DECOMPOSING UNITARY REPRESENTATIONS OF LOCALLY COMPACT GROUPS

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Let G be a locally compact group, which we will always assume to be Hausdorff and second countable. Let D be the collection of equivalence classes of irreducible unitary representations π such that for any (hence all) nonzero $u, v \in \mathcal{H}_\pi$ the function $\varphi_{\pi,u,v} : x \mapsto \langle \pi(x)u, v\rangle_\pi$ is in $L^2(G)$. If $\pi \in D$, then $\varphi_{\pi,u,v}$ generates a subrepresentation (a closed invariant subspace) of the right regular representation R in $L^2(G)$ that is isomorphic to π. Let $\mathcal{E}_\pi$ denote the image in $L^2(G)$ of the functions $\varphi_{\pi,u,v}$. Define two subspaces of $L^2(G)$:

$$L^2_{\text{disc}}(G) = \bigoplus_{\pi \in D} \mathcal{E}_\pi \qquad \text{(Hilbert space direct sum)}$$
$$L^2_{\text{cts}}(G) = (L^2(G)_{\text{disc}})^\top,$$

We call these the discrete and continuous parts of $L^2(G)$ respectively. By construction, $L^2_{\text{disc}}(G)$ is absolutely reducible: if $C(\pi, R)$ denotes the space of unitary operators intertwining π and R, set $m(\pi) = \dim C(\pi, R)$. Then

$$L^2_{\text{disc}}(G) \approx \bigoplus_{\pi \in D} \pi^{\oplus m(\pi)}.$$

Whereas, by construction $L^2_{\text{cts}}(G)$ has no irreducible subrepresentations.

In these notes, I follow Folland (A course in abstract harmonic analysis) showing that $L^2_{\text{cts}}(G)$ decomposes as a *direct integral* (a generalization of direct *sum*) over irreducible representations.

1 Direct integrals

1.1 ...of trivial Hilbert bundles

Let $(A, \mathcal{M}, \mu)$ be a measure space, and $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}}, |\cdot|_{\mathcal{H}})$ a Hilbert space which we will always assume to be separable. We want to form a new Hilbert space 'over A.' A naive approach is to make the underlying vectorspace

$$\left[\int_A^{\oplus} \mathcal{H}\, \mathrm{d}\mu\right]_{\text{Naive}} = \bigoplus_{\alpha \in A} \mathcal{H}.$$

The elements of this space are then finitely supported $\mathcal{H}$ valued functions on A with norm given by $|f|^2_{\text{Naive}} = \sum_{\alpha \in A} |f(\alpha)|^2_{\mathcal{H}}$. In the case that A is uncountable, there is no way that this will be complete.

The missing ingredient from the naive approach was the measure-structure on A. Rather than requiring that the functions be finitely *supported*, we instead require that the functions be *square-integrable*, in the sense that the function

$$\alpha \mapsto |f(\alpha)|_{\mathcal{H}}^2$$

is square-integrable over A. This is the right idea: define the **direct integral** of $\mathcal{H}$ over A with respect to μ to be

$$\int_A^{\oplus} \mathcal{H}\,\mathrm{d}\mu = L^2(A,\mathcal{H},\mu) = \left\{ f : A \to \mathcal{H} \mid |f|^2 = \int_A |f(\alpha)|_{\mathcal{H}}^2\,\mathrm{d}\mu(\alpha) < \infty \right\} / \ker(|\cdot|^2).$$

This is the direct integral of the **trivial Hilbert bundle** in the sense that the Hilbert space $\mathcal{H}$ is fixed as we vary over A. We need to generalize this to allow a family of spaces H_α for $\alpha \in A$.

