Research Statement

Justin Katz

Introduction The primary focus of my research is the spectral geometry of Riemannian manifolds, with an emphasis on the locally symmetric spaces associated to arithmetic lattices in Lie groups. My work makes essential use of geometric topology, harmonic analysis, algebraic groups, number theory, arithmetic/algebraic/differential geometry, and automorphic forms. This topic interacts with class field theory, Galois cohomology and K–forms of linear algebraic groups, Bruhat–Tits theory, and the geometry/topology of symmetric and locally symmetric spaces. It has direct connections to Riemannian geometry, geometric analysis, algebraic/arithmetic geometry, and the Langlands' program.

Background The interplay of number theory and Riemanian geometry has a long history of producing incisive results in both fields. Underlying many of these interactions is a rich analogy between on the one hand: *a compact negatively curved Riemannian manifold M along with its system of finite covering spaces*; and on the other: *a ring* $\mathbf{o}$ *of integers inside an algebraic number field* $\mathbf{k}$*, along with their finite extensions*. Under this analogy *the prime ideals* $\mathbf{p}$ *of* $\mathbf{o}$ find their geometric counterparts among *the primitive closed geodesics γ in M*. Just as one can *measure the size of a prime* $\mathbf{p}$ *in* $\mathbf{o}$ via the absolute norm map $N_{\mathbf{k}}(\mathbf{p}) = \#(\mathbf{o}/\mathbf{p})$, so too can one *measure the size of a closed geodesic γ on M*, via its length $\ell_M(\gamma)$.

Associated to such an $\mathbf{o} \leq \mathbf{k}$, one introduces the **Dedekind zeta function,** defined in a suitable right half-plane as a product over prime ideals $\mathbf{p}$ in $\mathbf{o}$

$$\zeta_{\mathbf{k}}(s) = \prod_{\mathbf{p}} (1 - N_{\mathbf{k}}(\mathbf{p})^{-s})^{-1},$$

which encodes structural information about the asymptotics of the *prime counting function* $\pi_{\mathbf{k}}(x) = \#\{\mathbf{p} : N_{\mathbf{k}}(\mathbf{p}) < x\}$ in terms of the nature of the singularity of $\zeta_{\mathbf{k}}(s)$ at $s = 1$. This amounts to the Landau's extension of Euler's prime number theorem: $\pi_{\mathbf{k}}(x)$ is asympotically $x/\log x$. In his epoch-making paper [**?**], Selberg introduced a zeta function associated to a compact negatively curved riemannian manifold M, defined entirely by analogy, for s in a suitable right half plane, as a product over the set of primitive closed geodesics γ on M:

$$\zeta_M(s) = \prod_{\gamma} \prod_{k=0}^{\infty} (1 - \exp^{-(s+k)\ell_M(\gamma)}).$$

Exactly as before, one finds that the asymptotics of the *primitive geodesic counting function* $\pi_M(x) = \#\{\gamma : \ell_M(\gamma) < x\}$ are encoded in the behavior of $\zeta_M(s)$ as s approaches the abcissa of absolute convergence of ζ_M. As an inaugural application of his trace formula, he showed that for M a compact Riemann surface, one has a *prime geodesic theorem*: just as for primes in number fields, one has asymptotically $\pi_M(x) = x/\log x$. In the intervening years, Huber [5], Sarnak [10], and Margulis [6] proved, for increasingly general M, that these prime geodesic theorems persist.

The analytic properties of number theoretic zeta and L functions are deeply mysterious. Many foundational questions (some of which are several centuries old, and carry a hefty monetary bounty on their solution) remain unanswered. By contrast, the analytic properties of Selberg's zeta functions are inherent to their design. In particular, there is a structural description of the zeroes of $\zeta_M(s)$ in terms of the harmonic analysis on M. If Δ_M is the Laplace-Beltrami operator on M, then for $s \in \mathbb{C}$ away from the spectrum of Δ_M, the resolvent $(\Delta_M - s)^{-1}$ is a compact, self adjoint operator for s away from the spectrum of Δ_M. Selberg's insight in [**?**] is that, up to some topological fudge-factors, his zeta function $\zeta_M(s)$ is precisely the Fredholm determinant $\det(\Delta_M - s)^{-1})$. Many analytic features of $\zeta_M(s)$ are now evident: at least formally,

