Spectral rigidity of some arithmetic surfaces

Justin Katz

Purdue University

November 15, 2022

Abstract nonsense

- Generally speaking, stuff is hard to understand.
- A framework: introduce an invariant

$$F : \textbf{Stuff} \to \textbf{Things}$$

- For a thing $\in$ **Things**, study the fiber

$$F^{-1}(\text{thing}) \subset \textbf{Stuff}$$

- Rigidity: $F^{-1}(\text{thing})$ is a singleton

Abstract nonsense

- Generally speaking, stuff is hard to understand.
- A framework: introduce an invariant

$$F : \textbf{Stuff} \to \textbf{Things}$$

- For a thing $\in$ **Things**, study the fiber

$$F^{-1}(\text{thing}) \subset \textbf{Stuff}$$

- Rigidity: $F^{-1}(\text{thing})$ is a singleton

Abstract nonsense

- Generally speaking, stuff is hard to understand.
- A framework: introduce an invariant

$$F : \textbf{Stuff} \to \textbf{Things}$$

- For a thing $\in$ **Things**, study the fiber

$$F^{-1}(\text{thing}) \subset \textbf{Stuff}$$

- Rigidity: $F^{-1}(\text{thing})$ is a singleton

Abstract nonsense

- Generally speaking, stuff is hard to understand.
- A framework: introduce an invariant

$$F : \textbf{Stuff} \to \textbf{Things}$$

- For a thing $\in$ **Things**, study the fiber

$$F^{-1}(\text{thing}) \subset \textbf{Stuff}$$

- Rigidity: $F^{-1}(\text{thing})$ is a singleton

Abstract nonsense

- Generally speaking, stuff is hard to understand.
- A framework: introduce an invariant

$$F : \textbf{Stuff} \to \textbf{Things}$$

- For a thing $\in$ **Things**, study the fiber

$$F^{-1}(\text{thing}) \subset \textbf{Stuff}$$

- Rigidity: $F^{-1}(\text{thing})$ is a singleton

Abstract nonsense

- Generally speaking, stuff is hard to understand.
- A framework: introduce an invariant

$$F : \textbf{Stuff} \to \textbf{Things}$$

- For a thing $\in$ **Things**, study the fiber

$$F^{-1}(\text{thing}) \subset \textbf{Stuff}$$

- Rigidity: $F^{-1}(\text{thing})$ is a singleton

Abstract nonsense

- Generally speaking, stuff is hard to understand.
- A framework: introduce an invariant

$$F : \textbf{Stuff} \to \textbf{Things}$$

- For a thing $\in$ **Things**, study the fiber

$$F^{-1}(\text{thing}) \subset \textbf{Stuff}$$

- Rigidity: $F^{-1}(\text{thing})$ is a singleton

Abstract nonsense

- Generally speaking, stuff is hard to understand.
- A framework: introduce an invariant

$$F : \textbf{Stuff} \to \textbf{Things}$$

- For a thing $\in$ **Things**, study the fiber

$$F^{-1}(\text{thing}) \subset \textbf{Stuff}$$

- Rigidity: F^{-1}(thing) is a singleton

Abstract nonsense

- Generally speaking, stuff is hard to understand.
- A framework: introduce an invariant

$$F : \textbf{Stuff} \to \textbf{Things}$$

- For a thing $\in$ **Things**, study the fiber

$$F^{-1}(\text{thing}) \subset \textbf{Stuff}$$

- Rigidity: $F^{-1}(\text{thing})$ is a singleton

Abstract nonsense

- Generally speaking, stuff is hard to understand.
- A framework: introduce an invariant

$$F : \textbf{Stuff} \to \textbf{Things}$$

- For a thing $\in$ **Things**, study the fiber

$$F^{-1}(\text{thing}) \subset \textbf{Stuff}$$

- Rigidity: $F^{-1}(\text{thing})$ is a singleton

Abstract nonsense

- Generally speaking, stuff is hard to understand.
- A framework: introduce an invariant

$$F : \textbf{Stuff} \to \textbf{Things}$$

- For a thing $\in$ **Things**, study the fiber

$$F^{-1}(\text{thing}) \subset \textbf{Stuff}$$

- Rigidity: $F^{-1}(\text{thing})$ is a singleton (bonus points: when F is not-so-complicated and **Things** not so mysterious.)

Concrete nonsense

In this talk:

- **Stuff** := closed Riemannian manifolds (M, g)
- **Things** := discretely supported measures on $\mathbb{R}_{\geq 0}$

Concrete nonsense

In this talk:

- **Stuff** $:=$ closed Riemannian manifolds (M, g)
- **Things** $:=$ discretely supported measures on $\mathbb{R}_{\geq 0}$

Concrete nonsense

In this talk:

- **Stuff** $:=$ closed Riemannian manifolds (M, g)
- **Things** $:=$ discretely supported measures on $\mathbb{R}_{\geq 0}$

Concrete nonsense

In this talk:

- **Stuff** $:=$ closed Riemannian manifolds (M, g)
- **Things** $:=$ discretely supported measures on $\mathbb{R}_{\geq 0}$
- $F(\cdot) := \mathrm{spec}_{(\cdot)}$, where

$$\mathrm{spec}_{M,g}(\{\lambda\}) = \dim \ker_{L^2(M, \mathrm{d}\,\mathrm{vol}_g))}(\Delta_{M,g} - \lambda)$$

Concrete nonsense

In this talk:

- **Stuff** := closed Riemannian manifolds (M, g)
- **Things** := discretely supported measures on $\mathbb{R}_{\geq 0}$
- $F(\cdot) := \mathsf{spec}_{(\cdot)}$, where

$$\mathsf{spec}_{M,g}(\{\lambda\}) = \dim \ker_{L^2(M, \mathrm{d}\,\mathrm{vol}_g))}(\Delta_{M,g} - \lambda)$$

$F(\cdot) := \mathrm{Lspec}_{(\cdot)}$, where

$$\mathsf{spec}_{M,g}(\{\ell\}) = \#\{\gamma \subset \pi_1(M) : \ell_g(\gamma) = \ell\}$$

Concrete nonsense

In this talk:

- **Stuff** $:=$ closed Riemannian manifolds (M, g)
- **Things** $:=$ discretely supported measures on $\mathbb{R}_{\geq 0}$
- $F(\cdot) := \mathsf{spec}_{(\cdot)}$, where

$$\mathsf{spec}_{M,g}(\{\lambda\}) = \dim \ker_{L^2(M, \mathrm{d}\,\mathrm{vol}_g))}(\Delta_{M,g} - \lambda)$$

$F(\cdot) := \mathrm{Lspec}_{(\cdot)}$, where

$$\mathsf{spec}_{M,g}(\{\ell\}) = \#\{\gamma \subset \pi_1(M) : \ell_g(\gamma) = \ell\}$$

Concrete nonsense

In this talk:

- **Stuff** := closed Riemannian manifolds (M, g)
- **Things** := discretely supported measures on $\mathbb{R}_{\geq 0}$
- $F(\cdot) := \mathrm{spec}_{(\cdot)}$, where

$$\mathrm{spec}_{M,g}(\{\lambda\}) = \dim \ker_{L^2(M,\,\mathrm{d}\,\mathrm{vol}_g))}(\Delta_{M,g} - \lambda)$$

$F(\cdot) := \mathrm{Lspec}_{(\cdot)}$, where

$$\mathrm{spec}_{M,g}(\{\ell\}) = \#\{\gamma \subset \pi_1(M) : \ell_g(\gamma) = \ell\}$$

Theorem

Certain hyperbolic 2 manifolds are absolutely spectrally rigid. Principal congruence covers of surfaces arising from maximal orders in quaternion algebras with type number 1 are spectrally rigid.

