

## 1. PSEUDO-INTRODUCTION

Let  $M$  be a compact Riemannian manifold. The eigenvalues of the Laplace operator  $\Delta$  acting on  $L^2(M)$  form a discrete sequence  $0 = \lambda_0 \leq \lambda_1 \leq \dots$ . The multiset of eigenvalues  $\{\lambda_i\}$  is the **Laplace spectrum** of  $M$ . We say that two manifolds are **isospectral** if their Laplace spectra coincide. A manifold is **absolutely spectrally rigid** if the only manifolds to which it is isospectral are in fact isometric.

In this article, we prove the following

**Theorem 1.** *Let  $k$  be a totally real number field with ring of integers  $R$ , and  $A$  a quaternion algebra over  $k$  with type number 1. Further, suppose that there is a unique real place of  $k$  at which  $A$  is unramified. Let  $\mathcal{O}$  be a maximal order in  $A$ ,  $\mathcal{O}^1$  the multiplicative group of norm 1 units. For an ideal  $I$  in  $R$ , let  $\mathcal{O}^1(I) \leq \mathcal{O}^1$  be the kernel of the reduction map. Denote by  $\Gamma$  (resp.  $\Gamma(I)$ ) the image of  $\mathcal{O}^1$  (resp.  $\mathcal{O}^1(I)$ ) in  $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{Isom}^+(\mathbb{H})$ . Suppose that  $I$  is not divisible by any prime in  $R$  at which  $A$  is ramified. Then the Riemann surface  $\Gamma(I) \backslash \mathbb{H}$  is absolutely spectrally rigid.*

## 2. AUDIBLE PROPERTIES

**2.1. Heat invariants.** Let  $M$  be a compact Riemannian manifold. Enumerate the spectrum of its Laplace-Beltrami operator as  $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ , where each eigenvalue is repeated according to its multiplicity. Pick an  $L^2(M)$ -orthonormal basis  $\{\varphi_j\}$  of corresponding eigenfunctions. The **heat kernel** of  $M$  is the function  $K : \mathbb{R}_{>0} \times M \times M$  defined by

$$(1) \quad K(t, x, y) = \sum_j e^{-t\lambda_j} \varphi_j(x) \varphi_j(y).$$

**2.2. Scope: Riemannian manifolds.**

**Theorem 2.** *Volume is an audible property.*

*TODO: use asymptotics of  $\Theta_M$ .*

□

**Theorem 3.** *Dimension is an audible property.*

*TODO: use asymptotics of  $\Theta_M$ .*

□

**2.3. Scope: smooth, compact Riemannian surfaces.**

**Theorem 4.** *Suppose  $\dim M = 2$  and that  $M$  is smooth. The homeomorphism class of  $M$  is audible.*

*TODO: gauss-bonnet.*

□

**Theorem 5.** *Suppose  $\dim M = 2$ , and that  $M$  is smooth. The property of having constant curvature is audible.*

*TODO: GCB the metric of constant curvature minimizes topological entropy, which is a spectral invariant.*

□

#### 2.4. Scope: Riemann surfaces.

**Theorem 6.** *Suppose  $M$  is a compact Riemann surface. Then arithmeticity is audible.*

*TODO: apply Takeuchi.*

□

#### 2.5. Scope: Arithmetic Riemann surfaces.

**Theorem 7.** *Suppose  $M$  is an arithmetic Riemann surface. The commensurability class of  $M$  is audible.*

*TODO: apply Reid.*

□

### 3. THE PROOF

Now let us fix all of data  $(k, A, \mathcal{O}, I)$  as in the main theorem. Sequentially applying the theorems of the preceeding section, we may assume that a manifold  $M$  isospectral to  $\Gamma(I) \backslash \mathbb{H}$  is of the form  $M = \Lambda \backslash \mathbb{H}$  for some subgroup  $\Lambda$  of  $\Gamma$  with the same index as  $\Gamma(I)$ .

To prove the theorem, it suffices to show that  $\Lambda$  is in fact conjugate (in  $A^\times$ ) to  $\Gamma(I)$ . To this end, we convert the problem into a local one.

[TODO: get to the prime power setting].

