SPECTRAL RIGIDITY OF HURWITZ SURFACES

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1 Invariants

Let $\Gamma \leq \mathrm{PSL}(2,\mathbb{R})$ be a cofinite fuchsian group. We define several invariants of Γ:

- $N_\Gamma = \{N(\gamma) : \gamma \in \Gamma\}$, where $N(\gamma)$ is the larger of the two roots of $x^2 - tx + 1$. For each $\varepsilon \in N_\gamma$, set $N_\Gamma(\varepsilon) = \{\gamma \in \Gamma : N(\gamma) = \varepsilon\}$.

Γ acts on each $N_\Gamma(\varepsilon)$ by conjugation, and we denote by $\widetilde{N}_\Gamma(\varepsilon)$ the quotient by this action.

Suppose $\Gamma = \mathcal{O}^1$ for some maximal order $\mathcal{O}$ in an indefinite division quaternion algebra B over $\mathbb{Q}$. Let trd and nrd denote the reduced trace and norm in B. In this case, $\mathrm{tr}(\gamma) = \mathrm{trd}(\gamma) \in \mathbb{Z}$ and $\mathrm{nrd}(\gamma) = \det(\gamma) = 1$ for any $\gamma \in \Gamma$. Consequently, each $\varepsilon \in N_\gamma \setminus \{1\}$ is a unit in the real quadratic field $\mathbb{Q}(\varepsilon)$. Let d_ε be the field discriminant of $\mathbb{Q}(\varepsilon)$. By Dirichlet's unit theorem, the unit group of the ring of integers $\wr_{d_\varepsilon}$ in $\mathbb{Q}(\varepsilon)$, modulo torsion, is infinite cyclic. Thus, there is a unique $\varepsilon_o \in \wr_{d_\varepsilon}$ and integer j_ε such that $\varepsilon = \varepsilon_o^{j_\varepsilon}$.

Let Δ denote the set of positive field discriminants, and for $d \in \Delta$ let $\varepsilon_d = t_d + u_d\sqrt{d}$ be the fundamental unit of $\mathbb{Q}(\sqrt{d})$, and for each $j > 1$ define $(t_{d,j}, u_{d,j})$ via $\varepsilon_d^j = t_{d,j} + u_{d,j}\sqrt{d}$.

The map $(d,j) \mapsto \varepsilon_d^j$ establishes a bijection $\Delta \times \mathbb{Z} \to \varepsilon_d^j$. Set $\widetilde{N}_\Gamma(d,j) = \widetilde{N}_\Gamma(\varepsilon_d^j)$.

Let ρ be a finite dimensional representation of Γ. For each $d \in \Delta$, the local Selberg zeta function attached to (Γ,ρ) at d is

$$Z_d(s,\Gamma,\rho) := \prod_{j=1}^{\infty} \prod_{\gamma \in \widetilde{N}_\Gamma(j,d)} \prod_{k=0}^{\infty} \det(1 - \rho(\gamma)\varepsilon_d^{-j(s+k)})$$

Its logarithmic derivative $\Phi_d(s,\Gamma,\rho)$ has an expression as a generalized Dirichlet series

$$\Phi_d(s,\Gamma,\rho) = -\log(\varepsilon_d) \sum_{n=1}^{\infty} \frac{c_d(n,\rho)}{\varepsilon_d^{-ns}}$$

Where

$$c_d(n,\rho) := \sum_{f|n} f \sum_{\gamma \in \widetilde{N}_\Gamma(d,f)} \mathrm{tr}\,\rho(\gamma^{n/f})$$

Theorem 1. Let Γ be the norm 1 units of a maximal order in a indefinite quaternion algebra A over a totally real field K satisfying the Eichler condition (unique unramified real place). Further, suppose that this quaternion algebra has type number 1 (has a unique conjugacy class of maximal orders). Let $\mathfrak{p}$ be a prime of O_K such that the reduction mod $\mathfrak{p}$ map $\pi_\mathfrak{p} : \Gamma \to \mathrm{PSL}(2, O_K/\mathfrak{p}) := G(\mathfrak{p})$ is surjective and set $\Gamma(p) = \ker(\pi_\mathfrak{p})$. Then the surface $Y := \Gamma(p)\backslash\mathfrak{H}$ is spectrally rigid.

Proof. Suppose $Y' := \Lambda\backslash\mathfrak{H}$ is isospectral to Y. By Reid (ref IOU) $\Gamma(p)$ and Λ are commensurable. Since $\Gamma(p)$ is arithmetic, so too is Λ and they have the common invariant quaternion algebra A. Furthermore, since Y and Y' are isospectral, they are length isospectral, and so the trace spectra agree $\operatorname{tr}\Gamma(p) = \operatorname{tr}\Lambda$. In particular, since $\Gamma(p)$ has integral traces, so does Λ. Consequently, there exists a maximal order O in A such that Λ embeds in O^1. Since A has type number 1, we can take O to be the order such that $O^1 = \Gamma$. Consequently, $\Lambda \leq \Gamma$. This inclusion induces a cover $\tau : Y' \to X_o := \Lambda\backslash\mathfrak{H}$. Since Y and Y' are isospectral they have the same area, so $\pi_{\mathfrak{p}}$ and τ have the same degree.

