REMARKS, OBSERVATIONS, KEEPSAKES

JUSTIN KATZ

- $\mathrm{GL}(n,R)$ is the set of ordered bases, as a free R module, of R^n.
- Elements of a quotient G/K are subsets of G. So $G/K \subset 2^{2^G}$.
- $M(2,\mathbb{Z}) \cap \mathrm{GL}(2,\mathbb{R}) = \{A \in M(2,\mathbb{Z}) : \det(A) \in \mathbb{Z}\}$, which is strictly larger than $\cup_{n\in\mathbb{Z}} \left[\begin{smallmatrix} n & 0 \\ 0 & n \end{smallmatrix}\right] \mathrm{GL}(2,\mathbb{Z})$.
- $\mathrm{SL}(2,\mathbb{Z})$ and $\Gamma(2)$ are, respectively, (finite index) subgroups of the mapping class group of a genus 1 surface $S_{1,0}$ and a genus 1 surface with 1 puncture, $S_{1,1}$.
- $\mathrm{GL}(2,\mathbb{Q})$ acts transitively on $\mathbb{Q}^2 - \{(0,0)\}$.
- The submonoid $\mathrm{GL}(2,\mathbb{Q}) \cap M(2,\mathbb{Z})$ acts transitively on $\mathbb{Z}^2$.
- There is a one to one correspondence between primitive geodesics on $\mathrm{PSL}(2,\mathbb{Z})\backslash H$ and ideal classes in real quadratic orders. To specify a class function supported on primitive hyperbolic elements of $\mathrm{PSL}(2,\mathbb{Z})$ is to define a function on all ideal class groups of all real quadratic orders.
- For a real quadratic field K, the idele class group is $K^\times\backslash\mathbb{I}_K$. The norm 1 units $K^\times\backslash\mathbb{I}_K^1$ is compact, and surjects onto every class group of every quadratic order contained in K. Consequently, any $\mathrm{SL}(2,\mathbb{Z})$ class function induces a $K^\times$ invariant function on $\mathbb{I}_K^1$ for every real quadratic field K of $\mathbb{Q}$, though it need not be continuous. Will it at least be integrable? When the class function is the character of a representation of $\mathrm{SL}(2,\mathbb{Z})$ which factors through a finite index subgroup, one knows at the very least that it takes only finitely many values.
- For any field K, any order $\mathcal{O}$ of K,and any fractional ideal J of $\mathcal{O}$, and any integral ideal a of $\mathcal{O}$, the quotient K/aJ^{-1} is an $\mathcal{O}$ module, and is (as an abelian group) isomorphic to $\mathbb{Q}^2/\Lambda$ for some lattice Λ.
- For a real quadratic field K, with real embeddings i_1 and i_2, the most natural way to embed K in $\mathbb{R}$ is via $x \mapsto (i_1(x), i_2(x))$. It is precisely in this embedding that the norm one units $\mathcal{O}^\times$ act by translation along hyperbolas with standard basis vectors as axes. There is an element of $\mathrm{GL}(2,K) \subset \mathrm{GL}(2,\mathbb{R})$ which conjugates the image of this embedding to $\mathbb{Q}^2$.
- The fundamental domain for the cyclic group generated by a hyperbolic element η of $\mathrm{SL}(2,\mathbb{R})$ acting linearly on $\mathbb{R}^2$ is the union of half open sectors spanned by v and ηv, and ηv and v for any nonzero v away from the eigenspaces of η.
- A subset of $\mathbb{Z}_p$ is open/closed if and only if it is the preimage under reduction mod p^n of some subset A of $\mathbb{Z}_p/p^n\mathbb{Z}_p$. Given any subset A of $\mathbb{Z}_p$, one can certainly look at its image under reduction mod p^n, its preimage under each such reduction as an open/closed approximation.
- Kronecker Limit formula for real quadratic K of (fundamental) discriminant d (from "On a zeta function associated to a Quadratic order"):

$$\zeta_K(s) = \frac{2h_K \log \varepsilon_d}{\sqrt{d}(s-1)} - \frac{1}{6\sqrt{d}} \sum_{Q \in PBQF(d)/\Gamma} \int_{-\log\varepsilon_d}^{\log\varepsilon_d} \log(y_Q(v)^6 |\Delta(z_Q(v))|)\, \mathrm{d}v$$
$$+ \frac{h_K \log \varepsilon_d}{\sqrt{d}}(4\gamma - \log d) + O(s-1)$$

where Δ is the modular discriminant and $z_Q(v) = \frac{\alpha_Q - i\alpha_Q' e^{-v}}{1 - ie^{-v}}$ with α_Q, α_Q' the roots of dehomogenized Q.

- Tanaka duality manifesting itself for groups: Let Γ be a residually finite group, and A a commutative ring. Let $\mathrm{Rep}_A(\Gamma)$ be the category of A-representations of Γ. Then a morphism $u : \Gamma_1 \to \Gamma_2$ between groups should induce an isomorphism between profinite completions if and only if the dual (restriction) map $u* : \mathrm{Rep}_A(\Gamma_2) \to \mathrm{Rep}_A(\Gamma_1)$ is an equivalence of categories, for every A.
- There's a hierarchy of information vis-a-vis understanding the rep theory of a group. For now let G be finite. The first tier is knowing the number n of irreducible representations V_i of G and their degrees d_i, this is enough to give the polynomial ring generated by reps of G a natural grading. The second tier is knowing the splitting data of tensor products: $V_i \otimes V_j = \bigoplus_k m_{ijk} V_k$. This, apparently, is enough to give the whole character table of G. Note, this only tells us about the character of $V_i \times V_j$; it is not enough to tell us what the character of V_i times V_j is.

1 Reps and stuff

From Mostow, on "Representative functions": Let G be a group, V a vector space, $\rho : G \to \mathrm{Aut}(V)$ a representation over some field k.

- A representation, above all else, is an action of a group by automorphisms of an abelian group.
- Associate to each pair of vectors $v \in V$, $w \in V^*$, the k-valued function $\rho_{v,w}(g) = \langle \rho(g)v, w\rangle$, and call it a coefficient function, and write $[\rho]$ for the k-linear span of such functions.
- When V is finite dimensional, so is $[\rho]$. As a space of functions on G, $[\rho]$ has commuting left and right G actions, by translation. Set $\mathrm{Rep}_k[G]_X = \cup_{\rho \in X}[\rho]$ for some collection of representations X of G. Take as default, X the collection of finite dimensional reps over k. Then a k valued function f on G is in $\mathrm{Rep}[G]$ if and only if the k linear span of its G translates is finite dimensional.
- Note $\mathrm{Rep}[G]$ is a unital k-algebra.
- If V_ρ is the underlying space of some finite dimensional rep ρ of G, then V_ρ embeds as a G module into $[\rho] \oplus ... \oplus [\rho]$ ($\dim V_\rho$ summands), via the G map $v \mapsto \rho_{v,w_1} + ... + \rho_{v,w_{\dim(V_\rho)}}$, where $w_1, ..., w_{\dim V_\rho}$ is a basis for V_ρ^*.
- Let τ denote the right translation rep of G on $\mathrm{Rep}_k(G)$. Suppose P is a bi-invariant k-subalgebra of $\mathrm{Rep}_k(G)$ containing the constant function 1, and α an algebra endomorphism of P, a) preserving fixing the constant function 1, b)commuting with $\tau(G)$. Then every subspace of P which is $\tau(G)$ stable is also α stable.
- If, further, P is invariant under the involution $f \mapsto f'$ (inversion), then α preserves the constant functions, and α is actually an automorphism with inverse given by $(\alpha^{-1}f)(s) = \alpha(f \cdot s)'(1)$.
- It follows that there is a bijection between proper automorphisms of $\mathrm{Rep}_k(G)$ and $\mathrm{Hom}_{k-alg}(\mathrm{Rep}_k(G), k)$. Call this common space $A(G)$ or $\mathrm{Spec}(\mathrm{Rep}_k(G), k)$. One should regard $A(G)$ as a sort of underlying space, and $\mathrm{Rep}_k(G)$ as the algebra of coordinate functions on it. Note, $A(G)$ is actually a pro-algebraic group: it's the projective limit of algebraic groups.
- A homomorphism $\theta : G' \to G$ induces a k algebra homomorphism $\theta^t : R[G] \to R[G']$ (a transpose), and thus a homomorphism $\hat{\theta} : A(G') \to A(G)$.
- Let L be a LAG/$\mathbb{Q}$, and Γ a subgroup such that each everywhere defined rational $\mathbb{Q}$ function on L has bounded denominators on Γ. Let ρ be a $\mathbb{Q}$ rep of L on the $\mathbb{Q}$ space V. Then Γ stabilizes a lattice in $V_{\mathbb{Q}}$.

- The proalgebraic topology is akin to looking at the congruence topology relative to all possible arithmetic realizations.

2 Shintani stuff

From "The p-adic Shintani modular symbol and evil Eisenstein series."

- Let V denote $\mathbb{Q}^2$. Write $S(V) = S(\mathbb{A}_V^(\infty))$ as the group of test functions on the finite adeles of V. Concretely, $S(V)$ consists of the functions $f : \mathbb{Q}^2 \to \mathbb{Z}$ that are supported on a lattice, and also periodic with respect to some lattice, hence factor through a finite quotient of their support.
- Key point: continuity of such functions comes from factoring through a finite quotient of support, while compact supportedness comes from being supported on a single lattice. For example, the constant 1 function on $\mathbb{Q}^2$ is continuous but not compactly supported, hence not Schwartz. By contrast, the indicator function for $\mathbb{Z}[1/p]^2$ is neither continuous nor compactly supported.
- Shintani's setup: $G = \mathrm{GL}(2, \mathbb{Q})^+$ and X the space of rational symmetric two by two matrices. Let $\hat{G}$ and $\hat{X}$ be their congruence completions. Then $\hat{G}$ consists of those $g \in \mathrm{GL}(2, \mathbb{A}^f)$ with $\det g \in \mathbb{Q}_{>0}$. Here, $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ and $\hat{\Gamma}$ its closure in $\hat{G}$.
- There is a natural bijection $\hat{\Gamma}\backslash\hat{X}$ with $\Gamma\backslash X$ and $\hat{\Gamma}\backslash\hat{G}/\hat{\Gamma}$ with $\Gamma\backslash G/\Gamma$. Note that, under this identification, a function is compactly supported on the latter if and only it is finitely supported on the former.
- Picking a haar measure on $\hat{G}$ normalized so $\int_{\hat{\Gamma}} \mathrm{d}g = 1$, the action of the hecke algebra $H(\Gamma, G)$ on $C^\infty(\Gamma\backslash X)$ is by convolution. For $x \in X$, let Γ_x be stabilizer. Fix base point $x_o \in X$ which is primitve of conductor 1. Let $\mathrm{d}\mu$ be compatible measure on $\hat{\Gamma}\backslash\hat{X}$, normalized so $\Gamma\hat{\cdot} x \subset \hat{X}$ has measure 1.
- Then for any other $x \in X$, and picking $g_x \in G$ so that $g_x \cdot x = x_o$, one has

$$\mu(x) := \int_{\Gamma\hat{\cdot}x} \mathrm{d}\mu = |\Gamma_{x_o} : g_x \Gamma_x g_x^{-1}].$$