Background

A (very hard) question: Which measures μ arise as $\mathrm{spec}_{M,g}$ for some (M, g)?

Examples:

- For a flat torus $\mathbb{R}^n/L$.

$$\mathrm{spec}_{\mathbb{R}^n/L,\,\mathrm{d}x^2} = \sum_{x \in L^*} \delta_{-||x||^2}$$

  where L^* is the lattice dual to L. Case $L = \mathbb{Z}^n$: which integers are sums of n squares? In how many ways?

- For a round $n-1$ dimensional sphere S^{n-1}:

$$\mathrm{spec}_{S^{n-1}} = \sum_{a \geq 0} \frac{(a + n/2 - 1)\prod_{j=1}^{n-3}(a+j)}{(n/2-1)\cdot(n-3)!}\delta_{a^2+(n-2)a}$$

Generally speaking, one shouldn't expect a closed form expression for spectra.

Background

A (very hard) question: Which measures μ arise as $\mathrm{spec}_{M,g}$ for some (M, g)?

Examples:

- For a flat torus $\mathbb{R}^n/L$,

$$\mathrm{spec}_{\mathbb{R}^n/L,\,\mathrm{d}x^2} = \sum_{x \in L^*} \delta_{-||x||^2}$$

  where L^* is the lattice dual to L. Case $L = \mathbb{Z}^n$: which integers are sums of n squares? In how many ways?

- For a round $n-1$ dimensional sphere S^{n-1}:

$$\mathrm{spec}_{S^{n-1}} = \sum_{a \geq 0} \frac{(a + n/2 - 1) \prod_{j=1}^{n-3}(a + j)}{(n/2 - 1) \cdot (n-3)!} \delta_{a^2 + (n-2)a}$$

Generally speaking, one shouldn't expect a closed form expression for spectra.

A (very hard) question: Which measures μ arise as $\mathrm{spec}_{M,g}$ for some (M, g)?
Examples:

- For a flat torus $\mathbb{R}^n/L$.

$$\mathrm{spec}_{\mathbb{R}^n/L,\,\mathrm{d}x^2} = \sum_{x \in L^*} \delta_{-||x||^2}$$

where L^* is the lattice dual to L. Case $L = \mathbb{Z}^n$: which integers are sums of n squares? In how many ways?

- For a round $n-1$ dimensional sphere S^{n-1}:

$$\mathrm{spec}_{S^{n-1}} = \sum_{a \geq 0} \frac{(a + n/2 - 1)\prod_{j=1}^{n-3}(a + j)}{(n/2 - 1)\cdot(n-3)!}\delta_{a^2+(n-2)a}$$

Generally speaking, one shouldn't expect a closed form expression for spectra.

Background

A (very hard) question: Which measures μ arise as $\mathrm{spec}_{M,g}$ for some (M, g)?

Examples:

- For a flat torus $\mathbb{R}^n/L$.

$$\mathrm{spec}_{\mathbb{R}^n/L,\,\mathrm{d}x^2} = \sum_{x \in L^*} \delta_{-||x||^2}$$

  where L^* is the lattice dual to L. Case $L = \mathbb{Z}^n$: which integers are sums of n squares? In how many ways?

- For a round $n-1$ dimensional sphere S^{n-1}:

$$\mathrm{spec}_{S^{n-1}} = \sum_{a \geq 0} \frac{(a + n/2 - 1)\prod_{j=1}^{n-3}(a+j)}{(n/2-1)\cdot(n-3)!}\delta_{a^2+(n-2)a}$$

Generally speaking, one shouldn't expect a closed form expression for spectra.

Background

A (very hard) question: Which measures μ arise as $\mathrm{spec}_{M,g}$ for some (M, g)?
Examples:

- For a flat torus $\mathbb{R}^n/L$.

$$\mathrm{spec}_{\mathbb{R}^n/L,\,\mathrm{d}x^2} = \sum_{x \in L^*} \delta_{-||x||^2}$$

where L^* is the lattice dual to L. Case $L = \mathbb{Z}^n$: which integers are sums of n squares? In how many ways?

- For a round $n-1$ dimensional sphere S^{n-1}:

$$\mathrm{spec}_{S^{n-1}} = \sum_{a \geq 0} \frac{(a+n/2-1)\prod_{j=1}^{n-3}(a+j)}{(n/2-1)\cdot(n-3)!}\delta_{a^2+(n-2)a}$$

Generally speaking, one shouldn't expect a closed form expression for spectra.

Background

A (very hard) question: Which measures μ arise as $\mathrm{spec}_{M,g}$ for some (M,g)?
Examples:

- For a flat torus $\mathbb{R}^n/L$.

$$\mathrm{spec}_{\mathbb{R}^n/L,\,\mathrm{d}x^2} = \sum_{x \in L^*} \delta_{-||x||^2}$$

  where L^* is the lattice dual to L. Case $L = \mathbb{Z}^n$: which integers are sums of n squares? In how many ways?

- For a round $n-1$ dimensional sphere S^{n-1}:

$$\mathrm{spec}_{S^{n-1}} = \sum_{a \geq 0} \frac{(a+n/2-1)\prod_{j=1}^{n-3}(a+j)}{(n/2-1)\cdot(n-3)!}\delta_{a^2+(n-2)a}$$

Generally speaking, one shouldn't expect a closed form expression for spectra.

Background

A (very hard) question: Which measures μ arise as $\mathrm{spec}_{M,g}$ for some (M,g)?
Examples:

- For a flat torus $\mathbb{R}^n/L$.

$$\mathrm{spec}_{\mathbb{R}^n/L,\,\mathrm{d}x^2} = \sum_{x\in L^*} \delta_{-||x||^2}$$

  where L^* is the lattice dual to L. Case $L=\mathbb{Z}^n$: which integers are sums of n squares? In how many ways?
- For a round $n-1$ dimensional sphere S^{n-1}:

$$\mathrm{spec}_{S^{n-1}} = \sum_{a\geq 0} \frac{(a+n/2-1)\prod_{j=1}^{n-3}(a+j)}{(n/2-1)\cdot(n-3)!}\delta_{a^2+(n-2)a}$$

Generally speaking, one shouldn't expect a closed form expression for spectra.

Background

A (very hard) question: Which measures μ arise as $\mathrm{spec}_{M,g}$ for some (M, g)?
Examples:

- For a flat torus $\mathbb{R}^n/L$.

$$\mathrm{spec}_{\mathbb{R}^n/L,\, \mathrm{d}x^2} = \sum_{x \in L^*} \delta_{-||x||^2}$$

  where L^* is the lattice dual to L. Case $L = \mathbb{Z}^n$: which integers are sums of n squares? In how many ways?
- For a round $n-1$ dimensional sphere S^{n-1}:

$$\mathrm{spec}_{S^{n-1}} = \sum_{a \geq 0} \frac{(a + n/2 - 1) \prod_{j=1}^{n-3} (a+j)}{(n/2-1) \cdot (n-3)!} \delta_{a^2 + (n-2)a}$$

Generally speaking, one shouldn't expect a closed form expression for spectra.

Background

A (very hard) question: Which measures μ arise as $\mathrm{spec}_{M,g}$ for some (M, g)?