Let $\rho_\Lambda = \operatorname{ind}_\Lambda^\Gamma 1$ and $\rho_{\Gamma(p)} = \operatorname{ind}_{\Gamma(p)}^\Gamma 1$ and let χ_Λ and $\chi_{\Gamma(p)}$ denote their respective characters. Since Y and Y' are isospectral, $\Phi(s, \Gamma(p)) = \Phi(s, \Lambda)$. Thus $\Phi(s, \Gamma, \chi_\Lambda) = \Phi(s, \Gamma, \chi_{\Gamma(p)})$, and for each $d \in \Delta$ we have $\Phi_d(s, \Gamma, \chi_\Lambda) = \Phi_d(s, \Gamma, \chi_{\Gamma(p)})$, so for each $(d, n) \in \Delta \times \mathbb{Z}$, we have $c_d(n, \chi_\Lambda) = c_d(n, \chi_{\Gamma(p)})$. That is,

$$\sum_{f|n} f \sum_{\gamma \in \widetilde{N}_\Gamma(d,f)} \chi_\Lambda(\gamma^{n/f}) = \sum_{f|n} f \sum_{\gamma \in \widetilde{N}_\Gamma(d,f)} \chi_{\Gamma(p)}(\gamma^{n/f}).$$

The following lemma demonstrates that for most (d, n), $\chi_{\Gamma(p)}$ is constant along the set $\widetilde{N}_\Gamma(d, n)$

Lemma 1. Suppose $t_{d,n} \neq 2 \mod p$. Then $\chi_{\Gamma(p)}$ is identically zero on $\widetilde{N}_\Gamma(d, n)$.

Proof. First, note that $\Gamma(p)$ is normal in Γ, so $\chi_{\Gamma(p)}$ factors through the right regular representation of $G(p)$. That is,

$$\chi_{\Gamma(p)}(\gamma) = \begin{cases} |G(p)| & \text{if } \gamma \in \Gamma(p) \\ 0 & \text{else} \end{cases}$$

Thus, we seek to show that $\widetilde{N}_\gamma(d, n) \cap \Gamma(p) =$ when $t_{d,n} \neq 2$. Suppose that $\gamma \in \Gamma$ has $N(\gamma) = \varepsilon_d^n$. Observe that the characteristic polynomial $x^2 - t_{d,n}x + 1$ of γ as an automorphism of $\mathbb{R}^2$ reduces mod p to give the characteristic polynomial $x^2 - t_{d,n}x + 1 \mod p$ of $\pi_p(\gamma)$ as an automorphism of $\mathbb{F}_p^2$. For $g \in G(p)$, the characteristic polynomial $x^2 - \operatorname{tr}(g)x + 1$ of g completely determines the conjugacy class of g, provided $\operatorname{tr}(g) \neq 2 \mod p$. □

Consequently, whenever (d, n) is such that $t_{d,n} \neq 2 \mod p$, we have $c_d(n, \chi_{\Gamma(p)}) = c_d(n, \chi_\Lambda) = 0$. Thus, $\chi_d(n, \chi_\Lambda)$ is identically zero on $\widetilde{N}_\Gamma(d, n)$.

Let $\Gamma_1(p)$ be the preimage of the upper triangular unipotent subgroup $U(p) = \left[\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right]$ of $G(p)$ under π_p.

Lemma 2. $\Gamma(p)\Lambda$ is conjugate to $\Gamma_1(p)$.

Proof. Set $\Lambda(p) = \Gamma(p) \cap \Lambda$. Since this is the kernel of π_p restricted to Λ, it is normal in Λ. By the diamond isomorphism $H := \Lambda/\Lambda(p) \approx \Gamma(p)\Lambda/\Gamma(p) \leq G(p)$. Take $\gamma \in \Lambda$. Then $\operatorname{tr}(\gamma) = 2 \mod p$, so $\pi_p(\gamma)$ is unipotent. Any nonidentity subgroup of $G(p)$ consisting of unipotent elements is conjugate to $U(p)$. Since $\Gamma(p)\Lambda$ is the preimage of H under π_p, it is conjugate to $\Gamma_1(p)$. □

Thus, after a suitable conjugation, we may assume that $\Lambda \leq \Gamma_1(p)$.

For $\gamma \in \mathcal{O}^1$, let u_γ be the maximal integer u such that $2\gamma - \operatorname{trd}(\gamma) \in u\mathcal{O}$.

Lemma 3. $\gamma \in \Gamma(p)$ if and only if $p|u_\gamma$.