1.2 ... of general Hilbert bundles

Throughout this section, fix the following data and terminology:

- **The 'base space'**: $(A, \mathcal{M}, \mu)$ a measure space
- **The 'fibers'**: For each $\alpha \in A$ a separable Hilbert space $\mathcal{H}_\alpha$ with norm and inner product denoted $|\cdot|_\alpha$ and $\langle \cdot,\cdot\rangle_\alpha$ respectively.
- **A choice of 'basis sections'**: A countable set $\mathcal{E}$ of maps $e : A \to \coprod_{\alpha\in A} \mathcal{H}_\alpha$ such that
  (1) $e_\alpha \in \mathcal{H}_\alpha$ for all $\alpha \in A$ (think: section of a bundle).
  (2) For fixed $e, e' \in \mathcal{E}$, the map $\alpha \mapsto \langle e_\alpha, e'_\alpha\rangle$ is measurable.
  (3) For a fixed $\alpha \in A$, the set $\{e_\alpha\}$ is a Hilbert space basis of $\mathcal{H}_\alpha$.

This data defines a **Hilbert bundle** over A. A function $f : A \to \coprod_{\alpha\in A} \mathcal{H}_\alpha$ such that $f_\alpha \in \mathcal{H}_\alpha$ is called a **section**. A section is **measurable** if the map $\alpha \mapsto \langle f_\alpha, e_\alpha\rangle_\alpha$ is, for each $e \in \mathcal{E}$.

Remark 1. In practice, Hilbert spaces are typically spaces of functions. Since sections are then 'function valued functions,' I denote evaluation at $\alpha \in A$ as a subscript.

Definition 1. The **direct integral** of the Hilbert bundle above (with respect to μ) is

$$\int_A^{\oplus} \mathcal{H}_\alpha\,\mathrm{d}\mu(\alpha) = \{ f : A \to \coprod_{\alpha\in A} \mathcal{H}_\alpha | \quad f_\alpha \in \mathcal{H}_\alpha, \quad \int_A |f_\alpha|_\alpha^2\,\mathrm{d}\mu(\alpha) < \infty \} / \sim$$

where $\sim$ denotes agreement away from set of measure zero. The elements of the direct integral are called **square integrable sections** of the Hilbert bundle.

Verification that this is a Hilbert space, with the inner product $\langle f, g\rangle = \int_A \langle f_\alpha, g_\alpha\rangle_\alpha\,\mathrm{d}\mu(\alpha)$ is roughly the same argument that $L^2(A,\mu)$ is, along with Lebesgue's dominated convergence theorem.

The following are some quick exercises that I think are pretty important to grasp 'what's going on.'

Exercise 1.
- Let A be a countable set, μ the counting measure on A, with arbitrary (separable) $\mathcal{H}_\alpha$ for all $\alpha \in A$. Show

$$\int_A^{\oplus} \mathcal{H}_\alpha\,\mathrm{d}\mu(\alpha) \approx \bigoplus_{\alpha\in A} \mathcal{H}_\alpha.$$

- Define a map $H_{\alpha'} \to \int^{\oplus} \mathcal{H}_\alpha \, \mathrm{d}\mu(\alpha)$ by $v \mapsto f$ where f is the function on A such that $f_\alpha = 0$ unless $\alpha = \alpha'$, whereat $f_{\alpha'} = v$. For arbitrary μ on A, why is this not necessarily an embedding?

1.3 ... of operators

Consider a collection of unitary operators $T_\alpha \in \mathcal{L}(\mathcal{H}_\alpha)$, (the latter is the space of bounded operators on $\mathcal{H}_\alpha$) such that for any measurable section $\alpha \mapsto f_\alpha$, the section $\alpha \mapsto T_\alpha f_\alpha$ is measurable. Such a collection is called a **measurable field of operators**. Such a field gives rise to a unitary[1] operator on $\int_A^{\oplus} \mathcal{H}_\alpha \, \mathrm{d}\mu(\alpha)$ called the **direct integral of the field** T, denoted $\int_A^{\oplus} T_\alpha \, \mathrm{d}\mu(\alpha)$ or just $\int^{\oplus} T$ if the rest of the data is clear. This operator acts on square integrable sections fibre–wise,

$$\left[\left(\int_A^{\oplus} T_\alpha \, \mathrm{d}\mu(\alpha) \right) f \right]_\alpha = T_\alpha f_\alpha .$$

Exercise 2.
- Let A be a finite set, and μ be the counting measure. Fix a Hilbert space $\mathcal{H}$ of dimension 1. Characterize the operators in $\int_A^{\oplus} \mathcal{H} \, \mathrm{d}\mu(\alpha)$ that arise as the direct integral of $|A|$ operators on $\mathcal{H}$.