Examples:

- For a flat torus $\mathbb{R}^n/L$.

$$\mathrm{spec}_{\mathbb{R}^n/L,\,\mathrm{d}x^2} = \sum_{x \in L^*} \delta_{-||x||^2}$$

  where L^* is the lattice dual to L. Case $L = \mathbb{Z}^n$: which integers are sums of n squares? In how many ways?

- For a round $n-1$ dimensional sphere S^{n-1}:

$$\mathrm{spec}_{S^{n-1}} = \sum_{a \geq 0} \frac{(a + n/2 - 1)\prod_{j=1}^{n-3}(a+j)}{(n/2-1)\cdot(n-3)!}\delta_{a^2+(n-2)a}$$

Generally speaking, one shouldn't expect a closed form expression for spectra.

Background

A (very hard) question: Which measures μ arise as $\mathrm{spec}_{M,g}$ for some (M,g)?

Examples:

- For a flat torus $\mathbb{R}^n/L$.

$$\mathrm{spec}_{\mathbb{R}^n/L,\,\mathrm{d}x^2} = \sum_{x\in L^*} \delta_{-||x||^2}$$

  where L^* is the lattice dual to L. Case $L = \mathbb{Z}^n$: which integers are sums of n squares? In how many ways?

- For a round $n-1$ dimensional sphere S^{n-1}:

$$\mathrm{spec}_{S^{n-1}} = \sum_{a\geq 0} \frac{(a+n/2-1)\prod_{j=1}^{n-3}(a+j)}{(n/2-1)\cdot(n-3)!}\delta_{a^2+(n-2)a}$$

Generally speaking, one shouldn't expect a closed form expression for spectra.

Isospectrality

- Two Riemannian manifolds (M_1, g_1) and (M_2, g_2) are said to be **isospectral**, or belong to the same **spectral genus**, when $\mathrm{spec}_{M_1,g_1} = \mathrm{spec}_{M_2,g_2}$.
- (M, g) is **spectrally rigid** if its spectral genus is a singleton.
- Mark Kac asked in 1966 whether all Riemannian manifolds are spectrally rigid.
- Milnor, two years prior:

Isospectrality

- Two Riemannian manifolds (M_1, g_1) and (M_2, g_2) are said to be **isospectral**, or belong to the same **spectral genus**, when $\text{spec}_{M_1,g_1} = \text{spec}_{M_2,g_2}$.
- (M, g) is **spectrally rigid** if its spectral genus is a singleton.
- Mark Kac asked in 1966 whether all Riemannian manifolds are spectrally rigid.
- Milnor, two years prior:

- Two Riemannian manifolds (M_1, g_1) and (M_2, g_2) are said to be **isospectral**, or belong to the same **spectral genus**, when $\mathrm{spec}_{M_1,g_1} = \mathrm{spec}_{M_2,g_2}$.
- (M, g) is **spectrally rigid** if its spectral genus is a singleton.
- Mark Kac asked in 1966 whether all Riemannian manifolds are spectrally rigid.
- Milnor, two years prior:

Isospectrality

- Two Riemannian manifolds (M_1, g_1) and (M_2, g_2) are said to be **isospectral**, or belong to the same **spectral genus**, when $\mathrm{spec}_{M_1,g_1} = \mathrm{spec}_{M_2,g_2}$.
- (M, g) is **spectrally rigid** if its spectral genus is a singleton.
- Mark Kac asked in 1966 whether all Riemannian manifolds are spectrally rigid.
- Milnor, two years prior:

Isospectrality

- Two Riemannian manifolds (M_1, g_1) and (M_2, g_2) are said to be **isospectral**, or belong to the same **spectral genus**, when $\mathrm{spec}_{M_1,g_1} = \mathrm{spec}_{M_2,g_2}$.
- (M, g) is **spectrally rigid** if its spectral genus is a singleton.
- Mark Kac asked in 1966 whether all Riemannian manifolds are spectrally rigid.
- Milnor, two years prior:no!

EIGENVALUES OF THE LAPLACE OPERATOR ON CERTAIN MANIFOLDS

BY J. MILNOR

PRINCETON UNIVERSITY

Communicated February 6, 1964

To every compact Riemannian manifold M there corresponds the sequence $0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ of eigenvalues for the Laplace operator on M. It is not known just how much information about M can be extracted from this sequence.[1] This note will show that the sequence does not characterize M completely, by exhibiting two 16-dimensional toruses which are distinct as Riemannian manifolds but have the same sequence of eigenvalues.

By a *flat torus* is meant a Riemannian quotient manifold of the form R^n/L, where L is a lattice (= discrete additive subgroup) of rank n. Let L^* denote the dual lattice, consisting of all $y \in R^n$ such that $x \cdot y$ is an integer for all $x \in L$. Then each $y \in L^*$ determines an eigenfunction $f(x) = \exp(2\pi\, ix\cdot y)$ for the Laplace operator on R^n/L. The corresponding eigenvalue λ is equal to $(2\pi)^2 y\cdot y$. Hence, the number of eigenvalues less than or equal to $(2\pi r)^2$ is equal to the number of points of L^* lying within a ball of radius r about the origin.

According to Witt[2] there exist two self-dual lattices $L_1, L_2 \subset R^{16}$ which are distinct, in the sense that no rotation of R^{16} carries L_1 to L_2, such that each ball about the origin contains exactly as many points of L_1 as of L_2. It follows that the Riemannian manifolds R^{16}/L_1 and R^{16}/L_2 are not isometric, but do have the same sequence of eigenvalues.

In an attempt to distinguish R^{16}/L_1 from R^{16}/L_2 one might consider the eigenvalues of the Hodge-Laplace operator $\Delta = d\delta + \delta d$, applied to the space of differential p-forms. However, both manifolds are flat and parallelizable, so the identity

$$\Delta(f\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}) = (\Delta f) dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

shows that one obtains simply the old eigenvalues, each repeated $\binom{16}{p}$ times.

[1] Compare Avakumović, V., "Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten," *Math. Zeits.*, **65**, 327–344 (1956).

[2] Witt, E., "Eine Identität zwischen Modulformen zweiten Grades," *Abh. Math. Sem. Univ. Hamburg*, **14**, 323–337 (1941). See p. 324. I am indebted to K. Ramanathan for pointing out this reference.

Heat invariants

- A **spectral invariant**, or **audible property**, is one which is constant along spectral genera.
- A key source of spectral invariants: the trace of the **heat kernel**

$$\Theta_{M,g}(t) = \int e^{-\lambda t} \, \mathrm{d}\,\mathrm{spec}_{M,g}(\lambda)$$

Heat invariants

- A **spectral invariant**, or **audible property**, is one which is constant along spectral genera.
- A key source of spectral invariants: the trace of the **heat kernel**

$$\Theta_{M,g}(t) = \int e^{-\lambda t}\, \mathrm{d}\,\mathrm{spec}_{M,g}(\lambda)$$

Theorem (Minakshisundaram-Pleijel)

- *There are constants $a_k(M,g) = a_k$ such that, as $t \to 0^+$ one has*
- *The coefficients a_k are integrals of polynomials in curvatures and its derivatives. The first three are:*

Heat invariants

- A **spectral invariant**, or **audible property**, is one which is constant along spectral genera.
- A key source of spectral invariants: the trace of the **heat kernel**

$$\Theta_{M,g}(t) = \int e^{-\lambda t} \,\mathrm{d}\,\mathrm{spec}_{M,g}(\lambda)$$

Theorem (Minakshisundaram-Pleijel)