$$\zeta_M(s) = \det(\Delta_M - s)^{-1}$$

is the reciprocal of the characteristic polynomial of Δ_M. One expects, and indeed it is so, that the poles of $\zeta_M(s)$ must then occur at the eigenvalues of Δ_M, with the order at a pole λ coniciding with the dimension of the eigenspace E_λ. A convenient outcome of this simultaneous description of $\zeta_M(s)$ in terms of its Euler product and as a Fredholm determinant is that two (compact negatively curved) Riemannian manifolds M_1, M_2, are Laplace-isospectral (i.e. the functions $s \mapsto \dim\ker(\Delta_{M_1} - s)$ and $s \mapsto \dim\ker(\Delta_{M_2} - s)$ coincide identically on $\mathbb{C}$) if and only if their Selberg zeta functions coincide (i.e. the characteristic polynomials of Δ_{M_1} and Δ_{M_2} coincide).

In terms of Selberg's zeta function, the problem of spectral rigidity is easily posed:

Suppose M_1 and M_2 are two Riemannian manifolds such that their Selberg zeta functions coincide identically, i.e. $\zeta_{M_1} = \zeta_{M_2}$ as meromorphic functions. Must M_1 and M_2 be isometric?

Thinking of the assignment $\zeta_\cdot : M \mapsto \zeta_M$ as a meromorphic-function-valued function on the space of isometry classes of closed Riemannian manifolds, we say that M is **spectrally rigid** if $\zeta_\cdot$ is injective at M. Riemannian manifolds M which are not spectrally rigid are **spectrally flexible**. For flexible M, one wishes to understand the fiber of the $\zeta_\cdot$ at M. For example, what additional invariants must you compute in order to distinguish M from its spectral companions?

Past work: flexibility The failure of spectral rigidity was first observed in Milnor in 1964 [7], where he constructed a pair of isospectral, nonisometric 16 dimensional flat tori. In a sweeping generalization of Milnor's construction, Vignéras in [13] described a procedure for producing infinite families of compact hyperbolic 2 and 3 manifolds which fail to be spectrally rigid. Her methodology exploits a particular mode of failure of the local-to-global principle for rational conjugacy classes of maximal orders in a quaternion algebra over a number field with sufficiently complicated arithmetic. A critical step in her argument is an application of the Selberg trace formula to relate the eigenvalue spectrum on the associated locally symmetric spaces with their respective geodesic length spectrum; which in turn is closely related to the solution of an explicit family of diophantine equations. As her examples are arithmetic, it follows from work of Borel [2] that only finitely such pairs from her construction can have genus at most g.

In 1985, Sunada [12] constructed many more examples of isospectral, non-isometric manifolds; his construction produces positive dimensional subspaces of the moduli space of Riemann surfaces of genus g that fail to be spectrally rigid provided g is sufficiently large (e.g. if $g > 168$). His construction was motivated by an old construction by Gassman [4]. The construction is elementary, using finite covers and the existence of groups with pairs of subgroups satisfying a condition called **almost conjugate**.

In my first paper [1] , in collaboration with Donu Arapura, Partha Solapurkar, and Ben McReynolds, we applied a refined notion of almost conjugacy to constructed locally symmetric manifolds and complex projective surfaces that share many algebraic and analytic invariants. For example, we produce non-isometric closed hyperbolic n-manifolds, as covers of a fixed manifold, that have isomorphic integral cohomology in such a way that the isomorphism commute with the natural maps induced by the cover. We also produced arbitrarily large collections of pairwise non-isomorphic smooth projective surfaces where the isomorphisms are natural with respect to the Hodge structure, or as Galois modules. In particular, the projective surface have isomorphic Picard and Albanese varieties, and have isomorphic effective Chow motives. All of these examples also have the same eigenvalue and geodesic length spectrum for their associated Riemannian structures. The construction based on a refinement of Sunada's method, based on examples first discovered by L. Scott [11] and recently used by D. Prasad [9], in a construction that partly motivated ours.