1.4 ... of representations

Let G be a locally compact group and consider a collection of unitary representations $\pi_\alpha : G \mapsto \mathcal{L}(\mathcal{H}_\alpha)$. Further, suppose that for each $x \in G$ the field of operators $\pi_\alpha(x)$ is measurable, so that $\int^{\oplus} \pi_\alpha(x) \, \mathrm{d}\mu(\alpha)$ defines a unitary operator on $\int^{\oplus} \mathcal{H}_\alpha \, \mathrm{d}\mu(\alpha)$. We call such a collection a **field of unitary representations**. The map $x \mapsto \int^{\oplus} \pi(x) \, \mathrm{d}\mu(\alpha)$ then defines a unitary representation[2] of G on $\int^{\oplus} \mathcal{H}_\alpha \, \mathrm{d}\mu(\alpha)$.

2 Decomposing unitary representations

The theorem of this section, which decomposes a unitary representation into a direct integral of a field of representations, is phrased in terms of a choice of commutative (C^*, weakly closed) algebra of intertwining operators. Let's look at how this manifests in the simplest case:

Consider a finite group G, with an irreducible unitary representation of π on finite dimensional Hilbert space V. Now suppose τ is a unitary representation of G on W which is unitarily equivalent to $\pi \oplus \pi$ on $V \oplus V$. The algebra of intertwining operators $\mathrm{Hom}_\tau(W, W)$ is isomorphic to $M_2(\mathbb{C})$ (basically Schur's lemma).

Claim 1. A choice of isomorphism $\tau \approx \pi \oplus \pi$ is equivalent to a choice of maximal commutative subalgebra of $\mathrm{Hom}_\tau(W, W)$.

Proof. Given an isomorphism $\varphi : W \to V \oplus V$ intertwining τ and $\pi \oplus \pi$, take the preimage of the algebra of diagonal matrices in $\mathrm{End}(V) \oplus \mathrm{End}(V)$.

Given a maximal commutative subalgebra B of $\mathrm{Hom}_\tau(W, W)$, all operators in B can be simultaneously diagonalized. Since $\mathrm{Hom}_\tau(W, W)$ is isomorphic to $M_2(\mathbb{C})$, B must be two dimensional,

[1] Exercise: check this

[2] Exercise: check this.

so pick basis vectors T, S. Since T, S are linearly independent, there must be an eigenspace W_1 on which their eigenvalues are distinct. Since T, S are intertwining operators W_1 is a τ invariant subspace (which is necessarily proper). Conclude that W_1 is equivalent to V, and by Schur, the equivalence is unique up to scale. The same argument shows that the orthogonal complement W_2 to W_1 is also equivalent to V. After taking some linear combination of T and S, we may assume that T and S are projections onto W_1 and W_2 respectively. Then $T \oplus S$ provides a G isomorphism $W \to W_1 \oplus W_2 \to V \oplus V$. $\square$

Paraphrasing: to decompose W into its irreducible subrepresentations, one must make a choice of orthogonal projections onto its invariant subspaces, which is a commutative subalgebra of intertwining operators. Now suppose G is a locally compact group and $\pi = \int^{\oplus} \pi_\alpha$ on $\mathcal{H} = \int^{\oplus} \mathcal{H}_\alpha$. Take a measurable subset $E \subset A$ and let χ_E be its characteristic function. Define an operator on $\int^{\oplus} \mathcal{H}_\alpha$ by $[M_E f]_\alpha = \chi_E(\alpha) f_\alpha$. Then[3] M_E is a projection onto a closed π–invariant subspace of $\mathcal{H}$. Note that the collection of M_E for all measurable E commute with one another, and will generate a commutative C^* algebra of intertwining operators.

The reference to such algebras in the following theorem is what allows one to modulate the fine-ness of the direct integral decomposition. On one extreme, a maximal commutative subalgebra of intertwining operators implies that the representations appearing in the direct integral are *irreducible*. On the other, the zero subalgebra makes the theorem vacuous.