### 4. ARITHMETIC SUBGROUPS OF ALGEBRAIC GROUPS

A matrix group  $G \leq \mathrm{GL}(n, \mathbb{C})$  is said to be **algebraic** if it consists of all invertible matrices whose coefficients annihilate some set of polynomials on  $M(n, \mathbb{C})$ . If this set of polynomials can be taken with coefficients in some subring  $R \leq \mathbb{C}$ , then this group is said to be **defined over  $R$** .

**4.1. Base case.** Let  $X$  denote the **Bruhat-Tits tree** for  $\mathrm{SL}_2(k)$ . The vertices of  $X$  are homothety classes of lattices in  $k^2$ , and two vertices  $x, y \in X$  are adjacent if there exist lattices  $L_x \in x$  and  $L_y \in y$  so that  $L_y \leq L_x$  and  $L_y/\pi L_x$  is a  $\mathfrak{k}$ -line in  $L_x/\pi L_x \approx \mathfrak{k}^2$ . We write  $d : X \times X \rightarrow \mathbb{Z}_{\geq 0}$  for the graph-theoretic distance function on  $X$ .

For a lattice  $L \leq k^2$ , we write  $[L]$  for the vertex in  $X$  corresponding to  $L$ . For a subset  $A \subset X$  of vertices, write

$$\begin{aligned} B(A, r) &= \{y \in X : d(x, y) \leq r, \forall x \in A\} \\ S(A, r) &= \{y \in X : d(x, y) = r \forall x \in A\} \end{aligned}$$

for the ball and sphere about  $A$  of radius  $r$ , respectively.

A **geodesic** is a non-backtracking path in  $X$ . The length of a geodesic  $c$  is  $\sup_{x,y \in c} d(x,y) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ . For vertices  $x, y \in X$  write  $[x, y]$  for the unique geodesic starting at  $x$  and finishing at  $y$ .

For a subset  $A \subset X$  we say that a vertex  $x \in A$  is an **interior point** of  $A$  and write  $x \in A^{\text{int}}$  if  $B(x, 1) \subset A$ . Otherwise we say  $x$  is a **boundary point** of  $A$  and write  $x \in \partial A$ . We say  $x$  is an **extremal point** in  $A$  if either:  $A = \{x\}$  or  $\text{Card}(S(x, 1) \cap A) = 1$ .

Let  $\mathcal{G} = \text{SL}_2$ , and write  $G = \mathcal{G}(k)$  and  $K = \mathcal{G}(R)$ .  $G$  acts on the set of lattices in  $k^2$ , and this action passes to one on  $X$ . For a subgroup  $H \leq G$ , we write  $X^H$  for its fixed point set on  $X$ . For a subset  $A \subset X$ , and a subgroup  $H \leq G$ , we write  $H_A$  for its (pointwise) stabilizer in  $H$ .

For each vertex  $x$  and integer  $n \geq 1$ , we have a  $R/\mathcal{P}^n$ -representation  $\rho_x : G_x \rightarrow \text{SL}(L_x/\pi^n L_x) \approx (R/\mathcal{P}^n)$

Upon picking a basis  $e_1, e_2$  for  $L_o$  we may identify  $G_{x_o}$  with  $\text{SL}_2(R)$ . The action of  $\text{SL}_2(k)$  on  $X$  has exactly two orbits. Letting  $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ , the set  $\{x_o, \alpha x_o\}$  constitutes a fundamental domain for this action. The respective stabilizers  $G_{x_o}$  and  $G_{\alpha x_o} = \alpha G_{x_o} \alpha^{-1}$  are representatives of the two conjugacy classes of maximal compact subgroups. The following lemma is standard:

**Lemma 1.** *For a vertex  $x \in X$ , there is a  $G_x$  equivariant bijection between  $n$ -sphere  $S(x, n)$  and cyclic  $R/\mathcal{P}^n$ -submodules of  $L_x/\pi^n L_x$ . Using a basis for  $L_x$ , the latter set is naturally identified with  $\mathbb{P}^1(R/\mathcal{P}^n)$*

