To see this, let its Fourier coefficients be

$$\hat{F} : \mathbb{Z}^2 \to \mathbb{C}, \quad \hat{F}(w) = \int_{t \in \mathbb{R}^2/\mathbb{Z}^2} F(t) e^{-2\pi i \langle w,t \rangle} dt$$

so that F is expressable as

$$F(x) = \sum_{w \in Z^2} \hat{F}(w) e^{2\pi i \langle w,x \rangle}.$$

Combining the expression of F as a sum with these last two displays produces

$$\hat{F}(w) = \int_{t \in \mathbb{R}^2/\mathbb{Z}^2} \sum_{v \in \mathbb{Z}^2} f(t+v) e^{-2\pi i \langle w,t \rangle} dt = \int_{t \in \mathbb{R}^2} f(t) e^{-2\pi i \langle w,t \rangle} dt = \mathcal{F} f(w)$$

which combines with the expression of F as a sum of Fourier coefficients to show

$$\sum_{v \in \mathbb{Z}^2} f(x+v) = \sum_{w \in \mathbb{Z}^2} \mathcal{F} f(w) e^{2\pi i \langle w,x \rangle}.$$

Specializing to $x = 0$, we have exactly the formula for Poisson summation from the second display of this section. Next we study how the translated function f_g behaves under Fourier transformation.

$$\begin{aligned}
\mathcal{F} f_g(\xi) &= \int_{t \in \mathbb{R}^2} f_g(t) e^{-2\pi i \langle \xi,t \rangle} dt \\
&= \int_{t \in \mathbb{R}^2} f(tg) e^{-2\pi i \langle \xi g^{-\mathsf{T}}, tg \rangle} d(tg) \quad \text{since } \det g = 1 \\
&= \int_{t \in \mathbb{R}^2} f(t) e^{-2\pi i \langle \xi g^{-\mathsf{T}}, t \rangle} dt \\
&= \mathcal{F} f(\xi g^{-\mathsf{T}}).
\end{aligned}$$

Applying the basic Poisson summation formula to f_g with this result in mind, we now have

$$\sum_{v \in Z^2} f(vg) = \sum_{w \in Z^2} \mathcal{F} f(wg^{-\mathsf{T}}).$$

Note also that since the Haar measure on G is invariant under $g \to g^{-\mathsf{T}}$ and this automorphism stabilizes Γ,

$$\int_{\Gamma \backslash G} F(g)\, \mathrm{d}g = \int_{\Gamma \backslash G} F(g^{-\mathsf{T}})\, \mathrm{d}g.$$

Hence, we can rerun the last calculation of section 2.2 with the f and $\mathcal{F} f$ exchanged to find

$$\int_{\Gamma \backslash G} F(\bar{g})\, \mathrm{d}g = \mathcal{F} f(0) \cdot \mathrm{vol}(\Gamma \backslash G) + \zeta(2) f(0).$$

Combining the two results, we finally have[6]

$$\begin{aligned}\mathcal{F}f(0)\cdot\mathrm{vol}(\Gamma\backslash G)+\zeta(2)f(0) &= f(0)\cdot\mathrm{vol}(\Gamma\backslash G)+\zeta(2)\mathcal{F}f(0)\\ (f(0)-\mathcal{F}f(0))\cdot\mathrm{vol}(\Gamma\backslash G) &= (f(0)-\mathcal{F}f(0))\cdot\zeta(2)\\ \mathrm{vol}(\Gamma\backslash G) &= \zeta(2)\end{aligned}$$

yielding $\mathrm{vol}(\Gamma\backslash G)=\pi^2/6$.

2.4 Comparison with $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$

It is a classical result that the volume of the fundamental domain is

$$\mathrm{vol}\left\{z=x+iy\in\mathfrak{H}:|x|\leq\frac{1}{2},|z|\geq 1\right\}=\pi/3.$$

It is not immediately obvious how this result fits with our calculation that the volume of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})=\pi^2/6$, as we have seen that $G/K\to\mathfrak{H}$ by $gK\to g(i)$ is well-defined and so we expect the fact that $\Gamma\backslash G\approx\Gamma\backslash G/K\approx\Gamma\backslash\mathfrak{H}$ means the spaces have equivalent volumes. The dissonance arising from the result of $\pi^2/6$ can be resolved by studying what differences arose in the normalizations we made for the previous calculation, and what normalizations go into this classical calculation.

