Algebraic Properties of Multilinear Constraints

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Abstract

In this paper the different algebraic varieties that can be generated from multiple view geometry with uncalibrated cameras have been investigated. The natural descriptor, $\mathcal{V}_{\mathrm{n}}$, to work with is the image of $\mathbb{P}^3$ in $\mathbb{P}^2 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^2$ under a corresponding product of projections, $(A_1 \times A_2 \times \ldots \times A_m)$.

Another descriptor, the variety $\mathcal{V}_{\mathrm{b}}$, is the one generated by all bilinear forms between pairs of views, which consists of all points in $\mathbb{P}^2 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^2$ where all bilinear forms vanish. Yet another descriptor, the variety $\mathcal{V}_{\mathrm{t}}$, is the variety generated by all trilinear forms between triplets of views. It has been shown that when $m = 3$, $\mathcal{V}_{\mathrm{b}}$ is a reducible variety with one component corresponding to $\mathcal{V}_{\mathrm{t}}$ and another corresponding to the trifocal plane.

Furthermore, when $m = 3$, $\mathcal{V}_{\mathrm{t}}$ is generated by the three bilinearities and one trilinearity, when $m = 4$, $\mathcal{V}_{\mathrm{t}}$ is generated by the six bilinearities and when $m \geq 4$, $\mathcal{V}_{\mathrm{t}}$ can be generated by the $\binom{m}{2}$ bilinearities. This shows that four images is the generic case in the algebraic setting, because $\mathcal{V}_{\mathrm{t}}$ can be generated by just bilinearities. Furthermore, some of the bilinearities may be omitted when $m \geq 5$.

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1 Introduction

There has been an intensive research on multiple view geometry during the last years and there exists a lot of formulations of the structure from motion problem from many uncalibrated views. The purpose of these formulations is to reconstruct an unknown object from a number of its projective images. As a by-product the egomotion of the camera can be calculated. The first approaches were based on minimal point sets for reconstruction. These includes the reconstruction from seven points in two views, [19], and from six points in three views, [13] and [9].

Later on the interest focused on linear reconstruction techniques. Since the equations appearing are polynomial in the indeterminates, it is always possible to make linear reconstructions when the number of available data is large enough. One advantage of linear reconstructions is that they are less sensitive to noise than methods relying on solving polynomial equations. The first approaches to linear reconstruction where based on the fundamental matrix between two views. This matrix expresses the fact that corresponding points in two images obey the so called bilinear constraint, or epipolar constraint. It can be recovered linearly from at least eight point matches in two views and then reconstruction can also be made linearly from the fundamental matrix, see [4].

The next step was to consider three images at the same instant. At this stage the so called trilinear constraints appear, see [17], [18], [8] and [7]. The coefficients of the trilinearities are elements of the so called trifocal tensor. It was discovered that this tensor could be linearly estimated from at least seven point matches in three views and then reconstruction could be made, also linearly.

The obvious extension to the trilinear constraints was to consider four or more images at the same instant. It turns out that there exist quadrilinear constraints between four different views, see [5] and [20]. However, these constraints are algebraically dependent on the trilinear ones, see also [10]. It also became apparent that multilinear constraints between more than four views contained no new information.

A common theme for these linear reconstruction algorithms is that they are based on the following procedure.

1. Obtain point correspondences between the images.

2. Estimate multilinear constraints from these correspondences.

3. Reconstruct the object from the multilinear constraints.

The first step is based on low-level vision operations, e.g. corner detection and tracking. In the second step one has to choose a subset of all available multilinear constraints, when the number of images is large. The third step depends on the subset of estimated multilinear constraint. Thus an interesting question is what subset of multilinear constraints that should be chosen. To answer this question one has to know the relation between different multilinear constraints. This is the problem we will address, where we by relations mean algebraic relations. A somewhat different problem, not dealt with here, is what subset of multilinear constraints that should be used to obtain a robust reconstruction when using real data.

A surprising fact is that, for three different views, given the three bilinear constraints, it is in general possible to calculate the camera matrices and then the trilinear constraints, see [10] and [12]. Here, 'in general' alludes to the case when the camera motion is not linear. However, algebraically the trilinear constraints do not follow from the bilinear ones. This can easily be seen by considering points on the trifocal plane, defined by the three camera centers. The bilinear constraints impose only the condition that the three image points lie on the trifocal lines, but the trilinear ones impose one further condition. The question is now how the fact that the trilinear constraints are implied by the bilinear ones, via the camera matrices, can live together with the fact that the trilinear constraints do not follow algebraically from the bilinear ones, that is the trilinearities do not belong to the ideal generated by the bilinearities. This is the key question we will answer in this paper. We will clarify the meaning of the statement 'the trilinearities follows from the bilinearities, when the camera does not move on a line', where the statement is right or wrong depending of what kind of operations we are allowed to do on the bilinearities. The statement is true if we are allowed to pick out coefficients from the bilinearities and use them to calculate camera matrices and then the trilinearities, but the statement is false if we are just allowed to make algebraic manipulations of the bilinearities, where the image coordinates are considered as variables. However, as will be shown, the statement is also true algebraically with a slightly different interpretation of the allowed operations on the bilinearities, considered as generators of an ideal.

In order to understand this relation between the bilinearities and the trilinearities we will use some algebraic geometry and commutative algebra. A general reference for the former are [15], [14] and for the latter are [16], [1]. We will define different algebraic varieties as well as ideals and derive the relations between them. The following notations will be used. Algebraic varieties will be denoted by $\mathcal{V}$. Ideals in the coordinate ring $k[X]$, $X = (x_1, \ldots, x_n)$, with coefficients in the field k, will be denoted by $\mathcal{I}$. The ideal corresponding to a variety $\mathcal{V}$ will be denoted by $\mathcal{I}(\mathcal{V})$. The variety defined by the ideal $\mathcal{I}$ will be denoted by $\mathcal{V}(\mathcal{I})$. Often an ideal, $\mathcal{I}$, will be generated from a set of polynomials; $p_1, \ldots, p_n$, i.e. $I = (p_1, \ldots, p_n)$. The radical of an ideal, $\mathcal{I}$, will be denoted by $\sqrt{\mathcal{I}}$, i.e. $\sqrt{\mathcal{I}} = \{\, f \in k[X] \mid f^n \in \mathcal{I} \text{ for some } n \in \mathbb{N} \,\}$.

In Section 2 the multi imaging problem will be formulated in algebraic terms, the varieties will be defined and a coordinate system to describe the general situation will be introduced. In Section 3 the simplest case of one image is studied and in Section 4 the case of two images. The case of three images is studied in Section 5, where both the varieties and the ideals are investigated. The case of more than three images is studied in Section 6, before the conclusions are given in Section 7.

2 Problem Formulation

In computer vision a camera is usually modelled by the equation

$$\lambda \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_x & s & x_0 \\ 0 & \alpha_y & y_0 \\ 0 & 0 & 1 \end{bmatrix} [\, R \mid -Rt \,] \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \quad \Leftrightarrow \quad \lambda \mathbf{x} = K[\, R \mid -Rt \,]\mathbf{X} \ . \tag{1}$$

Here $\mathbf{X} = [\, X\, Y\, Z\, 1]^T$ denotes object coordinates in extended form and $\mathbf{x} = [\, x\, y\, 1]^T$ denotes extended image coordinates. The scale factor λ accounts for perspective effects and (R, t) represents a rigid transformation of the object, i.e. R denotes a 3×3 rotation matrix and t a 3×1 translation vector. Finally, the parameters in K represent intrinsic properties of the image formation system: α_x and α_y represent magnifications in the x- and y-directions in the light sensitive area, s represents the skew, i.e. nonrectangular arrays can be modelled, and (x_0, y_0) is called the principal point and is interpreted as the orthogonal projection of the focal point onto the image plane. The parameters in R and t are called **extrinsic parameters** and the parameters in K are called the **intrinsic parameters**. Observe that there are 6 extrinsic and 5 intrinsic parameters, totally 11, the same number as in an arbitrary 3×4 matrix defined up to a scale factor. If the extrinsic as well as the intrinsic parameters are allowed to vary between the different imaging instants, $i = 1, \ldots, m$, (1) can compactly be written

$$\lambda_i \mathbf{x}_i = A_i \mathbf{X} \ . \tag{2}$$

Using projective coordinates in the images and in the object, represented by $\mathbf{x}_i = (x_i, y_i, z_i)$ and $\mathbf{X} = (X, Y, Z, W)$, (2) can be written

$$\mathbf{x}_i \sim A_i \mathbf{X} \ , \tag{3}$$

where $\sim$ means equality up to scale.

We will consider the following problem. Given m images taken by uncalibrated cameras of a rigid object. Describe the possible connections between corresponding points in the different images. Algebraically, this can be formulated as follows. Let $\mathbb{P}^2$ and $\mathbb{P}^3$ denote the projective spaces of dimension 2 and 3 respectively. Let A_i be m different projective transformations from $\mathbb{P}^3$ to $\mathbb{P}^2$:

$$\begin{cases} A_1 : \mathbb{P}^3 \ni \mathbf{X} \mapsto A_1\mathbf{X} \in \mathbb{P}^2 \ , \\ A_2 : \mathbb{P}^3 \ni \mathbf{X} \mapsto A_2\mathbf{X} \in \mathbb{P}^2 \ , \\ \qquad \cdots \ , \\ A_m : \mathbb{P}^3 \ni \mathbf{X} \mapsto A_m\mathbf{X} \in \mathbb{P}^2 \ . \end{cases} \tag{4}$$

In (4) each A_i is described by a 3×4 matrix or rank 3. The mapping is undefined on the nullspace of this matrix, corresponding to the **focal point**, f_i, of camera i, given by

$$A_i f_i = 0 \ . \tag{5}$$

The equations in (4) can be regarded as one transformation, $\Phi_m = (A_1, A_2, \dots A_m)$, from $\dot{\mathbb{P}}^3$ to $\mathbb{P}^2 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^2 = (\mathbb{P}^2)^m$, and can be written

$$\Phi_m : \dot{\mathbb{P}}^3 \ni X \mapsto (A_1\mathbf{X}, A_2\mathbf{X}, \dots, A_m\mathbf{X}) \in (\mathbb{P}^2)^m \ , \tag{6}$$

where $\dot{\mathbb{P}}^3 = \mathbb{P}^3 \setminus \{f_1, f_2, \dots, f_m\}$, that is $\mathbb{P}^3$ with omission of the camera centres. This removal of a finite set of points from $\mathbb{P}^3$ gives a quasiprojective variety, i.e. an open subset of a projective variety. We want to describe the range of Φ_m as a subset in $(\mathbb{P}^2)^m$.

It can be shown that $(\mathbb{P}^2)^m$ is a projective variety. First consider the case $m = 2$. Using the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^8$, we can regard $\mathbb{P}^2 \times \mathbb{P}^2$ as a projective variety, by identifying it with a 4 dimensional subvariety, $\mathcal{S}_2$, in $\mathbb{P}^8$. This process can be repeated and shows that $(\mathbb{P}^2)^m$ can be embedded in $\mathbb{P}^{3^m-1}$ as a projective subvariety, denoted $\mathcal{S}_m$. We can think of it as m copies of $\mathbb{P}^2$ and need not bother about the actual embedding. Moreover, this fact has a very important implication on the functions and ideals generating varieties in $(\mathbb{P}^2)^n$. These functions must be homogeneous of the same degree in every triplet of variables corresponding to a factor $(\mathbb{P}^2)$, see [15], pp. 56.

2.1 Multilinear forms

Consider the equations, obtained from (4) and introduce scalars λ_i to take care of the projective equivalence. These equations can be written

$$Mu = 0 \ , \tag{7}$$

with

$$M = \begin{bmatrix} A_1 & \mathbf{x}_1 & 0 & 0 & \dots & 0 \\ A_2 & 0 & \mathbf{x}_2 & 0 & \dots & 0 \\ A_3 & 0 & 0 & \mathbf{x}_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_m & 0 & 0 & 0 & \dots & \mathbf{x}_m \end{bmatrix} , \qquad u = \begin{bmatrix} \mathbf{X} \\ -\lambda_1 \\ -\lambda_2 \\ -\lambda_3 \\ \vdots \\ -\lambda_m \end{bmatrix} . \tag{8}$$

Since M has a nontrivial nullspace, it follows that

$$\operatorname{rank}(M) \leq m + 3 \ . \tag{9}$$

The matrix M in (8) contains one block with three rows for each image. Observe that all determinants of $(m+3) \times (m+3)$ submatrices of M in (8) are **multihomogeneous** of degree $(1, 1, 1, \dots, 1)$, that is of the same degree in every triplet of image coordinates.