- Suppose $x \in X$ has conductor f, and discriminant d. Then $\mu(x) = [\mathcal{O}^1_{d,1} : \mathcal{O}^1_{d,f}]$.
- There is a bijection:

$$\{\alpha \in a | N(\alpha) > 0(\alpha, f) = 1\}/\mathcal{O}^1_f \approx \{b \in C^{-1} | b + ff\mathcal{O}_f = \mathcal{O}_f\}$$

  where a is an $\mathcal{O}_f$ ideal in the class $C \in Pic(\mathcal{O}_f)$. The identification is $\alpha \mapsto \alpha\overline{a}/N(a)$ where $\bar{\cdot}$ is galois conj.
- The fourier–eisenstein transform: let φ be a Γ invariant complex valued function on $\Gamma\backslash X$ which is supported on finitely many orbits. Set $F(\varphi)(s_1, s_2) = \int_{\hat{X}} \varphi(x)E(x; s_1, s_2)\,\mathrm{d}\mu(x)$, where $E(x, s_1, s_2) = \mu(x)^{-1} \sum_{v\in\mathbb{Z}^2/\,\mathrm{SO}(Q_x)} |Q_x(v)|^{-s_1-1/2} |\det x|^{-s_2+1/4}$ where Q_x is the BQF defined by X.
- When φ is supported on finitely many orbits, the fourier–eisenstein transform of φ is just a finite linear combination of eisenstein series: $F(\varphi)(s_1, s_2) = \sum_{x\in\Gamma\backslash X} \varphi(x)[\mathcal{O}^1 : \mathcal{O}^1_{f(x)}]E(x, s_1, s_2)$
- K is real quadratic field with discriminant D, let $X_{D,1} = X$ be two by two rational symmetric matrices with determinant $D/4$. Set $K' = K - \mathbb{Q}$. The map $\alpha \mapsto S_\alpha := \frac{\sqrt{D}}{\alpha - \overline{\alpha}} \begin{bmatrix} 1 & -\operatorname{tr}\alpha/2 \\ -\operatorname{tr}\alpha/2 & N\alpha \end{bmatrix}$ is a bijection K' with X.

- Letting $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ act on α by $g \cdot \alpha = \frac{d\alpha - c}{-b\alpha + a}$, one has $g \cdot S_\alpha = S_{g \cdot \alpha}$. This gives an identification of $C^\infty(\Gamma\backslash X)$ with $C^\infty(\Gamma\backslash K')$.
- Following Arakawa, define for $\alpha \in K'$: $\xi(s,\alpha) = \sum_{n=1}^\infty \frac{\cot \pi n \alpha}{n^s}$. The $\xi(s,\alpha)$ cvges absolutely for $\mathrm{Re}(s) > 1$, has meromorphic continuation to $\mathbb{C}$ with simple pole at $s = 1$. Let $c_{-1}(\alpha)$ denote the residue at $s = 1$.
- Apparently: $c_{-1}(\alpha)$ is left Γ invariant, and is a common $H(\Gamma, G)$ eigenfunction: for $f \in H(\Gamma, G)$ one has $f \cdot c_{-1} = \hat{f}(-1/2)c_{-1}$.

3 An effective grunwald wang

- K number field, $C_K = \mathbb{I}_K/K^\times$ idele class group. Grunwald: given a finite set S of places of K, and a family of characters χ^ν of $K_\nu^\times$ for $\nu \in S$ of orders m_ν (local characters), there exists a continuous character χ of C_K of finite order (global character) whose local component χ_ν at each $\nu \in S$ restrict to χ^ν on the copy of $K_\nu^\times$ in C_K. T
- Given the local data, there are infinitely many global characters satisfying Grunwald's conclusion. Most are highly ramified. The game: control the ramification of possible such.
- For χ_ν a continuous character of $K_\nu^\times$, define arithmetic conductor: $N(\chi_\nu)$ to be 1 if χ_ν is unramified or ν is archimedean, and q_ν^n when n is the smallest integer such that $(1+p_\nu^n)^\times \subset \ker(\chi_\nu)$, for p_ν the unique maximal ideal in $O_n u$. For global characters, take a product.
- A theorem (S version of multiplicity one for GL(1)): Let χ be a nontrivial global character of C_K of finite order. Set $A(\chi, S) = d_K N(\chi) N_S$. Then there exists a place ν of K with:
  - p_ν is not in S
  - $\chi_\nu \neq 1$ and is not ramified
  - $\log N(p_\nu)\ ll \log A(\chi, S)$
  - $N(p_\nu) \ll_{\varepsilon,K} N(\chi)^{1/2+\varepsilon} N_S^\varepsilon$ for every $\varepsilon > 0$
  - With GRH, $N(p_\nu) \ll (\log A(\chi, S))^2$.

4 Useful identities

- For $A \in M(2,R)$: $\mathrm{tr}(A^2) = (\mathrm{tr}\, A)^2 - 2\det(A)$.
- For $u, v \in \mathrm{SL}(2,R)$: $\mathrm{tr}(uv) = \mathrm{tr}(u)\,\mathrm{tr}(v) - \mathrm{tr}(u^{-1}v)$

5 notes on Goldfeld's textbook

Let (π, V) be an automorphic cuspidal representation of $\mathrm{GL}(2,\mathbb{A}_\mathbb{Q})$. Fix $f_c1, f_2 \in V$ and set, for $g \in \mathrm{GL}(2,\mathbb{A}_\mathbb{Q})$:

$$\beta(g) := \int_{\mathrm{GL}(1,\mathbb{A}_\mathbb{Q})\,\mathrm{GL}(2,\mathbb{Q})\backslash\,\mathrm{GL}(2,\mathbb{A}_\mathbb{Q})} f_1(hg)\overline{f_2(h)}\,\mathrm{d}^\times h,$$

Let φ be a Bruhat-Schwarz function. Define for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$ define

$$Z(s,\Phi,\beta) := \int_{\mathrm{GL}(2,\mathbb{A}_\mathbb{Q})} \Phi(g)\beta(g)|\det(g)|^{s+1/2}\,\mathrm{d}^\times g.$$

- An idea: take π to be the right regular representation on $L^2_0(\mathrm{GL}(2,\mathbb{Q}) \mod \mathrm{GL}(2,\mathbb{A}_{\mathbb{Q}}))$ as the automorphic cuspidal representation. Then the game becomes: picking vectors f_1 and f_2 which decay fast enough (on the spectral side)to make these integrals coverge.

6 notes on (local) densities

Every element of $S \in \mathrm{Sym}(2,\mathbb{Z}_p)$ gives rise to a binary quadratic form, and yields a function $\mathbb{Z}_p \to \mathbb{C}$ (really to $\mathbb{Q}$), $x \mapsto \alpha(S,x)$ the **local representation density of x by S**. Explictly, letting

$$A_t(S,x) = |\{X \in M(2,\mathbb{Z}_p/p^t\mathbb{Z}_p) : S[X] = x \mod p^t\mathbb{Z}_p\}|,$$

one has

$$\alpha(S,x) = \lim_{t\to\infty} p^{-t} A_t(S,x)$$

It turns out that this local density has an integral expression. Letting

$$W(S,x) := \int_{\mathbb{Q}_p}\int_{\mathbb{Z}_p^2} \psi(bS[v])\psi(-xb)\,\mathrm{d}v\,\mathrm{d}b,$$

one has $W(S,x) = \alpha(S,x)$ for all $S \in \mathrm{Sym}(2,\mathbb{Z}_p)$ and all $x \in \mathbb{Z}_p$.

Note: as a function of S, the value of $W(S,x)$ depends only on the $\mathrm{GL}_2(\mathbb{Z}_p)$ orbit of class of S.

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- Any open subgroup of of $\mathrm{PGL}(2,k)$, or $\mathrm{SL}(2,k)$, or $\mathrm{GL}(2,k)$, contains unipotent elements. Consequently, if Γ is arithmetic fuchsian (cocompact or not), the p-adic closure of Γ will contain unipotent elements at all primes p unramified in the ambient quat alg.
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8 Notes on: Clifford theory for representations, by Karpilovsky

- In this section, all rings are unital.
- For a (finite) family of rings $R_1, ..., R_n$, and a given ring R, one has $R \approx R_1 \times \cdots \times R_n$ (as rings) if and only if there exist pairwise orthogonal central idempotents $e_1, \cdots, e_n \in R$ such that
  a) $1 = e_1 + \cdots + e_n$, and
  b) $R_i \approx Re_i$ (as rings), for all i.
- For a ring R, denote the left R module with underlying abelian group R and R action given by multiplication on the left by ${}_RR$ (this is the left regular R module).
- If M is an (S,R) bimodule, and N a left R module, then $M \otimes_R N$ (which a priori is a priori only an abelian group) is naturally a left S module.
- To summarize: tensor product is a functor from the category of pairs (left R modules,right R) modules to the category of abelian groups.
- In particular, if R is a commutative ring, and M, N are R modules, then $M \otimes_R N$ is canonically an R module.

- Let M, M', M'' be right R modules, and N, N', N'' be left R modules. Suppose we have right R modules maps $u : M' \to M$ and $v : M \to M''$ such that v surjects M onto M'', and v is 0 on the image of M' in M under u (i.e. the sequence $M' \to M\ toM'' \to 0$ is exact), and similarly for maps $s : N' \to N$ and $t : N \to N''$. Then the sequence (of abelian groups)