The classical result of $\pi/3$ comes from integrating the measure $\mathrm{d}x\,\mathrm{d}y/y^2$, which arises as the invariant measure on $\mathfrak{H}$; we know the action of P^+ on $\mathfrak{H}$ is transitive, so $\mathfrak{H}$ inherits a unique P^+-invariant measure, which necessarily must be the G-invariant measure on $\mathfrak{H}$. To see that $\mathrm{d}x\,\mathrm{d}y/y^2$ is the Haar measure for $\mathfrak{H}$ we need only check that it is P^+-invariant, and we know this can be confirmed by studying its invariance under N and M^+. The measure is invariant under N's action, x-translation, and hence is N-invariant, so we need only understand the action of M^+ :

$$m_t(x+iy)=t^2(x+iy).$$

This suggests that in order for $\mathrm{d}\mu_{\mathfrak{H}}(m_t(x+iy))=\mathrm{d}\mu_{\mathfrak{H}}(x+iy)$, we have

$$\mathrm{d}\mu_{\mathfrak{H}}(m_t(x+iy))=\mathrm{d}(t^2x)\,\mathrm{d}(t^2y)\phi(t^2y)=t^2\,\mathrm{d}xt^2\,\mathrm{d}y\phi(t^2y)=\mathrm{d}x\,\mathrm{d}y\phi(y)=\mathrm{d}\mu_{\mathfrak{H}}(x+iy)$$

which immediately shows that $\phi(z)=z^{-2}$ and so the proper invariant measure is $\mathrm{d}x\,\mathrm{d}y/y^2$. Integrating this measure over the fundamental domain gives us $\pi/3$.

We first observe that coordinates $z=x+iy$ on $\mathfrak{H}$ correspond to coordinates

$$\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}\begin{bmatrix}\sqrt{y} & 0\\ 0 & 1/\sqrt{y}\end{bmatrix}$$

in this case, rather than to

$$\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}\begin{bmatrix}t & 0\\ 0 & 1/t\end{bmatrix}$$

[6]Choosing f such that $f(0)\neq\mathcal{F}f(0)$.

as on P^+ earlier. This change results in halving the y-coordinate's measure relative to the t-coordinate:

$$\frac{1}{t^2}\,\mathrm{d}x\frac{\mathrm{d}t}{t} = \frac{1}{y}\,\mathrm{d}x\frac{\mathrm{d}\sqrt{y}}{\sqrt{y}} = \frac{1}{2}\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

Integration over the respective spaces accounts for the remaining difference. We once again use the now-named projection map $\rho : G/K \to \mathfrak{H}$ by $gK \to g(i)$ since K is the isotropy group in G of $i \in \mathfrak{H}$. Taking a right K-invariant $f \in C_c^o(G)$, we can express integration by

$$\begin{aligned}
\int_G f(g)\,\mathrm{d}g &= \int_{G/K}\left(\int_K f(gk)\,\mathrm{d}k\right)\mathrm{d}\bar{g} \\
&= \int_{G/K}\left(\int_K f(g)\,\mathrm{d}k\right)\mathrm{d}\bar{g} \quad \text{(by right K-invariance)} \\
&= \int_{G/K} f(\bar{g})\int_K \mathrm{d}k\,\mathrm{d}\bar{g} \quad \text{(const wrt the K-integral)} \\
&= \mathrm{vol}(K)\cdot\int_{G/K} f(\bar{g})\,\mathrm{d}\bar{g}.
\end{aligned}$$

Using the isomorphism $gK \to g(i) = z$, we express $f(\bar{g})$ on G/K as $\tilde{f}(z)$ on $\mathfrak{H}$, so

$$\int_G f(g)\,\mathrm{d}g = \mathrm{vol}(K)\cdot\int_{\mathfrak{H}} f(z)\,\mathrm{d}\mu_{\mathfrak{H}}(z).$$

Thus by integrating on G/K instead of $\mathfrak{H}$ it appears that the answer of $\pi^2/6$ lost a factor of $\mathrm{vol}(K) = 2\pi$. In fact, the projection map $\rho : \Gamma\backslash G \to \Gamma\backslash G/K$ has fibers $\rho^{-1}(\Gamma gK) = \{\Gamma gk : k \in K\}$. The factor lost by integrating over G/K is $\mathrm{vol}(K)$ only if the fibers of ρ are exactly K, but in this case $\Gamma gk = -\Gamma gk$ because $-1 \in \Gamma$, so certainly there is some amount of collapsing for each pair $\pm k \in K$. For there to be further collapsing, it must be the case that $\Gamma gk_1 = \Gamma gk_2$ for distinct $k_1, k_2 \in K$, meaning that $\Gamma g = \Gamma gk_o$ for a nonidentity $k_o = k_2k_1^{-1} \in K$. If such k_o exists, then $gk_og^{-1} \in \Gamma - \{\pm 1\}$, and so there is some $g_i \in \Gamma, g_i \neq \pm 1$ such that g_i fixes $g(i)$. This happens exactly when g is an elliptic point of Γ.