Definition 2.1. The subdeterminants of size $(m+3)\times(m+3)$ from M in (8) are called the **multilinear constraints.** ■

The multilinear constraints obtained from submatrices containing all rows corresponding to two images and one row from each of the other images are called the **bilinear constraints.** The bilinear constraints between image i and image j can be written as a product of x-, y- and z-coordinates in the other images and

$$b_{i,j}(x_i, y_i, z_i, x_j, y_j, z_j) := \det \begin{bmatrix} A_i & \mathbf{x}_i & 0 \\ A_j & 0 & \mathbf{x}_j \end{bmatrix} = 0 \ . \tag{10}$$

Since the first factors consists of projective coordinates, some combination of these projective coordinates has a nonvanishing product and the bilinear constraints are equivalent to the constraint in (10), which is sometimes called the **epipolar constraint.**

The multlinear constraints obtained from submatrices containing all rows corresponding to one image, two rows each from two other images and one row from each of the other images are called the **trilinear**

constraints. The trilinear constraints between image i, j and k can be written as a product of x-, y- and z-coordinates in the other images and 6×6 subdeterminants of

$$\begin{bmatrix} A_i & \mathbf{x}_i & 0 & 0 \\ A_j & 0 & \mathbf{x}_j & 0 \\ A_k & 0 & 0 & \mathbf{x}_k \end{bmatrix} . \tag{11}$$

Again the first factors consists of projective coordinates, and some combination of these projective coordinates has a nonvanishing product, thus the trilinear constraints are equivalent to the constraints expressed by subdeterminants from (11).

The multilinear constraints obtained from submatrices containing two rows corresponding to each of three images and one row from each of the other images are called the **quadrilinear constraints.** The quadrilinear constraints between image i, j, k and l can be written as a product of x-, y- and z-coordinates in the other images and 7×7 subdeterminants of

$$\begin{bmatrix} A_i & \mathbf{x}_i & 0 & 0 & 0 \\ A_j & 0 & \mathbf{x}_j & 0 & 0 \\ A_k & 0 & 0 & \mathbf{x}_k & 0 \\ A_l & 0 & 0 & 0 & \mathbf{x}_l \end{bmatrix} . \tag{12}$$

Again the first factors consists of projective coordinates, and some combination of these projective coordinates has nonvanishing product, thus the quadrilinear constraints are equivalent to the constraints expressed by subdeterminants from (12).

By choosing a suitable coordinate system in the (unknown) object configuration it can be assumed that the first camera matrix can be written $A_1 = [\, I \,|\, 0 \,]$. Then an equivalent formulation can be obtained by eliminating the first three variables in (8) obtaining

$$\operatorname{rank} \begin{bmatrix} t_2 & Q_2\mathbf{x}_1 & \mathbf{x}_2 & 0 & \dots & 0 \\ t_3 & Q_3\mathbf{x}_1 & 0 & \mathbf{x}_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_m & Q_m\mathbf{x}_1 & 0 & 0 & \dots & \mathbf{x}_m \end{bmatrix} \leq m \ , \tag{13}$$

where we have used the notation $A_i = [\, Q_i \,|\, t_i \,]$, the same as the one found in [10] and [11].

Yet another formulation is common in the literature, see [5]. This time the projective equations in (3) are written in the form

$$(A_i^1 \cdot \mathbf{X}, A_i^2 \cdot \mathbf{X}, A_i^3 \cdot \mathbf{X}) \sim (x_i, y_i, z_i) \ , \tag{14}$$

where A_i^j means the j:th row of A_i and $u \cdot v$ means scalar product of u and v. Introducing a scalar factor to describe the projective equivalence and eliminating this scalar factor gives

$$\begin{bmatrix} z_i A_i^1 & -x_i A_i^3 \\ z_i A_i^1 & -y_i A_i^3 \end{bmatrix} \mathbf{X} = M\mathbf{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ , \tag{15}$$

which is similar to (7) and (8). The equations in (15) are equivalent to the equations in (14) if and only if $z_i \neq 0$. When $z_i = 0$ we have $y_i A_i^1 \cdot \mathbf{X} - x_i A_i^2 \cdot \mathbf{X} = 0$ from (14) but $x_i A_i^3 \cdot \mathbf{X} = 0$ and $y_i A_i^3 \cdot \mathbf{X} = 0$ from (15). Thus the equations in (15) do not describe the same conditions as (9) or (13).

2.2 The varieties and ideals

In the sequel we are going to investigate different subsets in $\mathcal{S}_m$. The first one is defined as follows.

Definition 2.2. The **natural descriptor**, $\mathcal{V}_\mathrm{n}$, is the range of Φ_m in (6), i.e.

$$\mathcal{V}_\mathrm{n} = \Phi_m(\dot{\mathbb{P}}^3) \subset \mathcal{S}_m \ . \tag{16}$$

■

$\mathcal{V}_{\mathrm{n}}$ is a constructible set, see [15], i.e. a rational image of a quasiprojective variety. The definition raises several questions

1. Is $\mathcal{V}_{\mathrm{n}}$ a projective variety?
2. If it is, how can it be described as the set of zeros to an ideal of polynomials?
3. Is $\mathcal{V}_{\mathrm{n}}$ irreducible?
4. If not, what are the irreducible components?

The second descriptor we will consider is connected to the bilinear or epipolar constraints and can, according to (10), for images i and j be written

$$b_{i,j}(x_i, y_i, z_i, x_j, y_j, z_j) = \begin{bmatrix} x_i & y_i & z_i \end{bmatrix} F_{i,j} \begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix} = 0 \ , \tag{17}$$

where $F_{i,j}$ can be expressed in A_i and A_j. $F_{i,j}$ is called the **fundamental matrix** between views i and j. Another way to write the bilinear constraints, that more easily generalises to higher order multilinear constraints, is

$$\sum_{k,l} (B_{i,j})_{k,l} P^k K^l = 0 \ , \tag{18}$$

where P^k is x_i, y_i or z_i according to if $k = 1$, $k = 2$ and $k = 3$, respectively and K^l is x_j, y_j or z_j according to if $l = 1$, $l = 2$ and $l = 3$, respectively. In this way the bilinearities between views i and j can be described by a tensor $(B_{i,j})_{k,l}$ that is covariant in both indices, see [20].

Definition 2.3. The **bilinear descriptor** or **bilinear variety**, $\mathcal{V}_{\mathrm{b}}$, is defined as the projective subvariety in $\mathcal{S}_m$, generated by all bilinear constraints. ■

Remark. As we have seen above, according to the Segre-embedding, all variables in $\mathcal{S}_m$ are products of m projective coordinates, one from every image. This means that a polynomial equation defining a variety in $\mathcal{S}_m$ must be multihomogeneous of the same degree in every triplet of projective coordinates. ■

Since it is defined by homogeneous polynomials, $\mathcal{V}_{\mathrm{b}}$ is a projective variety. But questions 3 and 4 above can be asked again together with the question of the connection between $\mathcal{V}_{\mathrm{b}}$ and $\mathcal{V}_{\mathrm{n}}$.

The third variety we will consider is connected to the trilinear constraints. These can, according to (11), for images i, j and k be written

$$t_{i,j,k}(x_i, y_i, z_i, x_j, y_j, z_j, x_k, y_k, z_k) := \sum_{l,m,n} (T_{i,j,k})^l_{m,n} P_l K^m L^n = 0 \ , \tag{19}$$

where P_l is x_i, y_i or z_i according to $l = 1$, $l = 2$ and $l = 3$ respectively. In the same way K^m and L^n are the first, second and third dual coordinates in the second and third image respectively. This can be interpreted as coordinates of an arbitrary line passing through a point in image j and k. We remark that the trilinear tensor $T_{i,j,k}$ between images i, j and k are covariant in index l and contravariant in indices m and n. It turns out that the trilinear constraints between three fixed images can in general be written as the vanishing of 3 polynomials, see [10].

Definition 2.4. The **trilinear descriptor** or **trilinear variety**, $\mathcal{V}_{\mathrm{t}}$, is defined as the projective subvariety in $\mathcal{S}_m$, defined by all trilinear constraints. ■

Again, since it is defined by homogeneous polynomials, $\mathcal{V}_{\mathrm{t}}$ is a projective variety, but the previous raised questions can be asked again.

It is also possible to generate a variety by combining the bilinear and trilinear forms.

Definition 2.5. The **bitrilinear descriptor** or **bitrilinear variety**, $\mathcal{V}_{\mathrm{bt}}$, is defined as the projective subvariety in $\mathcal{S}_m$, generated by all bilinear constraints and all trilinear constraints. ■

It follows that

$$\mathcal{V}_{\mathrm{bt}} = \mathcal{V}_{\mathrm{b}} \cap \mathcal{V}_{\mathrm{t}} \ .$$

When more than three images are available it is possible to generate a variety from the quadrilinear constraints. These can, according to (12), for images i, j, k and l be written

$$q_{i,j,k,l}(x_i, y_i, z_i, x_j, y_j, z_j, x_k, y_k, z_k, x_l, y_l, z_l) := \sum_{m,n,o,p} (T_{i,j,k,l})_{m,n,o,p} P^m K^n L^o M^p = 0 \ , \tag{20}$$

where P^m, K^n, L^o and M^p are the first, second and third dual coordinates in image i, j, k and l respectively, according to the value of m, n, o, and p. We remark that the quadrilinear constraints can be described by the quadrifocal tensor $(Q_{i,j,k,l})^{m,n,o,p}$, which is covariant in all indices, see [20]. Finally, we make the following definition

Definition 2.6. The **quadrilinear descriptor** or **quadrilinear variety**, $\mathcal{V}_{\mathrm{q}}$, is defined as the projective subvariety in $\mathcal{S}_m$, generated by all quadrilinear constraints in (20). ■

Again, it is obvious that $\mathcal{V}_{\mathrm{q}}$ is variety and it is natural to ask for the connections to $\mathcal{V}_{\mathrm{b}}$, $\mathcal{V}_{\mathrm{t}}$ and $\mathcal{V}_{\mathrm{bt}}$ and of irreducibility. It is furthermore possible to make other combinations of multilinear constraints in order to define different varieties. However, it will be evident later that nothing new is obtained.

We are also going to investigate different ideals in $\mathbb{R}[x_1, y_1, \ldots, y_m, z_m]$. These ideals generate varieties in $\mathcal{S}_m = (\mathbb{P})^m$ if and only if they consists of multihomogeneous polynomials of the same degree in every coordinate triplet. For example, there is no meaning in asking the question if the bilinear constraint, in the form (10), between two images is contained in some ideal generating a variety in $(\mathbb{P}^2)^3$ for three images. The reason for this is that variables from the third image are not present in this bilinearity between the first two images. Thus this bilinear constraint is not homogeneous of the same degree in every triplet of variables. This difficulty will be overcome in the sequel by considering every multilinear constraint in its homogenised forms, and when we speak of generators of an ideal, describing a variety in $(\mathbb{P}^2)^n$, we implicitly assume that the generators are replaced by their homogenised equivalents, as obtained directly from (9) and (8).

We state the formal definitions of the ideals:

Definition 2.7. The **natural ideal** $\mathcal{I}_{\mathrm{n}}$, in $\mathbb{R}[x_1, y_1, \ldots, y_m, z_m]$ is the ideal generated by all multihomogeneous functions that vanishes on the natural descriptor, $\mathcal{V}_{\mathrm{n}}$, i.e.

$$\mathcal{I}_{\mathrm{n}} = \mathcal{I}(\mathcal{V}_{\mathrm{n}}) \ .$$

■

Definition 2.8. The **bilinear ideal** $\mathcal{I}_{\mathrm{b}}$, in $\mathbb{R}[x_1, y_1, \ldots, y_m, z_m]$ is the ideal generated by all bilinearities. ■

The **trilinear ideal**, **bitrilinear ideal** and **quadrilinear ideal** are defined analogously.