Current work: spectral rigidity Despite the failure of spectral rigidity, there have been several positive results. In 1982, Wolpert [14] proved that a generic (in the Baire sense) Riemann surface is determined by its spectrum. Generalizing work of Kneser for flat tori, Wolpert proved that for a fixed Riemann surface, there are only finitely many Riemann surfaces with the same eigenvalue spectrum. In 1992, Reid proved that if X is an arithmetic Riemann surface, then any Riemann surface Y that is isospectral to X must be commensurable with X. In particular, Y must be arithmetic. The content of my thesis is the determination of a particular collection of arithmetic hyperbolic surfaces which are, in fact, spectrally rigid. Before I can state my theorem, we need some terminology.

First, we define arithmetic hyperbolic surfaces. Let $\mathbf{k}$ be a totally real number field with ring of integers $\mathbf{o}$, and $\mathbf{B}$ an indefinite quaternion algebra over $\mathbf{k}$ which is split at a unique real place of $\mathbf{k}$. That is to say, among all of the inclusions $\rho : \mathbf{k} \to \mathbb{R}$, there is a unique one ρ_o such that $\mathbf{B} \otimes_{\mathbf{k}} \mathbf{k}_{\rho_o}$ is isomorphic to the algebra of two by two matrices over $\mathbf{k}_{\rho_o} = \mathbb{R}$, the completion of $\mathbf{k}$ at ρ_o. We will also use the symbol ρ_o to denote the resulting inclusion $\mathbf{B} \to \mathbf{B} \otimes_{\mathbf{k}} \mathbf{k}_{\rho_o}$. An **order O** in $\mathbf{B}$ is an $\mathbf{o}$ subalgebra of $\mathbf{B}$ of maximal rank. Let $\mathbf{O}^1$ denote the multiplicative subgroup of $\mathbf{O}$ consisting of elements of reduced norm 1. Then ρ_o yields an identification $\mathbf{O}^1$ with a lattice $\Gamma_{\mathbf{O}}$ in $\mathrm{SL}(2,\mathbb{R})$. We say that a lattice Λ in $\mathrm{SL}(2,\mathbb{R})$ is **arithmetic** if there exists some $\mathbf{O}$ in some $\mathbf{B}$ over some $\mathbf{k}$ such that some $\mathrm{GL}(2,\mathbb{R})$ conjugate of Λ is commensurable with $\Gamma_{\mathbf{O}}$. For any lattice Λ in $\mathrm{SL}(2,\mathbb{R})$, we let $X(\Lambda)$ denote the associated hyperbolic orbifold $\Lambda\backslash \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. By the uniformization theorem, every hyperbolic surface M can be realized as a $X(\Lambda)$ for some $\Lambda \le \mathrm{SL}(2,\mathbb{R})$, unique up to $\mathrm{GL}(2,\mathbb{R})$ conjugation. Consequently arithmeticity is a well defined property of the underlying hyperbolic orbifold. We say that a quaternion algebra $\mathbf{B}$ over $\mathbf{k}$ has **type number one** if it has a unique $\mathbf{B}^\times$ conjugacy class of maximal orders. Let $\mathbf{I}$ be an ideal in $\mathbf{o}$, and for any subgroup Λ of $\Gamma_{\mathbf{O}}$, let $\Lambda(\mathbf{I})$ be the kernel of the reduction-mod -$\mathbf{I}$ map $\Gamma_{\mathbf{O}} \to \mathrm{SL}(2,\mathbf{o}/\mathbf{I})$, restricted to Λ. In particular $\Gamma_{\mathbf{O}}(\mathbf{I}) \le \Gamma_{\mathbf{O}}$ is the **principal congruence kernel mod I** . Note that normal inclusion of $\Gamma_{\mathbf{O}}(\mathbf{I})$ in $\Gamma_{\mathbf{O}}$ induces a regular cover $X(\Gamma_{\mathbf{O}}(\mathbf{I}))$ over $X(\Gamma_{\mathbf{O}}(\mathbf{I}))$, with deck group isomorphic to $\Gamma_{\mathbf{O}}(\mathbf{I})\backslash\Gamma_{\mathbf{O}} = \mathrm{SL}(2,\mathbf{o}/\mathbf{I})$. We are now prepared to state the theorem:

Theorem 1. *Let* $\mathbf{k},\mathbf{o},\mathbf{B}$ *be as above. Suppose further that* $\mathbf{B}$ *has* **type number** 1. *Let* $\mathbf{O}$ *be a representative of the single conjugacy class of maximal orders in* $\mathbf{B}$. *Then for any ideal* $\mathbf{I}$, *not divisble by any prime over which* $\mathbf{B}$ *is ramified, the surface* $X(\Gamma_{\mathbf{O}}(\mathbf{I}))$ *is absolutely spectrally rigid.*

This theorem is the first, to my knowledge, which produces infinitely many infinite families of Riemannnian manifolds which are demonstrably spectrally rigid. By carefully choosing $\mathbf{k}$, and $\mathbf{B}$, one can apply this theorem to the principal congruence Hurwitz surfaces, partially confirming a conjecture of Alan Reid.

Future projects: To study the relationship among Dedekind zeta functions of number fields, number theorists study their factorizations into Artin L-functions associated to Galois representations representations. Following their lead, we introduce the analogous L functions and study factorizations of Selberg's zeta function. Fix a compact negatively curved Riemannian manifold M, with fundamental group Γ. Let $\rho : \Gamma \to U(V)$ be a unitary representation on a finite dimensional Hilbert space V. Let

$$L_{M,\rho}(s) = \prod_{\gamma} \prod_{k=0}^{\infty} \det(\mathrm{id}_V - \rho(\gamma) \exp^{-(s+k)\ell_M(\gamma)}).$$

These functions satisfy the so-called Artin-Takgai formalism: for a cover $M' \to M$:

1. For ρ a unitary representation of $\Gamma' = \pi_1(M') \leq \Gamma$,

$$L_{M',\rho}(s) = L_{M,\mathrm{ind}_{\Gamma'}^{\Gamma}\rho}(s),$$

   and for a unitary representation σ of Γ,

$$L_{M,\sigma}(s) = L_{M',\mathrm{res}_{\Gamma'}^{\Gamma}\sigma}(s).$$

2. For a pair σ, σ' of unitary representations of Γ,

$$L_{M,\sigma\oplus\sigma'}(s) = L_{M,\sigma}(s)L_{M,\sigma'}(s)$$

Applying the second property above, we may extend the function $L_{M,\cdot}(s)$ from the set $\mathrm{URep}(M)$ of unitary reps of $\pi_1(M)$ to the additive group $\mathrm{VURep}(\pi_1(M))$ of virtual unitary representations. Let $T(M)$ be some reasonable space of negatively curved metrics on M. Then we may view the assignment $(\mu,\rho) \mapsto L_{\mu,\rho}$ as a meromorphic-function valued function on the space $T(M) \times \mathrm{VURep(M)}$ which is a homomorphism in the second variable.

Question 1. *For which metrics μ is the homomorphism $L_{\mu,\cdot}$ injective? When it isn't injective, how large is its kernel?*

This question has a meaningful analogue in number theory, whereat the answer is positive [3]. It has as an immediate consequence that all arithmetically equivalent number fields arise from Gassman equivalence[8]. A positive answer to question 1 will show that all instances of isospectrality within a commensurability class must arise from Sunada's construction.

Question 2. *Fix a twist ρ. For which spaces of metrics $T(M)$ on M is the function $L_{\cdot,\rho}$ injective on $T(M)$?*

When M is a compact surface of genus $g > 1$, a natural candidate is the space $\mathscr{M}_g$ of hyperbolic metrics.