$$M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0$$

is exact. Similarly for $M \otimes_R -$
- In particular, the homomorphism (of abelian groups) $v \otimes t : M \otimes_R N \to M'' \otimes_R N''$ is surjective, with kernel $\mathrm{Im}(u \otimes 1) + \mathrm{Im}(1 \otimes s)$
- Let M be an (S, R) bi-module. Then $M \otimes_R R \approx M$ as (left) S modules.
- If R is a subring of a ring S, and M is a free left R module with R basis $e_1, \cdots, e_n$, then $S \otimes_R M$ is a free left S module with S basis $\{1 \otimes e_1, \cdots, 1 \otimes e_n\}$.
- For M a right R module, and A a left ideal of R. Then $M \otimes_R A \to M$ via $m \otimes a \mapsto ma$ is the 'canonical' map. Its image is the (additive) subgroup MA of M consisting of sums $\sum_i m_i a_i$ for $m_i \in M$ and $a_i \in A$.
- The property that a right R module M is flat is equivalent to the canonical map $M \otimes_R A \to MA$ being an isomorphism (of abelian groups) for every left R ideal A.
- Definition: for a commutative ring R, an R-algebra is a ring A which is also an R module, in such a way that $r(xy) = (rx)y = x(ry)$ for all $x, y \in A$ and $r \in R$.
- Tensor products of R algebras are again R algebras. If A_1 and A_2 are R algebras with units e_1 and e_2 respectively, then the unit of $A_1 \otimes_R A_2$ is $e_1 \otimes e_2$. The maps $f_1 : A_1 \to A_1 \otimes_R A_2$ and $f_2 : A_2 \to A_1 \otimes_R A_2$ such that $f_1(x) = x \otimes e_2$ and $f_2(x) = e_1 \otimes x$ are homomorphisms of R algebras such that $f_1(a_1)f_2(a_2) = f_2(a_2)f_1(a_1)$ for all $a_i \in A_i$. The tensor product is the universal such object: so tensor products are universal in the *domain*.
- For any ring R, write $M_n(R)$ for the ring of $n \times n$ matrices with coefficients in R, and for $i, j \leq n$, let e_{ij} be the elt in $M_n(R)$ which is 1 in the i, jth spot and 0 elsewhere. The elements e_{ij} are referred to as the *matrix units*.
- Key point: the matrix units $\{e_{ij} : 1 \leq i, j \leq n\}$ satisfy the following properties:
  (1) $e_{ij}e_{ks} = 0$ if $j \neq k$ and $e_{ij}e_{ks} = e_{is}$ if $j = k$.
  (2) $\mathrm{id} = 1 = e_{11} + \cdots + e_{nn}$
  (3) The centralizer of $\{e_{ij} : 1 \leq i, j \leq n\}$ in $M_n(R)$ is $R\,\mathrm{id} = R$.
  (4) $R \approx e_{11}M_n(R)e_{11}$ (as rings).
- In fact, for any ring S containing elements v_{ij} for $1 \leq i, j \leq n$ satisfying the first three properties above, let R be the centralizer of $\{v_{ij} : 1 \leq i, j \leq n\}$ in S. Then the map $M_n(R) \to S$ defined by $(a_{ij}) \mapsto \sum a_{ij}v_{ij} \in S$ is an isomorphism of R algebras such that $R \approx v_{11}Sv_{11}$ (as rings).
- For any R module V and $n \geq 1$, one has $\mathrm{End}_R(V^n) \approx M_n(\mathrm{End}_R(V))$.
- Key point: the map $W \mapsto W^n$ is an isomorphism of the lattice of R submodules W of V onto the lattice of $M_n(R)$ submodules of V^n.
- $\mathrm{End}_{M_n(R)}(V^n) \approx \mathrm{End}_R(V)$
- The map $V \mapsto V^n$ induces a bijective correspondence between the isomorphism classes of R modules and $M_n(R)$ modules. The inverse map is given by $W \mapsto e_{11}W$.
- For an ideal I of R, set $M_n(I) = \{(a_{ij}) \in M_n(R) : a_{ij} \in I \,\mathrm{for}\, 1 \leq i, j \leq n\}$. Note, this is not a subring of $M_n(R)$ in the author's definition: unless $I = R$, $M_n(I)$ will not have a unit. It is, however, an $M_n(R)$ submodule of $M_n(R)$. That is, it is a (left and right) $M_n(R)$ ideal.
- In fact, the correspondence $I \mapsto M_n(I)$ is a bijection between ideals of R and $M_n(R)$. In particular, R is simple if and only if $M_n(R)$ is.
- The correspondence is natural: $M_n(R)/M_n(I) \approx M_n(R/I)$ (as rings)
- If $R = I_1 \oplus \cdots \oplus I_s$ is a two sided decomposition of R, then $M_n(R) = M_n(I_1) \oplus \cdots \oplus M_n(I_s)$.

- An R module is artinian if all descending chains of submodules terminates, and is noetherian if all ascending chains of submodules terminate. Example: $\mathbb{Z}$ is noetherian but not artinan, since $\cdots \leq p^n\mathbb{Z} \leq p^{n-1}\mathbb{Z} \leq \cdots \leq p\mathbb{Z} \leq \mathbb{Z}$ does not terminate.
- Say a ring R is artinian/noetherian if the left regular R module ${}_RR$ is artinian/noetherian.
- Say V is finitely generated if there exist $v_1, \cdots, v_n$ so that $V = Rv_1 + \cdots Rv_n$, and finitely co-generated if for every family $\{V_i : i \in I\}$ of submodules of V with $\bigcap_{i\in I} V_i = 0$, there exists a finite subset J of I such that $\bigcap_{j\in J} V_i = 0$.
- Artinian modules are characterized by the property: every nonempty set of submodules has a minimal element. Alternatively, every quotient is finitely cogenerated. Consequently, every nonzero artinian module has an irreducible submodule.
- noetherian modules are characterized by the property: every nonempty set of submodules has a maximal element. ALternatively, every submodule is finitely generated.
- Key point: a nonzero R module has a composition series if and only if it is both artinian and noetherian.
- Let W be an R submodule of an R module V. Then V is artinian/noetherian if and only if both W and V/W are artinian/noetherian.
- For finitely generated modules V over R: if R is artinian/noetherian then V is artinian/noetherian.
- If R is a ring and V a nonzero R module which is either artinian or noetherian. Then V is a direct sum of indecomposable submodules.
- If V is completely reducible, then every submodule is a homomorphic image of V, and every homomorphic image of V is isomorphic to a submodule of V. If $V \neq 0$ then V contains an irreducible submodule.
- For a nonzero R module V, the following are equivalent: 1) V is completely reducible, 2) V is the *direct* sum of irreducible submodules, 3) V is the sum (not necessarily direct) of irreducible submodules.
- Definition: a completely reducible R module V is *homogeneous* if it can be written as a sum of mutually isomorphic irreducible submodules. The sum of irreducible submodules isomorphic to a given irreducible one is called a homogeneous component.
- A submodule W of an R module V is called *fully invariant* if it is mapped into itself by all elts of $\mathrm{End}_R(V)$. In a completely reducible R module, a submodule is fully invariant if and only if it is a sum of a set of homogeneous components.
- If V is completely reducible and can be written as $V = \bigoplus_{j\in J} W_j$ for homogeneous components W_j, then $\mathrm{End}_R(V) \approx \prod \mathrm{End}_R(W_j)$ (as rings).

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- The radical of an R module V, written as $J(V)$ is the intersection of all maximal submodules of V (if there are no such submodules, the 'empty intersection' is V). If V is finitely generated and nonzero, then $J(V) \neq V$.
- The jacobson radical of a ring R is the radical of the left regular module ${}_RR$. Equivalently, this is the intersection of all maximal left ideals of R. Say R is semisimple if $J(R) = 0$.
- The annihilator of an R module V is the kernel of the module-defining-map $R \to \mathrm{End}(V)$. That is, $\mathrm{Ann}(R)$ is the collection of $r \in R$ which act as the zero operator on V. Then $\mathrm{Ann}(V)$ is an ideal in R, and V is naturally an $R/\mathrm{Ann}(V)$ module.
- Call an R module faithful if $\mathrm{Ann}(V) = 0$. That is, if for all $r \in R$, there is some $v \in V$ such that $rv \neq 0$.
- An R module is irreducible if and only if $V \approx R/X$ (as R modules) for some maximal left ideal X of R.
- Call an ideal I of R primitive if R/I (a ring) has a faithful irreducible module. Then I is primitive if and only if I is the annihilator of an irreducible R module.

- Homomorphisms of modules map jacobson radicals to jacobson radicals (surjectivity if and only if the homomorphism is surjective with kernel contained in jacobson radical of the image.)
- If $W \leq V$ as R modules, then $J(W) \leq J(V)$ and $(J(V)+W)/W \leq J(V/W)$.
- If $W \leq J(V)$ as R modules, then $J(V/W) = J(V)/W$.
- For any ring R and $x \in R$, then $x \in J(R)$ if and only if for all $r \in R$, $1 - rx$ is a left unit of R. This implies $J(R)$ contains no nonzero idempotents
- Key computation: suppose $I \leq R$ is a left nil ideal of R. Then for $x \in I$ and any $r \in R$, we have $rx \in I$, so $(rx)^n = 0$ for some n. In particular, the sequence of sums $1 + \cdots + (rx)^k$ stabilizes for $k \geq n$. The stable value is the left inverse of $1 - rx$.
- For an artinian ring R, $J(R)$ is nilpotent.
- For any ring R and any $n > 0$, $J(M_n(R)) = M_n(J(R))$.
- Schur: if V is an irreducible R module, then $\mathrm{End}_R(V)$ is a divison ring.
- For V an R module and $e \in R$ an idempotent. Then $\mathrm{Hom}_R(eR, V) \approx Ve$ as additive groups. Similarly for a right R module. Proof: the isomorphism is $f \mapsto f(e)$.

8.2 1.6: group algebras

- In this section R is a commutative ring. Key fact: if $H \leq G$ and T is a right/left transversal for H in G, then RG is a free left/right RH module, freely generated by T.
- For any groups G, H, $R(G \times H) \approx RG \otimes_R RH$.
- Key lemma: Let $H \leq G$ finite index subgroup, and $T \subset G$ a left transversal for H in G containing 1. Let V be an RG module, define $f : (V_H)^G \to V$ (subscript is restriction, superscript is induction) by $f(\sum_{t\in T} t \otimes v_t) = \sum_{t \in T} tv_t$ (for $v_t \in V$). Then f is a surjective homomorphism of RG modules, and $\ker(f) \leq (V_H)^G$ is a direct summand of $((V_H)^G)_H$.
- Key lemma: for H, G, T as above, and V, W both RG modules, then for any $f \in \mathrm{Hom}_{RH}(V_H, W_H)$ the map $f^* : V \to W$ defined by $f^*(v) = \sum_{t \in T}(tft^{-1})(v)$ is an RG homomorphism. Furthermore, a different choice of T does not change f^*.

9 Notes on Traces of Hecke Operator

- Let R denote the right regular representation of $G(\mathbb{A})$ on $L^2(\omega)$, the space of square integrable functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ which transform by ω on $Z(\mathbb{A})$. So $R(g)f(x) = f(gx)$ for $f \in L^2(\omega)$. The cuspidal subspace $L^2_o(\omega)$ consists of those $f \in L^2(\omega)$ such that $\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} f(ng) = 0$ for almost all $g \in G(\mathbb{A})$. Set $R_o = R(\cdot)|_{L^2_o(\omega)}$.
- Let $C_c(G(\mathbb{A}), \omega^{-1})$ be the subspace of continuous $f : G(\mathbb{A}) \to \mathbb{C}$ with compact-mod-center support such that $f(zg) = \omega^{-1}(z)f(g)$ for all central z. Extend the definition of R to $C_c(G(\mathbb{A}), \omega^{-1})$ by $R(f)\varphi(g) = \int_{\overline{G}(\mathbb{A})} f(x)\varphi(gx)\,\mathrm{d}x$.
- Then $R_o(f)$ is trace class for all such f (though $R(f)$ may not be).
- The following is the Arthur/Selberg trace formula for GL_2: where $v(g) = H(g) + H(wg)$ is the weight function, with H the height function and w the weyl element. The unipotent term is given by $F(a) = \int_K f(k^{-1}([\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}]\,k)\,\mathrm{d}k$, and $Z_F(s) = \int_{\mathbb{A}^\times} F(a)|a|^s\,\mathrm{d}^\times a$ is the tate zeta
- The remaining terms are 'noncuspidal.'
- Some observations regarding the congruence subgroups $\Gamma(N)$, $\Gamma_0(N)$, and $\Gamma_1(N)$: One has $\Gamma(1)/\Gamma(N) \approx \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, that $\Gamma_0(N)/\Gamma_1(N) \approx (\mathbb{Z}/N\mathbb{Z})^\times$ and that $\Gamma_1(N)/\Gamma(N) \approx \mathbb{Z}/N\mathbb{Z}$.
- Set $\psi(N) = |\Gamma(1) : \Gamma_0(N)|$. Then $\psi(N) = N\prod_{p|N}(1 + 1/p)$.