- *There are constants $a_k(M,g) = a_k$ such that, as $t \to 0^+$, one has*

$$\Theta_{M,g}(t) = (4\pi t)^{-\dim(M)/2} \sum_{k=0}^{\infty} a_k t^k.$$

- *The coefficients a_k are integrals of polynomials in curvatures and its derivatives. The first three are:*

$$a_0 = \mathrm{vol}(M),\ a_1 = \frac{1}{6}\int_M \tau,\ a_2 = \frac{1}{360}\int_M 5\tau^2 - 2|\mathrm{Ric}|^2 + 2|R|^2$$

Heat invariants

- A **spectral invariant**, or **audible property**, is one which is constant along spectral genera.
- A key source of spectral invariants: the trace of the **heat kernel**

$$\Theta_{M,g}(t) = \int e^{-\lambda t} \, \mathrm{d}\, \mathrm{spec}_{M,g}(\lambda)$$

Theorem (Minakshisundaram-Pleijel)

- *There are constants $a_k(M,g) = a_k$ such that, as $t \to 0^+$ one has*

$$\Theta_{M,g}(t) = (4\pi t)^{-\dim(M)/2} \sum_{k=1}^{\infty} a_k t^k.$$

- *The coefficients a_k are integrals of polynomials in curvatures and its derivatives. The first three are:*

$$a_0 = \mathrm{vol}(M), \quad a_1 = \frac{1}{6}\int_M \tau, \quad a_2 = \frac{1}{360}\int_M 5\tau^2 - 2|\mathrm{Ric}|^2 - 10|R|^2.$$

Heat invariants

- A **spectral invariant**, or **audible property**, is one which is constant along spectral genera.
- A key source of spectral invariants: the trace of the **heat kernel**

$$\Theta_{M,g}(t) = \int e^{-\lambda t}\,\mathrm{d}\,\mathrm{spec}_{M,g}(\lambda)$$

Theorem (Minakshisundaram-Pleijel)

- *There are constants $a_k(M,g) = a_k$ such that, as $t \to 0^+$ one has*

$$\Theta_{M,g}(t) = (4\pi t)^{-\dim(M)/2} \sum_{k=1}^{\infty} a_k t^k.$$

- *The coefficients a_k are integrals of polynomials in curvatures and its derivatives. The first three are:*

$$a_0 = \mathrm{vol}(M),\; a_1 = \frac{1}{6}\int_M \tau,\; a_2 = \frac{1}{360}\int_M 5\tau^2 - 2|\mathrm{Ric}|^2 - 10|R|^2.$$

Heat invariants

- A **spectral invariant**, or **audible property**, is one which is constant along spectral genera.
- A key source of spectral invariants: the trace of the **heat kernel**

$$\Theta_{M,g}(t) = \int e^{-\lambda t} \,\mathrm{d}\,\mathrm{spec}_{M,g}(\lambda)$$

Theorem (Minakshisundaram-Pleijel)

- *There are constants $a_k(M,g) = a_k$ such that, as $t \to 0^+$ one has*

$$\Theta_{M,g}(t) = (4\pi t)^{-\dim(M)/2} \sum_{k=1}^{\infty} a_k t^k.$$

- *The coefficients a_k are integrals of polynomials in curvatures and its derivatives. The first three are:*

$$a_0 = \mathrm{vol}(M),\ a_1 = \frac{1}{6}\int_M \tau,\ a_2 = \frac{1}{360}\int_M 5\tau^2 - 2|\mathrm{Ric}|^2 - 10|R|^2.$$

Heat invariants

- A **spectral invariant**, or **audible property**, is one which is constant along spectral genera.
- A key source of spectral invariants: the trace of the **heat kernel**

$$\Theta_{M,g}(t) = \int e^{-\lambda t}\, \mathrm{d}\, \mathrm{spec}_{M,g}(\lambda)$$

Theorem (Minakshisundaram-Pleijel)

- *There are constants $a_k(M,g) = a_k$ such that, as $t \to 0^+$ one has*

$$\Theta_{M,g}(t) = (4\pi t)^{-\dim(M)/2} \sum_{k=1}^{\infty} a_k t^k .$$

- *The coefficients a_k are integrals of polynomials in curvatures and its derivatives. The first three are:*

$$a_0 = \mathrm{vol}(M), \; a_1 = \frac{1}{6}\int_M \tau, \; a_2 = \frac{1}{360}\int_M 5\tau^2 - 2|\mathrm{Ric}|^2 - 10|R|^2 .$$

Heat invariants

- A **spectral invariant**, or **audible property**, is one which is constant along spectral genera.
- A key source of spectral invariants: the trace of the **heat kernel**

$$\Theta_{M,g}(t) = \int e^{-\lambda t} \, \mathrm{d}\, \mathrm{spec}_{M,g}(\lambda)$$

Theorem (Minakshisundaram-Pleijel)

- *There are constants $a_k(M,g) = a_k$ such that, as $t \to 0^+$ one has*

$$\Theta_{M,g}(t) = (4\pi t)^{-\dim(M)/2} \sum_{k=1}^{\infty} a_k t^k.$$

- *The coefficients a_k are integrals of polynomials in curvatures and its derivatives. The first three are:*

$$a_0 = \mathrm{vol}(M), \; a_1 = \frac{1}{6}\int_M \tau, \; a_2 = \frac{1}{360}\int_M 5\tau^2 - 2|\mathrm{Ric}|^2 - 10|R|^2.$$

Heat invariants

- A **spectral invariant**, or **audible property**, is one which is constant along spectral genera.
- A key source of spectral invariants: the trace of the **heat kernel**

$$\Theta_{M,g}(t) = \int e^{-\lambda t} \, \mathrm{d}\, \mathsf{spec}_{M,g}(\lambda)$$

Theorem (Minakshisundaram-Pleijel)

- *There are constants $a_k(M,g) = a_k$ such that, as $t \to 0^+$ one has*

$$\Theta_{M,g}(t) = (4\pi t)^{-\dim(M)/2} \sum_{k=1}^{\infty} a_k t^k.$$

- *The coefficients a_k are integrals of polynomials in curvatures and its derivatives. The first three are:*

$$a_0 = \mathrm{vol}(M),\ a_1 = \frac{1}{6}\int_M \tau,\ a_2 = \frac{1}{360}\int_M 5\tau^2 - 2|\mathrm{Ric}|^2 - 10|R|^2.$$

Heat invariants: consequences

- Evidently, dimension, volume, and total (scalar) curvature are spectral invariants.
- Consequently anything isospectral to a surface is a surface
- In fact: by Gauss-Bonnet a_1 for a surface is a topological invariant.
- Comparing a_1 and a_2, any variation in curvature is audible.

Heat invariants: consequences

- Evidently, dimension, volume, and total (scalar) curvature are spectral invariants.
- Consequently anything isospectral to a surface is a surface
- In fact: by Gauss-Bonnet a_1 for a surface is a topological invariant.
- Comparing a_1 and a_2, any variation in curvature is audible.

Heat invariants: consequences

- Evidently, dimension, volume, and total (scalar) curvature are spectral invariants.
- Consequently anything isospectral to a surface is a surface
- In fact: by Gauss-Bonnet a_1 for a surface is a topological invariant.
- Comparing a_1 and a_2, any variation in curvature is audible.

Heat invariants: consequences

- Evidently, dimension, volume, and total (scalar) curvature are spectral invariants.
- Consequently anything isospectral to a surface is a surface, **to which it is homeomorphic**!
- In fact: by Gauss-Bonnet a_1 for a surface is a topological invariant.
- Comparing a_1 and a_2, any variation in curvature is audible.