An elliptic point of a congruence subgroup is a point in $\mathfrak{H}$ whose isotropy sugroup does not fix $\{\pm I\}$ and so is nontrivial as a group of transformation. Γ is trivially a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and so has only finitely many elliptic points.[7] As such, there are only finitely many elliptic points $g \in \Gamma$ or $g \in \Gamma g_o$ (where $g_o(i)$ is the complex cube root of unity) for which additional collapsing occurs. Since there are only finitely many such points, this has no effect on the integral. Thus, the total volume of fibers relevant to the integral is $\{\pm 1\}\backslash K$. With all of this in mind, the calculation of volumes can be reconciled as

$$\mathrm{vol}(\Gamma\backslash G) \times \frac{1}{\mathrm{vol}(\{\pm 1\}\backslash K)} \times (\text{change in measure}) = \frac{\pi^2}{6} \times \frac{1}{\pi} \times 2 = \frac{\pi}{3}$$

as expected.

[7] For a reference, see *A First Course In Modular Forms* by Diamond-Shurman. Section 2.3 contains the invoked theorem, sections 1.2 and 1.5 provide useful background

Chapter 3

Generalizing to $\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})$

With the volume of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ in hand, we prove inductively that

$$\mathrm{vol}(\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})) = \prod_{i=2}^{n} \zeta(i),$$

building on the base case $\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})) = \zeta(2)$. This result entails normalizing the volumes of orthogonal groups. Throughout this chapter, we use the notation

$$G = \mathrm{SL}_n(\mathbb{R}), \quad \Gamma = \mathrm{SL}_n(\mathbb{Z}).$$

The work is similar to the base case, but here the group decomposition lowers the dimension by 1.

3.1 Generalizing ideas about $\mathrm{SL}(n)$

Many of the arguments from the base case can be inductively modified to give general case results. Integration on the quotient is again characterized over the groups, which are unimodular. Thus the Haar measures of the subgroup decompositions will describe integration. There will be a subgroup in this case not used in the base case because it is the identity when $n = 2$.

Since Γ is discrete, again by Lemma 1.1.1 when given a function $F \in C_c^o(\Gamma\backslash G)$ we produce a function $f \in C_c^o(G)$ satisfying

$$F(\bar{g}) = \sum_{\gamma\in\Gamma} f(\gamma g).$$

This allows us to again define

$$\int_{\Gamma\backslash G} F(\bar{g})\, \mathrm{d}\bar{g} = \int_G f(g)\, \mathrm{d}g$$

independent of choice of f, making the integration formula well-defined. If we can re-establish the generated-by-commutators result, Propositions 1.2.1 and 1.2.3 will show that specifying the Haar measure on G will determine a right G-invariant measure on $\Gamma\backslash G$.

Proposition 3.1.1 ($\mathrm{SL}_n(k)$ is unimodular for algebraically closed fields k)**.**

Proof. $\mathrm{SL}_n(k)$ can be expressed as the Bruhat[1] decomposition $\bigsqcup_{w\in W} P^+wP^+$ where W is the permutation matrices of $\mathrm{SL}_n(k)$ and P^+ is the parabolic, positive-diagonal subset of $\mathrm{SL}_n(k)$. To see that $\mathrm{SL}_n(k)$ is a product of commutators, we will prove that transposition matrices and scale-combine matrices are products of commutators. Since any $p \in P^+$ can be multiplied by permutations and scale-recombines into the identity, we will then have both that each w and p is a product of commutators. To that end, (denoting $n-2$ strings of 0's by ellipses)

$$\begin{bmatrix} a & \dots & 0 \\ \vdots & I_{n-2} & \vdots \\ 0 & \dots & a^{-1} \end{bmatrix} \begin{bmatrix} 1 & \dots & b \\ \vdots & I_{n-2} & \vdots \\ 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} a & \dots & 0 \\ \vdots & I_{n-2} & \vdots \\ 0 & \dots & a^{-1} \end{bmatrix}^{-1} \begin{bmatrix} 1 & \dots & b \\ \vdots & I_{n-2} & \vdots \\ 0 & \dots & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \dots & (a-1)b \\ \vdots & I_{n-2} & \vdots \\ 0 & ... & 1 \end{bmatrix}.$$

