SLIGHTLY MORE REFINED ROKS

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These are notes on the article [2] from the journal issue [4].

- A *dessin* is a hypermap on a compact oriented two-manifold. Equivalently, they are bipartite graphs embedded on a surfaces, with simply connected complementary regions.
- Dessins arise on all smooth complex projective curves defined over a numberfield. Belyi demonstrated that there exist nonconstant meromorphic functions $\beta : C \to \mathbb{CP}^1$ on a smooth curve C ramified at most over $0, 1, \infty \in \mathbb{CP}^1$ if and only if C can be defined over a numberfield. The dessin arises as the preimage of the interval $[0, 1]$ under β.
- Conversely any topological hypermap on a compact surface admits a unique conformal structure, corresponding to an algebraic curve defined over a number field, together with a belyi function producing that hypermap as a dessin.
- Belyi functions on a curve C correspond to inclusions of fuchsian groups $\Gamma \leq \Delta(r, s, t)$ (with finite index) and $\Delta = \Delta(r, s, t)$ is a fuchsian triangle group, such that $\Gamma\backslash\mathbb{H}$ identifies with $C(\mathbb{C})$. Identifying[1] the quotient $\Delta\backslash\mathbb{H}$ with the Riemann sphere $\mathbb{CP}^1$ (arranging the ramification points to occur at $0, 1, \infty$), the belyi function $C \to \mathbb{CP}^1$ identifies with the canonical quotient map $\Gamma\backslash\mathbb{H} \to \Delta\backslash\mathbb{H}$ induced by the inclusion $\Gamma \to \Delta$.
- A *regular* dessin is one for which the group of color–and–orientation preserving automorphisms acts transitively on the edges. The surfaces of genus $g > 1$ admitting regular dessins are called *quasi-platonic*. Such dessins on such surfaces are distinguished as arising from those Γ which are *normal* subgroups of the relevant triangle group.
- A *uniform* dessin is characterized by $\Gamma < \Delta$ being a *torsion free* finite index subgroup, though need not necessarily be normal.
- A (tantalizing) theorem in [3] seems to characterize those surfaces admitting several different uniform dessins *of the same signature* as those arising from those *contained in*[2] a congruence subgroup of an arithmetic triangle group.

Background on fields of moduli/fields of definition

Here, S is a smooth algebraic curve[3] and $k \subset \mathbb{C}$ is a field. Say that k is *a* **field of definition** for S if there exist J homogeneous polynomials $F_j \in k[x_0, \cdots, x_n]$ such that $S(\mathbb{C})$ and

$$S_F = \{\, x \in \mathbb{CP}^n \mid F_j(x) \ \text{ for all j} \,\}$$

are isomorphic[4].

Given a set F of homogeneous generators for the ideal defining S, and an element $\sigma \in \mathrm{Gal}(\overline{Q}/\mathbb{Q})$, one can define S_{F^σ} as above, with coefficients of the polynomials conjugated by σ. The **inertia subgroup** $I_S \leq \mathrm{Gal}(\overline{Q}/\mathbb{Q})$ consists of those $\sigma \in \mathrm{Gal}(\overline{Q}/\mathbb{Q})$ such that $S_{F^\sigma} \approx S_F$ as[5] complex analytic varieties. Apparently this subgroup is independent (perhaps up to conjugation?) of the choice of model. The fixed field $\overline{\mathbb{Q}}^{I_S} = \mathrm{Fix}(I_S)$ is the **field of moduli** for S, which we denote by

[1] In what sense are we making this identification? Surely, we are doing so as topological spaces, but what more?
[2] Note: contained *in* rather than *containing*
[3] over $\mathbb{C}$?
[4] complex analytically?
[5] presumably

$M(S)$. The field of moduli is contained in any field of definition, though not every curve is defined over its field of moduli. That is, $M(S)$ may not be a field of definition for S.