Heat invariants: consequences

- Evidently, dimension, volume, and total (scalar) curvature are spectral invariants.
- Consequently anything isospectral to a surface is a surface, **to which it is homeomorphic**!
- In fact: by Gauss-Bonnet a_1 for a surface is a topological invariant.
- Comparing a_1 and a_2, any variation in curvature is audible.

Heat invariants: consequences

- Evidently, dimension, volume, and total (scalar) curvature are spectral invariants.
- Consequently anything isospectral to a **hyperbolic** surface is a **hyperbolic** surface, **to which it is homeomorphic**!
- In fact: by Gauss-Bonnet a_1 for a surface is a topological invariant.
- Comparing a_1 and a_2, any variation in curvature is audible.

Hyperbolic surfaces: Fuchsian uniformization

- A compact hyperbolic surface admits a description as $\Gamma\backslash\mathbb{H}$ for a cocompact lattice $\Gamma \in \mathrm{PSL}(2,\mathbb{R}) \approx \mathrm{Isom}^+(\mathbb{H})$, where $\mathbb{H}$ is the hyperbolic plane.
- $\mathbb{H} \to \Gamma\backslash\mathbb{H}$ is the universal covering map, Γ is the deck group, and identifies with the fundamental group of the quotient.
- Elements of Γ are *pointed homotopy* classes of closed curves, conjugacy classes in Γ are *free homotopy* classes of closed curves on the compact surface.
- Within each such free homotopy class is a unique geodesic representative. Its length ℓ is related to the trace t of a representative matrix by $2e^{\ell/2} = t + \sqrt{t^2-4}$.
- Define the **length** and **trace** spectrum of $\Gamma\backslash\mathbb{H}$:

$$\mathrm{Lspec} = \sum_{\ell \in \ell(\Gamma)} m(\ell)\delta_\ell, \quad \mathrm{Tspec} = \sum_{t \in \mathrm{tr}(\Gamma)} m(t)\delta_t$$

where $m(\ell)$ and $m(t)$ are the number of geodesics on $\Gamma\backslash\mathbb{H}$ of length ℓ or trace t respectively.

Hyperbolic surfaces: Fuchsian uniformization

- A compact hyperbolic surface admits a description as $\Gamma\backslash\mathbb{H}$ for a cocompact lattice $\Gamma \in \mathrm{PSL}(2,\mathbb{R}) \approx \mathrm{Isom}^+(\mathbb{H})$, where $\mathbb{H}$ is the hyperbolic plane.
- $\mathbb{H} \to \Gamma\backslash\mathbb{H}$ is the universal covering map, Γ is the deck group, and identifies with the fundamental group of the quotient.
- Elements of Γ are *pointed homotopy* classes of closed curves, conjugacy classes in Γ are *free homotopy* classes of closed curves on the compact surface.
- Within each such free homotopy class is a unique geodesic representative. Its length ℓ is related to the trace t of a representative matrix by $2e^{\ell/2} = t + \sqrt{t^2 - 4}$.
- Define the **length** and **trace** spectrum of $\Gamma\backslash\mathbb{H}$:

$$\mathrm{Lspec} = \sum_{\ell \in \ell(\Gamma)} m(\ell)\delta_\ell, \quad \mathrm{Tspec} = \sum_{t \in \mathrm{tr}(\Gamma)} m(t)\delta_t$$

where $m(\ell)$ and $m(t)$ are the number of geodesics on $\Gamma\backslash\mathbb{H}$ of length ℓ or trace t respectively.

Hyperbolic surfaces: Fuchsian uniformization

- A compact hyperbolic surface admits a description as $\Gamma\backslash\mathbb{H}$ for a cocompact lattice $\Gamma \in \mathrm{PSL}(2,\mathbb{R}) \approx \mathrm{Isom}^+(\mathbb{H})$, where $\mathbb{H}$ is the hyperbolic plane.
- $\mathbb{H} \to \Gamma\backslash\mathbb{H}$ is the universal covering map, Γ is the deck group, and identifies with the fundamental group of the quotient.
- Elements of Γ are *pointed homotopy* classes of closed curves, conjugacy classes in Γ are *free homotopy* classes of closed curves on the compact surface.
- Within each such free homotopy class is a unique geodesic representative. Its length ℓ is related to the trace t of a representative matrix by $2e^{\ell/2} = t + \sqrt{t^2 - 4}$.
- Define the **length** and **trace** spectrum of $\Gamma\backslash\mathbb{H}$:

$$\mathrm{Lspec} = \sum_{\ell \in \ell(\Gamma)} m(\ell)\delta_\ell, \quad \mathrm{Tspec} = \sum_{t \in \mathrm{tr}(\Gamma)} m(t)\delta_t$$

where $m(\ell)$ and $m(t)$ are the number of geodesics on $\Gamma\backslash\mathbb{H}$ of length ℓ or trace t respectively.

Hyperbolic surfaces: Fuchsian uniformization

- A compact hyperbolic surface admits a description as $\Gamma\backslash\mathbb{H}$ for a cocompact lattice $\Gamma \in \mathrm{PSL}(2,\mathbb{R}) \approx \mathrm{Isom}^+(\mathbb{H})$, where $\mathbb{H}$ is the hyperbolic plane.
- $\mathbb{H} \to \Gamma\backslash\mathbb{H}$ is the universal covering map, Γ is the deck group, and identifies with the fundamental group of the quotient.
- Elements of Γ are *pointed homotopy* classes of closed curves, conjugacy classes in Γ are *free homotopy* classes of closed curves on the compact surface.
- Within each such free homotopy class is a unique geodesic representative. Its length ℓ is related to the trace t of a representative matrix by $2e^{\ell/2} = t + \sqrt{t^2-4}$.
- Define the **length** and **trace** spectrum of $\Gamma\backslash\mathbb{H}$:

$$\mathrm{Lspec} = \sum_{\ell \in \ell(\Gamma)} m(\ell)\delta_\ell, \quad \mathrm{Tspec} = \sum_{t \in \mathrm{tr}(\Gamma)} m(t)\delta_t$$

where $m(\ell)$ and $m(t)$ are the number of geodesics on $\Gamma\backslash\mathbb{H}$ of length ℓ or trace t respectively.

Hyperbolic surfaces: Fuchsian uniformization

- A compact hyperbolic surface admits a description as $\Gamma\backslash\mathbb{H}$ for a cocompact lattice $\Gamma \in \mathrm{PSL}(2,\mathbb{R}) \approx \mathrm{Isom}^+(\mathbb{H})$, where $\mathbb{H}$ is the hyperbolic plane.
- $\mathbb{H} \to \Gamma\backslash\mathbb{H}$ is the universal covering map, Γ is the deck group, and identifies with the fundamental group of the quotient.
- Elements of Γ are *pointed homotopy* classes of closed curves, conjugacy classes in Γ are *free homotopy* classes of closed curves on the compact surface.
- Within each such free homotopy class is a unique geodesic representative. Its length ℓ is related to the trace t of a representative matrix by $2e^{\ell/2} = t + \sqrt{t^2 - 4}$.
- Define the **length** and **trace** spectrum of $\Gamma\backslash\mathbb{H}$:

$$\mathrm{Lspec} = \sum_{\ell \in \ell(\Gamma)} m(\ell)\delta_\ell, \quad \mathrm{Tspec} = \sum_{t \in \mathrm{tr}(\Gamma)} m(t)\delta_t$$

where $m(\ell)$ and $m(t)$ are the number of geodesics on $\Gamma\backslash\mathbb{H}$ of length ℓ or trace t respectively.