- For $g \in G(\mathbb{R})^+$ (elts of $\mathrm{GL}_2(\mathbb{R})$ with positive determinant), write $j(g,z) = (cz+d)\det(g)^{-1/2}$. The action of $G(\mathbb{R})^+$ on $\mathfrak{H}$ by linear fractional transformations can be written in terms of the natural action of $G(\mathbb{R})^+$ on $\mathbb{C}^2$ via $g\left[\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right] = \det(g)^{1/2} j(g,z) \left[\begin{smallmatrix} gz \\ 1 \end{smallmatrix}\right]$
- For $z \in \mathfrak{H}$, one has $\mathrm{Im}(gz) = |j(g,z)|^{-2}\,\mathrm{Im}(z)$ and $j(g'g,z) = j(g',gz)j(g,z)$.
- For a dirichlet character $\omega' : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^\times$, there is a unique positive integer $N_{\omega'}$ minimal among those such that ω' factors through $\mathbb{Z}/N_{\omega'}\mathbb{Z}$. Call this the conductor of ω'.
- For a dirichlet character ω' mod N, extend first to $\mathbb{Z}/N\mathbb{Z}$ by 0 off of $(\mathbb{Z}/N\mathbb{Z})^\times$, and then to $\mathbb{Z}$ by $\mathcal{N}\mathbb{Z}$ invariance.
- Then ω' determines a character of $\Gamma_0(N)$ by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mapsto \omega'(d)$.
- $G(\mathbb{R})^+$ acts on functions $h : \mathfrak{H} \to \mathbb{C}$ from the right (with action dependent on a choice of $k \geq 0$: $h_\delta(z) = j(\delta,z)^{-k} h(\delta z)$. A weight k modular form is a fixed vector under the aforementioned action. Write $W_k(\Gamma)$ and $M_k(\Gamma)$ the weak modular forms and modular forms respectively.
- For a congruence subgroup Γ and a finite order character $\chi : \Gamma \to \mathbb{C}^\times$, set $\Gamma_\chi = \ker(\chi)$. Let $W_k(\Gamma,\chi) \subset W_k(\Gamma_\chi)$ be those h satisfying $h_\gamma = \chi(\gamma)^{-1}h$.
- For $\delta \in G(\mathbb{R})^+$ let $\chi_\delta : \delta^{-1}\Gamma\delta \to \mathbb{C}^\times$ be defined by $\chi_\delta(\alpha) = \chi(\delta\alpha\delta^{-1})$ for $\alpha \in \delta^{-1}\Gamma\delta$. Then $h \mapsto h_\delta$ is an isomorphism $W_k(\Gamma,\chi) \to W_k(\delta^{-1}\Gamma\delta, \chi_\delta)$. When $\Gamma = \Gamma_0(N)$ and $\chi = \omega'$ write $W_k(N,\omega')$.
- In this case, $\Gamma_1(N) \subset \Gamma_{\omega'} \subset \Gamma_0(N)$. Set $M_k(N,\omega') = W_k(N,\omega') \cap M_k(\Gamma_1(N))$. Generally speaking: $W_k(\Gamma_1(N)) = \bigoplus_{\omega'} W_k(N,\omega')$ with the sum over dirichlet characters ω' of conductor dividing N.
- Let N be the unipotent radical of B, the borel of upper triangular matrices. Then for $\Gamma \leq \Gamma(1)$ there exists a positive integer M such that $\Gamma \cap N(\mathbb{Z}) = N(M\mathbb{Z})$.
- If $A = \{\delta\}$ is transversal of $\Gamma\backslash\Gamma(1)$ in $\Gamma(1)$, then $\{\delta\infty : \delta \in A\}$ contains a set of representatives of the cusps of Γ.
- Suppose $\Gamma(N) \leq \Gamma$. Then since $\Gamma(N)$ is normal in $\Gamma(1)$, one has $\tau\left[\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right]\tau^{-1} \in \Gamma(N) \leq \Gamma$ for all $\tau \in \Gamma(1)$.
- Given $\delta \in G(\mathbb{Q})^+$ let $s = \delta\infty$, and choose $\tau \in \Gamma(1)$ so that $\tau\infty = s$. Then $\tau^{-1}\delta\infty = \infty$, so $(\delta^{-1}\tau\left[\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right]\tau^{-1}\delta) \in (\delta^{-1}\Gamma\delta)_\infty$ is upper triangular. In fact, $\delta^{-1}\tau\left[\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right]\tau^{-1}\delta = \left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]$ for some $b \in \mathbb{Q}^\times$.
- Ultimately, for Γ a congruence subgroup and $\delta \in G(\mathbb{Q})^+$, there exists an $M_\delta = M_\delta(\Gamma) \in \mathbb{Q}_{>0}$ so that $N(\mathbb{Q}) \cap \delta^{-1}\Gamma\delta = N(M_\delta\mathbb{Z})$.
- If $\alpha \in M_2(\mathbb{Z})$ with $\det(\alpha) = m \neq 0$, then $m\alpha^{-1} \in M_2(\mathbb{Z})$.
- If $\{\gamma_i\}$ is a set of coset representatives for $(\alpha^{-1}\Gamma\alpha \cap \Gamma)\backslash\Gamma$, then $\Gamma\alpha\Gamma = \bigsqcup_i \Gamma\alpha\gamma_i$.
- If $|\Gamma\backslash\Gamma\alpha\Gamma| = |\Gamma\alpha\Gamma/\Gamma|$ then there is a common set of representatives in $\Gamma\alpha\Gamma$ for $\Gamma\backslash\Gamma\alpha\Gamma$ and $\Gamma\alpha\Gamma/\Gamma$.
- For $g \in G(\mathbb{Q})$, $g\left[\begin{smallmatrix} \mathbb{Z} \\ \mathbb{Z} \end{smallmatrix}\right]$ is the lattice spanned by the columns of g.
- For $g \in G(\mathbb{Q})$, then $g \in \Gamma_0^{\pm}(N)$ if and only if both $g\left[\begin{smallmatrix} \mathbb{Z} \\ \mathbb{Z} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbb{Z} \\ \mathbb{Z} \end{smallmatrix}\right]$ and $g\left[\begin{smallmatrix} \mathbb{Z} \\ N\mathbb{Z} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbb{Z} \\ N\mathbb{Z} \end{smallmatrix}\right]$
- Let $L = \left[\begin{smallmatrix} \mathbb{Z} \\ \mathbb{Z} \end{smallmatrix}\right]$. If $\alpha \in M_2(\mathbb{Z})$ with $\alpha L \subset L$ and $\det(\alpha) \neq 0$, then $|L : \alpha L| = |\det\alpha|$.
- There is an involution $\cdot^\ell$ on $\Delta_0(N)$ given by $\alpha^\ell = \left[\begin{smallmatrix} 1 & 0 \\ 0 & N \end{smallmatrix}\right]^t \alpha \left[\begin{smallmatrix} 1 & 0 \\ 0 & N \end{smallmatrix}\right]^{-1}$. This amounts to $\left[\begin{smallmatrix} w & x \\ Ny & z \end{smallmatrix}\right]^\ell = \left[\begin{smallmatrix} w & y \\ Nx & z \end{smallmatrix}\right]$.
- Set $T(n) = \{\alpha \in \Delta_0(N) : \det(\alpha) = n\}$. Then $T(n) = \bigsqcup_{ad=n, a>0, a|d, \gcd(a,N)=1} \Gamma_0(N)\left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right]\Gamma_0(N)$.
- Suppose that π_o and π are representations of G on V_o and V respectively. Let $T : V_o \to V$ be a bounded linear map. Suppose $W_o \subset V_o$ is a G stable subspace, and that for each $w_o \in W_o$ the function $g \mapsto \pi(g)T\pi_o(g^{-1})w_o$ is an integral V valued function on G w/r/t the left Haar measure $\mathrm{d}g$. Then $L : W_o \to V$ defined by $Lw_o = \int_G \pi(g)T\pi_o(g^{-1})w_o\,\mathrm{d}g$ intertwines $\pi_o|_{W_o}$ and π.

10 Notes on GGP-S

- Let $G_o = \{g_{a,b} := \left[\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right] : a \in K^\times, b \in K\} \leq G = \mathrm{GL}_2(K)$. Then every unitary irreducible representation of G restricts to an irreducible representation of G_o.
- All but one unitary irreducible representation of G_o is one dimensional: given by multiplicative characters on $K^\times$. The remaining unitary irreducible representation is infinite dimensional, given by $L^2(K^\times, \mathrm{d}^\times x)$, and is of the form $U(g_{a,b})\varphi(x) = \chi(bx)\varphi(ax)$ where χ is a fixed nontrivial additive character of K.

11 Notes on Complex analytic stuff (Werner Basler)

- Wronski's identity: suppose $X : G \to M_n(\mathbb{C})$ is a holomorphic function on $G \subset \mathbb{C}$ such that $X'(z) = A(z)X(z)$ for all $z \in G$. Set $w(z) = \det X(z)$ and $a(z) = \operatorname{tr} A(z)$. Then
$$w(z) = w(z_o) \exp\left(\int_{z_o}^{z} a(u)\,\mathrm{d}u\right)$$
for any $z_o \in G$. In particular, either $w(z) = \det X(z)$ is identically zero on G or is nonvanishing on G.
- Given $A : G \to M_n(\mathbb{C})$, an $X : G \to M_n(\mathbb{C})$ as above is called a fundmanetal solution for A if $\det X(z) \neq 0$ for some, hence all, $z \in G$.
- Inhomogeneous systems: G simply connected region in $\mathbb{C}$, and $A : G \to M_n(\mathbb{C})$ and $b : G \to \mathbb{C}^n$ holomorphic. Consider the equation $x' = A(z)x + b(z)$ for $z \in G$. Call $x' = A(z)x$ the associated homogeneous system.
- Variation of constants formula: the general solution to the inhomogeneous equation in the preceding bullet can be obtained from the fundamental solution X to the homogeneous problem via
$$x(z) = X(z)\left(c + \int_{z_o}^{z} X^{-1}(u)b(u)\,\mathrm{d}u\right)$$
where $z_o \in \mathbb{C}$ and $c \in C^n$ arbitrary.
- Homogeneous systems with a first order pole at z_o: given a fundamental solution $X : G \to M_n(\mathbb{C})$ find an $M \in M_n(\mathbb{C})$ such that $X(z) = S(z)(z - z_o)^M$ with $S(z)$ holomorphic and single valued for z in a punctured disk about z_o.
- The equation we consider is
$$zx' = A(z)x, \quad A(z) = \sum_{n=0}^{\infty} A_n z^n, \quad |z| < \rho.$$
Say this equation has good spectrum if none of the eigenvalues of A_o differ by a natural number. Equivalently, as n runs through $\mathbb{Z}_{>0}$, the spectrum of $A_o + nI$ are pairwise disjoint.
- When the equation has good spectrum, it has a fundamental solution of the form
$$X(z) = S(z)z^{A_o}, \quad S(z) = \sum_{n=0}^{\infty} S_n z^n, \quad S_o = I, \quad |z| < \rho$$
with the coefficients S_n determined by the relations
$$S_n\left(A_o + nI\right) - A_o S_n = \sum_{m=0}^{n-1} A_{n-m} S_m$$
.