2.3 Choice of coordinates

Since we are only interested in algebraic relations between different ideals, we have the freedom to choose coordinates in $\mathbb{P}^3$ and in every $\mathbb{P}^2$ as we like. Consider the m projective transformations A_i in (4) and the n focal points $f_i \in \mathbb{P}^3$ defined by (5). Assuming that the first five focal points f_i are projectively independent, we choose coordinates in $\mathbb{P}^3$ such that

$$\begin{cases} f_1 = (0,0,0,-1) \ , \\ f_2 = (1,0,0,-1) \ , \\ f_3 = (0,1,0,-1) \ , \\ f_4 = (0,0,1,-1) \ , \\ f_5 = (1,1,1,-1) \ , \end{cases} \tag{21}$$

where the minus sign in the fourth component will be convenient later. Furthermore, we choose coordinates in each $\mathbb{P}^2$ such that each A_i can be written

$$A_i = \begin{bmatrix} 1 & 0 & 0 & \times \\ 0 & 1 & 0 & \times \\ 0 & 0 & 1 & \times \end{bmatrix} \quad , \tag{22}$$

where $\times$ denotes an unknown entry. Because of (5), this means that the projection matrices can be written

$$\begin{aligned} A_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} , \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} , \quad A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} , \\ A_4 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} , \quad A_5 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} , \quad A_k = \begin{bmatrix} 1 & 0 & 0 & a_k \\ 0 & 1 & 0 & b_k \\ 0 & 0 & 1 & c_k \end{bmatrix} , \quad k \geq 6 , \end{aligned} \tag{23}$$

with $A_k f_k = 0$, where

$$f_k = (-a_k, -b_k, -c_k, 1), \quad k \geq 6 \ .$$

Definition 2.9. The coordinate system chosen in (23) will be called a **normalised coordinate system** for the multiple view geometry.

Remark. This choice of coordinates can be done if the matrices A_i are assumed to be in general position. ■

The epipoles, epipolar lines, trifocal planes and trifocal lines will be needed later. We give the formal definitions.

Definition 2.10. The **epipole**, $e_{i,j}$, from camera j in image i is defined by

$$e_{i,j} = A_i f_j \ , \tag{24}$$

see Figure 1. ■

Definition 2.11. The **trifocal plane**, $TP_{i,j,k}$, for cameras i, j and k is the plane containing f_i, f_j and f_k, see Figure 1. ■

Definition 2.12. The **epipolar line**, $EL_{i,j}$, is the line in $\mathbb{P}^3$ containing f_i and f_j, see Figure 1. ■

Definition 2.13. The **trifocal line** $tl_{i,j,k}$, in image i from the triplet of images i, j and k is the intersection of the trifocal plane, $TP_{i,j,k}$, and image plane i, see Figure 1. ■

3 One Image

There is really not much to say when we have only one image, but one thing need to be pointed out. Consider the map

$$\Phi_1 : \dot{\mathbb{P}}^3 \ni \mathbf{X} \mapsto A_1 \mathbf{X} \in \mathbb{P}^2 \ , \tag{25}$$

The mapping Φ_1 is a rational function from the quasiprojective variety $\dot{\mathbb{P}}^3$ to the (quasi)projective variety $\mathbb{P}^2$. Since the preimage, $\Phi_1^{-1}(\mathbf{x})$, is a quasiprojective subvariety of dimension 1 in $\dot{\mathbb{P}}^3$, Φ_1 is not a birational equivalence.

4 Two Images

Things are more complicated already in the case of two images. Now we have the following map:

$$\Phi_2 : \dot{\mathbb{P}}^3 \ni \mathbf{X} \mapsto (A_1\mathbf{X}, A_2\mathbf{X}) \in \mathbb{P}^2 \times \mathbb{P}^2 \quad . \tag{26}$$

Considering a suitable restriction of Φ_2, we get the following theorem, which is well known in a different formulation, see [4].

Theorem 4.1 (Fundamental Theorem of Epipolar Geometry).

$$\widetilde{\Phi}_2 : \mathbb{P}^3 \setminus \{EL_{1,2}\} \ni \mathbf{X} \mapsto (A_1\mathbf{X}, A_2\mathbf{X}) \in \left(\mathbb{P}^2 \setminus \{e_{1,2}\}\right) \times \left(\mathbb{P}^2 \setminus \{e_{2,1}\}\right) \ , \tag{27}$$

where $EL_{1,2}$ denotes the epipolar line and $e_{1,2}$ and $e_{2,1}$ denotes the epipoles, defined in Definition 2.12 *and Definition* 2.10 *respectively, is a birational map between the quasiprojective varieties $\mathbb{P}^3 \setminus \{EL_{1,2}\}$ and $\left((\mathbb{P}^2 \setminus \{e_{1,2}\}) \times (\mathbb{P}^2 \setminus \{e_{2,1}\})\right) \cap \mathcal{V}_b$.*

Proof. $\left((\mathbb{P}^2 \setminus \{e_{1,2}\}) \times (\mathbb{P}^2 \setminus \{e_{2,1}\})\right) \cap \mathcal{V}_b$ is a quasiprojective variety since it is the intersection of two quasiprojective varieties. Obviously, $\widetilde{\Phi}_2$ is rational. It remains to prove that its inverse exists and is rational. This can be seen from the fact that given a point $(\mathbf{x}_1, \mathbf{x}_2)$ in the range of $\widetilde{\Phi}_2$, it is possible to reconstruct the corresponding point in $\mathbb{P}^3 \setminus \{EL_{1,2}\}$, by intersecting the rays $A_1^{-1}(\mathbf{x}_1)$ and $A_2^{-1}(\mathbf{x}_2)$.The reconstructed point can be written as the intersection of these lines, which is clearly a rational map. ∎

Definition 4.1. The mapping, $\widetilde{\Phi}_2$, in (27) is called the **birational restriction** of Φ_2. ∎

Note that the inverse image of Φ_2 is 1-dimensional at the point $(e_{1,2}, e_{2,1})$, because every point on the epipolar line projects to $(e_{1,2}, e_{2,1})$. This means that Φ_2 is not bijective between $\dot{\mathbb{P}}^3$ and $\mathcal{V}_b$.

Next we will characterise the set of points in $\mathcal{V}_n$. The natural descriptor $\mathcal{V}_n \subset \mathbb{P}^2 \times \mathbb{P}^2$ consists of the following pairs of points:

- One arbitrary point, (x_1, y_1, z_1), in the first $(\mathbb{P}^2 \setminus \{e_{1,2}\})$ and one point, (x_2, y_2, z_2), in the second $(\mathbb{P}^2 \setminus \{e_{2,1}\})$, on the line $b_{1,2} = y_1z_2 - z_1y_2 = 0$ (or vice versa).
- The epipole $e_{1,2} = (1,0,0)$ in the first $\mathbb{P}^2$ and the epipole $e_{2,1} = (1,0,0)$ in the second $\mathbb{P}^2$.

Here the first set of points corresponds to images of points not on the epipolar line and the second set of points corresponds to images of points on the epipolar line.

We now turn to the bilinear variety $\mathcal{V}_b \subset \mathbb{P}^2 \times \mathbb{P}^2$, generated by the bilinear forms. For two images there is just one bilinear form,

$$b_{1,2}(\mathbf{x}_1, \mathbf{x}_2) = b_{1,2}(x_1, y_1, z_1, x_2, y_2, z_2) = y_1z_2 - z_1y_2 \ . \tag{28}$$

The projective subvariety in $\mathbb{P}^2 \times \mathbb{P}^2$ generated by $b_{1,2}$ consists of the following pairs of points:

- One arbitrary point, (x_1, y_1, z_1), in the first $(\mathbb{P}^2 \setminus \{e_{1,2}\})$ and one point, (x_2, y_2, z_2), in the second $(\mathbb{P}^2 \setminus \{e_{2,1}\})$, on the line $b_{1,2} = y_1z_2 - z_1y_2 = 0$ (or vice versa).
- The epipole $e_{1,2} = (1,0,0)$ in the first $\mathbb{P}^2$ and an arbitrary point in the second $\mathbb{P}^2$,
- The epipole $e_{2,1} = (1,0,0)$ in the second $\mathbb{P}^2$ and an arbitrary point in the first $\mathbb{P}^2$.

This includes also the point $(e_{1,2}, e_{2,1})$, consisting of the two epipoles and shows the following theorem

Theorem 4.2. *For two images we have, with strict inclusion,*

$$\mathcal{V}_n \subset \mathcal{V}_b. \tag{29}$$

Proof. See the descriptions above. ∎

We now return to the natural descriptor $\mathcal{V}_n \in \mathbb{P}^2 \times \mathbb{P}^2$. Consider the natural ideal, i.e.

$$\mathcal{I}_n = \mathcal{I}(\mathcal{V}_n) \subset \mathbb{R}[x_1, y_1, z_1, x_2, y_2, z_2] \ .$$

Thus $\mathcal{I}_n$ is the ideal generated by the polynomial functions in $(x_1, y_1, z_1, x_2, y_2, z_2)$ that vanish at all points in $\mathcal{V}_n$. Since $\mathcal{V}_n$ is a subset in $\mathbb{P}^2 \times \mathbb{P}^2$, these functions are bihomogeneous of the same degree in (x_1, y_1, z_1) and (x_2, y_2, z_2), see [15], pp. 56. The bilinearity in (28) is a function of degree $(1, 1)$ in $\mathcal{I}_n$. Actually, this bilinearity generates $\mathcal{I}_n$ according to the first part of the following theorem.

Theorem 4.3. *The ideal $\mathcal{I}_n$ generated by the natural descriptor, can be described as*

$$\mathcal{I}_n = (b_{1,2}) \ ,$$

which means that it is the principal ideal generated by the bilinearity. The natural descriptor $\mathcal{V}_n \in \mathbb{P}^2 \times \mathbb{P}^2$ is not a variety, in the sense that it can not be described as the common zeroes to a system of polynomial equations. Furthermore,

$$\overline{\mathcal{V}_n} = \mathcal{V}_b \ ,$$

i.e. the closure of $\mathcal{V}_n$ (in the Zariski topology) equals $\mathcal{V}_b$.

Proof. By definition, $\mathcal{V}_n$ is the range of Φ_2, that is all points $(x_1, y_1, z_1, x_2, y_2, z_2)$, fulfilling for some X, Y, Z, W, λ_1, λ_2,

$$\begin{cases} x_1 = \lambda_1 X \\ y_1 = \lambda_1 Y \\ z_1 = \lambda_1 Z \end{cases} , \qquad \begin{cases} x_2 = \lambda_2 (X + W) \\ y_2 = \lambda_2 Y \\ z_2 = \lambda_2 Z \end{cases} ,$$

according to the chosen coordinates in (23). Elimination of the parameters X, Y, Z, W, λ_1 and λ_2, gives that $\mathcal{I}_n$ is characterised by

$$y_1 z_2 - z_1 y_2 = 0 \ .$$

This proves the first assertion.

For every variety, $\mathcal{V}$, $\mathcal{V} = \mathcal{V}(\mathcal{I}(\mathcal{V}))$ holds. However, Theorem 4.2 and the first part of this Theorem shows that

$$\mathcal{V}_n \subset \mathcal{V}_b = \mathcal{V}(\mathcal{I}(\mathcal{V}_n)) \ , \tag{30}$$

with strict inclusion. ■

The following example illustrates the fact that the image of a quasiprojective variety need not itself be a variety.