Hence every element $\begin{bmatrix} 1 & \dots & * \\ \vdots & I_{n-2} & \vdots \\ 0 & \dots & 1 \end{bmatrix}$ is a commutator. Replacing $\begin{bmatrix} 1 & \dots & b \\ \vdots & I_{n-2} & \vdots \\ 0 & \dots & 1 \end{bmatrix}$ with $\begin{bmatrix} 1 & \dots & 0 \\ \vdots & I_{n-2} & \vdots \\ b & \dots & 1 \end{bmatrix}$, every element $\begin{bmatrix} 1 & \dots & 0 \\ \vdots & I_{n-2} & \vdots \\ * & \dots & 1 \end{bmatrix}$ is a commutator under the same condition, and similarly we can show that each matrix of this form (the identity with one value replacing a zero) is a commutator as well. Since all of the products have been of upper diagonal matrices, we can make the simple scaling-up agrument that for diagonal matrices with subblocks A and B, $\begin{bmatrix} A & * \\ 0^k & a \end{bmatrix}\begin{bmatrix} B & * \\ 0^k & b \end{bmatrix} = \begin{bmatrix} AB & * \\ 0^k & ab \end{bmatrix}$ and so all of the previous matrix manipulations scale up inductively with one extra row and column added. Running the same matrix arguments, W and P^+ are generated by commutators, meaning in turn $\mathrm{SL}_n(k) = \bigsqcup_{w\in W} P^+wP^+$ is generated by commutators and is unimodular. □

We will once again describe the Haar measure on G by subgroup decomposition. Here we let $K = \mathrm{SO}(n)$ and $P^+ = \{m \in M_{n,n}(\mathbb{R}) : m_{i,i} > 0, m_{i>j} = 0\}$, the collection of $n\times n$ upper-triangular real matrices with positive diagonals, the respective n-dimensional analogues of the previous subgroups. Hence $P^+ \cap K = I_n$ and we recover the Iwasawa decomposition $G = P^+ \cdot K$. Once we have specified the left Haar measure on P^+ and the Haar measure[2] on K, we will have the Haar integral

$$f \to \int_{P^+}\int_K f(pk)\,\mathrm{d}k\,\mathrm{d}p$$

for $f \in C_c^o(G)$.

In order to evaluate this, we now investigate the measures needed on P^+ and K. By compactness of $\mathrm{SO}(n)$ we will normalize K to have volume 1; if we sought to produce a classical result for the volume of $\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})$, the classical normalization of $\mathrm{SO}(n)$'s

[1] For a reference, see chapter IV.14 of *Linear Algebraic Groups* second edition by A. Borel.

[2] K is unimodular, so specifying the Haar measure gives us the desired right Haar measure

volume to $\prod_{i=1}^{n-1} \mathrm{vol}(S^i)$ would be necessary, but for our purposes this would be unnecessary clutter. To understand the measure on P^+, we will again use decomposition, but with an additional subgroup this time for greater control over the elements at play. Introduce

$$N = \left\{ \begin{bmatrix} I_{n-1} & v \\ 0 & 1 \end{bmatrix} : v \in \mathbb{R}^{n-1} \right\}$$
$$M = \left\{ \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix} : h \in \mathrm{SL}_{n-1}(\mathbb{R}) \right\}$$
$$A^+ = \left\{ \begin{bmatrix} (t^{\frac{1}{n-1}}) I_{n-1} & 0 \\ 0 & t^{-1} \end{bmatrix} : t > 0 \right\}$$

and note that $P^+ = NMA^+$, and further define $N_{\mathbb{Z}} = N \cap \Gamma$ and $M_{\mathbb{Z}} = M \cap \Gamma$.

3.2 Calculating the volume of $\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})$

The calculation again uses a rotation-invariant, periodicized test function as the integrand. Fourier transformation and Poisson summation scale up identically, allowing for a simple evaluation of the integral.

Let f be a Schwartz function on $\mathbb{R}^n$ and define a left Γ-invariant function F on G by

$$F(g) = \sum_{v \in \mathbb{Z}^n} f(vg)$$

with the intent to also consider this as a function on $\Gamma\backslash G$ by its Γ-invariance, $F(\bar{g}) = F(g)$. We again establish a bijection between $\mathbb{Z}^n$ (where the sum is defined) and a quotient of Γ (where the calculation takes place). To do so, consider the subset Q of G,

$$Q = \{ \begin{bmatrix} h & * \\ 0 & 1 \end{bmatrix} : h \in \mathrm{SL}_{n-1}(\mathbb{R}) \}.$$

Note both that $Q = NM$ and that Q is exactly the isotropy subgroup of $e = (0, 0, ..., 0, 1)$ in G, the analogue of the isotropy group of i from the $n = 2$ case. We let $Q_{\mathbb{Z}} = Q \bigcap \Gamma$ so that $Q_{\mathbb{Z}}\backslash\Gamma$'s action on $\mathbb{Z}^n - \{0\}$ generates orbits which can be scaled to each point in $\mathbb{Z}^n$. This gives us the relation

$$\mathbb{Z}^n - \{0\} = \sum_{l \in \mathbb{Z}_{>0}} \sum_{\gamma \in Q_{\mathbb{Z}}\backslash\Gamma} l \cdot e \cdot \gamma.$$

With this machinery now in place, we once again attempt to integrate F over $\Gamma\backslash G$ by

$$\int_{\Gamma\backslash G} F(\bar{g})\,\mathrm{d}\bar{g} = \int_{\Gamma\backslash G} f(0)\,\mathrm{d}g + \sum_{l} \int_{\Gamma\backslash G} \sum_{\gamma} f(le\gamma g)\,\mathrm{d}g.$$