- Confluent hypergeometric systems: for $A, B \in M_n(\mathbb{C})$, consider the equation
$$zx' = (zA + B)\,x$$
- Hypergeometric systems: for $A, B \in M_n(\mathbb{C})$, consider
$$(A - zI)\,x' = Bx$$
. For now, suppose $A = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ is diagonal. By making a change of variable $z = au + b$ for some $a \neq 0$, one can achieve $\lambda_1 = 0$ and $\lambda_n = 1$.
- Suppose $z \mapsto T(z) \in \mathrm{GL}_n(\mathbb{C})$ is holomorphic and nonvanishing on a neighborhood of the origin in $\mathbb{C}$. Then x is a solution to $zx' = A(z)x$ if and only if for $x = T(z)\widetilde{x}$ one has $\widetilde{x}$ is a solution to $z\widetilde{x}' = B(z)\widetilde{x}$ where $B(z)$ given by $zT'(z) = A(z)T(z) - T(z)B(z)$.

12 Notes on Kobayashi's Complex Hyperbolic Spaces

- For a bounded domain $X \subset \mathbb{C}^n$ define the Caratheodory distance $c_X(p, q)$ to be the supremum over all holomorphic $f : X \to D$ (where D is a disk in $\mathbb{C}$) of $\rho(f(p), f(q))$ where ρ is the poincare metric on D. Then any holomorphic map $(X, c_X) \to (Y, c_Y)$ is distance decreasing.
- Arzela-Ascoli: suppose X is locally compact seperable and Y is a locally compact metric space with distance d_Y. Then a family $\mathcal{F} \subset C(X, Y)$ is relatively compact in $C(X, Y)$ if and only if both $\mathcal{F}$ is equicontinuous at every point $x \in X$ and for ever $x \in X$, the set $\{f(x) : f \in \mathcal{F}\} \subset Y$ is relatively compact.
- $D(X, Y)$ is the set of distance decreasing maps $X \to Y$.
- V an n dimensional complex vector space, and V^* its dual space. Let F be nonegative real defined on a subset of V such that if F is defined at v then it is defined at tv for all $t \in \mathbb{C}$ and $F(tv) = |t|F(v)$.
- Let X be a Riemann surface and $\mathrm{d}\sigma^2 = 2\lambda\,\mathrm{d}z\,\mathrm{d}\overline{z}$ be a hermitian pseudo metric on X and let $\omega = i\lambda\,\mathrm{d}z \wedge \mathrm{d}\overline{z}$ be its associated Kahler form.
- Set $\mathrm{d}^c = i(\mathrm{d}'' - \mathrm{d}')$ so that $\mathrm{d}\,\mathrm{d}^c = 2i\,\mathrm{d}'\,\mathrm{d}''$.
- The Ricci form associated to ω is $\mathrm{Ric}(\omega) = -\mathrm{d}\,\mathrm{d}^c \log\lambda = 2K\omega$. where
$$K = -\frac{1}{\lambda}\frac{\partial^2 \log\lambda}{\mathrm{d}z\,\mathrm{d}\overline{z}}$$
is the Gaussian curvature of $\mathrm{d}\sigma$.
- Let D_a be the open disk of radius a in $\mathbb{C}$. The poincare metric is
$$\mathrm{d}s_a^2 \frac{4a^2\,\mathrm{d}z\,\mathrm{d}\overline{z}}{A(a^2 - |z|^2)^2}$$
. This metric is complete and has curvature $-A$. Take $a = 1$ so that $A = 1$ and let $\mathrm{d}s^2 = \mathrm{d}s_1^2$. Let φ be the Kahler form for $\mathrm{d}s^2$. Then $\mathrm{Ric}(\varphi) = -2\varphi$ since $K = -1$.
- Generalization of Ahlfors–Pick: let $\mathrm{d}\sigma^2$ be any hermitian pseudometric on D with curvature bounded above by -1. Then $\mathrm{d}\sigma^2 \leq \mathrm{d}s^2$.
- Let X be a Riemann surface with Hermitian pseudometric $\mathrm{d}s_X^2$ with curvature bounded above by -1. Then every holomorphic $f : D \to X$ is distance decreasing: $f^*\,\mathrm{d}s_X^2 \leq \mathrm{d}s^2$.
- Let H be the upper half plane. Then $w \mapsto \frac{i-w}{i+w}$ is biholomorphism $H \to D$. The pulled back poincare metric is $\mathrm{d}s_H^2 = \frac{\mathrm{d}w\,\mathrm{d}\bar{w}}{v^2}$ for $w = u + iv$.
- D^* is the punctured disk, and $p : H \to f^*$ the covering defined by $z = p(w) = e^{2\pi i w}$. Then
$$\mathrm{d}s_{D^*}^2 = \frac{4\,\mathrm{d}z\,\mathrm{d}\overline{z}}{|z|^2(\log 1/|z|^2)^2}$$

is a complete metric on D^* of curvature -1. Its area element (i.e. its Kahler form) is

$$\mu_{D^*} = \frac{i\,\mathrm{d}z \wedge \mathrm{d}\bar{z}}{|z|^2(\log|z|^2)^2}.$$

13 Complex Analytic stuff

- A concrete Riemann surfaces is a 'branched Riemann domain' $\pi : X \to \mathbb{C}$ or $\pi : X \to \mathbb{P}^1(\mathbb{C})$.
- Serre duality: If X is a compact Riemann surface and V is a holomorphic vector bundle over X, then $H^1(X, V)$ and $H^0(X, K_X \otimes V^*)$ are finite dimensional vectorspaces of equal dimension.
- In particular, try $V = \mathcal{O}_X = X \times \mathbb{C}$, then $H^1(X, \mathcal{O}_X) \approx H^0(X, K_X)$.

14 Notes on Wolpert's book

- Margulis' lemma for the hyperbolic plane: elements of a discrete torsion free group acting on the hyperbolic plane by isometries which move a base point a distance < 2 are contained in a cyclic subgroup.
- For a hyperbolic subgroup, an area $2\ell \cot \ell/2$ collar about a geodesic of length ℓ will embed into the quotient.
- For a parabolic subgroup, an area 2 neighborhood of a cusp will embed.
- Consider the fibration $\pi : P = \{(z, w, t)|zw = t,\ |z|, |w|, |t| < 1\} \to D = \{\, t \mid |t| < 1\,\}$. The differential of $zw - t$ is nowhere vanishing on $\mathbb{C}^3$ so P is a smooth complex submanifold. Since $\mathrm{d}z$, $\mathrm{d}w$, and $\mathrm{d}(zw - t)$ are $\mathbb{C}$ linearly independent, (z, w) form global coordinates for P. Then $\pi(z, w) = zw = t$ and $\mathrm{d}\pi = z\,\mathrm{d}w + w\,\mathrm{d}z$, which vanishes only at the origin. The fibers of π away from the origin in $\mathbb{C}^3$ make up the family of hyperbolas in $\mathbb{C}^2$ with coordinates (z, w) which limit to the union of coordinate axes.
- For $t \neq 0$, a fiber projected to z axis is $\{\ |\, t| < |z| < 1\}$, or projected to the w axis is $\{\ |\, t| < |w| < 1\,\}$. For $t = 0$, the fiber $\{\,(z, 0) \mid |z| < 1\,\} \cup \{\,(0, w) \mid |w| < 1\,\} \subset \mathbb{C}^2$.
- A vector v in $\mathbb{C}^3$ is tangent to P provided $v(zw - t) = 0$ and is tangent to a fiber of $\pi : P \to D$ provided $\mathrm{d}\pi(v) = v(zw) = 0$.
- The *relative cotangent bundle* of a fiber is

$$\frac{\text{cotangents of } P}{\text{pullback of contangents of } D} = \frac{\mathcal{O}(\,\mathrm{d}z) + \mathcal{O}(\,\mathrm{d}w)}{\mathcal{O}(\pi^*\,\mathrm{d}t)}$$

- Look for nonsingular change of basis $f\,\mathrm{d}z + g\,\mathrm{d}w = a\omega + b\,\mathrm{d}t$ (for a suitable differential ω) of $\mathcal{O}(\,\mathrm{d}z) + \mathcal{O}(\,\mathrm{d}w)$.
- When $\omega = \frac{\mathrm{d}z}{z} - \frac{\mathrm{d}w}{w}$ the change of basis is

$$\begin{bmatrix} z/2 & -w/2 \\ 1/w & 1/z \end{bmatrix} \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right].$$

  withe determinant 1.
- This gives a direct sum decomposition $\mathcal{O}(w\,\mathrm{d}z) + \mathcal{O}(z\,\mathrm{d}w) = \mathcal{O}(\omega) \bigoplus \mathcal{O}(\pi^*\,\mathrm{d}t)$. So the relative cotangent bundle is $\mathcal{O}(\omega)$.
- F is a fixed reference Riemann surface of genus g with n distinguished points. $S(F)$ is the set of conformal structures on F. $\mathrm{Diff}^+(F)$ is the group of orientation preserving diffeomorphisms of F which fix the marked points. The subgroup $\mathrm{Diff}_o(F)$ is the normal subgroup consisting of those diffeomorphisms homotopic to the identity. $M(F) = S(F)/\,\mathrm{Diff}^+(F)$ is

the moduli space of conformal structures on F with labeled distinguished points. $T(F) = S(F)/\operatorname{Diff}_o(F)$ is the Teichmuller space of marked such structures. There is a branched covering map $T(F) \to M(F)$ with deck group MCG $= \operatorname{Diff}^+(F)$ the pure mapping class group.

- For a map f of Riemann surfaces, given in local coordinates as $w(z)$ the complex differential of f is $\partial f = w_z\,\mathrm{d}z + w_{\overline{z}}\,\mathrm{d}\overline{z}$. Setting $\mu = w_{\overline{z}}/w_z$ this is then $\partial f = w_z(\,\mathrm{d}z + \mu\,\mathrm{d}\overline{z})$. Then $\arg(\,\mathrm{d}z + \mu\,\mathrm{d}\overline{z})$ is a well defined angle measure. For the deformation given by μ, the direction $(\arg\mu)/2$ resp. $(\arg\mu + \pi)/2$ is the direction of maximal resp. minimal stretch.
- A beltrami differential on a Riemann surface R is a section of $K^{-1} \otimes \overline{K}$ with $|\mu|_\infty$ finite. Write $B(R)$ for the C banach space of L^∞ such sections. Write $B_1(R)$ for the open unit ball in that Banach space.
- The schwarzian derivative (an infinitesimal form of the cross ratio) for a holomorphic map h is
$$\{h\} := \left(\frac{h''}{h'}\right)' - \frac{1}{2}\left(\frac{h''}{h'}\right)^2$$
which satisfies $\{h \circ A\} = \{h\} \circ \left(A \cdot (A')^2\right)$ for all linear fractional A.
- Here's how to define a map $\Phi : B_1(R) \to Q(\overline{R})$. Given μ lift to a tensor $\widetilde{\mu}$ on $\mathbb{H}$, the upper half plane, and extend by 0 to $\mathbb{L}$ the lower halfplane. The equation $w_{\overline{z}} = \widetilde{\mu} w_z$ has a unique solution $w^{\widetilde{\mu}}$ fixing 0, 1, and ∞. THe map Φ sends μ to $\{w^{\widetilde{\mu}}\}|_{\mathbb{L}} \in Q(\mathbb{L}/\Gamma)$ where $R = \mathbb{H}/\Gamma$.
- There is a canonical pairing b/w $B(R)$ and $Q(R)$ given by integration: $(\mu, \varphi) \mapsto \int_R \mu\varphi$. Let $Q(R)^\top \subset B(R)$ be the annihilator.