Example 4.1. Consider the map

$$f : \mathbb{A}^2 \ni (x, y) \mapsto (x, xy) \in \mathbb{A}^2 \ , \tag{31}$$

where $\mathbb{A}^2$ denotes the 2-dimensional affine space, which clearly is a quasiprojective variety. It can be seen that

$$\mathrm{Im} f = \{\, (x, y) \in \mathbb{A}^2 \,|\, x \neq 0 \,\} \cup \{\, (0, 0) \} \ ,$$

which is not a quasiprojective variety. ■

Remark. $\mathcal{V}_b$ can also be described as the range of an extension of the map Φ_2 to a multivalued map $\hat{\Phi}_2$ from the whole of $\mathbb{P}^3$, defined by

$$\hat{\Phi}_2(\mathbf{X}) = \begin{cases} (A_1\mathbf{X}, A_2\mathbf{X}) & ; A_1\mathbf{X} \neq 0,\ A_2\mathbf{X} \neq 0 \ , \\ \{(A_1\mathbf{X}, \mathbf{x}_2) \,|\, \mathbf{x}_2 \in \mathbb{P}^2\} & ; A_2\mathbf{X} = 0 \ , \\ \{(\mathbf{x}_1, A_2\mathbf{X}) \,|\, \mathbf{x}_1 \in \mathbb{P}^2\} & ; A_1\mathbf{X} = 0 \ . \end{cases} \tag{32}$$

Then the range of $\hat{\Phi}_2$ equals exactly the variety, $\mathcal{V}_b$, generated by the bilinear constraint. ■

Remark. The underlying mechanism here is that the image of a quasiprojective variety under a rational function, f, needs not be a variety, even if f is nonsingular everywhere. Note that Φ_2 is nonsingular everywhere on $\dot{I\!P}^3$, because it is a linear function. ■

The conclusion is that we have to remove the epipoles from the image planes in order to make $\mathcal{V}_{\text{n}}$ a quasiprojective variety. Then we can take the projective closure in the Zariski topology of the quasiprojective variety obtained if the epipoles are removed. This closure is the quasiprojective variety $\mathcal{V}_{\text{b}}$. In the opposite direction, given $\mathcal{V}_{\text{b}}$ we can identify the singular point $(e_{1,2}, e_{2,1})$ corresponding to the epipoles, remove all points of the form $(e_{1,2}, \mathbf{x}_2)$ and $(\mathbf{x}_1, e_{2,1})$, replace them with the point pair $(e_{1,2}, e_{2,1})$ and obtain the natural descriptor $\mathcal{V}_{\text{n}}$. The singular points can be calculated from

$$\frac{\partial b_{1,2}}{\partial X_1} = 0 \ , \quad \frac{\partial b_{1,2}}{\partial Y_1} = 0 \ , \quad \dots \ , \quad \frac{\partial b_{1,2}}{\partial Z_2} = 0 \ , \tag{33}$$

which shows that the only singular point on $\mathcal{V}_{\text{b}}$ is $(e_{1,2}, e_{2,1})$. It is also obvious that $\mathcal{V}_{\text{b}}$ is irreducible, because it is generated by a single irreducible polynomial. Thus we have answered all questions raised above for the case of two images.

5 Three Images

Things becomes even more complicated in the case of three images. We will first consider the descriptors defined above and relations between them and then how these relations can be expressed in terms of ideals.

5.1 Varieties

Let

$$\Phi_3 : \dot{I\!P}^3 \ni \mathbf{X} \mapsto (A_1\mathbf{X}, A_2\mathbf{X}, A_3\mathbf{X}) \in I\!P^2 \times I\!P^2 \times I\!P^2 \ . \tag{34}$$

Considering a suitable restriction of Φ_3, we get the following theorem.

Theorem 5.1 (Fundamental Theorem of Trifocal Geometry).

$$\widetilde{\Phi}_3 : \begin{cases} I\!P^3 \setminus \{TP_{1,2,3}\} \to (I\!P^2 \setminus \{tl_{1,2,3}\}) \times (I\!P^2 \setminus \{tl_{2,1,3}\}) \times (I\!P^2 \setminus \{tl_{3,1,2}\}) \\ \qquad \mathbf{X} \mapsto (A_1\mathbf{X}, A_2\mathbf{X}, A_3\mathbf{X}), \end{cases} \tag{35}$$

where $TP_{1,2,3}$ denotes the trifocal plane and $tl_{1,2,3}$, $tl_{2,1,3}$ and $tl_{3,1,2}$ denotes the trifocal lines, cf. Definition 2.11 and Definition 2.13, is a birational map between the quasiprojective varieties $I\!P^3 \setminus \{TP_{1,2,3}\}$ and $\left((I\!P^2 \setminus \{tl_{1,2,3}\}) \times (I\!P^2 \setminus \{tl_{2,1,3}\}) \times (I\!P^2 \setminus \{tl_{3,1,2}\})\right) \cap \mathcal{V}_{\text{b}}$.

Proof. $\left((I\!P^2 \setminus \{tl_{1,2,3}\}) \times (I\!P^2 \setminus \{tl_{2,1,3}\}) \times (I\!P^2 \setminus \{tl_{3,1,2}\})\right) \cap \mathcal{V}_{\text{b}}$ is a quasiprojective variety since it is the intersection between two quasiprojective varieties. Obviously, $\widetilde{\Phi}_3$ is rational. It remains to prove that its inverse exists and is rational. This can be seen from the fact that given a point $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ in the range of $\widetilde{\Phi}_3$, it is possible to reconstruct the point in $I\!P^3 \setminus \{TP_{1,2,3}\}$ by intersecting the lines $A_1^{-1}(\mathbf{x}_1)$, $A_2^{-1}(\mathbf{x}_2)$ and $A_3^{-1}(\mathbf{x}_3)$. The reconstructed point can be written as the intersection of these lines, defining a rational map. ■

Remark. It is also possible to make a birational restriction by removing just the epipolar line, $EL_{1,2}$, between f_1 and f_2 from $I\!P^3$ and the corresponding image points. This follows from the fact that it is possible to construct an inverse rational map by considering the intersection of the rays from the first two images, when the epipolar line $EL_{1,2}$ has been excluded. ■

Definition 5.1. The mapping, $\widetilde{\Phi}_3$, in (35) is called the **birational restriction** of Φ_3. ■

It is always possible to reconstruct a point in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, which is in the range of Φ_3, using a rational map, defined by intersecting two lines. However, we can not in advance choose a function for this. For example, if the point $\mathbf{X}$ is on the epipolar line, $EL_{1,2}$, we can not use just A_1 and A_2 to make a reconstruction. We have to use A_3 also. This indicates why it is impossible to find an inverse rational map which would give a birational equivalence between $\dot{\mathbb{P}}^3$ and $\mathcal{V}_n$.

Next we will characterise the set of points in $\mathcal{V}_n$. The natural descriptor $\mathcal{V}_n \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ consists of the following pairs of points, see Figure 2 for an illustration:

- One arbitrary point, (x_1, y_1, z_1), in the first $(\mathbb{P}^2 \setminus \{tl_{1,2,3}\})$, one point, (x_2, y_2, z_2), in the second $(\mathbb{P}^2 \setminus \{tl_{2,1,3}\})$ on the line $b_{1,2} = 0$ and one point, (x_3, y_3, z_3), in the third $(\mathbb{P}^2 \setminus \{tl_{3,1,2}\})$ on the intersection between the lines $b_{1,3} = 0$ and $b_{2,3} = 0$ (and any permutation of the three images).
- One arbitrary point, $(x_1, y_1, 0)$, on $(tl_{1,2,3} \setminus \{e_{1,3}, e_{1,2}\})$ in the first $\mathbb{P}^2$, one arbitrary point, $(x_2, y_2, 0)$, on $(tl_{2,1,3} \setminus \{e_{2,3}, e_{2,1}\})$ in the second $\mathbb{P}^2$ and a unique point, $(x_3, y_3, 0)$, on $(tl_{3,1,2} \setminus \{e_{3,1}, e_{3,2}\})$ in the third $\mathbb{P}^2$ given by the trilinear constraints, or as a projection of the reconstructed point from image 1 and image 2 onto the third image (and any permutation of the three images).
- The epipole $e_{1,2} = (1, 0, 0)$ in the first $\mathbb{P}^2$, the epipole $e_{2,1} = (1, 0, 0)$ in the second $\mathbb{P}^2$ and an arbitrary point, $(x_3, y_3, 0)$, on the trifocal line $tl_{3,1,2}$ in the third $(\mathbb{P}^2 \setminus \{e_{3,1}, e_{3,2}\})$ (and any permutation of the three images).

These correspond in turn to images of points not on the trifocal plane, points in the trifocal plane, not on an epipolar line and points on an epipolar line.

We now turn to the variety generated by the bilinear forms. In this case each image pair contributes with a bilinear form,

$$
\begin{aligned}
b_{1,2}(\mathbf{x}_1, \mathbf{x}_2) &= b_{1,2}(x_1, y_1, z_1, x_2, y_2, z_2) = y_1 z_2 - z_1 y_2 \ , \\
b_{1,3}(\mathbf{x}_1, \mathbf{x}_3) &= b_{1,3}(x_1, y_1, z_1, x_3, y_3, z_3) = x_1 z_3 - z_1 x_3 \ , \\
b_{2,3}(\mathbf{x}_2, \mathbf{x}_3) &= b_{2,3}(x_2, y_2, z_2, x_3, y_3, z_3) = x_2 z_3 - z_2 x_3 + y_2 z_3 - z_2 y_3 = (x_2 + y_2) z_3 - z_2 (x_3 + y_3) \ .
\end{aligned}
\tag{36}
$$

The projective subvariety, $\mathcal{V}_b \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, generated by these forms consists of the following triplets of points, see Figure 3 for an illustration:

- One arbitrary point, (x_1, y_1, z_1), in the first $(\mathbb{P}^2 \setminus \{tl_{1,2,3}\})$, one point, (x_2, y_2, z_2) in the second $(\mathbb{P}^2 \setminus \{tl_{1,2,3}\})$ on the line $b_{1,2} = 0$ and one point, (x_3, y_3, z_3), in the third $(\mathbb{P}^2 \setminus \{tl_{3,1,2}\})$ on the intersection of the lines $b_{1,3} = 0$ and $b_{2,3} = 0$ (and any permutation of the three images).
- One arbitrary point, $(x_1, y_1, 0)$, on $tl_{1,2,3}$ in the first $\mathbb{P}^2$, one arbitrary point, $(x_2, y_2, 0)$, on $tl_{2,1,3}$ in the second $\mathbb{P}^2$ and one arbitrary point, $(x_3, y_3, 0)$, on $tl_{3,1,2}$ in the third $\mathbb{P}^2$ (and any permutation of the three images).
- The epipole $e_{1,3} = (0, 1, 0)$ in the first $\mathbb{P}^2$, the epipole $e_{2,3} = (1, -1, 0)$ in the second $\mathbb{P}^2$ and an arbitrary point, (x_3, y_3, z_3), in the third $\mathbb{P}^2$ (and any permutation of the three images).

Finally, consider the variety $\mathcal{V}_t \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ generated by the trilinear forms

$$
t_{i,j,k}(x_i, y_i, z_i, x_j, y_j, z_j, x_k, y_k, z_k,) = \sum_{l,m,n} (T_{i,j,k})_l^{m,n} P_l K^m L^n \ , \tag{37}
$$

where $(i, j, k) = (1, 2, 3)$. The projective subvariety in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ corresponding to these forms is given by the following triplets of points, see Figure 4 for an illustration:

- One arbitrary point, (x_1, y_1, z_1), in the first $(\mathbb{P}^2 \setminus \{tl_{1,2,3}\})$, one point, (x_2, y_2, z_2), in the second $(\mathbb{P}^2 \setminus \{tl_{2,1,3}\})$ on the line $b_{1,2} = 0$ and one point, (x_3, y_3, z_3), in the third $(\mathbb{P}^2 \setminus \{tl_{3,1,2}\})$, given by the trilinear constraints (and any permutation of the three images).
- One arbitrary point, $(x_1, y_1, 0)$, on $(tl_{1,2,3} \setminus \{e_{1,2}, e_{1,3}\})$ in the first $\mathbb{P}^2$, one arbitrary point, $(x_2, y_2, 0)$, on $(tl_{2,1,3} \setminus \{e_{2,1}, e_{2,3}\})$ in the second $\mathbb{P}^2$ and a unique point, (x_3, y_3, z_3), on $(tl_{3,1,2} \setminus \{e_{3,1}, e_{3,2}\})$ in the third $\mathbb{P}^2$ given by the trilinear constraints, or as a projection of the reconstructed point from image 1 and image 2 onto the third image (and any permutation of the three images).

- The epipole $e_{1,2} = (1,0,0)$ in the first $\mathbb{P}^2$, the epipole $e_{2,1} = (1,0,0)$ in the second $\mathbb{P}^2$ and an arbitrary point, $(x_3, y_3, 0)$, on $tl_{3,1,2}$ in the third $\mathbb{P}^2$ (and any permutation of the three images).
- The epipole $e_{1,3} = (0,1,0)$ in the first $\mathbb{P}^2$, the epipole $e_{2,3} = (1,-1,0)$ in the second $\mathbb{P}^2$ and an arbitrary point, (x_3, y_3, z_3), in the third $\mathbb{P}^2$ (and any permutation of the three images).