Unwinding this integral gives us

$$\int_{\Gamma\backslash G} F(\bar{g})\,\mathrm{d}\bar{g} = \mathrm{vol}(\Gamma\backslash G) f(0) + \sum_{l} \int_{Q_{\mathbb{Z}}\backslash G} f(leg)\,\mathrm{d}g.$$

We renormalize the invariant integral on $Q_{\mathbb{Z}}\backslash G$ so that for left-invariant functions ϕ,

$$\int_{Q_{\mathbb{Z}}\backslash G} \phi(\bar{g})\,\mathrm{d}\bar{g} = \mathrm{vol}(S^{n-1}) \cdot \int_{A^+}\int_{Q_{\mathbb{Z}}\backslash NM}\int_K \phi(nmak)t^{-n}\,\mathrm{d}k\,\mathrm{d}n\,\mathrm{d}m\,\mathrm{d}a.$$

To see that $t^{-n}\,\mathrm{d}k\,\mathrm{d}n\,\mathrm{d}m\,\mathrm{d}a$ is a left Haar measure on P^+, we study how elements of the respective subgroups interact.

$$\begin{aligned} n_v m_h &= m_h n_{hv} \quad \text{(where } hv \text{ is matrix multiplication)}\\ m_h a_t &= a_t m_h \quad \text{(the matrices commute)}\\ a_t n_v &= n_{t^{n/(n-1)}v} a_t \quad \text{(where } t^{n/(n-1)}v \text{ is scaling } v \text{ by } t^{n/(n-1)}\text{)} \end{aligned}$$

$$\begin{aligned} n_{v_0} m_{h_0} a_{t_0} n_v m_h a_t &= n_{v_0} m_{h_0} n_{t_0^{n/(n-1)}v} a_{t_0} m_h a_t\\ &= n_{v_0} n_{h_0^{-1} t_0^{n/(n-1)}v} m_{h_0} m_h a_{t_0} a_t\\ &= n_{v_0 + h_0^{-1} t_0^{n/(n-1)}v} m_{h_0 h} a_{t_0 t}. \end{aligned}$$

We want $\mathrm{d}_{\mu_{P^+}}(n_{v_0} m_{h_0} a_{t_0} n_v m_h a_t) = \mathrm{d}_{\mu_{P^+}}(n_v m_h a_t)$, meaning that for some correction ξ,

$$\begin{aligned} \mathrm{d}_{\mu_{P^+}}(n_{v_0 + h_0^{-1} t_0^{n/(n-1)}v} m_{h_0 h} a_{t_0 t}) &= \xi \cdot \mathrm{d}_{\mu_{P^+}}(n_{v_0 + h_0^{-1} t_0^{n/(n-1)}v})\,\mathrm{d}_{\mu_{P^+}}(m_{h_0 h})\,\mathrm{d}_{\mu_{P^+}}(a_{t_0 t})\\ &= \xi t_0^n \cdot \mathrm{d}n\,\mathrm{d}m\,\mathrm{d}a\\ &= \xi t_0^n \cdot \mathrm{d}_{\mu_{P^+}}(n_v)\,\mathrm{d}_{\mu_{P^+}}(m_h)\,\mathrm{d}_{\mu_{P^+}}(a_t) = \mathrm{d}_{\mu_{P^+}}(n_v m_h a_t). \end{aligned}$$

Thus the correction factor is t^{-n} where t is the entry at play in a_t, making the measure described left invariant. Applying this normalization now to the integral, we have

$$\int_{\Gamma\backslash G} F(\bar{g})\,\mathrm{d}\bar{g} = \mathrm{vol}(\Gamma\backslash G) f(0) + \mathrm{vol}(S^{n-1}) \sum_l \int_{A^+}\int_{Q_{\mathbb{Z}}\backslash NM}\int_K f(l\cdot e\cdot nmak)t^{-n}\,\mathrm{d}k\,\mathrm{d}n\,\mathrm{d}m\,\mathrm{d}a.$$

Since $NM = Q$ has no action on e, the integrand is invariant under NM and so for the integral over $Q_{\mathbb{Z}}\backslash NM = N_{\mathbb{Z}}M_{\mathbb{Z}}\backslash NM$, we note that $\mathrm{vol}(N_{\mathbb{Z}}\backslash N) = \mathrm{vol}(\mathbb{R}/\mathbb{Z}) = 1$, leaving a factor of $\mathrm{vol}(M_{\mathbb{Z}}\backslash M) = \mathrm{vol}(\mathrm{SL}_{n-1}(\mathbb{Z})\backslash\mathrm{SL}_{n-1}(\mathbb{R}))$.

