Spectral rigidity of some arithmetic surfaces

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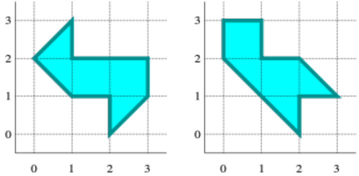

Figure: Some isospectral drums

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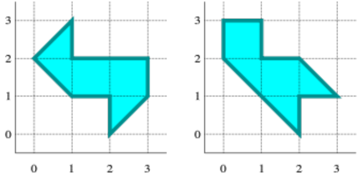

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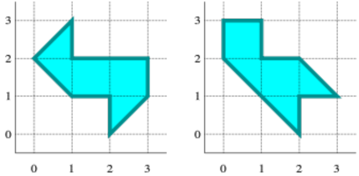

Figure: Some isospectral drums

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Figure: A very special (expensive) drum

Hurwitz surfaces: the most symmetric of drums

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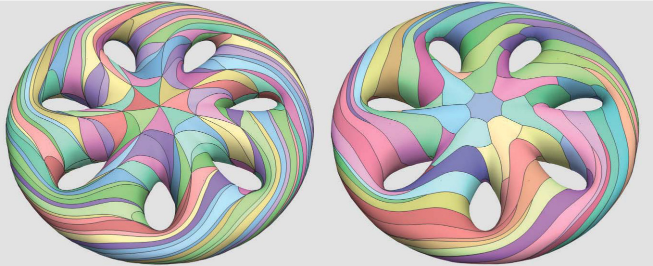

Figure: A genus 7 Hurwitz (Macbeath) surface [van Wijk, 2009]

Specific speculation

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Figure: not a genus 7 Hurwitz (Macbeath) surface

The following theorem answers in the affirmative infinitely many times (though, not all!)

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- For an ideal I of R_k, let $\Gamma(I)$ be the kernel of the reduction map $\Gamma(1) \to \mathrm{PSL}(2, R_k/I)$.

Theorem (McReynolds, Katz *in progress*)

Suppose that B has **type number 1**. *Then for all but finitely many primes $\mathcal{P}$ of R_k, and for all $n \geq 0$, the hyperbolic surface $\Gamma(\mathcal{P}^n)\backslash\mathbb{H}$ is spectrally rigid.*

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- Hurwitz surfaces arise from normal subgroups of the $2, 3, 7$ triangle group, which arises as the norm 1 units of a quaternion algebra over the field $\mathbb{Q}(2\cos 2\pi/7)$, which has class number 1. Hence the theorem applies: **principal congruence Hurwitz surfaces are spectrally rigid.**

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Traces, lengths, axes and commensurability.
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