Hyperbolic surfaces: Fuchsian uniformization

- A compact hyperbolic surface admits a description as $\Gamma\backslash\mathbb{H}$ for a cocompact lattice $\Gamma \in \mathrm{PSL}(2,\mathbb{R}) \approx \mathrm{Isom}^+(\mathbb{H})$, where $\mathbb{H}$ is the hyperbolic plane.
- $\mathbb{H} \to \Gamma\backslash\mathbb{H}$ is the universal covering map, Γ is the deck group, and identifies with the fundamental group of the quotient.
- Elements of Γ are *pointed homotopy* classes of closed curves, conjugacy classes in Γ are *free homotopy* classes of closed curves on the compact surface.
- Within each such free homotopy class is a unique geodesic representative. Its length ℓ is related to the trace t of a representative matrix by $2e^{\ell/2} = t + \sqrt{t^2-4}$.
- Define the **length** and **trace** spectrum of $\Gamma\backslash\mathbb{H}$:

$$\mathrm{Lspec} = \sum_{\ell \in \ell(\Gamma)} m(\ell)\delta_\ell, \quad \mathrm{Tspec} = \sum_{t \in \mathrm{tr}(\Gamma)} m(t)\delta_t$$

where $m(\ell)$ and $m(t)$ are the number of geodesics on $\Gamma\backslash\mathbb{H}$ of length ℓ or trace t respectively.

Hyperbolic surfaces: Fuchsian uniformization

- A compact hyperbolic surface admits a description as $\Gamma\backslash\mathbb{H}$ for a cocompact lattice $\Gamma \in \mathrm{PSL}(2,\mathbb{R}) \approx \mathrm{Isom}^+(\mathbb{H})$, where $\mathbb{H}$ is the hyperbolic plane.
- $\mathbb{H} \to \Gamma\backslash\mathbb{H}$ is the universal covering map, Γ is the deck group, and identifies with the fundamental group of the quotient.
- Elements of Γ are *pointed homotopy* classes of closed curves, conjugacy classes in Γ are *free homotopy* classes of closed curves on the compact surface.
- Within each such free homotopy class is a unique geodesic representative. Its length ℓ is related to the trace t of a representative matrix by $2e^{\ell/2} = t + \sqrt{t^2 - 4}$.
- Define the **length** and **trace** spectrum of $\Gamma\backslash\mathbb{H}$:

$$\mathrm{Lspec} = \sum_{\ell \in \ell(\Gamma)} m(\ell)\delta_\ell, \quad \mathrm{Tspec} = \sum_{t \in \mathrm{tr}(\Gamma)} m(t)\delta_t$$

where $m(\ell)$ and $m(t)$ are the number of geodesics on $\Gamma\backslash\mathbb{H}$ of length ℓ or trace t respectively.

Hyperbolic surfaces: Fuchsian uniformization

- A compact hyperbolic surface admits a description as $\Gamma\backslash\mathbb{H}$ for a cocompact lattice $\Gamma \in \mathrm{PSL}(2,\mathbb{R}) \approx \mathrm{Isom}^+(\mathbb{H})$, where $\mathbb{H}$ is the hyperbolic plane.
- $\mathbb{H} \to \Gamma\backslash\mathbb{H}$ is the universal covering map, Γ is the deck group, and identifies with the fundamental group of the quotient.
- Elements of Γ are *pointed homotopy* classes of closed curves, conjugacy classes in Γ are *free homotopy* classes of closed curves on the compact surface.
- Within each such free homotopy class is a unique geodesic representative. Its length ℓ is related to the trace t of a representative matrix by $2e^{\ell/2} = t + \sqrt{t^2-4}$.
- Define the **length** and **trace** spectrum of $\Gamma\backslash\mathbb{H}$:

$$\mathrm{Lspec} = \sum_{\ell\in\ell(\Gamma)} m(\ell)\delta_\ell, \quad \mathrm{Tspec} = \sum_{t\in\mathrm{tr}(\Gamma)} m(t)\delta_t$$

where $m(\ell)$ and $m(t)$ are the number of geodesics on $\Gamma\backslash\mathbb{H}$ of length ℓ or trace t respectively.

Hyperbolic surfaces: Fuchsian uniformization

- A compact hyperbolic surface admits a description as $\Gamma\backslash\mathbb{H}$ for a cocompact lattice $\Gamma \in \mathrm{PSL}(2,\mathbb{R}) \approx \mathrm{Isom}^+(\mathbb{H})$, where $\mathbb{H}$ is the hyperbolic plane.
- $\mathbb{H} \to \Gamma\backslash\mathbb{H}$ is the universal covering map, Γ is the deck group, and identifies with the fundamental group of the quotient.
- Elements of Γ are *pointed homotopy* classes of closed curves, conjugacy classes in Γ are *free homotopy* classes of closed curves on the compact surface.
- Within each such free homotopy class is a unique geodesic representative. Its length ℓ is related to the trace t of a representative matrix by $2e^{\ell/2} = t + \sqrt{t^2 - 4}$.
- Define the **length** and **trace** spectrum of $\Gamma\backslash\mathbb{H}$:

$$\mathrm{Lspec} = \sum_{\ell \in \ell(\Gamma)} m(\ell)\delta_\ell, \quad \mathrm{Tspec} = \sum_{t \in \mathrm{tr}(\Gamma)} m(t)\delta_t$$

where $m(\ell)$ and $m(t)$ are the number of geodesics on $\Gamma\backslash\mathbb{H}$ of length ℓ or trace t respectively.

Hyperbolic surfaces: Fuchsian uniformization

- A compact hyperbolic surface admits a description as $\Gamma\backslash\mathbb{H}$ for a cocompact lattice $\Gamma \in \mathrm{PSL}(2,\mathbb{R}) \approx \mathrm{Isom}^+(\mathbb{H})$, where $\mathbb{H}$ is the hyperbolic plane.
- $\mathbb{H} \to \Gamma\backslash\mathbb{H}$ is the universal covering map, Γ is the deck group, and identifies with the fundamental group of the quotient.
- Elements of Γ are *pointed homotopy* classes of closed curves, conjugacy classes in Γ are *free homotopy* classes of closed curves on the compact surface.
- Within each such free homotopy class is a unique geodesic representative. Its length ℓ is related to the trace t of a representative matrix by $2e^{\ell/2} = t + \sqrt{t^2 - 4}$.
- Define the **length** and **trace** spectrum of $\Gamma\backslash\mathbb{H}$:

$$\mathrm{Lspec} = \sum_{\ell \in \ell(\Gamma)} m(\ell)\delta_\ell, \quad \mathrm{Tspec} = \sum_{t \in \mathrm{tr}(\Gamma)} m(t)\delta_t$$

where $m(\ell)$ and $m(t)$ are the number of geodesics on $\Gamma\backslash\mathbb{H}$ of length ℓ or trace t respectively.