$$\int_{\Gamma\backslash G} F(\bar{g})\,\mathrm{d}\bar{g} = \mathrm{vol}(\Gamma\backslash G) f(0) + \mathrm{vol}(S^{n-1})\mathrm{vol}(\mathrm{SL}_{n-1}(\mathbb{Z})\backslash\mathrm{SL}_{n-1}(\mathbb{R})) \sum_l \int_{A^+}\int_K f(l\cdot e\cdot ak)t^{-n}\,\mathrm{d}k\,\mathrm{d}a.$$

Since f is rotation-invariant it is right K-invariant and so this term passes through the K-integral with a factor of $\mathrm{vol}(K) = 1$, giving us

$$\int_{\Gamma\backslash G} F(\bar{g})\,\mathrm{d}\bar{g} = \mathrm{vol}(\Gamma\backslash G) f(0) + \mathrm{vol}(S^{n-1})\mathrm{vol}(\mathrm{SL}_{n-1}(\mathbb{Z})\backslash\mathrm{SL}_{n-1}(\mathbb{R})) \sum_l \int_{A^+} f(lea)t^{-n}\,\mathrm{d}a.$$

The only effect of a on e is to scale it by t^{-1} so we may replace the integral over A^+ with the corresponding

$$\int_{\Gamma\backslash G} F(\bar{g})\,\mathrm{d}\bar{g} = \mathrm{vol}(\Gamma\backslash G) f(0) + \mathrm{vol}(S^{n-1})\mathrm{vol}(\mathrm{SL}_{n-1}(\mathbb{Z})\backslash\mathrm{SL}_{n-1}(\mathbb{R})) \sum_l \int_0^\infty f(let^{-1})t^{-n}\,\frac{\mathrm{d}t}{t}.$$

Replacing t by lt^{-1} does not change the integral or differential element, and gives us the suggestive form

$$\int_{\Gamma\backslash G} F(\bar{g})\,\mathrm{d}\bar{g} = \mathrm{vol}(\Gamma\backslash G) f(0) + \mathrm{vol}(S^{n-1})\mathrm{vol}(\mathrm{SL}_{n-1}(\mathbb{Z})\backslash\mathrm{SL}_{n-1}(\mathbb{R}))\sum_{l}\frac{1}{l^n}\int_0^\infty f(et)t^n\frac{\mathrm{d}t}{t}$$

so we again use the rotation-invariance of f to turn this into a space-integral:

$$\begin{aligned}\int_{\Gamma\backslash G} F(\bar{g})\,\mathrm{d}\bar{g} &= \mathrm{vol}(\Gamma\backslash G) f(0) + \mathrm{vol}(\mathrm{SL}_{n-1}(\mathbb{Z})\backslash\mathrm{SL}_{n-1}(\mathbb{R}))\zeta(n)\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x\\ &= \mathrm{vol}(\Gamma\backslash G) f(0) + \mathrm{vol}(\mathrm{SL}_{n-1}(\mathbb{Z})\backslash\mathrm{SL}_{n-1}(\mathbb{R}))\zeta(n)\mathcal{F}f(0).\end{aligned}$$

This result is analogous to the previous one, so now we again use Poisson summation to extract more information from this answer.

3.3 Poisson summation revisited

Again the presence of $f(0)$ and $\mathcal{F}f(0)$ make the calculation incomplete, and so Poisson summation will show that they are interchangeable and hence removable. The Poisson summation formula used previously is identical for the general case, and so is again useful.

From the basic Poisson summation formula,

$$F(g) = \sum_{v\in\mathbb{Z}^n} f(vg) = \sum_{w\in\mathbb{Z}^n}\mathcal{F}f(wg^{-\mathsf{T}}) = F(g^{-\mathsf{T}}).$$

Because the automorphism $g \to g^{-\mathsf{T}}$ on G is measure preserving and stabilizes Γ, and $\mathcal{F}(\mathcal{F}f)(0) = f(0)$, the previous section's calculation holds with $\mathcal{F}f$ and f exchanged,

$$\int_{\Gamma\backslash G} F(\bar{g})\,\mathrm{d}\bar{g} = \mathrm{vol}(\Gamma\backslash G)\mathcal{F}f(0) + \mathrm{vol}(\mathrm{SL}_{n-1}(\mathbb{Z})\backslash\mathrm{SL}_{n-1}(\mathbb{R}))\zeta(n)f(0).$$