Theorem 1. Let G be a locally compact group, π be a unitary representation on a separable Hilbert space $\mathcal{H}$, and B a weakly closed commutative C^* algebra of intertwining operators $\mathcal{H} \to \mathcal{H}$. Then there exists

- a measure space $(A, \mathcal{M}, \mu)$,
- a field of Hilbert spaces $\mathcal{H}_\alpha$ over A,
- a field of representations $\pi_\alpha \in \mathcal{L}(\mathcal{H}_\alpha)$ over A,
- a unitary isomorphism $U : \mathcal{H} \to \int_A^{\oplus} \mathcal{H}_\alpha \, \mathrm{d}\mu(\alpha)$

such that

$$U \pi U^{-1} = \int_A^{\oplus} \pi_\alpha \, \mathrm{d}\mu(\alpha)$$

and

$$UBU^{-1} \text{ is the algebra of diagonal operators on } \int^{\oplus} \mathcal{H}_\alpha \, \mathrm{d}\mu(\alpha)$$

By analogy to the Euclidean Fourier transform (of which this is a direct generalization), denote $U(\cdot)$ by $\hat{\cdot}$. Remind yourself that this means $\hat{v}$ is a function on A, taking a value in H_α at $\alpha \in A$. This theorem says that the action of π on v is given by

$$a\pi(x)v = U^{-1} \left(\alpha \mapsto \pi_\alpha(x)[\hat{v}(\alpha)] \right)$$

and for any operator $T \in B$, there exists a $\varphi \in L^\infty(A)$ such that

$$Tv = U^{-1} \left[\alpha \mapsto \varphi(\alpha) \hat{f}(\alpha) \right].$$

[3] Exercise: check this

2.1 A brief interlude on the unitary dual

As a set, denote by $\hat{G}$ the collection of irreducible unitary representations modulo unitary equivalence. In this section, I will briefly describe a σ–algebra of subsets of $\hat{G}$ which, when G is 'good,' will let $\hat{G}$ serve as A universally in the theorem above.

For each $n < \infty$, let $\mathcal{H}_n$ denote a fixed Hilbert space of dimension n (say $\mathbb{C}^n$ with the Euclidean inner product), and let $\mathcal{H}_\infty$ denote a fixed infinite dimensional separable Hilbert space (say $\ell^2(\mathbb{Z})$). Now for each n let $\mathrm{Irr}_n(G)$ denote the collection of irreducible unitary representations of G on $\mathcal{H}_n$. Note: we are not identifying unitarily equivalent representations *yet*. Let X_n denote the collection of matrix coefficient functions on $\mathrm{Irr}_n(G)$; i.e. maps $\mathrm{Irr}_n(G) \to \mathbb{C}$ of the form

$$\pi \mapsto \langle \pi(x)u, v\rangle.$$

Then set B_n to be the σ-algebra on $\mathrm{Irr}_n(G)$ generated by X_n; i.e. the smallest σ–algebra containing *all* preimages of subsets of $\mathbb{C}$ under *all* maps in B_n.

Now let

$$\mathrm{Irr}(G) = \bigcup_{n\leq\infty} \mathrm{Irr}_n(G).$$

Because we have not made any identifications, this union is disjoint. Define a σ-algebra on B by

$$E \in B \iff E \cap \mathrm{Irr}_n(G) \in B_n \text{ for all } n.$$

Now let $\cdot \mapsto [\cdot]$ be the map taking a unitary representation to its unitary equivalence class, which is a surjection $\mathrm{Irr}(G) \to \hat{G}$. Then define a σ–algebra, called the **Borel–Mackey** structure on $\hat{G}$, by pulling back along $[\cdot]$.

Aside: there is also a topology on $\hat{G}$, called the Fell topology. The σ–algebra of Borel subsets with respect to the Fell topology is coarser than the Borel–Mackey algebra. In particular, singletons are in the Borel–Mackey algebra, but need not be in the Borel–Fell algebra. In any case, one should expect that the Fell topology is not Hausdorff at a handful of points, arising when a 'continuous' family of unitary irreps degenerate.