Hyperbolic surfaces: Arithmeticity

- A hyperbolic surface $\Gamma\backslash\mathbb{H}$ is **arithmetic** if there exists
  - A totally real number field k,
  - a quaternion algebra A over k such that there is a unique real place $\rho : k \to \mathbb{R}$ such that $A \otimes_\rho \mathbb{R} = M(2,\mathbb{R})$, and
  - a maximal order $\mathcal{O} \leq A$ with norm 1 units $\mathcal{O}^1$

  such that Γ is commensurable to the image of $\rho(\mathcal{O}^1)$ in $\mathrm{PSL}(2,\mathbb{R})$.
- Classic example:

$$k = \mathbb{Q},\ A = M(2,\mathbb{Q}),\ \mathcal{O} = M(2,\mathbb{Z}),\ \Gamma = \mathrm{PSL}(2,\mathbb{Z})$$

- Key fact: the isomorphism class of the quternion algebra A is a complete invariant of the commensurability class of an arithmetic surface.

- A hyperbolic surface $\Gamma\backslash\mathbb{H}$ is **arithmetic** if there exists
  - A totally real number field k,
  - a quaternion algebra A over k such that there is a unique real place $\rho : k \to \mathbb{R}$ such that $A \otimes_\rho \mathbb{R} = M(2, \mathbb{R})$, and
  - a maximal order $\mathcal{O} \leq A$ with norm 1 units $\mathcal{O}^1$

  such that Γ is commensurable to the image of $\rho(\mathcal{O}^1)$ in $\mathrm{PSL}(2, \mathbb{R})$.
- Classic example:

$$k = \mathbb{Q},\ A = M(2, \mathbb{Q}),\ \mathcal{O} = M(2, \mathbb{Z}),\ \Gamma = \mathrm{PSL}(2, \mathbb{Z})$$

- Key fact: the isomorphism class of the quternion algebra A is a complete invariant of the commensurability class of an arithmetic surface.

Hyperbolic surfaces: Arithmeticity

- A hyperbolic surface $\Gamma\backslash\mathbb{H}$ is **arithmetic** if there exists
  - A totally real number field k,
  - a quaternion algebra A over k such that there is a unique real place $\rho : k \to \mathbb{R}$ such that $A \otimes_\rho \mathbb{R} = M(2, \mathbb{R})$, and
  - a maximal order $\mathcal{O} \leq A$ with norm 1 units $\mathcal{O}^1$

  such that Γ is commensurable to the image of $\rho(\mathcal{O}^1)$ in $\mathrm{PSL}(2, \mathbb{R})$.
- Classic example:

$$k = \mathbb{Q},\ A = M(2, \mathbb{Q}),\ \mathcal{O} = M(2, \mathbb{Z}),\ \Gamma = \mathrm{PSL}(2, \mathbb{Z})$$

- Key fact: the isomorphism class of the quternion algebra A is a complete invariant of the commensurability class of an arithmetic surface.

Hyperbolic surfaces: Arithmeticity

- A hyperbolic surface $\Gamma\backslash\mathbb{H}$ is **arithmetic** if there exists
  - A totally real number field k,
  - a quaternion algebra A over k such that there is a unique real place $\rho : k \to \mathbb{R}$ such that $A \otimes_\rho \mathbb{R} = M(2, \mathbb{R})$, and
  - a maximal order $\mathcal{O} \leq A$ with norm 1 units $\mathcal{O}^1$

  such that Γ is commensurable to the image of $\rho(\mathcal{O}^1)$ in $\mathrm{PSL}(2, \mathbb{R})$.
- Classic example:

$$k = \mathbb{Q},\ A = M(2, \mathbb{Q}),\ \mathcal{O} = M(2, \mathbb{Z}),\ \Gamma = \mathrm{PSL}(2, \mathbb{Z})$$

- Key fact: the isomorphism class of the quternion algebra A is a complete invariant of the commensurability class of an arithmetic surface.

Hyperbolic surfaces: Arithmeticity

- A hyperbolic surface $\Gamma\backslash\mathbb{H}$ is **arithmetic** if there exists
  - A totally real number field k,
  - a quaternion algebra A over k such that there is a unique real place $\rho : k \to \mathbb{R}$ such that $A \otimes_\rho \mathbb{R} = M(2, \mathbb{R})$, and
  - a maximal order $\mathcal{O} \leq A$ with norm 1 units $\mathcal{O}^1$

  such that Γ is commensurable to the image of $\rho(\mathcal{O}^1)$ in $\mathrm{PSL}(2, \mathbb{R})$.
- Classic example:

$$k = \mathbb{Q},\ A = M(2, \mathbb{Q}),\ \mathcal{O} = M(2, \mathbb{Z}),\ \Gamma = \mathrm{PSL}(2, \mathbb{Z})$$

- Key fact: the isomorphism class of the quternion algebra A is a complete invariant of the commensurability class of an arithmetic surface.

Hyperbolic surfaces: Arithmeticity

- A hyperbolic surface $\Gamma\backslash\mathbb{H}$ is **arithmetic** if there exists
  - A totally real number field k,
  - a quaternion algebra A over k such that there is a unique real place $\rho : k \to \mathbb{R}$ such that $A \otimes_\rho \mathbb{R} = M(2, \mathbb{R})$, and
  - a maximal order $\mathcal{O} \leq A$ with norm 1 units $\mathcal{O}^1$

  such that Γ is commensurable to the image of $\rho(\mathcal{O}^1)$ in $\mathrm{PSL}(2, \mathbb{R})$.
- Classic example:

$$k = \mathbb{Q},\ A = M(2, \mathbb{Q}),\ \mathcal{O} = M(2, \mathbb{Z}),\ \Gamma = \mathrm{PSL}(2, \mathbb{Z})$$

- Key fact: the isomorphism class of the quternion algebra A is a complete invariant of the commensurability class of an arithmetic surface.

Hyperbolic surfaces: Arithmeticity

- A hyperbolic surface $\Gamma\backslash\mathbb{H}$ is **arithmetic** if there exists
  - A totally real number field k,
  - a quaternion algebra A over k such that there is a unique real place $\rho : k \to \mathbb{R}$ such that $A \otimes_\rho \mathbb{R} = M(2, \mathbb{R})$, and
  - a maximal order $\mathcal{O} \leq A$ with norm 1 units $\mathcal{O}^1$

  such that Γ is commensurable to the image of $\rho(\mathcal{O}^1)$ in $\mathrm{PSL}(2, \mathbb{R})$.
- Classic example:

$$k = \mathbb{Q},\ A = M(2, \mathbb{Q}),\ \mathcal{O} = M(2, \mathbb{Z}),\ \Gamma = \mathrm{PSL}(2, \mathbb{Z})$$

- Key fact: the isomorphism class of the quternion algebra A is a complete invariant of the commensurability class of an arithmetic surface.

- A hyperbolic surface $\Gamma\backslash\mathbb{H}$ is **arithmetic** if there exists
  - A totally real number field k,
  - a quaternion algebra A over k such that there is a unique real place $\rho : k \to \mathbb{R}$ such that $A \otimes_\rho \mathbb{R} = M(2, \mathbb{R})$, and
  - a maximal order $\mathcal{O} \leq A$ with norm 1 units $\mathcal{O}^1$

  such that Γ is commensurable to the image of $\rho(\mathcal{O}^1)$ in $\mathrm{PSL}(2, \mathbb{R})$.
- Classic example:

$$k = \mathbb{Q},\ A = M(2, \mathbb{Q}),\ \mathcal{O} = M(2, \mathbb{Z}),\ \Gamma = \mathrm{PSL}(2, \mathbb{Z})$$

- Key fact: the isomorphism class of the quternion algebra A is a complete invariant of the commensurability class of an arithmetic surface.

Selberg Trace formula: for a closed hyperbolic surface, knowledge of eigenvalue spectrum is equivalent to knowledge of the length spectrum, equivalent to knowledge of trace spectrum.