Inside of VURep(M) is the subgroup FVUrep(M) consisting of those virtual representations which admit a continuous extension to the profinite completion of $\pi_1(M)$).

Question 3. *Supposing the answer to question 1 is negative, must the kernel intersect* FVURep(M) *nontrivially?*

Now suppose that M is an arithmetic hyperbolic 2 or 3 manifold of the form $\mathbf{G}(k)\backslash\mathbf{G}(\mathbb{A}_k)/\prod_{\mathfrak{p}<\infty}\mathbf{G}(\mathscr{O}_\mathfrak{p})$ for some inner form $\mathbf{G}$ of SL_2 and write Γ for the corresponding Fuchsian or Kleinian lattice in $G := \mathbf{G}(k^\infty)$. The unitary representations of Γ which factor through a finite *congruence quotient* are precisely those which extend continuously to $K := \prod_{\mathfrak{p}<\infty}\mathbf{G}(\mathscr{O}_\mathfrak{p})$. Write $\mathrm{CFVURep}(M)$ the subgroup of $\mathrm{FVURep}(M)$ generated by these representations.

Question 4. *Supposing the answer to question 3 is negative, must the kernel intersect* CFVURep(M) *nontrivially?*

As mentioned in **??**, Alan Reid proved that isospectral arithmetic hyperbolic 2 and 3 manifolds must be commensurable.

Question 5. *Let M and M' be isospectral nonarithmetic hyperbolic* 2 *or* 3 *manifolds M and M' be commensurable?*

If this is so, then if paired with a positive answer to question 1, one would have a complete characterization of isospectrality for such manifolds.

References

[1] D. Arapura, J. Katz, D. B. McReynolds, and P. Solapurkar. Integral gassman equivalence of algebraic and hyperbolic manifolds. *Mathematische Zeitschrift*, 291(1-2):179–194.

[2] Armand Borel and Gopal Prasad. Finiteness theorems for discrete subgroups of bounded covolume in semi-simple groups. *Inst. Hautes Études Sci. Publ. Math.*, (69):119–171.

[3] Takeo Funakura. On artin theorem of induced characters. *Comment, math. Univ. St. Pauli*, 27(1):51–58.

[4] Fritz Gassmann. Bemerkungen zu der vorstehenden arbeit von hurwitz (comments on the article über beziehungen zwischen den primidealen eines algebraischen körpers und den substitutionen seiner gruppe by hurwitz) math. *Z*, 25:655–665.

[5] Heinz Huber. Zur analytischen theorie hyperbolischer raumformen und bewegungsgruppen. *Mathematische Annalen*, 138(1):1–26.

[6] Grigorii Aleksandrovich Margulis. Applications of ergodic theory to the investigation of manifolds of negative curvature. *Funktsional'nyi Analiz i ego Prilozheniya*, 3(4):89–90.

[7] J. Milnor. Eigenvalues of the laplace operator on certain manifolds. *Proceedings of the National Academy of Sciences of the United States of America*, 51(4):542.

[8] Robert Perlis. On the equation $\zeta_k(s) = \zeta_{k'}(s)$. *Journal of Number Theory*, 9(3):342–360, 1977.

[9] Dipendra Prasad. A refined notion of arithmetically equivalent number fields, and curves with isomorphic jacobians. *Advances in Mathematics*, 312:198–208.

[10] Peter Sarnak. Asymptotic behavior of periodic orbits of the horocycle flow and eisenstein series. *Communications on Pure and Applied Mathematics*, 34(6):719–739.

[11] L Scott. Integral equivalence of permutation representations. *Group theory (Granville, OH, 1992)*, 262274.

[12] Toshikazu Sunada. Riemannian coverings and isospectral manifolds. *Annals of Mathematics*, 121(1):169–186.

[13] Marie-France Vignéras. Variétés riemanniennes isospectrales et non isométriques. *Annals of Mathematics*, 112(1):21–32.

[14] Scott Wolpert. The eigenvalue spectrum as moduli for flat tori. *Transactions of the American Mathematical Society*, 244:313–321.