2.2 Some representation theoretic definitions

To strengthen the decomposition theorem above, we will need to isolate a class of groups for which $\hat{G}$ is 'good.' We need some definitions to make this precise.

A measurable space $(X, \mathcal{M})$ is **standard** if it is measurably isomorphic to a Borel subset of a complete separable metric space[4].

A unitary representation π of G is **primary** if the center of its algebra of intertwining operators is trivial (scalar multiples of the identity). Schur says that irreducible representations are primary, and if π is completely reducible (as a direct sum) then it is primary if and only if all of its irreducible subrepresentations are unitarily equivalent (the example of τ on W above is an example).

[4]Astonishingly (to me at least), there are only two options for such spaces: either X is countable and $\mathcal{M} = 2^X$, or X is measurably isomorphic to $[0, 1]$ with its Euclidean topology and its σ-algebra of Borel sets.

A group G is said to be **type I** if every primary representation is a direct *sum* of some irreducible subrepresentation. *These are the 'good' groups*

Theorem 2. A locally compact group G is type I if and only if its Borel-Mackey σ–algebra is standard.

2.3 Strengthening the decomposition

For each $n \leq \infty$, let $\hat{G}_n \subset G$ be the collection of unitary equivalence classes of n dimensional irreducible unitary representations of G. As before $\mathcal{H}_n$ is a choice of fixed Hilbert space of dimension n. Then, with respect to the Borel–Mackey structure, there is a canonical 'locally trivial' measurable field of Hilbert spaces $\{H_p\}$ sitting over $\hat{G}$. That is, $\hat{G} = \coprod_{n\leq\infty} \hat{G}_n$ is a partition into measurable subsets such that $\mathcal{H}_p = \mathcal{H}_n$ for $p \in \hat{G}_n$.

Further, essentially by definition, there is a measurable field of representations $\{\pi_p\}$ over $\hat{G}$ acting on the canonical field of Hilbert spaces $\{H_p\}$ such that $\pi_p \in p$ for all equivalence classes $p \in \hat{G}$. One must be a little careful in showing that the choice of representatives π_p can be made measurably in p.

Theorem 3. Suppose that G is locally compact and type 1, π a unitary representation of G on a separable Hilbert space, and $\mathcal{H}_p$, π_p are the fields over $\hat{G}$ discussed in the preceding two paragraphs. Then there exist pairwise disjointly supported finite measures $\mu_1, ..., \mu_\infty$ on $\hat{G}$ such that

$$\pi \approx \bigoplus_{n\leq\infty} n \cdot \rho_n$$

where $n\rho_n$ denotes n copies of $\int^{\oplus} \pi_p \,\mathrm{d}\mu_n(p)$.

2.4 Plancherel

Let $J^1 = L^1(G) \cap L^2(G)$ and let J^2 be the linear span of $f * g$ for $f, g \in J^1$. For $f \in J^1$ define the Fourier transform

$$\hat{f}(\pi) = \int f(x)\pi(x^{-1})\,\mathrm{d}x.$$

We think of $\hat{f}(\pi)$ as element of $\mathcal{H}_\pi \otimes \overline{\mathcal{H}_\pi}$. This is the same as viewing $\hat{f}(\pi)$ as a trace class operator on $\mathcal{H}_\pi$. Then we have the Plancherel's theorem:

Theorem 4. Let G be type 1 and unimodular. Then there is a measure μ on $\hat{G}$, uniquely determined by a choice of Haar on G, with the following properties:

- The Fourier transform maps J^1 unitarily into $\int^{\oplus} \mathcal{H}_\pi \otimes \mathcal{H}_{\overline{\pi}} \,\mathrm{d}\mu$ which extends to a unitary isomorphism on $L^2(G)$.
- The Fourier transform intertwines the two sided regular representation τ with $\int^{\oplus} \pi \otimes \overline{\pi}\,\mathrm{d}\mu(\pi)$.
- For $f, g \in J^1$

$$\int_G f(x)\overline{g}(x)\,\mathrm{d}x = \int_{\hat{G}} \mathrm{tr}[\hat{f}(\pi)\hat{g}(\pi)^*]\,\mathrm{d}\mu(\pi)$$