**Lemma 2.** *Let  $H \leq G$  be a subgroup such that  $X^H$  is nonempty.*

- (1) *For a subgroup  $H \leq G$  and a vertex  $x \in X^H$ , the bijection in lemma 1 restricts to one between  $H$ -fixed vertices  $X^H \cap S(x, n)$ , and  $H$ -stable  $R/\mathcal{P}^n$ -submodules of  $L_x/\pi^n L_x$ .*
- (2) *For  $x \in X^H$ , one has  $B(x, n) \leq X^H$  if and only if  $H$  acts by scalars under  $\rho_{x,n}$ .*
- (3) *A vertex  $x$  is a non-extremal boundary point of  $X^H$  if and only if  $\rho_{x,1}$  decomposes as a direct sum of (distinct) nontrivial characters.*
- (4) *A vertex  $x$  is extremal in  $X^H$  if and only if the  $\mathfrak{k}$ -representation  $\rho_{x,1}$  is reducible but indecomposable.*

*Proof.* item 1 and item 2 follow immediately from the definitions and the equivariance in lemma 1.

item 3: Suppose  $x$  is a nonextremal boundary point of  $X^H$ . There are then exactly two vertices  $y, z \in X^H$  adjacent to  $x$ . Under the identification of  $S(x, 1)$  with  $\mathbb{P}^1(\mathfrak{k})$ , we see that  $H$  must act as a group of hyperbolic transformations with common fixed points  $y, z$ .

item 4 Suppose  $x$  is extremal in  $X^H$ . Then  $x$  has exactly one neighbor in  $X^H$ . Under the identification of  $S(x, 1)$  with  $\mathbb{P}^1(\mathfrak{k})$ , we find that  $H$  must act as a group of parabolic transformations.  $\square$

**Proposition 1.** *[ref ribetGL2] Let  $g \in \text{SL}_2(R)$  and  $x \in \partial X^{(g)}$ . For an integer  $n > 0$ , the following are equivalent.*

- (1) *The characteristic polynomial  $p_g(T)$  is reducible mod  $\mathcal{P}^n$ .*
- (2) *There exists a geodesic of length  $n$  based at  $x$  and contained in  $X^{(g)}$ .*

*Proof.* Assuming item 1, pick  $\alpha, \beta \in R$  such that  $p_g(T) \equiv (T - \alpha)(T - \beta) \pmod{\mathcal{P}^n}$ . Then since  $x$  is a boundary point,  $g$  does not act as a scalar on  $L_x/\pi L_x$ . Consequently,  $L_x/\pi L_x$  is a cyclic  $R[g]$  module. Pick a vector  $v \in L_x$  such that  $\{v, gv\}$  projects to a basis of  $L_x/\pi L_x$ . By Nakayama's lemma,  $\{v, gv\}$  projects to a basis of  $L_x/\pi^n L_x$ .

Let  $w = (g - \beta)v$ . Then, compute modulo  $\mathcal{P}^n$ :

$$\begin{aligned} gw &\equiv (g^2 - g\beta)v \\ &\equiv ((\alpha + \beta)g - \alpha\beta g - g\beta)v \\ &\equiv \alpha(g - \beta)v \\ &\equiv \alpha w \end{aligned}$$

Thus, the cyclic  $R/\mathcal{P}^n$ -submodule of  $L_x/\pi^n L_x$  spanned by  $w$  is  $g$  stable. The corresponding vertex lies in  $X^{(g)}$  and is at distance  $n$  from  $x$  as desired.

Now assume that  $y \in X^{(g)}$  has  $d(x, y) = n$ . Pick a lattice  $L_y$  representing  $y$  with  $\pi^n L_x < L_y < L_x$ . Then  $L_y$  projects to a  $g$ -stable free rank 1  $R/\mathcal{P}^n$  submodule of  $L_x/\pi^n L_x$  on which  $g$  must act by an element  $\alpha \in (R/\mathcal{P}^n)^\times$ . It follows that  $(t - \alpha)$  divides  $p_g(t)$  modulo  $\mathcal{P}^n$ , proving reducibility.  $\square$

**Proposition 2.** *Let  $g \in \mathrm{SL}_2(R)$ . For an integer  $n > 0$ , the following are equivalent:*

- (1)  $\mathrm{tr}(g) = \pm 2 \pmod{\mathcal{P}^{2n}}$
- (2) *There is a point  $x \in X^g$  such that  $B(x, n) \subset X^g$ .*

*Proof.* The argument is by induction on  $n$ .