Theorem 1. The main theorem of Ahlfors–Bers deformation theory: The quotient $T(R) = B_1(R)/\operatorname{Diff}_o(R)$ is a complex manifold with $\Phi : T(R) \to Q(\overline{R})$ a holomorphic embedding, with image containing the $|\cdot|_\infty$ ball of radius 1/2 and contained in the ball of radius 3/2. At the origin, the $\mathbb{C}$ tangent space of $T(R)$ is $B(R)/Q(R)^\top$ and the $\mathbb{C}$ cotangent space is $Q(R)$, with $(\cdot,\cdot)$ the tangent-cotangent pairing.

- The coset representatives for $B(R)/Q(R)^\top$ are given by harmonic beltrami differentials $:= H(R)$. These are of the form $\mu = \overline{\varphi}(\,\mathrm{d}s^2)^{-1}$ with $\varphi \in Q(R)$ and $\mathrm{d}s^2$ the R hyperbolic metric.
- Beltrami differentials are dense in the space of square integrable sections $L^2(K^{-1} \otimes \overline{K})$.
- Let $\mathcal{A}$ be the concentric annulus with inner radius $\exp\left(-2\pi^2/\log\lambda\right)$ (for $\lambda > 1$) and outer radius 1. Then $\mathbb{H}$ covers $\mathbb{A}$ via the map $z \mapsto w = \exp(2\pi i \log z/\log\lambda)$. The deck group of this cover is infinite cyclic generated by $z \mapsto \lambda z$
- Let $\mathcal{H}$ be the horizontal strip, consisting of ζ with $0 < \operatorname{Im}\zeta < \pi$. Then $\mathcal{H}$ is also the universal cover of $\mathcal{A}$ via the map $z \mapsto \zeta = \log z$. Its deck group is infinite cyclic, generated by $\zeta \mapsto \zeta + \ell$ where $\lambda = \exp(\ell)$. We seek to determine the deformation of $\mathcal{A}$ under the variation $\zeta \mapsto \zeta + \ell + \varepsilon$.
- On the left side of the fundamental domain $\{0 < \operatorname{Re}(\zeta) < \ell\}$ use the coordinate ζ, and on the right side use the coordinate $\zeta_1 = \zeta - \ell$. Introduce the overlap $\zeta_1 = p(\zeta) = \zeta + \varepsilon$ identification. The first order variation of p is the vectorfield $\partial/\partial\zeta_1$ on the overlap of charts.
- Take a unit step function φ which is 0 close to 0 and 1 close to ℓ. On the fundamental domain, consider $f(\zeta) = \zeta + \varepsilon\varphi(\operatorname{Re}(\zeta))$ and extend by periodicity. Then f satisfies $f(\zeta+\ell) = f(\zeta)+\ell+\varepsilon$, so conjugates $\zeta \mapsto \zeta+\ell$ to $\zeta \mapsto \zeta+\ell+\varepsilon$. The first order variation is $\varphi(\operatorname{Re}\zeta)\partial/\partial\zeta$. Note that f is a smooth function, though not holomorphic.
- f has $\mathbb{C}$ derivatives $f_\zeta = 1 + \frac{\varepsilon}{2}\varphi'(\operatorname{Re}\zeta)$ and $f_{\bar{\zeta}} = \frac{\varepsilon}{2}\varphi'(\operatorname{Re}\zeta)$. So the beltrami differentials are $\mu_f = \frac{\frac{\varepsilon}{2}\varphi'}{1+\frac{\varepsilon}{2}\varphi'}$. Its first variation is $\dot{\mu} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mu_f\,|_{\varepsilon=0} = \frac{1}{2}\varphi'$.

- A quasi conformal map satisfies $f_{\bar{\zeta}} = \mu f_{\zeta}$. For a family of beltrami differentials $\mu(\varepsilon)$, and a variation from the identity map: $f(\zeta;\varepsilon) = \zeta + \varepsilon f_1(\zeta) + \ldots$, substituting into the preceeding equation gives $\dot{f}_{\bar{\zeta}} = \dot{\mu}$ (this is the Kodaira spencer story apparently).
- Variation of translation length: for a horizontal strip $\mathcal{H}$, one has

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\ell(\varepsilon) = \frac{1}{\pi}\int_{F_\ell} \frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}(u^\varepsilon_x - v^\varepsilon_y)\,\mathrm{d}E = \frac{2}{\pi}\operatorname{Re}\int_{F_\ell}\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\frac{\partial}{\partial\bar{\zeta}}h^\varepsilon\,\mathrm{d}E$$

  where $\mathrm{d}E$ is the Euclidean volume element, F_ℓ is the fundamental domain consisting of $0 \leq \operatorname{Re}\zeta \leq \ell$ and $0 \leq \operatorname{Im}\zeta \leq \pi$, and $h^\varepsilon = u^\varepsilon + iv^\varepsilon : \mathcal{H} \to \mathcal{H}$ is such that $h^\varepsilon(\zeta+\ell) = h^\varepsilon(\zeta) + \ell(\varepsilon)$.
- For $n = 1$ (so that we are looking at first variation), and substituting $z = e^\zeta$ we get Gardiner's formula:

$$\dot{\ell} = \frac{2}{\pi}\operatorname{Re}\int_{F_\ell}\dot{\mu}\,\mathrm{d}E = \frac{2}{\pi}\operatorname{Re}\int_{|z|=1}^{e^\ell}\int_{\arg z=0}^{\pi}\dot{\mu}(\frac{\mathrm{d}z}{z})^2\,\mathrm{d}E$$

  where, formally, $\mathrm{d}\ell\frac{2}{\pi}(\mathrm{d}\zeta)^2 = \frac{2}{\pi}(\frac{\mathrm{d}z}{z})^2 \in Q$.
- The Teichmuller space of the annulus is one $\mathbb{R}$ dimensional, parameterized by core length. Thus, this is not quite an exemplar for the computations on the teichmuller spaces of more complicated surfaces. Note, the geodesic length of the core geodesic determines only the conformal type of the annulus.
- We can extend the Teichmuller space of the annulus to one $\mathbb{C}$ dimensional by allowing for the consideration of maps of $\mathcal{H}$ which are translations on-or-near the boundaries. These are twist deformations.

14.1 Geodesic lengths, twists, and symplectic geometry

- Let F be a smooth surface of genus g, and R a Riemann surface diffeomorphic to F. A point in $\mathcal{T}(F)$ represents an isomorphism of $\pi_1(F)$ with the deck group $\pi_1(R)$ of the universal covering map $\mathbb{H} \to R$.
- For a free homotopy class $[\alpha]$, viewed as a conjugacy class in $\pi_1(F)$. The length of the geodesic on R in the free homotopy class of $[\alpha]$ is denoted $\ell_\alpha(R)$.
- Let $\Gamma \leq \mathrm{PSL}(2,\mathbb{R})$ be the uniformizing group for R. Pick a representative $\alpha \in [\alpha]$ and let Γ_α be its centralizer, arranged so that the axis of α is the imaginary line.
- Define the α-petersson series

$$\Theta_\alpha = \frac{2}{\pi}\sum_{\gamma\in\Gamma_\alpha\backslash\Gamma}\gamma^*\left(\frac{\mathrm{d}z}{z}\right)^2$$

- Key point: $\Theta_\alpha \in \mathcal{Q}(R)$ and is $\mathrm{d}\ell_\alpha$ (i.e. represents a cotangent vector to the point R in $T(F)$).
- The length-length formula: for geodesics α, β on a surface R of finite type,

$$\langle \operatorname{grad}\ell_\alpha, \operatorname{grad}\ell_\beta\rangle = \frac{2}{\pi}\left(\ell_\alpha\delta_{\alpha\beta} + \sum_{\Gamma_\alpha\backslash\Gamma/\Gamma_\beta}{}^{'}\, u\log|\frac{u+1}{u-1}| - 2\right)$$

  where for $C \in \Gamma$, $u(\widetilde{\alpha}, C(\widetilde{\beta}))$ is either $\cos\theta$ provided $\widetilde{\alpha}$ and $C(\widetilde{\beta})$ intersect, and $\cosh d(\widetilde{\alpha}, C(\widetilde{\beta}))$ otherwise.
- When α and β are disjoint, the terms in the sum are all positive. It follows that the gradient for a simple geodesic is nowhere vanishing.

- A key computation: for $h \in \mathrm{Diff} / \mathrm{Diff}_o$, one has $\ell_\alpha \circ [h] = \ell_{h^{-1}(\alpha)}$. Consequently, for any finite subgroup $G \subset$ MCG and maximal collection of disjoint simple closed curves $\{\alpha_j\}$, the function $\mathcal{L} = \sum_{g \in G} \sum_j \ell_{h(\alpha_j)}$. Then $\mathcal{L}$ is proper, strictly convex, and G invariant (as a function on teichmuller space). Thus $\mathcal{L}$ has a unique minimum, which must be G fixed point.

15 Notes on Surfaces, Circles, and Solenoids (Robert Penner)

- The **lambda length** of a pair of disjoint horocycles in the upper half plane centered at $u, v \in \mathbb{R}$, with euclidean diameters c, d respectively is $\sqrt{\frac{2}{cd}}|u - v|$, which is roughly the exponential of the hyperbolic distance between those horocycles.
- $G = \mathrm{PSL}(2, \mathbb{Z})$, and $\hat{G}$ is its profinite completion, $\mathbb{D}$ is the open unit disk and $\mathcal{H}$ is $(\mathbb{D} \times \hat{G})/G$ where $\gamma \in G$ acts on $(z, t) \in \mathbb{D} \times \hat{G}$ by $\gamma(z, t) = (\gamma z, t\gamma^{-1})$.
- Let $\langle \cdot, \cdot \rangle$ denote the Minkowski form (with signature $2, 1$) on $\mathbb{R}^3$, and

$$\mathbb{H} = \{\, u \in \mathbb{R}^3 : \langle u, u \rangle = -1 \text{ and } z > 0 \,\}$$