- For $h \in J^2$ one has

$$h(x) = \int \mathrm{tr}[\hat{\pi}(x)\hat{h}(\pi)]\,\mathrm{d}\mu(\pi).$$

3 explicit examples

In all the examples, G is of type 1. We apply the theorem to the unitary representation of G on $L^2(G)$ acting via right translation. The output of the theorem is always a measurable field of representations π_p on the canonical field of Hilbert spaces $\mathcal{H}_p$ over $\hat{G}$, along with a measure $\mathrm{d}\mu$. The measure is what we'll be looking at. Further, the collection of subrepresentations (not necessarily irreducible) correspond to elements of the direct integral supported on a measurable subset of the $\hat{G}$. The *irreducible* subspaces correspond to points in $\hat{G}$ with nonzero measure.

3.1 $L^2(\mathbb{R})$

Let $G = \mathbb{R}$. By Schur's lemma, any irreducible unitary representation of G must be one dimensional. For each $t \in \mathbb{R}$, define the character $\xi_t : \mathbb{R} \mapsto \mathrm{Aut}(\mathbb{C}) = \mathbb{C}^\times$ by $\xi_t(x) = e^{-2\pi i x t} =: \langle \xi_t, x\rangle$. This defines an irreducible unitary representation of G on $\mathbb{C}$, viewed as a 1 dimensional complex vectorspace. It is classical that $\hat{G} = \{\xi_t \mid t \in \mathbb{R}\} \approx \mathbb{R}$ where the identification is as topological groups (hence also as measure spaces). Let $\mathcal{H}_t = \mathbb{C}$ for all $t \in \mathbb{R} = \hat{G}$, and μ be the Lebesgue measure. Then for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ define a map U, taking values in $\int^\oplus \mathcal{H}_t\,\mathrm{d}\mu(t)$ by

$$[Uf]\,(t) = \int_{\mathbb{R}} \xi_t(x) f(x)\,\mathrm{d}\mu(x). \tag{1}$$

U is unitary on $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and extends to a unitary isomorphism $L^2(\mathbb{R}) \to \int^\oplus \mathcal{H}_t\,\mathrm{d}\mu(t)$, which we still denote by U. Then for any $f \in L^2(\mathbb{R})$, and $t \in \mathbb{R}$,

$$[UR_y f][(t) = \int^\oplus \xi_t(y)\,[Uf]\,(t)\,\mathrm{d}\mu(t) = [\xi_t(y)Uf](t)] = e^{2\pi i t y}\,[Uf]\,(t).$$

With respect to the Lebesgue measure, all points in $\hat{G}$ have measure zero, which is one explanation for why $L^2(\mathbb{R})$ has no *irreducible* subrepresentations. The non-irreducible subrepresentations are all of the form $U^{-1}\int_E^\oplus \mathcal{H}_t\,\mathrm{d}\mu(t) = \left\{f \in L^2(\mathbb{R}) \mid Uf \text{ is supported on } E\right\}$ where $E \subset \mathbb{R}$ is measurable. One should think about why the sequence of representations $U^{-1}\int_{(-1/n,1/n)}^\oplus \mathcal{H}_t\,\mathrm{d}\mu(t)$ does not 'converge' in any meaningful sense.

3.2 $L^2(\mathbb{R}/\mathbb{Z})$

We'll look at $L^2(\mathbb{R}/\mathbb{Z})$ as a unitary representation in two different ways. First, $\mathbb{R}/\mathbb{Z}$ is itself a compact abelian group, so the theorem applies to decompose $L^2(\mathbb{R}/\mathbb{Z})$ with respect to unitary irreps of $\mathbb{R}/\mathbb{Z}$. Second, $\mathbb{R}$ acts on $\mathbb{R}/\mathbb{Z}$ by translation, which induces a unitary action of $\mathbb{R}$ on $L^2(\mathbb{R}/\mathbb{Z})$. The theorem applies here too.