If $G < \mathrm{Aut}(S)$ say that k is a field of definition for the pair (S, G) if there exists a model S_F of S over k and an isomorphism $\varphi : S_F \to S$ such that $\varphi^{-1}G\varphi < \mathrm{Aut}(S_F)$ is also defined over k. Then the inertia group of the pair is

$$I_{(S,G)} = \left\{ \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \;\middle|\; \begin{array}{l}\text{there exists an isomorphism } f_\sigma : S \to S^\sigma \text{ such} \\ \text{that } \alpha^\sigma \circ f_\sigma = f_\sigma \circ \alpha \text{ for all } \alpha \in G\end{array} \right\}.$$

Write $M(S, G)$ for the field of moduli for the pair.

Now, given a belyi function $\beta : S \to \mathbb{CP}^1$ defining a dessin $\mathcal{D}$ on S, define the group of automorphisms of (S, β) to be the subgroup of $\mathrm{Aut}(S)$ such that $\beta \circ f = \beta$ (equality of functions $S \to \mathbb{CP}^1$). We also write $\mathrm{Aut}(\mathcal{D})$ for $\mathrm{Aut}(S, \beta)$.

For any model S_F as above, $\beta \circ \varphi$ is a rational function on S_F. Say that k is a field of definition of (S, β) (or $\mathcal{D}$) if there exists a model S_F of S such that both F and the covering $\beta \circ \varphi$ are defined over k. Define the inertia subgroup similarly to above and define $M(\mathcal{D}) = M(S, \beta)$ to be its fixed field: the field of moduli for $\mathcal{D}$. Note $I_(S, \beta) < I_S$ so $M(S) < M(S, \beta)$

Say a belyi function is regular if it defines a normal covering $S \to S/G \approx \mathbb{CP}^1$, for some subgroup G of $\mathrm{Aut}(S)$.

Lemma 1. If S is quasiplatonic with surface group $\Gamma \triangleleft \Delta$ for a maximal triangle group Δ, then $M(S) = M(S, \beta)$ for the belyi function $\beta : \Gamma\backslash\mathbb{H} \to \Delta\backslash\mathbb{H}$.

If Δ is not maximal, (say, for example, $r = s \neq t$) then there exist a $\sigma \in I_S$ fixing S but exchanging the zero set of its belyi function B and the zero set of $1 - B$ such that $M(S, B)$ (can be) a quadratic extension of $M(S)$.

Theorem 1. Let S be a quasiplatonic curve of genus $g > 1$ with full automorphism group $G = \mathrm{Aut}(S)$. If $Z(G) \leq G$ is a direct factor off G: i.e. $G = G' \times Z(G)$, then (S, G) can be defined over its field of moduli $M(S, G)$.

Note: this theorem applies to PGL_2 and PSL_2 and direct products of them with C_2's. A consequence: if $U \leq G$ and $C = U\backslash S$ (a cover of $G\backslash S$, covered by S), then C can be defined over $M(S, G)$. Further, all such S can be simultaneously defined over $M(S, G)$ in the sense that all the function fields $M(S, G)(C)$ are subfields of $M(S, G)(S)$.

Hurwitz groups and surfaces

Say G is a hurwitz group if it is a quotient of $\Delta(2, 3, 7)$ by a finite index torsion free normal subgroup. A theorem of Macbeath characterizes those q for which $\mathrm{PSL}_2(\mathbb{F}_q)$ is a hurwitz group. The cases are as follows:

- $q = 7$,
- $q = p$ for p prime ± 1 mod 7.
- $q = p^3$ for p prime and $p = \pm 2$ or ± 3 mod 7.

See [1] for details on the following arithmetic construction of Macbeath's curves. Given a number field k let $\mathcal{O}_k$ be its ring of integers. Let A be the indefinite quaternion algebra defined over $k = \mathbb{Q}(\cos \pi/7)$ unramified at all finite places. Then $\Delta(2, 3, 7)$ is the image of the norm 1 subgroup of A under canonical inclusion $A \to \mathrm{GL}(2, \mathbb{R})$ induced by the unique split real place, followed by the surjection $\mathrm{GL}(2, \mathbb{R}) \to \mathrm{PGL}(2, \mathbb{R})$. Further, Δ can be realized as a subgroup of $\mathrm{PSL}(2, \mathcal{O}_L)$ for an (at worst) quadratic extension of k.