Using these characterisations of the points in $\mathcal{V}_\mathrm{t}$ and $\mathcal{V}_\mathrm{b}$ gives the following theorem.

Theorem 5.2. *For three images we have, with strict inclusions,*

$$\mathcal{V}_\mathrm{n} \subset \mathcal{V}_\mathrm{t} \subset \mathcal{V}_\mathrm{b} \ . \tag{38}$$

Furthermore, the variety $\mathcal{V}_\mathrm{b}$ is reducible and can be written as a union of two irreducible varieties

$$\mathcal{V}_\mathrm{b} = \mathcal{V}_\mathrm{t} \cup \mathcal{V}_\mathrm{tp}, \tag{39}$$

where $\mathcal{V}_\mathrm{tp}$ is the variety containing one point on each trifocal line. Finally, the bitrilinear variety, $\mathcal{V}_\mathrm{bt}$, and the trilinear variety, $\mathcal{V}_\mathrm{t}$, coincide, i.e.

$$\mathcal{V}_\mathrm{bt} = \mathcal{V}_\mathrm{t} \ .$$

Proof. See the descriptions above for the first part. The decomposition of $\mathcal{V}_\mathrm{b}$ follows from the characterisations above. The subset $\mathcal{V}_\mathrm{tp}$ is a variety since it can be generated by the polynomials $p_1 = z_1$, $p_2 = z_2$, and $p_3 = z_3$. From Theorem 5.6 below, it can be seen that $\mathcal{V}_\mathrm{t}$ and $\mathcal{V}_\mathrm{tp}$ are irreducible. It follows from Theorem 5.2 that $\mathcal{V}_\mathrm{t} \subset \mathcal{V}_\mathrm{b}$. Together with $\mathcal{V}_\mathrm{bt} = \mathcal{V}_\mathrm{t} \cap \mathcal{V}_\mathrm{b}$, this gives $\mathcal{V}_\mathrm{bt} = \mathcal{V}_\mathrm{t}$. ■

Remark. Just as in the case of two images, we can describe the variety, $\mathcal{V}_\mathrm{t}$ as the range of an extension of the map Φ_3 to a multivalued map $\hat{\Phi}_3$ from the whole of $\mathbb{P}^3$, defined analogously to (32). Then the range of $\hat{\Phi}_3$ equals the variety, $\mathcal{V}_\mathrm{t}$, generated by the trilinear constraints. ■

We now return to the natural descriptor $\mathcal{V}_\mathrm{n} \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. Consider the ideal

$$\mathcal{I}_\mathrm{n} = \mathcal{I}(\mathcal{V}_\mathrm{n}) \subset \mathbb{R}[x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3] \ .$$

Since $\mathcal{V}_\mathrm{n}$ is a subset in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, the functions in $\mathcal{I}_\mathrm{n}$ are trihomogeneous in (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), see [15], pp. 56. The bilinearities in (36) are functions of degree $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$, which can be extended to functions of degree $(1,1,1)$ as described above. There are also trilinear functions of degree $(1,1,1)$. Furthermore, we have the following theorem.

Theorem 5.3. *The natural ideal, defined by the natural descriptor, can be described as*

$$\mathcal{I}_\mathrm{n} = \mathcal{I}(\mathcal{V}_\mathrm{t}) \ .$$

Moreover, the natural descriptor $\mathcal{V}_\mathrm{n} \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ is not a variety, in the sense that it can not be described as the common zeroes to a system of polynomial equations and

$$\overline{\mathcal{V}_\mathrm{n}} = \mathcal{V}_\mathrm{t} \ .$$

Proof. By definition, $\mathcal{V}_\mathrm{n}$ is the range of Φ_3, that is all points

$$(x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3) \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$$

fulfilling, for some X, Y, Z, W, λ_1, λ_2, λ_3,

$$\begin{cases} x_1 = \lambda_1 X \\ y_1 = \lambda_1 Y \\ z_1 = \lambda_1 Z \end{cases}, \qquad \begin{cases} x_2 = \lambda_2 (X + W) \\ y_2 = \lambda_2 Y \\ z_2 = \lambda_2 Z \end{cases}, \qquad \begin{cases} x_3 = \lambda_3 X \\ y_3 = \lambda_3 (Y + W) \\ z_3 = \lambda_3 Z \end{cases},$$

according to the chosen coordinates in (23). Elimination of X, Y, Z, W, λ_1, λ_2 and λ_3 can be done by calculating a Gröbner basis. The second part is analogous to Theorem 4.3.

■

5.2 Ideals

First we will show how the bilinear constraints can be obtained from the trilinear ones by simple matrix computations. All multilinear constraints, for three images, are obtained from (7). Computing subdeterminants gives the 3 bilinearities $b_{1,2}$, $b_{1,3}$ and $b_{2,3}$. The trilinearities can be denoted $t_{i,j}$, where $t_{i,j}$ denotes the subdeterminant obtained after removing rows i and j.

Theorem 5.4. *The bilinear constraints follow from the trilinear ones, in the sense that*

$$x_i b_{j,k}, y_i b_{j,k} \text{ and } z_i b_{j,k} \in (t_{1,4}, \ldots, t_{6,9}) \ ,$$

for (i, j, k) *equal to any permutation of* $(1, 2, 3)$.

Proof. It follows from (7) by duplication of the first, second and fourth column, respectively and expanding by the duplicated column. ■

Remark. In [6] it is shown that it is possible to derive the bilinear constraints, and all trilinear ones from just two trilinear constraints. However, this requires that the two trilinear constraints are known not just up to a scale factor, but exactly as they come out as subdeterminants. If the trilinear constraints are estimated from point correspondences between three images this information is not obtained, because the trilinear constraints can only be estimated up to an unknown scale factor. The same thing can be done if two bilinear constraints are known exactly, that is with appropriate scale factors. It has been shown in [10] that in order to compute the camera geometry we only need two bilinear constraints and one scale factor describing the relation between the two translational vectors obtained from the bilinear constraints. ■

Now we are going to study different ways of generating $\mathcal{I}_\mathrm{b}$ and $\mathcal{I}_\mathrm{t}$. It is obvious that $\mathcal{I}_\mathrm{b}$ can be generated by the three bilinearities, but that no two of them are sufficient to generate $\mathcal{I}_\mathrm{b}$. Things are more complicated for $\mathcal{I}_\mathrm{t}$. Although the trilinear constraints, locally, can be written as the vanishing of 3 trilinear forms among all trilinearities, we need four forms to generate $\mathcal{I}_\mathrm{t}$. Consider first the following simple example, which reveals the difference between the number of elements in a minimal generating set and the codimension of the corresponding variety.

Example 5.1. The condition that two vectors, $u = (a, b, c)$ and $v = (d, e, f)$, in $\mathbb{R}^3$ are parallel can be written

$$\operatorname{rank} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} < 2 \ , \tag{40}$$

which is equivalent to

$$\begin{cases} p_1(a, b, c, d, e, f) = ae - bd = 0 \ , \\ p_2(a, b, c, d, e, f) = bf - ce = 0 \ , \\ p_3(a, b, c, d, e, f) = cd - af = 0 \ . \end{cases} \tag{41}$$

Introduce the ideal $\mathcal{I}_\mathrm{ex} = (p_1, p_2, p_3) \subset \mathbb{R}[a, b, c, d, e, f]$. The codimension of the variety $\mathcal{V}(\mathcal{I}_\mathrm{ex})$ is 2, since the conditions in (40) can in general be obtained from 2 polynomial equations. This can be seen from

$$fp_1 + dp_2 + ep_3 = 0 \ .$$

However, it is not possible to generate the ideal (p_1, p_2, p_3) by two of the polynomials, say p_1 and p_2, because $u = (1, 0, 1)$ and $v = (1, 0, 2)$ obeys both $p_1 = 0$ and $p_2 = 0$ but $p_3 = -1$. This means that the codimension of $\mathcal{V}(\mathcal{I}_\mathrm{ex})$ is 2 and $\{p_1, p_2, p_3\}$ is a minimal generating set for $\mathcal{I}_\mathrm{ex}$. In fact, the dimension of $\mathcal{V}(\mathcal{I}_\mathrm{ex})$ equals the maximum number of algebraically independent functions in the coordinate ring $\mathbb{R}[a, b, c, d, e, f]/\mathcal{I}_\mathrm{ex}$. ■

Theorem 5.5. *The ideal* $\mathcal{I}_\mathrm{t}$ *can be generated by the three bilinearities in (36) and one trilinearity,* $t_{6,9} = x_1 y_2 x_3 + x_1 y_2 y_3 - y_1 x_2 x_3 - y_1 y_2 x_3$. $\mathcal{I}_\mathrm{t}$ *can not be generated by three multilinear functions.*

Proof. This is easily seen by computing a Gröbner basis for $(b_{1,2}, b_{1,3}, b_{2,3}, t_{6,9})$ and computing the normal form of the other trilinearities with respect to this Gröbner basis, for example by means of Maple. The normal form of a trilinearity is 0 if and only if the trilinearity is in the ideal described by the Gröbner basis. By calculating Gröbner bases for triplets of multilinear functions and normal forms of the resulting multilinear functions, it can be seen that it is not possible to generate $\mathcal{I}_{\mathrm{t}}$ by three multilinear functions. ■

Remark. Using our choice of coordinates, the trilinearity needed apart from the bilinearities can be any of $t_{3,6}$, $t_{3,9}$ or $t_{6,9}$ (they are in fact the same polynomial). Using an arbitrary coordinate system, it is in general possible to choose an arbitrary trilinearity. The condition that must be fulfilled is that the trilinearity does not vanish on the trifocal lines. ■

We are now ready to prove our key result describing the relations between $\mathcal{I}_{\mathrm{b}}$ and $\mathcal{I}_{\mathrm{t}}$. First observe that if $z_1 = z_2 = z_3 = 0$, then all bilinearities in (36) vanish but the trilinearity $t_{6,9}$ does not vanish. This means that the trilinear constraint $t_{6,9} = 0$ imposes further conditions on the other image coordinates. The conditions $z_1 = z_2 = z_3 = 0$ describe the intersection of the trifocal plane with the three images and this indicates that these constraints correspond to an irreducible component of $\mathcal{V}_{\mathrm{b}}$.

Theorem 5.6 (Primary Decomposition of the Bilinear Ideal). *The ideal $\mathcal{I}_{\mathrm{b}}$ is reducible and can be decomposed as*

$$\mathcal{I}_{\mathrm{b}} = \mathcal{I}_{\mathrm{t}} \cap \mathcal{I}_{\mathrm{tp}} \ , \tag{42}$$

where $\mathcal{I}_{\mathrm{tp}} = (z_1, z_2, z_3)$ is the ideal corresponding to the trifocal plane. In (42), $\mathcal{I}_{\mathrm{t}}$ and $\mathcal{I}_{\mathrm{tp}}$ are prime ideals and thus are irreducible.