Combining these results,

$$\begin{aligned}\int_{\Gamma\backslash G} F(\bar{g})\,\mathrm{d}\bar{g} &= \mathrm{vol}(\Gamma\backslash G) f(0) + \mathrm{vol}(\mathrm{SL}_{n-1}(\mathbb{Z})\backslash\mathrm{SL}_{n-1}(\mathbb{R}))\zeta(n)\mathcal{F}f(0)\\ &= \mathrm{vol}(\Gamma\backslash G)\mathcal{F}f(0) + \mathrm{vol}(\mathrm{SL}_{n-1}(\mathbb{Z})\backslash\mathrm{SL}_{n-1}(\mathbb{R}))\zeta(n)f(0)\end{aligned}$$

and so, rearranging,

$$\mathrm{vol}(\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R}))(f(0)-\mathcal{F}f(0)) = \mathrm{vol}(\mathrm{SL}_{n-1}(\mathbb{Z})\backslash\mathrm{SL}_{n-1}(\mathbb{R}))\zeta(n)(f(0)-\mathcal{F}f(0)).$$

Choosing f such that $f(0)\neq\mathcal{F}f(0)$,

$$\mathrm{vol}(\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})) = \mathrm{vol}(\mathrm{SL}_{n-1}(\mathbb{Z})\backslash\mathrm{SL}_{n-1}(\mathbb{R}))\zeta(n)$$

and so we have inductively proved that

$$\mathrm{vol}(\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})) = \prod_{i=2}^{n}\zeta(i).$$

Appendix A

Open set manipulation

The surjectivity proofs of this thesis tacitly invoked basic technique with topological groups, similar to $\varepsilon/2$ arguments. This appendix supplies the specifics.

Lemma A.0.1 (Translate)**.** *Given any neighborhood U of $g \in G$, by continuity of inversion there is a neighborhood $V = g^{-1}U$ of 1 such that $U = gV$.*

Proof. For any $h \in G$, the map $g \to gh$ is an automorphism of G. $\square$

Lemma A.0.2 (Shrink)**.** *Given an open neighborhood U of 1 in G, there is an open neighborhood V of 1 such that $V^2 \subset U$. Here squaring denotes $V^2 = \{gh : g, h \in V\}$.*

Proof. By continuity of $G \times G \to G$, the inverse image W of U is open and contains (1,1). Under the product topology, W contains an open of the form $V_1 \times V_2$, a product of opens containing 1. Taking $V = V_1 \cap V_2$, we have $V^2 \subset V_1 \cdot V_2 \subset U$. $\square$

Lemma A.0.3 (Symmetrize)**.** *Given a neighborhood V of 1, there is a subneighborhood V_s satisfying $V_s^{-1} = V_s$.*

Proof. Since group inversion $g \to g^{-1}$ is a continuous involution, the image $V^{-1} = \{g^{-1} : g \in V\}$ is open. Thus $V_s = V \cap V^{-1}$ is the subneighborhood in question. $\square$

Lemma A.0.4 (Closure)**.** *Given a set E in G,* closure $E = \bigcap_U E \cdot U$ *where U runs over open neighborhoods of 1.*

Proof. $g \in G$ is in the closure of E if and only if every neighborhood of g meets E. That is, gU meets E for every open neighborhood U of 1. Hence $g \in E \cdot U^{-1}$ for every U. Since inversion is an automorphism of G, the map $U \to U^{-1}$ is a self-bijection on neighborhoods of 1. Hence g is in closure E if and only if $g \in E \cdot U$ for every U. $\square$

Lemma A.0.5 (Compactify)**.** *Given a neighborhood U of* 1 *in G, there is a neighborhood V of* 1 *such that $\overline{V}$ is compact and $\overline{V} \subset U$.*

Proof. By continuity of $G \times G \to G$, there is V_o with $V_o \cdot V_o \subset U$. By previous lemma, $\overline{V_o} \subset V_o \cdot V_o \subset U$. By local compactness of G, there is a local basis at 1 of opens with compact closure. Take a neighborhood W of 1 with compact closure, so that $V = V_o \cap W$ has compact closure and still $\overline{V} \subset U$. $\square$

Lemma A.0.6 (Combination)**.** *For an open $U \subset G$, given $g \in U$, there is a neighborhood V of 1 with compact closure such that $g\overline{V}^2 \subset U$.*

Proof. $g^{-1}U$ is an open containing 1, so we may take $g = 1$ and instead produce $\overline{V}^2 \subset U$. By the shrinking result there is an open W containing 1 such that $W^2 \subset U$. By the compactifying result, there is a neighborhood V of 1 with $\overline{V}$ compact and $V \subset W$, so $V^2 \subset W^2 \subset U \implies \overline{V}^2 \subset U$ as desired. □

Lemma A.0.7 (Countable cover)**.** *Given a neighborhood V of 1 in G, there is a countable list $\{g_1, g_2, ...\}$ in G such that $G = \bigcup_i g_i V$.*