Proof. If $\gamma \in \Gamma(p)$, write $\gamma = \mathrm{id} + p\delta$ for some $\delta \in \mathcal{O}$. Then $\mathrm{trd}(\gamma) = 2 + p\,\mathrm{trd}(\delta) \in 2 + p\mathbb{Z}$, so that $2\gamma - \mathrm{trd}(\gamma) = p(2\delta - \mathrm{trd}(\delta)) \in p\mathcal{O}$. So $p|u_\gamma$.

Suppose $p|u_\gamma$. Then $\mathrm{nrd}(2\gamma - \mathrm{trd}(\gamma)) = 4 - \mathrm{trd}(\gamma)^2 \in u_\gamma^2\mathbb{Z}$. In particular, $\mathrm{trd}(\gamma)^2 = 4 \mod p^2$, so $\mathrm{trd}(\gamma) = 2 \mod p$. Since $2\gamma - \mathrm{trd}(\gamma) \in u_\gamma\mathcal{O} \subset p\mathcal{O}$, and $\mathrm{trd}(\gamma) = 2 \mod p$, we see $\gamma - \mathrm{id} \in p\mathcal{O}$. □

Lemma 4. Suppose $N(\gamma) = \varepsilon_d^n$, then $u_\gamma|u_{d,n}$. Such a γ is primitive in Γ if and only if $u_\gamma|u_{d,n}$ but $u_\gamma \nmid u_{d,k}$ for all $k|n$.

Consider the following set

$$\Delta_p := \{d \in \Delta : p|d \quad \text{ but } p \nmid u_{d,1}\}$$

Lemma 5. Suppose $d \in \Delta_p$. Then $p|u_{d,p}$ and $p \nmid u_{d,k}$ for $k < p$.

Proof. The relative class number for the order $\wr_d[k]$ of index k in $\wr_d$ is

$$\frac{h(k^2 d)}{h(d)} = \frac{\psi_d(k)}{\varphi_p(k)}$$

where

$$\psi_d(k) = k\prod_{q|k}(1 - \left(\frac{d}{q}\right)q^{-1}), \qquad \varphi_d(k) = \min\{j : k|u_{d,j}\}.$$

In particular, $\varphi_d(k)$ divides $\psi_d(k)$. Since $p|d$, $\psi_d(p) = p$ so $\varphi_d(p)$ is 1 or p. Since we assumed $d \in \Delta_p$, $\varphi_d(p) = p$. □

Lemma 6. If $d \in \Delta_p$ and $\gamma \in \widetilde{N}_\Gamma(d, n)$ for $n < p$, then $\chi_{\Gamma(p)}(\gamma) = 0$.

Proof. Since $u_\gamma|u_{d,n}$ when $N(\gamma) = \varepsilon_d^n$, and since $d \in \Delta_p$, we find $p \nmid u_\gamma$. Thus $\gamma \notin \Gamma(p)$, so $\chi_{\Gamma(p)}(\gamma) = 0$. □

Lemma 7. Suppose $\gamma \in \widetilde{N}_\Gamma(d, 1)$ for $d \in \Delta_p$. Then $\gamma^p \in \Lambda$.

Proof. After conjugating by Γ, we can assume $\gamma \in \Gamma_1(p)$. Pick a set of representatives $g_1, ..., g_p$ for $\Gamma_1(p)/\Lambda$. Then decompose the $\Gamma_1(p)$ conjugacy class into its Λ orbits $\gamma^{\Gamma_1(p)} = \gamma^{g_1\Lambda} \cup ... \cup \gamma^{g_p\Lambda}$.

For each $i \leq p$, let $f(i)$ be the minimal integer f such that $\gamma^f \in \Lambda^{g_i}$. Denote the number of i such that $f(i) = j$ by $\nu(j)$. Then $\sum_{j=1}^p j\nu(j) = p = |\Gamma_1(p) : \Lambda|$.

Recalling that $\chi_\Lambda^{\Gamma_1(p)}(\gamma)$ is the number of $g \in \Gamma_1(p)\backslash\Lambda$ such that $\gamma \in \Lambda^g$, we see that $\chi_\Lambda^{\Gamma_1(p)}(\gamma^k) = \sum_{j|k} \nu(j, \gamma)$. By mobius inversion, $\nu(k, \gamma) = \sum_{j|k} \mu(k/j)\chi_\Lambda^{\Gamma_1(p)}(\gamma^j)$.

Since $N(\gamma^j) = \varepsilon_d^j$, and $d \in \Delta_p$, we know that $\chi_\Lambda^{\Gamma_1(p)}(\gamma^j) = 0$ for all $j < p$, so $\nu(j, \gamma) = 0$ for all $j < p$. Finally, since $p = \sum_{j=1}^p j\nu(j) = p\nu(p)$. We see that $\nu(p) = 1$. That is, there is a unique g_i such that $\gamma^p \in \Lambda^{g_i}$. □

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References