Proof. We first show that the decomposition in (42) is valid. Consider a polynomial, $f \in \mathcal{I}_{\mathrm{b}}$:

$$\begin{aligned} f =& p_1 b_{1,2} + p_2 b_{1,3} + p_3 b_{2,3} = \\ =& p_1(y_1 z_2 - z_1 y_2) + p_2(x_1 z_3 - z_1 x_3) + \\ &+ p_3(x_2 z_3 - z_2 x_3 + y_2 z_3 - z_2 y_3) = \\ =& (-p_1 y_2 - p_2 x_3) z_1 + (p_1 y_1 - p_3 x_3 - p_3 y_3) z_2 + \\ &+ (p_2 x_1 + p_3 x_2 - p_3 y_2) z_3 \in I_{\mathrm{tp}} \ , \end{aligned} \tag{43}$$

when $p_i \in \mathbb{R}[x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3]$. Consider a polynomial, $g \in \mathcal{I}_{\mathrm{t}} \cap \mathcal{I}_{\mathrm{tp}}$. Since $g \in \mathcal{I}_{\mathrm{t}}$, by Theorem 5.5, it can be written

$$\begin{aligned} g =& q_1 b_{1,2} + q_2 b_{1,3} + q_3 b_{2,3} + q_4 t_{6,9} = \\ =& q_1(y_1 z_2 - z_1 y_2) + q_2(x_1 z_3 - z_1 x_3) + \\ &+ q_3(x_2 z_3 - z_2 x_3 + y_2 z_3 - z_2 y_3) + \\ &+ q_4(x_1 y_2 x_3 + x_1 y_2 y_3 - y_1 x_2 x_3 - y_1 y_2 x_3) \ , \end{aligned} \tag{44}$$

where $q_i \in \mathbb{R}[x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3]$. Moreover, since $g \in \mathcal{I}_{\mathrm{tp}}$,

$$g = h_1 z_1 + h_2 z_2 + h_3 z_3 \tag{45}$$

holds. (44) and (45) imply that

$$q_4 = z_1 r_1 + z_2 r_2 + z_3 r_3 \ , \tag{46}$$

where $r_i \in \mathbb{R}[x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3]$, since the other terms are in (z_1, z_2, z_3) and $t_{6,9}$ do not contain any of z_1, z_2 or z_3. We now have

$$\begin{aligned} z_1 t_{6,9} &\equiv z_1(x_1 y_2 x_3 + x_1 y_2 y_3 - y_1 x_2 x_3 - y_1 y_2 x_3) \equiv \\ &\equiv x_1 x_3 z_1 y_2 + x_1 y_3 z_1 y_2 - y_1 x_2 z_1 x_3 - y_1 y_2 z_1 x_3 \equiv \\ &\equiv x_1 x_3 y_1 z_2 + x_1 y_3 y_1 z_2 - y_1 x_2 x_1 z_3 - y_1 y_2 x_1 z_3 \equiv \\ &\equiv x_1 y_1 (z_2 x_3 + z_2 y_3) - y_1 z_3 (x_2 x_1 + y_2 x_1) \equiv \\ &\equiv x_1 y_1 (y_2 z_3 + x_2 z_3) - y_1 z_3 (x_2 x_1 + y_2 x_1) \equiv \\ &\equiv x_1 y_1 (y_2 z_3 + x_2 z_3 - x_2 z_3 - y_2 z_3) \equiv 0 \mod \mathcal{I}_{\mathrm{b}} \ , \end{aligned} \tag{47}$$

where we have used the reduction relations from the bilinearities taken in the order $b_{1,2}$, $b_{2,3}$. In the same way it can be shown that $z_2 t_{6,9} \in \mathcal{I}_{\mathrm{b}}$ and that $z_3 t_{6,9} \in \mathcal{I}_{\mathrm{b}}$. This shows that $g \in \mathcal{I}_{\mathrm{b}}$.

It is obvious that $\mathcal{I}_{\mathrm{tp}} = (z_1, z_2, z_3)$ is a prime ideal. To show that $\mathcal{I}_{\mathrm{t}}$ is a prime ideal we note that

$$\mathcal{I}_{\mathrm{t}} = \mathcal{I}(\mathcal{V}_{\mathrm{n}})$$

and $\mathcal{V}_{\mathrm{n}}$ is parametrically defined. This means, see [3], pp. 197, that the closure, $\overline{\mathcal{V}_{\mathrm{n}}}$, is an irreducible variety and since

$$\mathcal{I}_{\mathrm{t}} = \mathcal{I}(\overline{\mathcal{V}_{\mathrm{n}}})$$

$\mathcal{I}_{\mathrm{t}}$ is a prime ideal. ■

Finally, we have

Corollary 5.1. *The ideals $\mathcal{I}_{\mathrm{b}}$, $\mathcal{I}_{\mathrm{t}}$ and $\mathcal{I}_{\mathrm{tp}}$ are their own radicals, which implies*

$$\mathcal{I}(\mathcal{V}_{\mathrm{b}}) = \mathcal{I}_{\mathrm{b}} \ , \quad \mathcal{I}(\mathcal{V}_{\mathrm{t}}) = \mathcal{I}_{\mathrm{t}} \ . \tag{48}$$

Furthermore, the ideal $\mathcal{I}_{\mathrm{bt}}$ generated by the bilinearities and the trilinearities is the same as the ideal $\mathcal{I}_{\mathrm{t}}$ generated by the trilinearities,

$$\mathcal{I}_{\mathrm{bt}} = \mathcal{I}_{\mathrm{t}} \ .$$

Proof. The ideals $\mathcal{I}_{\mathrm{t}}$ and $\mathcal{I}_{\mathrm{tp}}$ are their own radicals because they are prime, see [16]. (48) is valid because $\mathcal{I}(\mathcal{V}) = \sqrt{\mathcal{I}}$, see [15]. $\mathcal{I}_{\mathrm{b}}$ is prime because,

$$\mathcal{I}_{\mathrm{b}} = \mathcal{I}_{\mathrm{t}} \cap \mathcal{I}_{\mathrm{tp}} \quad \Longrightarrow \quad \sqrt{\mathcal{I}_{\mathrm{b}}} = \sqrt{\mathcal{I}_{\mathrm{t}}} \cap \sqrt{\mathcal{I}_{\mathrm{tp}}} = \mathcal{I}_{\mathrm{t}} \cap \mathcal{I}_{\mathrm{tp}} = \mathcal{I}_{\mathrm{b}} \ .$$

Finally, theorem 5.6 gives $\mathcal{I}_{\mathrm{b}} \subset \mathcal{I}_{\mathrm{t}}$. This together with $\mathcal{I}_{\mathrm{bt}} = \mathcal{I}_{\mathrm{b}} \cap \mathcal{I}_{\mathrm{t}}$ proves the last assertion. ■

We conclude this section with the observation that the projective dimension of the varieties $\mathcal{V}_{\mathrm{t}}$ and $\mathcal{V}_{\mathrm{b}}$ is $3 = 9 - 3 - 3$. The number of variables is 9, they are divided into three groups of projective vectors and the constraints can in general be written as the vanishing of 3 polynomial equations. The dimension can also be calculated from the tangent spaces. This means that the codimension is 3 and we would like to have three polynomials to generate the variety $\mathcal{V}_{\mathrm{t}}$, unfortunately this is not possible.

6 More Than Three Images

Now we have the following map

$$\Phi_m : \dot{\mathbb{P}}^3 \ni \mathbf{X} \mapsto (A_1\mathbf{X}, \ldots, A_m\mathbf{X}) \in (\mathbb{P}^2)^m, \quad m > 3 \ . \tag{49}$$

It is possible to consider different birational restrictions in this case too. What we have do to is to take away a trifocal plane and the corresponding image points exactly as in the case of three images. It is also possible to remove one epipolar line and the corresponding image points, because then it is possible to select two cameras in advance to reconstruct the point in $\dot{\mathbb{P}}^3$ giving an inverse rational map.

The natural descriptor $\mathcal{V}_{\mathrm{n}}$ consists only of the m-tuples of points of the form

- One arbitrary point, (x_1, y_1, z_1), in the first $(\mathbb{P}^2 \setminus \{e_{1,2}, \ldots, e_{1,m}\})\}$, one point, (x_2, y_2, z_2), in the second $(\mathbb{P}^2 \setminus \{e_{1,2}, \ldots, e_{1,m}\})$ on the line $b_{1,2} = 0$ and one point in the other images corresponding to the projection of the reconstructed point from image 1 and 2 (and any permutation of the images).

Consider first the variety, $\mathcal{V}_{\mathrm{b}}$, for $m = 4$, generated by the bilinear forms. In this case there are 6 bilinearities,

$$\begin{aligned}
b_{1,2} &= y_1 z_2 - z_1 y_2 \ , \\
b_{1,3} &= x_1 z_3 - z_1 x_3 \ , \\
b_{2,3} &= x_2 z_3 - z_2 x_3 + y_2 z_3 - z_2 y_3 = (x_2 + y_2) z_3 - z_2 (x_3 + y_3) \ , \\
b_{1,4} &= x_1 y_4 - y_1 x_4 \ , \\
b_{2,4} &= x_2 y_4 - y_2 x_4 - y_2 z_4 + z_2 y_4 = (x_2 + z_2) y_4 - y_2 (x_4 + z_4) \ , \\
b_{3,4} &= x_3 y_4 + x_3 z_4 - y_3 x_4 - z_3 x_4 = x_3 (y_4 + z_4) - (y_3 + z_3) x_4 \ .
\end{aligned} \tag{50}$$

Using Gröbner bases, it turns out that the variety $\mathcal{V}_{\mathrm{b}}$ consists of the following points.

- One arbitrary point, (x_1, y_1, z_1), in the first $(\mathbb{P}^2 \setminus \{e_{1,2}, e_{1,3}, e_{1,4}\})$, one point, (x_2, y_2, z_2), in the second $(\mathbb{P}^2 \setminus \{e_{2,1}, e_{2,3}, e_{2,4}\})$ on the line $b_{1,2} = 0$ and one point in the other images corresponding to the projection of the reconstructed point from image 1 and 2 (and any permutation of the images).
- The epipoles $e_{1,4} = (0, 0, 1)$, $e_{2,4} = (1, 0, -1)$, $e_{3,4} = (0, 1, -1)$ and an arbitrary point in the fourth image (and any permutation of the images).

In general the variety $\mathcal{V}_\mathrm{b}$ have $\binom{m}{2}$ bilinearities and consists of the following m-tuples of points:

- One arbitrary point, (x_1, y_1, z_1), in the first $(\mathbb{P}^2 \setminus \{e_{1,2}, , e_{1,3}, \ldots, e_{1,m}\})$, one point, (x_2, y_2, z_2), in the second $(\mathbb{P}^2 \setminus \{e_{2,1}, e_{2,3}, \ldots, e_{2,m}\})$ on the line $b_{1,2} = 0$ and one point in the other images corresponding to the projection of the reconstructed point from image 1 and 2 (and any permutation of the images).
- The epipoles $\{e_{1,m}, \ldots, e_{m-1,m}\}$ and an arbitrary point in the m:th image (and any permutation of the images).

Theorem 6.1. *For m images, with $m > 3$, we have,*

$$\mathcal{V}_\mathrm{n} \subset \mathcal{V}_\mathrm{b} = \mathcal{V}_\mathrm{t} = \mathcal{V}_\mathrm{q} = \overline{\mathcal{V}_\mathrm{n}} \ , \tag{51}$$

where the inclusion is strict.

Proof. The theorem follows from Gröbner basis calculations for $m = 4$. For $m > 4$ all trilinearities between triplets of views can be generated by the six bilinearities between these three views and another arbitrary view. The quadrilinearities follow in the same way from the bilinearities and from the trilinearities, and vice versa. Finally, the first inclusion follows from the descriptions above. ■

Remark. Geometrically, this can be seen as follows for $m = 4$. When we have three images the bilinear constraints fail on the trifocal plane, but when we have another image outside the trifocal plane it is possible to resolve this failure by using the three new bilinear constraints involving the fourth image. ■

In the same way as before it is possible to extend Φ_m to a multivalued map $\hat{\Phi}_m$ such that the range of $\hat{\Phi}_m$ equals exactly the variety $\mathcal{V}_\mathrm{b}$.

A similar result can be obtained for the ideals.