  is the upper sheet. Then projection to the open unit disk at height 0 about the origin in $\mathbb{R}^3$ establishes an isometry b/w $\mathbb{H}$ and $\mathbb{D}$.
- The open positive light cone $L^+ = \{u \in \mathbb{R}^3 : \langle u, u \rangle = 0 \text{ and, } z > 0\}$ identifies with the collection of horocycles in $\mathbb{H}$ via 'duality' $u \mapsto h(u) = \{w \in \mathbb{H} : \langle w, u \rangle = -1\}$. Then L^+ identifies with the boundary S^1 of $\mathbb{D}$, via the map Π which sends a horocycle in L^+ to its center in S^1.
- A **decorated geodesic** is an unordered pair $\{h_0, h_1\}$ of horocycles with distinct centers: so there is a well defined geodesic connecting the centers of h_0 and h_1.
- There is a well defined signed distance δ associated to the decorated geodesic $\{h_0, h_1\}$ with sign positive if and only if the horocycles are disjoint. Define the **lambda length** of the decorated geodesic $\{h_0, h_1\}$ to be $\sqrt{2 \exp \delta}$
- Then $\lambda(h(u_0), h(u_1)) = \sqrt{-\langle u_0, u_1 \rangle}$ with h the duality defined above (where u_0, u_1 are in L^+).
- Let $F = F_g^s$ be a smooth surface of genus g with $s \geq 1$ punctures (with $2 - 2g - s < 0$.) Let $G = \pi_1(F)$, and let $\mathcal{T}(F) = \mathrm{hom}'(G, \mathrm{PSL}(2, \mathbb{R}))/\mathrm{PSL}(2, \mathbb{R})$ be its teichmuller space, where the prime indicates that we're looking only at those discrete faithful representations such that no element is elliptic, and loops about punctures are parabolic.
- Define the **decorated teichmuller space** $\widetilde{\mathcal{T}}(F) \to \mathcal{T}(F)$ to be the trivial $\mathbb{R}^s_{>0}$ bundle such that the fiber over a point is the set of all $s-$tuples of horocycles, one about each puncture, parameterized by hyperbolic length.
- An **arc family** in F is an isotopy class of a family of essential arcs disjointly embedded in F and connecting the punctures, w/ not two elements in the family are homotopic (relative to the punctures). If α is a maximal arc family, so that each component of $F - \alpha$ is a triangle, then we say that α is an ideal triangulation of F.
- A theorem: Fix an ideal triangulation τ of F. Then the assignment of λ lengths is a surjective homeomorphism $\mathcal{T}(F) \to \mathbb{R}^\tau_{>0}$.

15.1 Coordinates for the solenoid

- G is now a finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$, and $M = \mathbb{D}/G$. Let $\mathcal{C}_M$ be the category of finite sheeted orbifold covers $\pi : F \to M$, where F is a punctured Riemann surface. Then $\mathcal{C}_M$ is a directed set.
- The **punctured solenoid** is $\mathcal{H}_M = \varprojlim \mathcal{C}_M$. So a point of $\mathcal{H}_M$ is a sequence of points $y_i \in F_i$, for each cover $\pi_i : F_i \to M$, chosen compatibly with the projections.
- Apparently, finite index subgroups yield homeomorphic solenoids.
- G has characteristic subgroups G_N, which for $N \in \mathbb{Z}_{\geq} 0$ consist of the intersection of all subgroups of index $\leq N$, which form a nested sequence $G_{N+1} < G_N < G_{N-1} < \cdots G_2 < G_1 = G$.
- Define a metric $G \times G \to \mathbb{R}$ via $\gamma \times \delta \mapsto \min N^{-1} : \gamma\delta^{-1} \in G_N$. Then the profinite completion of G is the metric completion with respect to this metric.
- As above, form $\mathcal{H}_G = \mathbb{D} \times_G \hat{G}$. Then $\mathcal{H}_G$ is homeomorphic to $\mathcal{H}_{\mathrm{PSL}_2(\mathbb{Z})}$ for any finite index subgroup G of $\mathrm{PSL}_2(\mathbb{Z})$. Henceforth, $G = \mathrm{PSL}_2(\mathbb{Z})$.
- Let $\hom'(G \times \hat{G}, \mathrm{PSL}_2(\mathbb{R}))$ denote the collection of continuous functions such that:

  1) For all $\gamma_1, \gamma_2 \in G$, and $t \in \hat{G}$, we have
  $$\rho(\gamma_1 \circ \gamma_2, t) = \rho(\gamma_1, t\gamma_2^{-1}) \circ \rho(\gamma_2, t),$$
  2) For every $t \in \hat{G}$, there is a quasiconformal $\varphi_t : \mathbb{D} \to \mathbb{D}$ which conjugates the action of G on $\mathbb{D} \times \hat{G}$ from
  $$\gamma_1 : (z, t) \mapsto (\gamma z, t\gamma^{-1})$$
  to
  $$\gamma_\rho : (z, t) \mapsto (\rho(\gamma, t)z, t\gamma^{-1}).$$
- Then we let $G_\rho = \{\gamma_\rho : \gamma \in G\}$ and set $\mathcal{H} = \mathbb{D} \times_\rho \hat{G}$.
- Write $\mathrm{Cont}(\hat{G}, \mathrm{PSL}_2(\mathbb{R}))$ for group (under pointwise composition) of continuous functions $\hat{G} \to \mathrm{PSL}_2(\mathbb{R})$. Then $\mathrm{Cont}(\hat{G}, \mathrm{PSL}_2(\mathbb{R}))$ acts continuously on $\hom'(G \times \hat{G}, \mathrm{PSL}_2(\mathbb{R}))$ via $(\alpha\rho)(\gamma, t) = \alpha^{-1}(t\gamma^{-1}) \circ \rho(\gamma, t) \circ \alpha(t)$.
- Key structural theorem: there is a natural homeomorphism of $\mathcal{T}(\mathcal{H})$ with
$$\hom'(G \times \hat{G}, \mathrm{PSL}_2(\mathbb{R}))/\operatorname{Cont}(\hat{G}, \mathrm{PSL}_2(\mathbb{R})).$$

16 Notes on Will Harvey's essay on Teichmuller spaces, triangle groups, and dessins.

- The real affine linear map $\widetilde{f_\tau} : \mathbb{C} \to \mathbb{C}$ which sends $x + yi$ to $x + y\tau$ and induces the homeomorphism $f_\tau : C/\Lambda_i \to \mathbb{C}/\Lambda_\tau$ is **extremal** in its homotopy class (in teich's sense) in the sense that it has the least overall distortion measured by taking the supremum of the local stretching function on X. In this case, the local distortion is $\frac{\bar{\partial} f_\tau}{\partial f_\tau} = \frac{1+i\tau}{1-i\tau}\frac{\mathrm{d}\bar{z}}{\mathrm{d}z}$
- Teichmuller geodesic (disks): start with a Riemann surface X_o, and a quadratic differential form φ on X_o. This gives a (complex) one parameter family of deformations of X_o.
- First description: away from the zeroes of φ, write $\varphi = \mathrm{d}w^2$ to get a local parameter w up to transition functions of the form $w \mapsto \pm w + c$. Equivalently, set $w = \int_{z_o}^{z} \sqrt{\varphi(t)}$. Then, for each ε with $|\varepsilon| < 1$, define a new structure on the underlying topological surface S_o by rotating each chart through $\arg \varepsilon$ and expanding the real foliation of $\mathbb{R}^2 = \mathbb{C}$ while contracting the imaginary foliation via the mapping $z = x + iy \mapsto w = K_\varepsilon^{1/2} x + iK_\varepsilon^{-1/2} y$ where $K_\varepsilon = (1 + |\varepsilon|)/(1 - |\varepsilon|)$. Note: if $\arg \varepsilon = 0$ so that $\varepsilon \in (0, 1)$ then the family of such structures is called the **teichmuller ray** at X_o in the direction φ.

- Second description: write $\nu_\varepsilon(z) = \varepsilon\bar{\varphi}(z)/|\varphi(z)|$. Then solve the **beltrami equation**
$$\frac{\partial w}{\partial \bar{z}} = \nu_\varepsilon \frac{\partial w}{\partial z}.$$
Then conjugating X_o by the one parameter family of solutions gives a holomorphic curve of deformations.
- Given a $\varphi \in \Omega^2(X)$, nonzero, obtain (via Teichmuller deformation) a holomorphic mapping e_φ of $\mathcal{U}$ into $\mathcal{T}(X)$ via the map $e_\varphi(\varepsilon) = \varepsilon\frac{\bar{\varphi}}{|\varphi|} \in B_1(X) \subset L^\infty_{-1,1}(X)$ for $|\varepsilon| < 1$. The global metric on teichmuller space is in fact realized by the poincare measure for points in a teichmuller geodesic disk.
- Veech's examples: let X_n with $n \geq 5$ and odd, be the genus $g = (n-1)/2$ hyperelliptic surface with affine equation $y^2 = 1 - x^n$, and consider the holomorphic 1 form $\omega = \mathrm{d}x/y$ on it.
- ω has a zero of order $2g-2$ at the single point such that $x = \infty$. Then $q = \omega^2$ defines a teichmuller disk in $\mathcal{T}_g$.
- Let $\zeta = \exp(2\pi i/n)$ and let T_ζ be the triangle with vertices at $0, 1$ and ζ. Let $P = \cup_{\ell=0}^{n-1}\zeta^\ell T_\zeta$. Then multiplication by ζ induces a cyclic group of euclidean symmetries of P such that the quotient mapping is an n fold covering of the plane near 0.
- Repeat this construction but with $-T_\zeta$ to produce a (distinct, since n is odd) polygon Q. Then identify pointwise in $P \cup Q$ the pairs of outside edges in corresponding triangles using a translation.
- This gives a closed surface, with local complex structure away from the corners. Filling this in conformally gives a compact riemann surface X_ζ (with nontrivial symmetry group) having a local structure over $\mathbb{P}^1$ given by the map $p_2(z) = f \circ p_1 : X_\zeta \to X_\zeta/\langle z \mapsto \pm\zeta z\rangle$, where p_1 is the projection $X_\zeta \to \mathbb{CP}^1 = X_\zeta/\langle z \mapsto \pm z\rangle$ and $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ is the mapping $z \mapsto z^n$.
- The resulting order n symmetry of X induced by lifting $z \mapsto \zeta z$ fixes *three* points: the two center points, and the single orbit of corner points.
- Big theorem of Veech: the stabilizer in the mapping class group (of genus g) of the Teichmuller disk determined by the differential $q = \omega^2$ is a fuchsian triangle group $H_n = \langle \begin{bmatrix} 1 & 2\cot\pi/n \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \cos 2\pi/n & -\sin 2\pi/n \\ \sin 2\pi/n & \cos 2\pi/n \end{bmatrix} \rangle = \langle \sigma, \beta \rangle$, which is isomorphic to $\langle x_1^n = x_2^2 = 1 \rangle$ via $x_1^{g+1}x_2 = \sigma$ and $x_1 = \beta$.
- Let $C_n = [H_n, H_n] \leq H_n$ be the commutator subgroup. It is finite index and normal in H_n with quotient $\mathbb{Z}/2oplus\mathbb{Z}/n$. Harvey claims that the quotient of $\mathcal{U}$ by H_n or by C_n yield "the same" surface X_n. How is this so?

17 Notes on Goldman's bit in the teichmuller handbook

A lovely lemma:

Lemma 1. For $x, y \in \mathrm{SL}_2(\mathbb{C})$, the following are equivalent:

- x, y generate an irreducible representation on $\mathbb{C}^2$.
- $\mathrm{tr}(xyx^{-1}y^{-1}) \neq 2$ (multiplicative commutator)
- $\det(xy - yx) \neq 0$ (Lie algebra commutant)
- The pair (x, y) is not $\mathrm{SL}_2(\mathbb{C})$ conjugate to a representation by upper triangular matrices.
- Either the group $\langle x, y \rangle$ is not solvable, or there exists a splitting $\mathbb{C}^2 = L_1 \oplus L_2$ into an invariant pair of lines L_i such that one of x or y interchanges L_1 and L_2.