Theorem (Takeuchi)

A closed hyperbolic surface $\Gamma\backslash\mathbb{H}$ is arithmetic and derived from a quaternion algebra if and only if

- $k = \mathbb{Q}(\mathrm{tr}(\Gamma))$ *is a number field,*
- $\mathrm{tr}(\Gamma)$ *is contained in its integers, and*
- *for any nonidentity embedding* $\varphi : k \to \mathbb{C}$*, the image* $\varphi(\mathrm{tr}(\Gamma))$ *is bounded.*

Anything isospectral to a compact hyperbolic **arithmetic** surface is a compact hyperbolic **arithmetic** surface

Hyperbolic surfaces: audibility of arithmeticity

Selberg Trace formula: for a closed hyperbolic surface, knowledge of eigenvalue spectrum is equivalent to knowledge of the length spectrum, equivalent to knowledge of trace spectrum.

Theorem (Takeuchi)

A closed hyperbolic surface $\Gamma\backslash\mathbb{H}$ is arithmetic and derived from a quaternion algebra if and only if

- *$k = \mathbb{Q}(\mathrm{tr}(\Gamma))$ is a number field,*
- *$\mathrm{tr}(\Gamma)$ is contained in its integers, and*
- *for any nonidentity embedding $\varphi : k \to \mathbb{C}$, the image $\varphi(\mathrm{tr}(\Gamma))$ is bounded.*

Anything isospectral to a compact hyperbolic **arithmetic** surface is a compact hyperbolic **arithmetic** surface

Hyperbolic surfaces: audibility of arithmeticity

Selberg Trace formula: for a closed hyperbolic surface, knowledge of eigenvalue spectrum is equivalent to knowledge of the length spectrum, equivalent to knowledge of trace spectrum.

Theorem (Takeuchi)

A closed hyperbolic surface $\Gamma\backslash\mathbb{H}$ is arithmetic and derived from a quaternion algebra if and only if

- $k = \mathbb{Q}(\mathrm{tr}(\Gamma))$ *is a number field,*
- $\mathrm{tr}(\Gamma)$ *is contained in its integers, and*
- *for any nonidentity embedding* $\varphi : k \to \mathbb{C}$*, the image* $\varphi(\mathrm{tr}(\Gamma))$ *is bounded.*

Anything isospectral to a compact hyperbolic **arithmetic** surface is a compact hyperbolic **arithmetic** surface

Hyperbolic surfaces: audibility of arithmeticity

Selberg Trace formula: for a closed hyperbolic surface, knowledge of eigenvalue spectrum is equivalent to knowledge of the length spectrum, equivalent to knowledge of trace spectrum.

Theorem (Takeuchi)

A closed hyperbolic surface $\Gamma\backslash\mathbb{H}$ is arithmetic and derived from a quaternion algebra if and only if

- $k = \mathbb{Q}(\mathrm{tr}(\Gamma))$ *is a number field,*
- $\mathrm{tr}(\Gamma)$ *is contained in its integers, and*
- *for any nonidentity embedding $\varphi : k \to \mathbb{C}$, the image $\varphi(\mathrm{tr}(\Gamma))$ is bounded.*

Anything isospectral to a compact hyperbolic **arithmetic** surface is a compact hyperbolic **arithmetic** surface

Selberg Trace formula: for a closed hyperbolic surface, knowledge of eigenvalue spectrum is equivalent to knowledge of the length spectrum, equivalent to knowledge of trace spectrum.

Theorem (Takeuchi)

A closed hyperbolic surface $\Gamma\backslash\mathbb{H}$ *is arithmetic and derived from a quaternion algebra if and only if*

- $k = \mathbb{Q}(\mathrm{tr}(\Gamma))$ *is a number field,*
- $\mathrm{tr}(\Gamma)$ *is contained in its integers, and*
- *for any nonidentity embedding* $\varphi : k \to \mathbb{C}$*, the image* $\varphi(\mathrm{tr}(\Gamma))$ *is bounded.*

Anything isospectral to a compact hyperbolic **arithmetic** surface is a compact hyperbolic **arithmetic** surface

Selberg Trace formula: for a closed hyperbolic surface, knowledge of eigenvalue spectrum is equivalent to knowledge of the length spectrum, equivalent to knowledge of trace spectrum.

Theorem (Takeuchi)

A closed hyperbolic surface $\Gamma\backslash\mathbb{H}$ is arithmetic and derived from a quaternion algebra if and only if

- $k = \mathbb{Q}(\mathrm{tr}(\Gamma))$ *is a number field,*
- $\mathrm{tr}(\Gamma)$ *is contained in its integers, and*
- *for any nonidentity embedding* $\varphi : k \to \mathbb{C}$*, the image* $\varphi(\mathrm{tr}(\Gamma))$ *is bounded.*

Anything isospectral to a compact hyperbolic **arithmetic** surface is a compact hyperbolic **arithmetic** surface

Selberg Trace formula: for a closed hyperbolic surface, knowledge of eigenvalue spectrum is equivalent to knowledge of the length spectrum, equivalent to knowledge of trace spectrum.

Theorem (Takeuchi)

A closed hyperbolic surface $\Gamma\backslash\mathbb{H}$ is arithmetic and derived from a quaternion algebra if and only if

- $k = \mathbb{Q}(\mathrm{tr}(\Gamma))$ *is a number field,*
- $\mathrm{tr}(\Gamma)$ *is contained in its integers, and*
- *for any nonidentity embedding* $\varphi : k \to \mathbb{C}$*, the image* $\varphi(\mathrm{tr}(\Gamma))$ *is bounded.*

Anything isospectral to a compact hyperbolic **arithmetic** surface is a compact hyperbolic **arithmetic** surface

Hyperbolic surfaces: audibility of arithmeticity

Selberg Trace formula: for a closed hyperbolic surface, knowledge of eigenvalue spectrum is equivalent to knowledge of the length spectrum, equivalent to knowledge of trace spectrum.

Theorem (Takeuchi)

A closed hyperbolic surface $\Gamma\backslash\mathbb{H}$ is arithmetic and derived from a quaternion algebra if and only if

- $k = \mathbb{Q}(\mathrm{tr}(\Gamma))$ *is a number field,*
- $\mathrm{tr}(\Gamma)$ *is contained in its integers, and*
- *for any nonidentity embedding $\varphi : k \to \mathbb{C}$, the image $\varphi(\mathrm{tr}(\Gamma))$ is bounded.*

Anything isospectral to a compact hyperbolic **arithmetic** surface is a compact hyperbolic **arithmetic** surface **to which it is commensurable**

Hyperbolic surfaces: audibility of arithmeticity

Selberg Trace formula: for a closed hyperbolic surface, knowledge of eigenvalue spectrum is equivalent to knowledge of the length spectrum, equivalent to knowledge of trace spectrum.

Theorem (Takeuchi)

A closed hyperbolic surface $\Gamma \backslash \mathbb{H}$ is arithmetic and derived from a quaternion algebra if and only if

- $k = \mathbb{Q}(\mathrm{tr}(\Gamma))$ *is a number field,*
- $\mathrm{tr}(\Gamma)$ *is contained in its integers, and*
- *for any nonidentity embedding* $\varphi : k \to \mathbb{C}$, *the image* $\varphi(\mathrm{tr}(\Gamma))$ *is bounded.*

Anything isospectral to a compact hyperbolic **arithmetic** surface is a compact hyperbolic **arithmetic** surface **to which it is commensurable**, (by a theorem due to Reid).