For a rational prime $p \in \mathbb{Z}$, the ideal $p\mathcal{O}_k$ is:

- ramified if (and only if) $p = 7$; in this case $p\mathcal{O}_k = \mathfrak{p}^3$ for some prime $\mathfrak{p} \leq \mathcal{O}_k$ with $N(\mathfrak{p}) = 7$,

- split if (and only if) $p = \pm 1 \bmod 7$; in this case $p\mathcal{O}_k = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_2$ with primes $\mathfrak{p}_i \leq \mathcal{O}_k$ of norm p
- inert if(and only if) $p = \pm 2$ or $\pm 3 \bmod 7$; in this case $p\mathcal{O}_k = \mathfrak{p}$ remains prime, and has norm p^3.

Now let $\Delta = \Delta(r, s, t)$ be any *arithmetic* triangle group, realized as the norm 1 units $\mathcal{M}^1$ of a maximal order $\mathcal{M}$ of a quaternion algebra A over a totally real field[6] k. Let $A_\mathfrak{p}$ be the local quaternion algebra defined over the $\mathfrak{p}$-adic field $k_\mathfrak{p}$.

So long as $\mathfrak{p}$ doesnt divide the discriminant of A, the local quaternion alg $A_\mathfrak{p}$ is isomorphic to $M_2(k_\mathfrak{p})$. Let $\Delta(\mathfrak{p})$ denote the principal congruence subgroup of level $\mathfrak{p}$. Then all of the surfaces $\Delta(\mathfrak{p})\backslash\mathbb{H}$ have a regular belyi function $\beta : S = \Delta(\mathfrak{p})\backslash\mathbb{H} \to \Delta\backslash\mathbb{H}$ and has automorphism group of order $|Aut(S)| = |N(\Delta(\mathfrak{p}))/\Delta(\mathfrak{p})|$ where N denotes the $\mathrm{PSL}(2, \mathbb{R})$ normalizer of $\Delta(\mathfrak{p})$.

The principal congruence subgroups of level $\mathfrak{p}^n$ of the local quaternion algebra correspond to the intersections of certain collections of maximal orders. More precisely, the principal congruence subgroup of level $\mathfrak{p}^n$ corresponds to (norm one units of) the intersection of all of the vertices in the Bruhat-Tits tree of distance $\leq n$ from the root.

Let $\mathcal{E}_\mathfrak{p}$ denote the Eichler order of level $\mathfrak{p}$ in $A_\mathfrak{p}$: that is, the intersection of the two maximal orders $\mathcal{M}_\mathfrak{p}$ and $\gamma^{-1}\mathcal{M}_\mathfrak{P}\gamma$, where $\gamma = \left[\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Let $\Phi_0(\mathfrak{p}) = \mathcal{E}_\mathfrak{p}^1$ denote the norm 1 subgroup of $\mathcal{E}_\mathfrak{p}$. The **fricke element** $\left[\begin{smallmatrix} 0 & \pi^{-1} \\ -1 & 0 \end{smallmatrix}\right] \in A_\mathfrak{p}$ conjugates one maximal order into the other, and so preserves their intersection $\mathcal{E}_\mathfrak{p}$, and induces an involution on $\Phi_0(\mathfrak{p})$. Globally, the fricke involution may give rise to a matrix in $\mathrm{GL}(2, \mathbb{R})$ which conjugates two triangle groups. By the rigidity of triangle groups, such a conjugation can be realized inside $\mathrm{PSL}(2, \mathbb{R})$, hence gives rise to an involution $\alpha_\mathfrak{p}$ which generates a C_2 extension $\Delta_f(\mathfrak{p}) = \langle \alpha_\mathfrak{p}, \Delta_0(\mathfrak{p})\rangle$ of $\Delta_0(\mathfrak{p})$, called the Fricke extension.