All irreducible finite dimensional unitary reps of $\mathbb{R}/\mathbb{Z}$ are 1 dimensional. Take a smooth function $f \in L^2(\mathbb{R}/\mathbb{Z})$ and identify it with a smooth $\mathbb{Z}$ periodic function on $\mathbb{R}$ which we still call f. Then observe that for any $x \in \mathbb{R}$ $\Delta f(x+t) = [\Delta f]\,(x+t)$ for each $t \in \mathbb{R}$, where $\Delta = d^2/dx^2$ is the one dimensional Laplacian. Since Δ commutes with translation, it stabilizes the irreducible subrepresentations. Since those subspaces are 1-dimensional, they must be eigenspaces for Δ. Integration by parts twice, on $\mathbb{R}/\mathbb{Z}$, shows that eigenvalues must be nonpositive real. That is to say, if f spans an irreducible subrepresentation for $\mathbb{R}/\mathbb{Z}$ in $L^2(\mathbb{R}/\mathbb{Z})$ then its lift to $\mathbb{R}$ must satisfy the following ordinary differential equation with initial conditions

$$f'' = -\lambda^2 f \quad \text{for some } \lambda^2 \in \mathbb{R}, \qquad f(0) = f(1).$$

All solutions to the differential equation on $\mathbb{R}$ are of the form $f(x) = e^{\pm i\lambda x}$, and the initial condition forces $e^{\pm i\lambda} = 1$, so $\lambda \in 2\pi i\mathbb{Z}$. If $\lambda = 2\pi i n \neq 0$ then $\xi_n(x) = e^{2\pi i n x}$ is an eigenfunction and spans an irreducible representation. If $\lambda = 0$ then f satisfies the differential equation $f'' = 0$ which means f' is constant. The only smooth periodic function f such that f' is constant is itself a constant function. We have shown that $\{[\xi_n] : n \in \mathbb{Z}\} \subset \widehat{\mathbb{R}/\mathbb{Z}}$. To show the opposite containment, use the fact that any unitary rep of an abelian group acts as a unitary character. Thus $\widehat{\mathbb{R}/\mathbb{Z}}$ is in bijection with $\mathbb{Z}$ and we'll take for granted that this is actually an isomorphism of topological groups. L:et $\mathcal{H}_n = \mathbb{C}$ for all $n \in \widehat{\mathbb{R}/\mathbb{Z}} = \mathbb{Z}$ and let η denote the counting measure. Then for $f \in L^2$, define an operator $U : L^2(\mathbb{R}/\mathbb{Z}) \to \int_{\mathbb{Z}}^{\oplus} \mathcal{H}_n \, \mathrm{d}\eta(n) = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n$ (Hilbert space direct sum) by

$$[Uf](n) = \int_{\mathbb{R}/\mathbb{Z}} \xi_n(x) f(x) \, \mathrm{d}\eta(x).$$

Then for any $f \in L^2(\mathbb{R}/\mathbb{Z})$ and $y \in \mathbb{R}/\mathbb{Z}$ we have

$$[UR_y f](n) = \xi_n(x) \, [Uf](n).$$

With respect to the counting measure all points in $\hat{G}$ have measure one, so all irreducible finite dimensional unitary representations of G appear in $L^2(\mathbb{R}/\mathbb{Z})$ as direct summands. All subrepresentations can be obtained as (Hilbert space) direct sums of irreducibles.

Now we look at $L^2(\mathbb{R}/\mathbb{Z})$ as a unitary representation of $\mathbb{R}$. The parameter space in the direct integral decomposition $\int_{\mathbb{R}}^{\oplus} \mathcal{H}_t \, \mathrm{d}\eta$ remains $\mathbb{R}$, but the measure $\mathrm{d}\eta$ will not be Lebesgue. Instead, it must be supported on the representations ξ_t which are invariant under translation by the subgroup $\mathbb{Z}$. That is, $\xi_t(x+1) = \xi_t(x)$ for all $x \in \mathbb{R}$. This is to say that $e^{2\pi i t(1+x)} = e^{2\pi i t x}$, whence we recover $t = n \in \mathbb{Z}$. Let η be the counting measure on this copy of $\mathbb{Z} \subset \mathbb{R} = \hat{G}$. The definition of our operator U is identical to that in equation (1), with $\mathrm{d}\eta$ replacing $\mathrm{d}\mu$. Now lets look at the 0 element of the direct integral. It is an equivalence class:

$$0 = \left\{ f : \mathbb{R} \to \mathbb{C} \mid \int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}\eta(x) = 0 \right\}.$$