Proof. Let $U_1, U_2, ...$ be a countable basis of G. Hence for $g \in G$,

$$gV = \bigcup_{U_i \subset gV} U_i.$$

For each $g \in G$ there is some index $j(g)$ such that $g \in U_{j(g)} \subset gV$. Since there are countably many such indices i appearing as $j(g)$, for each such i let g_i be an element of G with $j(g_i) = i$ so that

$$g_i \in U_i \subset g_i V.$$

Hence for each $g \in G$ there is an index i such that

$$g \in U_{j(g)} = U_i \subset g_i V$$

and so

$$\bigcup g_i V = G.$$

□

Appendix B

Isomorphisms of Hausdorff spaces

When a group G acts transitively on a set X, and G_x denotes the isotropy subgroup of a point x, the quotient space G/G_x is set-bijective with X. This appendix shows that when G is a topological group and X a Hausdorff space, the set bijection is a Hausdorff space isomorphism.

Lemma B.0.8. *For a topological group G acting transitively on a Hausdorff space X, $G/G_x \approx X$ is not only a set bijection but is an isomorphism of Hausdorff spaces.*

Proof. Let G be a locally compact, Hausdorff topological group and X a locally compact, Hausdorff topological space, with a continuous transitive action of G on X. Given a fixed $x \in X$, the isotropy group in G of $x \in X$ is

$$G_x = \{g \in G : g(x) = x\}.$$

We will denote right G_x cosets as $g \cdot G_x = \bar{g}$. As such, there is a natural map

$$\jmath : G/G_x \longrightarrow X, \quad \jmath(\bar{g}) = g(x)$$

with natural inverse

$$\imath : X \longrightarrow G/G_x, \quad \imath(y) = \bar{g} \quad \text{(where } y = g(x) \text{ for some } g \in \bar{g})^1$$

which are well defined since G_x is the isotropy group of x. To see that $\jmath$ is is a set bijection, we need only verify $(\imath \circ \jmath)(\bar{g}) = \imath(\bar{g}(x)) = \bar{g}$ and $(\jmath \circ \imath)(y) = \jmath(\bar{g}) = y$ for arbitrary $\bar{g} \in G/G_x$ and $y \in X$.

For $\jmath$ to further be an isomorphism of Hausdorff spaces, we must show that both $\jmath$ and $\imath$ are continuous, which can be accomplished by showing that $\jmath$ is continuous and open. To see that $\jmath$ is continuous, consider the map $G \times X \to X$ by $g \times x \to g(x)$. This map is continuous by the continuity of the group action. Fixing x, the action is still continuous, so the restriction map $G \times \{x\} \to X$ by $g \to gx$ is again continuous and hence for fixed x we introduce $\wp : G \to X$ by $\wp(g) = g(x)$, also continuous. The quotient topology is the unique topology on G/G_x such that any continuous map $G \to X$ that is constant on G_x factors through the quotient map $q : G \to G/G_x$. As such, by a commutative diagram

[1] because G's action is transitive

argument there is a unique continuous (induced) map $j : G/G_x \to X$ satisfying $j \circ q = \wp$. Since the earlier natural map $\jmath$ satisfies this, by uniqueness $\jmath$ is the continuous map we desire. To see that $\jmath$ is open, we invoke a variant of the Baire category theorem: A locally compact Hausdorff topological space is not a countable union of nowhere dense sets, i.e. sets whose closure contain no non-empty open set. For invocations regarding open sets and neighborhoods in this proof, see appendix A. Given an open set U in G and $g \in U$, let V be a compact neighborhood of 1 such that $gV^2 \subset U$. Take a countable set of points $g_1, g_2, ...$ in G such that $G = \bigcup_i g_i V$, and define

$$W_n = g_n V x \subset X.$$

Hence $X = \bigcup_i W_i$ by transitivity of G's action. From the previous observation that $\wp$ is continuous, the W_n's are again compact, and as compact subsets of a Hausdorff space the W_n's are in turn closed. By the variant of Baire category theorem, at least one $W_m = g_m V x$ contains a non-empty open set S of X. For $h \in V$ so that $g_m h x \in S$,

$$gx = g(g_m h)^{-1}(g_m h)x \in gh^{-1}g_m^{-1}S.$$

Since the group action of each element of G is a homeomorphism[2] $X \to X$ and S is open in X, $gh^{-1}g_m^{-1}S$ is also open in X. Noting that

$$gh^{-1}g_m^{-1}S \subset gh^{-1}g_m^{-1}g_m V x \subset gh^{-1}Vx \subset gV^{-1}Vx \subset Ux,$$

gx is an interior point of Ux. This argument holds for all $g \in U$, so Ux is open and hence $\jmath$ is open. Since the inverse of an open map is continuous and the inverse of a continuous map is open, $\imath$ is thus continuous and open as well. Hence we have shown that $G/G_x \approx X$ is an isomorphism of Hausdorff spaces. □

[2]Since inversion in the group is continuous, the continuous inverse of the action of g on X is g^{-1}

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