Now $\alpha_\mathfrak{p}$ normalizes $\Delta_0(\mathfrak{p})$ but *not* Δ, though $\alpha_\mathfrak{p}^2 \in \Delta_0(\mathfrak{p})$. Conjugation by $\alpha_\mathfrak{p}$ induces an involution on the curve $\Delta_0(\mathfrak{p})\backslash\mathbb{H}$. Note: $\alpha_\mathfrak{p} \in \mathrm{PSL}(2, \mathbb{R})$ may not have order two, despite it inducing an involution on the curve $\Delta_0(\mathfrak{p})\backslash\mathbb{H}$.

Consequently, any group $K \leq \Delta_0(\mathfrak{p})$ is a subgroup of both Δ and $\alpha_\mathfrak{p}\Delta\alpha_\mathfrak{p}^{-1}$, and consequently possess two distinct uniform dessins on $K\backslash\mathbb{H}$.

The discussion also applies verbatim for prime powers: we obtain subgroups $\Delta_0(\mathfrak{p}^j) \leq \Delta_0(\mathfrak{p})$, involutions $\alpha_{\mathfrak{p}^j}$, and extensions $\Delta_{fr}(\mathfrak{p}^j) = \langle \Delta_0(\mathfrak{p}^j), \alpha_{\mathfrak{p}^j}\rangle$. Now, $\Delta_0(\mathfrak{p}^j)$ is normal in $\Delta_{fr}(\mathfrak{p}^j)$, but none of $\Delta_0(\mathfrak{p}^\ell)$ is normal in $\Delta_{fr}(\mathfrak{p}^j)$ for $j < \ell$. Nor are any of its Δ conjugates $\Delta_0^i(\mathfrak{p}^\ell)$.

Lemma 2.
- For each $j = 1, \cdots, n$ there are $q^{j-1}(q+1)$ congruence subgroups $\Delta_0^i(\mathfrak{p}^j)$, each conjugate to $\Delta_0(\mathfrak{p}^j)$ (in Δ) for $i = 0, \cdots, q^{j-1}(q+1) - 1$.
- Each of them is contained in Δ, and in j different triangle groups conjugate to Δ, **in which** $\Delta(\mathfrak{p}^n)$ **is included non-normally**.
- Every $\Delta_0^i(\mathfrak{p}^j)$ is the intersection of Δ with a conjugate triangle group $\Delta^{j,i}$, and for fixed j, the different $\Delta^{i,j}$ form an orbit under Δ conjugation.

Proof. Proceed by induction on j. The group $\Delta_0(\mathfrak{p})$ has index $q + 1$ in Δ, and for each class $\rho_i \in \Delta$ modulo $\Delta_0(\mathfrak{p})$ for $i = 0, \cdots, q$ set $\Delta_0^i(\mathfrak{p}) := \rho_i\Delta_0(\mathfrak{p})\rho_i^{-1}$ for $\rho_0 = \mathrm{id}, \cdots, \rho_q$ such that $\Delta(\mathfrak{p}^n) \triangleleft \Delta_0^i(\mathfrak{p}) < \Delta$.

For each of them, let $\alpha_i := \rho_i\alpha_\mathfrak{p}\rho_i^{-1}$ be its fricke involution (where $\alpha_\mathfrak{p}$ is the one for $\Delta_0(\mathfrak{p})$). Then form the corresponding Fricke extension $\Delta_{fr}^i(\mathfrak{p})$ which is properly contained in Δ.

The conjugacy of the different $\Delta_0^i(\mathfrak{p}^j)$ for fixed j comes from the following interpretation: consider the **fake projective line** $\mathbb{P}^1(\mathcal{O}_k/\mathfrak{p}^j)$ consisting of pairs $(x, y) \in (\mathcal{O}_k/\mathfrak{p}^j)^2$ not both coordinates divisible by $\mathfrak{p}$, modulo the diagonal action of $(\mathcal{O}_k/\mathfrak{p}^j)^\times$. The action of Δ on this fake projective line is transitive, and the subgroups $\Delta_0^i(\mathfrak{p}^j)$ are the point stabilizers. □

[6]apparently, as first observed by Takeuchi, all of the arithmetic triangle groups have trace fields with class number one. Consequently, for all such k, every ideal $\mathfrak{p} = \pi\mathcal{O}_k$ for some prime element π of $\mathcal{O}_k$.

References

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