Theorem 6.2. *For m images, $m \geq 3$, we have*

$$\mathcal{I}_\mathrm{n} = \mathcal{I}_\mathrm{b} = \mathcal{I}_\mathrm{t} = \mathcal{I}_\mathrm{q} \ . \tag{52}$$

Proof. For $m = 4$ the theorem follows from Gröbner basis calculations. For $m > 4$ the last two equalities follows from the fact that they are true for four images. The first equality can be shown as follows. By definition, $\mathcal{V}_\mathrm{n}$ is the range of Φ_m, that is all points $(x_1, \ldots, z_m) \in (\mathbb{P}^2)^m$ fulfilling, for some X, Y, Z, W, λ_1, $\lambda_2, \ldots, \lambda_m$,

$$\begin{cases} x_1 = \lambda_1 X \\ y_1 = \lambda_1 Y \\ z_1 = \lambda_1 Z \end{cases}, \quad \begin{cases} x_2 = \lambda_2 (X + W) \\ y_2 = \lambda_2 Y \\ z_2 = \lambda_2 Z \end{cases}, \quad \ldots, \quad \begin{cases} x_m = \lambda_m (X + a_m W) \\ y_m = \lambda_m (Y + b_m W) \\ z_m = \lambda_m (Z + c_m W) \end{cases}, \tag{53}$$

according to the chosen coordinates in (23). It is possible to eliminate X, Y, Z, W, λ_1, λ_2, ..., λ_m. Since all equation in (53) are linear in X, Y, Z, W, λ_1, $\lambda_2, \ldots, \lambda_m$, the elimination ideal can be described by (9), which is exactly the ideal generated by all multilinear forms. ■

Corollary 6.1. *The natural descriptor $\mathcal{V}_\mathrm{n} \in (\mathbb{P}^2)^m$ is not a variety for any m.*

Proof. The proof is analogous to the proof of Theorem 5.3. ■

Consider now the case $m = 4$. Although the varieties $\mathcal{V}_b$ and $\mathcal{V}_t$ are the same, the situation is not fully satisfactory. This is because the varieties $\mathcal{V}_b$ and $\mathcal{V}_t$ have dimension 3, making it desirable to have 5 generators of the ideals instead of 6. Using only 5 of the 6 bilinearities, we get a slightly larger variety. To see this consider the ideal, $\widetilde{\mathcal{I}}_b$ generated by the first 5 bilinearities, that is

$$\widetilde{\mathcal{I}}_b = (b_{1,2}, b_{1,3}, b_{2,3}, b_{1,4}, b_{2,4}) \ . \tag{54}$$

Looking at the bilinearities in (50) we find that

- if $z_1 = z_2 = z_3 = y_1 = y_2 = y_4 = 0$ then $b_{1,2} = b_{1,3} = b_{2,3} = b_{1,4} = b_{2,4} = 0$, which corresponds to the image of a point on the epipolar line, $EL_{1,2}$, defined by camera 1 and 2.
- if $z_1 = z_2 = z_3 = b_{1,4} = b_{2,4} = 0$ then $b_{1,2} = b_{1,3} = b_{2,3} = b_{1,4} = b_{2,4} = 0$, which corresponds to the image of a point in the trifocal plane, $TP_{1,2,3}$, defined by camera 1, 2 and 3.
- if $y_1 = y_2 = y_4 = b_{1,3} = b_{2,3} = 0$ then $b_{1,2} = b_{1,3} = b_{2,3} = b_{1,4} = b_{2,4} = 0$, which corresponds to the image of a point in the trifocal plane, $TP_{1,2,4}$, defined by camera 1, 2 and 4.

This means that $\mathcal{V}(\widetilde{\mathcal{I}}_b)$ consists of the quadruples of points in $\mathcal{V}_b$ and:

- One arbitrary point on each of the trifocal lines $tl_{3,1,2}$ and $tl_{4,1,2}$ in image 3 and 4 respectively together with $e_{1,2}$ and $e_{2,1}$.
- One arbitrary point on each of the trifocal lines $tl_{1,2,3}$, $tl_{2,1,3}$ and $tl_{3,1,2}$ in image 1, 2 and 3 respectively and the unique point in image 4 defined by the bilinear constraints $b_{1,4} = 0$ and $b_{2,4} = 0$.
- One arbitrary point on each of the trifocal lines $tl_{1,2,4}$, $tl_{2,1,4}$ and $tl_{4,1,2}$ in image 1, 2 and 4 respectively and the unique point in image 3 defined by the bilinear constraints $b_{1,3} = 0$ and $b_{2,3} = 0$.

We thus have the following theorem:

Theorem 6.3. *The ideal $\mathcal{I}_b$ can be generated by all six bilinearities but not by any five multilinear functions.*

In the general case, since the dimension of $\mathcal{V}_b$ is 3, we would like to generate $\mathcal{I}_b$ by $2m-3$ bilinearities, but this is not possible, at least for five images, according to the next theorem.

Theorem 6.4. *The bilinear ideal $\mathcal{I}_b$ for five images can not be generated by 7 bilinear forms.*

Proof. Consider, for example, the ideal generated by $b_{1,2}$, $b_{1,3}$, $b_{2,3}$, $b_{2,4}$, $b_{3,4}$, $b_{3,5}$ and $b_{4,5}$

$$\begin{aligned}
b_{1,2} &= y_1 z_2 - z_1 y_2 \ , \\
b_{1,3} &= x_1 z_3 - z_1 x_3 \ , \\
b_{2,3} &= x_2 z_3 - z_2 x_3 + y_2 z_3 - z_2 y_3 = (x_2 + y_2) z_3 - z_2 (x_3 + y_3) \ , \\
b_{2,4} &= x_2 y_4 - y_2 x_4 - y_2 z_4 + z_2 y_4 = (x_2 + z_2) y_4 - y_2 (x_4 + z_4) \ , \\
b_{3,4} &= x_3 y_4 + x_3 z_4 - y_3 x_4 - z_3 x_4 = x_3 (y_4 + z_4) - (y_3 + z_3) x_4 \ , \\
b_{3,5} &= x_3 y_5 - y_3 x_5 + y_3 z_5 - z_3 y_5 = (x_3 - z_3) y_5 - y_3 (x_5 - z_5) \ , \\
b_{4,5} &= x_4 z_5 - y_4 z_5 - z_4 x_5 + z_4 y_5 = (x_4 - y_4) z_5 - z_4 (x_5 - y_5) \ .
\end{aligned} \tag{55}$$

Observe that $x_2 + y_2 + z_2 = x_3 + y_3 + z_3 = x_4 + y_4 + z_4 = 0$ implies $b_{2,3} = b_{2,4} = b_{3,4} = 0$ and (x_1, y_1, z_1) and (x_5, y_5, z_5) can be chosen such that $b_{1,2} = b_{1,3} = b_{3,5} = b_{4,5} = 0$. This corresponds to three arbitrary points on the trifocal lines in image 2, 3 and 4. However, using the trilinear constraints between views 2, 3 and 4, or the other bilinear constraints, impose further conditions on (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4). This shows that it is not sufficient to use only these 7 bilinearities in this case. However, it can be shown that it is not possible in the other cases either by considering other collections of 7 bilinearities. ∎

Remark. It can be shown, using Gröbner bases, that it is possible to generate the ideal $\mathcal{I}_{\mathrm{b}}$ for five images from 9 of the available 10 bilinearities. This means that all homogenised forms of one bilinear constraint follow algebraically from the other. However, it can be shown, using Gröbner bases, that it is not possible to generate $\mathcal{I}_{\mathrm{b}}$ for five images from 8 bilinearities. ■

We conclude this section with two conjectures.

Conjecture 6.1. *It is not possible to generate $\mathcal{I}_{\mathrm{t}}$ for $m > 3$ images by $2m-3$ bilinearities or by $2m-3$ other multilinear functions.*

Conjecture 6.2. *Let*

$$\widetilde{\mathcal{I}_{\mathrm{b}}} = (b_{i,i+1}, b_{i,i+2}) \enspace ,$$

see Figure 5 for an illustration. Then $\widetilde{\mathcal{I}_{\mathrm{b}}}$ has a primary decomposition

$$\widetilde{\mathcal{I}_{\mathrm{b}}} = \mathcal{I}_{\mathrm{t}} \cap \dots \enspace .$$

That is one primary component is the trilinear ideal, $\mathcal{I}_{\mathrm{t}}$.

Remark. This means that the variety $\widetilde{\mathcal{V}_{\mathrm{b}}} = \mathcal{V}(\widetilde{\mathcal{I}_{\mathrm{b}}})$ is irreducible and can be written

$$\widetilde{\mathcal{V}_{\mathrm{b}}} = \mathcal{V}_{\mathrm{t}} \cup \dots \enspace ,$$

where the trilinear variety $\mathcal{V}_{\mathrm{t}}$ is one irreducible component. ■

7 Conclusions

In this paper we have shown that the image of $\dot{\mathbb{P}}^3$ in $\mathbb{P}^2 \times \mathbb{P}^2 \times \dots \times \mathbb{P}^2$ under an m-tuple of projections $\Phi_m = (A_1, A_2, \dots, A_m)$ is not an algebraic variety, i.e. it can not be described as the set of common zeros to a system of polynomial equations. To obtain an algebraic variety it is necessary to take one of two different approaches. The first one is to extend Φ_m to a multivalued map, defining the image of the focal point f_i of camera i to the set of points corresponding to the actual epipoles in the other images and an arbitrary point in image i. The other is to restrict Φ_m by removing an epipolar line or a trifocal plane and the corresponding image points.

For two image we have, with strict inclusion,

$$\begin{aligned} \mathcal{V}_{\mathrm{n}} \subset \mathcal{V}_{\mathrm{b}} &= \overline{\mathcal{V}_{\mathrm{n}}} \enspace , \\ \mathcal{I}_{\mathrm{n}} &= \mathcal{I}_{\mathrm{b}} \enspace . \end{aligned}$$

For three images, we have shown that the bilinearities follow algebraically from the trilinearities in the sense that the bilinearities belong to the trilinear ideal. Furthermore, we have shown that the bilinear variety is reducible and can be written as a union of two irreducible varieties; the trilinear variety and a variety corresponding to the trifocal plane. For the ideals the situation can be described by saying that the bilinear ideal can be written in a primary decomposition as an intersection of two prime ideals; the trilinear ideal and an ideal corresponding to the trifocal plane. Furthermore, the bitrilinear ideal is the same as the trilinear ideal. Thus for three images, with strict inclusions,

$$\begin{aligned} \mathcal{V}_{\mathrm{n}} \subset \mathcal{V}_{\mathrm{b}} \subset \mathcal{V}_{\mathrm{t}} = \mathcal{V}_{\mathrm{bt}} &= \overline{\mathcal{V}_{\mathrm{n}}} \enspace , \\ \mathcal{I}_{\mathrm{n}} = \mathcal{I}_{\mathrm{t}} = \mathcal{I}_{\mathrm{bt}} &\subset \mathcal{I}_{\mathrm{b}} \enspace , \\ \mathcal{V}_{\mathrm{b}} &= \mathcal{V}_{\mathrm{t}} \cup \mathcal{V}_{\mathrm{tp}} \enspace , \\ \mathcal{I}_{\mathrm{b}} &= \mathcal{I}_{\mathrm{t}} \cap \mathcal{I}_{\mathrm{tp}} \enspace . \end{aligned}$$

Moreover, the trilinear ideal can be generated by the three bilinearities and one trilinearity.

Finally, if four or more images are available the bilinear ideal is the same as the trilinear ideal. This means that it is possible to use only bilinearities to generate the algebraic variety defined by all multilinear forms. We have also shown that the quadrilinear ideal is the same as the trilinear ideal. Thus for four or more images, with strict inclusions,

$$\begin{aligned} \mathcal{V}_{\mathrm{n}} \subset \mathcal{V}_{\mathrm{b}} = \mathcal{V}_{\mathrm{t}} = \mathcal{V}_{\mathrm{bt}} = \mathcal{V}_{\mathrm{q}} &= \overline{\mathcal{V}_{\mathrm{n}}} \enspace , \\ \mathcal{I}_{\mathrm{n}} = \mathcal{I}_{\mathrm{t}} = \mathcal{I}_{\mathrm{bt}} = \mathcal{I}_{\mathrm{b}} &= \mathcal{I}_{\mathrm{q}} \enspace . \end{aligned}$$

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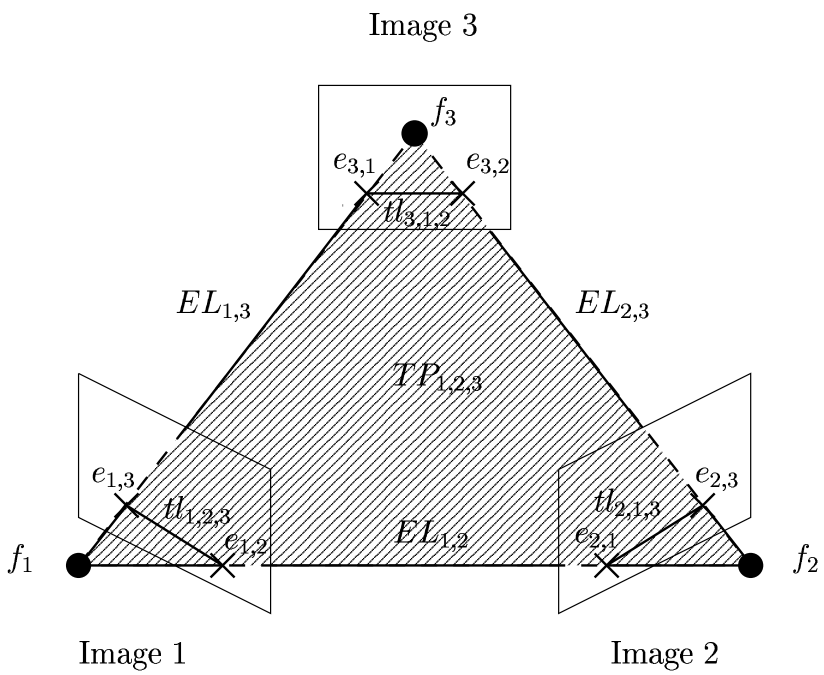

Figure 1: Illustration of the trifocal plane, the epipolar lines, the trifocal lines and the epipoles for three images.