- $\{\mathrm{id}, x, y, xy\}$ is a $\mathbb{C}$ basis for $M_2(\mathbb{C})$.

The fundamental group of the *three holed sphere* $\Sigma_{0,3}$ is free on two generators, but has a *redundant geometric presentation:*

$$\pi = \pi_1(\Sigma_{0,3}) = \langle X, Y, Z \mid XYZ = 1\rangle$$

where X, Y, Z correspond to the three components of $\partial\Sigma_{0,3}$. Denote the corresponding trace functions (on the representation variety $\hom(\pi, G)$) by the lower case letters. Here's a theorem:

Theorem 2. The equivalence class of a flat $\mathrm{SL}_2(\mathbb{C})$ bundle over $\Sigma_{0,3}$ with irreducible holonomy is determined by the equivalence class of its restrictions to the three components of $\partial\Sigma_{0,3}$. Any triple of isomorphism class of flat $\mathrm{SL}_2(\mathbb{C})$ bundles over the (disconnected, 1 dimensional space) $\partial\Sigma_{0,3}$ whose holonomy traces satisfy $x^2 + y^2 + z^2 - xyz \neq 4$ extends to a flat $\mathrm{SL}_2(\mathbb{C})$ bundle over $\Sigma_{0,3}$.

Every irreducible representation ρ of π in $\mathrm{SL}_2(\mathbb{C})$ apparently corresponds to an object in $\mathbb{H}^3$: a triple of geodesics. Any two of these geodesics admits a unique common perpendicular geodesic. These perpendiculars cut off a hexagon bounded by geodesic segments, with all (six) right angles.

The surface $\Sigma_{0,3}$ admits an orientation *reversing* involution ι_{Hex} whose restriction to each boundary component is a reflection. The quotient $\Sigma_{0,3}/\iota_{\mathrm{Hex}}$ is topologically a disk, combinatorially a hexagon. The three boundary components map to three intervals $\partial_i(\mathrm{Hex})$ in the boundary of Hex. The other three edges in $\partial\,\mathrm{Hex}$ correspond to the three arcs comprising the fix point set of ι_{Hex}. The orbifold fundamental group of Hex is $\mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$ (free product). Denote the generators by ι_{YZ}, ι_{ZX}, and ι_{XY}. The covering map $\Sigma_{0,3} to\,\mathrm{Hex}$ induces the embedding of fundamentals $\pi_1(\Sigma_{0,3}) \to \pi_1\mathrm{Hex}$ defined on generators by

$$\begin{aligned} X &\mapsto \iota_{ZX}\iota_{XY} \\ Y &\mapsto \iota_{XY}\iota_{YZ} \\ Z &\mapsto \iota_{YZ}\iota_{ZY}. \end{aligned}$$

Some observations about involutions: we're looking at projective transformations of $\mathbb{CP}^1$ which have order exactly two. Any such is given by a matrix $g \in \mathrm{GL}_2(\mathbb{C})$ such that $\det(g) = 1$ and $\mathrm{tr}(g) = 0$. Let $\widetilde{\mathrm{Inv}} = \mathrm{SL}_2(\mathbb{C}) \cap \mathfrak{sl}_2(\mathbb{C})$ denote the set of all such matrices. Since $\widetilde{\mathrm{Inv}}$ is invariant under multiplication by $\pm\,\mathrm{id}$, we can take its quotient, call it Inv. So we view $\mathrm{Inv} \subset \mathrm{PGL}_2(\mathbb{C})$, and it consists precisely of the collection of involutions of $\mathbb{CP}^1$. It naturally identifies with the set of unordered pairs of distinct points in $\mathbb{CP}^1$. We identify $\widetilde{\mathrm{Inv}}$ as the space of *oriented* geodesics in $\mathbb{H}^3$.

Every nonidentity element g of $\mathrm{PGL}_2(\mathbb{C})$ stabilizes a unique element ι_g of $\bar{\mathrm{Inv}}$ (the closure of Inv in $\mathbb{P}(\mathfrak{sl}_2(\mathbb{C}))$.) If g is semisimple, then ι_g is the unique involution with the same fixed points as g. Otherwise, g is parabolic, and ι_g corresponds to the line $\mathrm{Fix}(\mathrm{Ad}(g)) = \ker(\mathrm{id} - \mathrm{Ad}(g)) \subset \mathfrak{sl}_2(\mathbb{C})$, which is the Lie algebra centralizer of g in $\mathfrak{sl}_2(\mathbb{C})$.

If $g \in \mathrm{SL}_2(\mathbb{C})$ is nonidentity semisimple, then the two lifts of $\iota_g \in \mathrm{Inv}$ to $\widetilde{\mathrm{Inv}} \subset \mathrm{SL}_2(\mathbb{C})$ differ by $\pm\,\mathrm{id}$. Let $g' = g - \frac{1}{2}\mathrm{tr}(g)\,\mathrm{id}$ be the *traceless projection* of g. Then $\mathrm{tr}(g') = 0$, g' commmutes with g, and $\det(g') \neq 0$ (since assumed semisimple). Choose $\delta \in \mathbb{C}^\times$ so that $\delta^2 = \det(g') = \frac{4-\mathrm{tr}(g)^2}{4}$. Then $\delta^{-1}g' \in \widetilde{\mathrm{Inv}}$ and represents the involution ι_g centralizing g:

$$\widetilde{\iota_g} = \pm\frac{2}{\sqrt{4 - \mathrm{tr}(g)}}\left(g - \frac{\mathrm{tr}(g)}{2}\,\mathrm{id}\right)$$

Elements of even order $2k$ in $\mathrm{PGL}(2,\mathbb{C})$ correspond to elements in $\mathrm{GL}(2,\mathbb{C})$ of order $4k$, while elements of odd order $2k+1$ in $\mathrm{PGL}(2,\mathbb{C})$ have two lifts to $\mathrm{GL}(2,\mathbb{C})$, one of order $2k+1$ and one of order $2(2k+1)$.

Upper halfspace model of hyperbolic three space. Let $\mathbb{H}$ be hamilton's quaternions, generated by i and j with $i^2 = j^2 = -1$ and $ij + ji = 0$. Inside $\mathbb{H}$ is the subalgebra $\mathbb{C}$ with $\mathbb{R}$ basis $\{1, i\}$. Write $\mathcal{H}^3 = \{\, z + uj \mid z \in \mathbb{C}, u \in \mathbb{R}_{>0} \,\}$ and let $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ act by

$$z + uj \mapsto (a(z + uj) + b)(c(z + uj) + d)^{-1}.$$

Then $\mathrm{PGL}(2,\mathbb{C})$ is the group of orientation preserving isometries of $\mathcal{H}^3$ under the Poincare metric $u^{-2}(|\,\mathrm{d}z|^2 + \mathrm{d}u^2)$.

For $g \in \mathrm{SL}(2,\mathbb{C})$ noncentral, the following are equivalent:

- g has two distinct eigenvalues
- $\operatorname{tr} g \neq \pm 2$,
- the corresponding collineation of $\mathbb{CP}^1$ has two fixed points
- the corresponding orientation preserving isometry of $\mathcal{H}^3$ leaves invariant a unique geodesic ℓ_g each of whose endpoints are fixed
- a unique involution ι_g centralizes g

Set $\mathrm{Lie}(g,h) = gh - hg$ for $g, h \in \mathrm{SL}(2,\mathbb{C})$. Then $\mathrm{Lie} : \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to \mathfrak{sl}(2,\mathbb{C})$ and vanishes at (g,h) if and only if g and h commute. Further, $\mathrm{Lie}(g,h) \in \mathrm{GL}(2,\mathbb{C})$ if and only if $\langle g, h\rangle$ acts irreducibly on $\mathbb{C}^2$.

For g, h such that $\langle g, h\rangle$ acts irreducibly, let λ be the image of $\mathrm{Lie}(g,h)$ in $\mathrm{PSL}(2,\mathbb{C})$. Then, since $\operatorname{tr}(\mathrm{Lie}(g,h)) = 0$, the isometry λ has order two. Further, $\operatorname{tr}(g\,\mathrm{Lie}(g,h)) = 0$ so that $g\lambda$ also has order two, so that $\lambda g \lambda^{-1} = \lambda g \lambda = g^{-1}$. Similarly of h. We find that λ acts by reflection on the invariant axes ℓ_g, ℓ_h of g and h. So its fixed axis ℓ_λ is orthogonal to ℓ_g and ℓ_h. To summarize:

Theorem 3. If $g, h \in \mathrm{GL}(2,\mathbb{C})$, then the Lie product $\mathrm{Lie}(g,h) = gh - hg$ represents the common orthogonal geodesic $\perp (\ell_{\mathbb{P}(g)}, \ell_{\mathbb{P}(g)})$ to the invariant axes of $\mathbb{P}(g)$ and $\mathbb{P}(h)$.

A real character $(x,y,z) \in \mathbb{R}^3$ corresponds to a representation of a rank two free group into one of the two *real forms* $\mathrm{SU}(2)$ or $\mathrm{SL}(2,\mathbb{R})$ of $\mathrm{SL}(2,\mathbb{C})$. Geometrically, the $\mathrm{SU}(2)$ representations correspond to those which fix a point in $\mathcal{H}^3$ and $\mathrm{SL}(2,\mathbb{R})$ representations correspond to those which preserve a plane $\mathcal{H}^2 \subset \mathcal{H}^3$ (as well as an orientation on that plane).

Theorem 4. Let $(x,y,z) \in \mathbb{R}^3$, and $\kappa(x,y,z) := x^2 + y^2 + z^2 - xyz - 2$. Let $\rho : \pi \to \mathrm{SL}(2,\mathbb{C})$ be a representation with character (x,y,z) such that $\kappa(x,y,z) \neq 2$.

- If $x, y, z \in [-2, 2]$ and $\kappa < 2$, then $\rho(\pi)$ fixes a unique point in $\mathcal{H}^3$ and is conjugate to an $\mathrm{SU}(2)$ representation.
- Otherwise, $\rho(\pi)$ preserves a unique plane in $\mathcal{H}^3$ and the restriction to that plane preserves orientation.

If $\kappa(x,y,z) = 2$, then ρ is reducible and one of the following must occur:

- $\rho(\pi)$ acts identically on $\mathcal{H}^3$, so $\rho(\pi) \subset \pm\,\mathrm{id}$ has central image.
- $\rho(\pi)$ fixes a line in $\mathcal{H}^3$ in which case $x, y, z \in [-2, 2]$ and ρ is conjugate to a representation taking values in $\mathrm{SO}(2) = \mathrm{SU}(2) \cap \mathrm{SL}(2,\mathbb{R})$.

- $\rho(\pi)$ acts by transvections along a unique line in $\mathcal{H}^3$ in which case $x, y, z \in \mathbb{R} - (-2, 2)$, and ρ is conjugate to a representation taking values in $\mathrm{SO}(1,1) \subset \mathrm{SL}(2, \mathbb{R})$. Note, $\mathrm{SO}(1,1) \approx \mathbb{R}^\times$
- $\rho(\pi)$ fixes a unique point on $\partial_\infty \mathcal{H}^3$.

18 Notes on Mukai's introduction to moduli and invariants

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