Base case: suppose  $\mathrm{tr}(g) = 2 \pm 2 \pmod{\mathcal{P}^2}$ . By proposition 1 there is a geodesic  $\gamma$  of length 2 contained in  $X^g$ . Conjugating by an element of  $\mathrm{GL}_2(k)$  if needed, we may assume that  $\gamma$  takes the form  $(y, x_o, z)$ . Since  $g$  has two fixed points in the 1 neighborhood of  $x_o$ , it acts semisimply on  $L_{x_o}/\pi L_{x_o}$  say with eigenvalues  $\alpha, \beta \in R/\mathcal{P}$ . These eigenvalues must satisfy  $\alpha + \beta = \pm 2 \pmod{\mathcal{P}}$  and  $\alpha\beta = 1 \pmod{\mathcal{P}}$ , but then  $g$  acts as  $\pm \mathrm{id}$  and the claim follows.

Suppose now that for all  $k < n$  that  $\mathrm{tr}(g) = \pm 2 \pmod{\mathcal{P}^{2k}}$  iff and only if  $g$  fixes some  $k$  ball in  $X$ . Let  $\mathrm{tr}(g) = \pm 2 \pmod{\mathcal{P}^{2n}}$ . Applying proposition 1 as before, there is some geodesic  $\gamma$  passing through  $x_o$  of length  $2n$  which is fixed pointwise by  $g$ .

$\square$

**4.2. Some projective geometry.** Let  $F$  be a field, let  $V$  be a two dimensional vectorspace over  $F$ . The **projective line** over  $F$  is the set of  $F$ -lines in  $V$ . There are several useful coordinate systems that one can put on  $\mathbb{P}^1(F)$ .

The first system requires no choices: the association of a nonzero vector  $v$  with the  $F$ -line that it spans yeilds a surjection  $V \setminus \{0\} \rightarrow \mathbb{P}^1(F)$ . The multiplicative group  $F^\times$  acts transitively on the fibers, yeilding our first identification:  $\mathbb{P}^1(F) = F^\times / (V - \{0\})$

A bijection  $\mathbb{P}^1(F) \rightarrow \mathbb{P}^1(F)$  is a **projective transformation** if it lifts to a linear automorphism of  $V$ . Write  $G'$  for the group of projective transformations.

**Lemma 3.** *A nonidentity projective transformation can have at most 2 fixed points on  $\mathbb{P}^1(F)$*

*Proof.* The points in  $\mathbb{P}^1(F)$  fixed by  $g$  correspond to eigenlines of its lifts to  $\mathrm{GL}(V)$ . Since  $V$  is two dimensional, it supports at most two linearly independent lines.  $\square$

We classify nonidentity transformations accordingly:

- say  $g$  is **hyperbolic** if it fixes two points on  $\mathbb{P}^1(F)$
- say  $g$  is **parabolic** if it fixes exactly one point on  $\mathbb{P}^1(F)$
- say  $g$  is **elliptic** if it fixes no points in  $\mathbb{P}^1(F)$ .

The following lemma demonstrates how to determine the category of a projective transformation in terms of its lifts. For a linear transformation  $\tilde{g} \in \mathrm{GL}(V)$  write  $p_{\tilde{g}}(T) = \det(\tilde{g} - T \mathrm{id}) \in F[T]$  for its characteristic polynomial and  $\delta(\tilde{g}) = \mathrm{tr}(\tilde{g})^2 - 4 \det(\tilde{g})$  for its discriminant.

**Lemma 4.** *Let  $g$  be a nonidentity projective transformation.*

- *$g$  is hyperbolic if and only if any lift  $\tilde{g}$  is semisimple and diagonalizable over  $F$ , for any lift  $\tilde{g}$ . This is so if and only if  $\delta(\tilde{g}) \in (F^\times)^2$ .*
- *$g$  is parabolic if and only if  $\tilde{g}$  is not semisimple for any lift  $\tilde{g}$ . This is so if and only if  $\delta(\tilde{g}) = 0$ .*
- *$g$  is elliptic if and only if  $\tilde{g}$  is not semisimple for any lift  $\tilde{g}$ . This is so if and only if  $\delta(\tilde{g}) \in F^\times \setminus (F^\times)^2$ .*

**Lemma 5.** *For a subgroup  $H \leq \mathrm{SL}(2, k)$ , suppose  $X(H)$  contains a geodesic segment  $[y, z]$  of length two. Denote its midpoint  $x$ . Then there exists a character  $\chi : H \rightarrow \mathfrak{f}^\times$  such that  $\rho_x \approx \chi \oplus \chi^{-1}$ .*