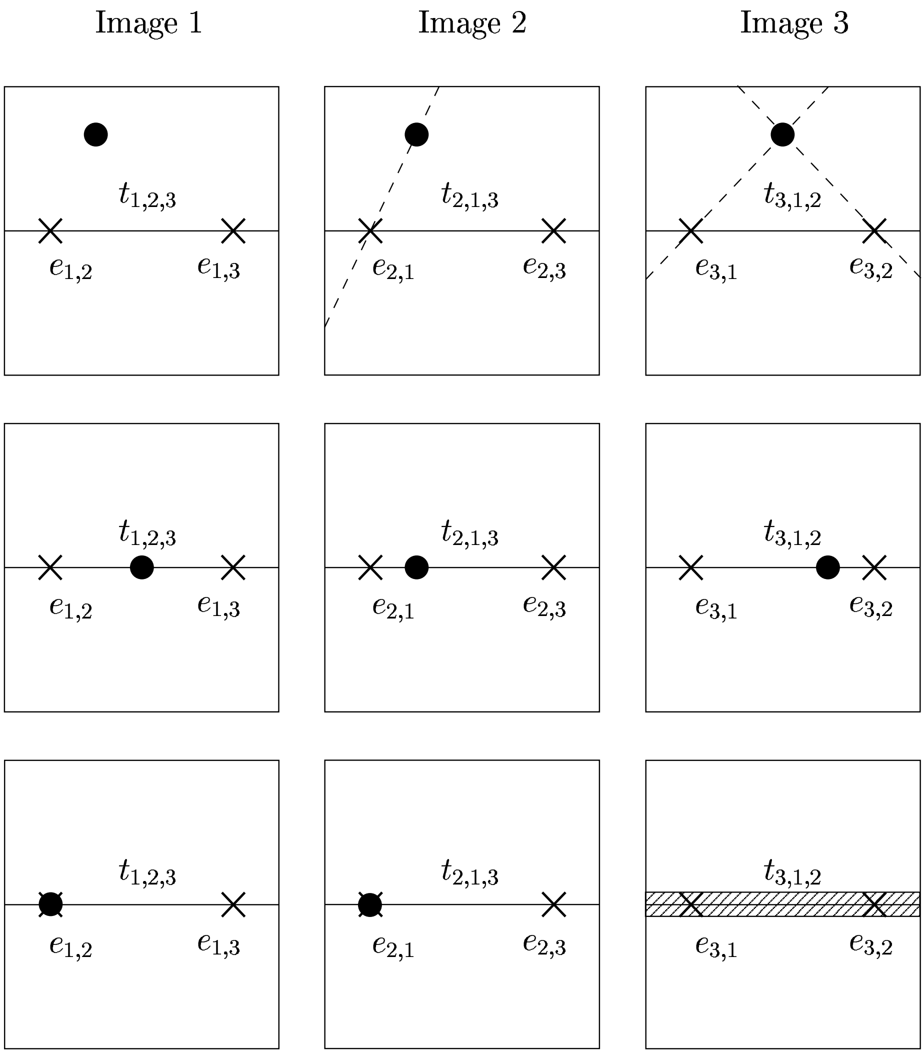

Figure 2: Illustration of the points in $\mathcal{V}_{\mathrm{n}}$ for three images.

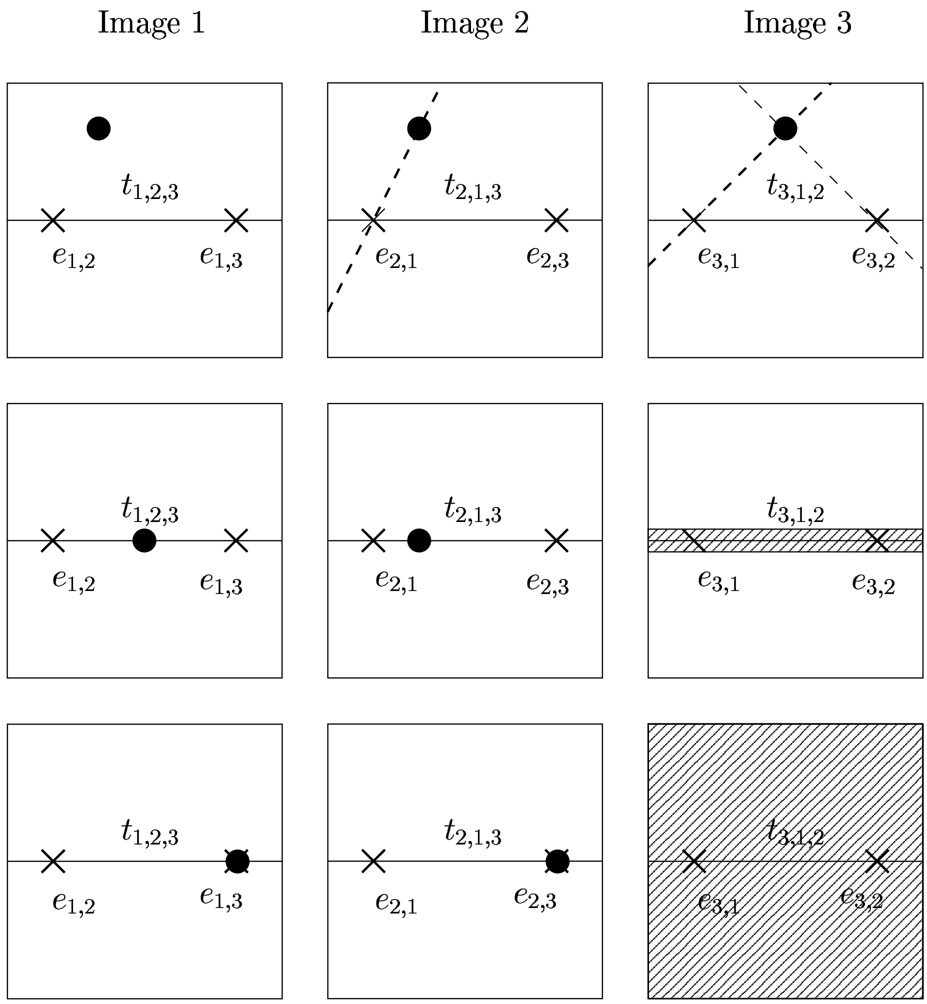

Figure 3: Illustration of the points in $\mathcal{V}_{\mathrm{b}}$ for three images.

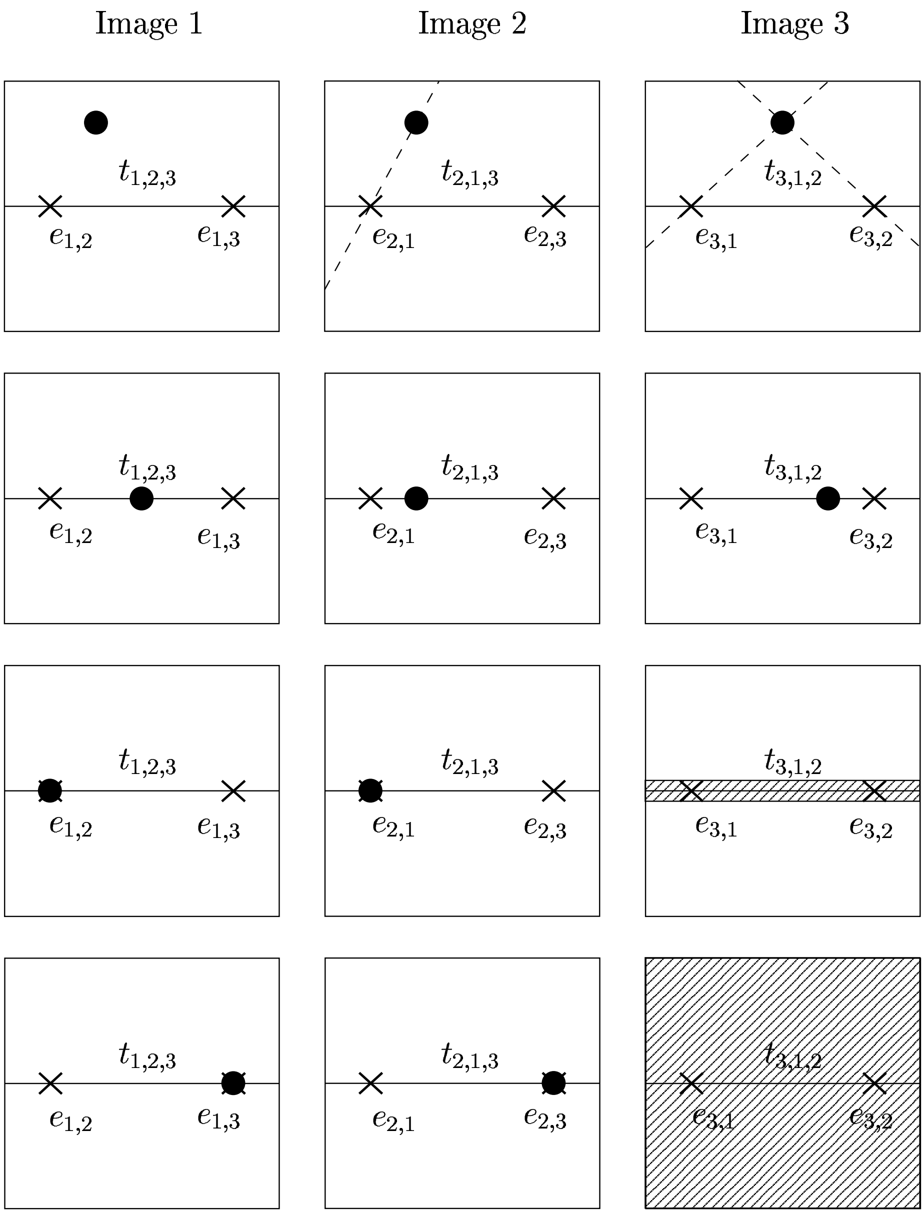

Figure 4: Illustration of the points in $\mathcal{V}_{\mathrm{t}}$ for three images.

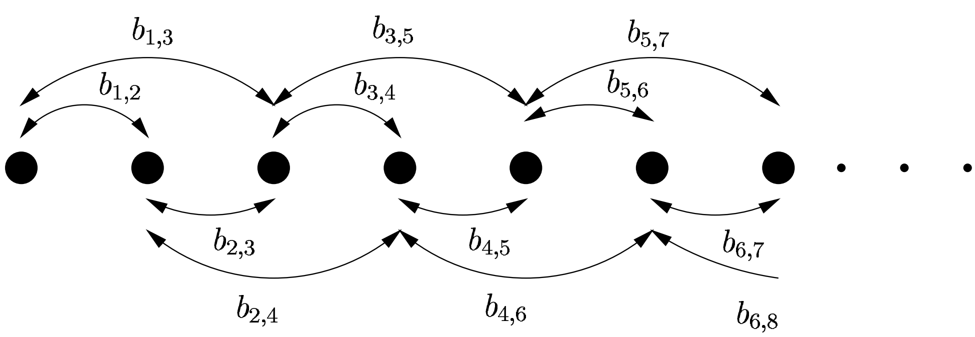

Figure 5: Illustration of the ideal $\widetilde{\mathcal{I}_{\mathrm{b}}}$.