These are precisely the functions supported away from $\mathbb{Z}$. Thus each class in $\int_{\mathbb{R}}^{\oplus} \mathcal{H}_t \, \mathrm{d}\eta(t)$ has exactly one representative supported on $\mathbb{Z}$. This provides the bijection between this decomposition and the preceding.

Note: The second example is the one which will apply to the decomposition of $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}))$ as a unitary rep of $G(\mathbb{A})$. The key point is that looking at functions on a *quotient* of G corresponds to a measure supported on a *subspace* of $\hat{G}$.

3.3 $L^2(\mathrm{SL}_2(\mathbb{R}))$

Let $G = \mathrm{SL}_2(\mathbb{R})$ and define subgroups

$$\begin{aligned} A &= \left\{ a_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \mid t \in \mathbb{R} \right\} \\ N &= \left\{ N_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \mid x \in \mathbb{R} \right\}, \\ K &= \left\{ r_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \mid \theta \in [0, 2\pi) \right\} \\ P &= AN \end{aligned}$$

The following is an exhaustive non-redundant list of irreducible unitary reps of G

- The trivial representation,
- The discrete series, $\delta_n^\pm$ $(n \geq 2)$. Let H_n^+ be the collection of holomorphic f on the upper half plane $\mathfrak{h}$ equipped with the norm $|f|_n^2 = \int_h |f(x+iy)|^2 y^n \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$. Define the representation by

$$\delta_n^+ \begin{bmatrix} a & b \\ c & d \end{bmatrix} f(\tau) = (cz+d)^{-n} f(\frac{az+b}{cz+d}).$$

  The action δ_n^- is the same as δ_n^+, but is on the space of *anti*-holomorphic functions.
- The mock discrete series (or limit of discrete series): Let $H_1^\pm$ be the space of holomorphic (resp. antiholomorphic) functions on $\mathfrak{h}$ such that

$$|f|_1^2 = \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^2 \,\mathrm{d}x < \infty.$$

  The representation is defined by the formula above.
- The unramified spherical principal series: for $s \geq 0$ define a character χ_t on P by $\chi_s(a_t n_x) = |e^t|^{is}$ and let $\pi_{is} = \mathrm{ind}_P^G(\chi_s)$.
- The aspherical principal series: same as before, but $s > 0$ and χ_s is twisted by the sign character.
- The complementary series: for $s \in (0, 1)$ let the space of k_s be all $f : \mathbb{R} \to \mathbb{C}$ so that

$$|f|_s^2 = s/2 \int f(x)\overline{f}(y)|x-y|^{s-1} \,\mathrm{d}x\,\mathrm{d}y < \infty$$

  and define the action

$$k_s \begin{bmatrix} a & b \\ c & d \end{bmatrix} f(x) = |cx+d|^{-1-s} f(\frac{az+b}{cz+d}).$$

So

$$\hat{G} = \{1\} \cup \{\delta_n^\pm : n \geq 1\} \cup \{\pi_{is}^+ : s \geq 0\} \cup \{\pi_{is}^- : s > 0\} \cup \{k_s : 0 < s < 1\}.$$

The plancherel measure is given by

$$\begin{aligned}
\mathrm{d}\mu(\pi_{it}^{+}) &= \frac{t}{2}\tanh\frac{\pi t}{2}\,\mathrm{d}t\\
\mathrm{d}\mu(\pi_{it}^{-}) &= \frac{t}{2}\coth\frac{\pi t}{2}\,\mathrm{d}t\\
\mathrm{d}\mu(\delta_n^{+}) &= \mathrm{d}\mu(\delta_n^{-}) = n-1.
\end{aligned}$$