Intergral Equivalence of Hyperbolic Manifolds

Justin Katz

Purdue University

Katz9@purdue.edu

May 15, 2021

Overview

- An ideal inverse theorem in geometry takes the form: Suppose M_1 and M_2 are Riemannian manifolds such that $F(M_1) = F(M_2)$ for some geometric invariant (or collection of such) F, then M_1 and M_2 are isometric.

Overview

- An ideal inverse theorem in geometry takes the form: Suppose M_1 and M_2 are Riemannian manifolds such that $F(M_1) = F(M_2)$ for some geometric invariant (or collection of such) F, then M_1 and M_2 are isometric.
- The natural **obstruction** to such theorems is a **construction** of nonisometric Riemannian manifolds M_1 and M_2 which have identical invariant(s): $F(M_1) = F(M_2)$.

Overview

- An ideal inverse theorem in geometry takes the form: Suppose M_1 and M_2 are Riemannian manifolds such that $F(M_1) = F(M_2)$ for some geometric invariant (or collection of such) F, then M_1 and M_2 are isometric.
- The natural **obstruction** to such theorems is a **construction** of nonisometric Riemannian manifolds M_1 and M_2 which have identical invariant(s): $F(M_1) = F(M_2)$.
- The focus of this talk will be on a procedure for constructing arbitrarily large families of pairwise non-isometric covers of a hyperbolic n manifold which are isospectral, and further have compatibly isomorphic **integral** cohomology.

Overview

- An ideal inverse theorem in geometry takes the form: Suppose M_1 and M_2 are Riemannian manifolds such that $F(M_1) = F(M_2)$ for some geometric invariant (or collection of such) F, then M_1 and M_2 are isometric.
- The natural **obstruction** to such theorems is a **construction** of nonisometric Riemannian manifolds M_1 and M_2 which have identical invariant(s): $F(M_1) = F(M_2)$.
- The focus of this talk will be on a procedure for constructing arbitrarily large families of pairwise non-isometric covers of a hyperbolic n manifold which are isospectral, and further have compatibly isomorphic **integral** cohomology.
- The mechanism for producing these families is a refinement of a nearly 100 year old technique which uses configurations of finite groups as combinatorial "seeds" for stitching together manifolds.

Gassmann equivalence

- We begin with three characterizations of a configuration of a finite group G and two subgroups G_1 and G_2 which serve as the input to Sunada's method for constructing isospectral manifolds.

Gassmann equivalence: Almost Conjugate Subgroups

Definition

Given a finite group G and a pair of subgroups $G_1, G_2 \leq G$, we say that G_1 and G_2 are **almost conjugate in** G if for each $g \in G$,

$$|G_1 \cap g^G| = |G_2 \cap g^G|$$

where g^G denotes the G conjugacy class of g.

Gassmann equivalence: Almost Conjugate Subgroups

Definition

Given a finite group G and a pair of subgroups $G_1, G_2 \leq G$, we say that G_1 and G_2 are **almost conjugate in** G if for each $g \in G$,

$$|G_1 \cap g^G| = |G_2 \cap g^G|$$

where g^G denotes the G conjugacy class of g.

- If G_1 and G_2 are conjugate in G, then G_1 and G_2 are clearly almost conjugate.

Gassmann equivalence: Almost Conjugate Subgroups

Definition

Given a finite group G and a pair of subgroups $G_1, G_2 \leq G$, we say that G_1 and G_2 are **almost conjugate in** G if for each $g \in G$,

$$|G_1 \cap g^G| = |G_2 \cap g^G|$$

where g^G denotes the G conjugacy class of g.

- If G_1 and G_2 are conjugate in G, then G_1 and G_2 are clearly almost conjugate.
- Our interest will be in non-conjugate, almost conjugate subgroups.

Gassmann equivalence: Fixed Point Equivalence

Definition

Given a finite group G and a pair of subgroups $G_1, G_2 \leq G$, we say that G_1 and G_2 are **fixed point equivalent** if, for any irreducible unitary representation $\rho : G \to \mathrm{GL}(V)$, one has

$$\dim \mathrm{Fix}_{G_1}(V) = \dim \mathrm{Fix}_{G_2}(V)$$

.

Gassmann equivalence: $\mathbb{Q}$ equivalence

Definition

Given a finite group G and a pair of subgroups $G_1, G_2 \leq G$, we say that G_1 and G_2 are $\mathbb{Q}$ **equivalent** if there is an isomorphism

$$\mathbb{Q}[G/G_1] \approx \mathbb{Q}[G/G_2]$$

as $\mathbb{Q}[G]$ modules.

Gassmann Equivalence

Lemma

If G is a finite group, and $G_1, G_2 \leq G$, then the following are equivalent:

- G_1 and G_2 are almost conjugate
- G_1 and G_2 are fixed point equivalent.
- G_1 and G_2 are $\mathbb{Q}$-equivalent.

If these conditions are satisfied, we call (G, G_1, G_2) a Gassmann triple.

Gassmann Equivalence

Lemma

If G is a finite group, and $G_1, G_2 \leq G$, then the following are equivalent:

- G_1 and G_2 are almost conjugate
- G_1 and G_2 are fixed point equivalent.
- G_1 and G_2 are $\mathbb{Q}$-equivalent.

If these conditions are satisfied, we call (G, G_1, G_2) a Gassmann triple.

- The equivalence of these three concepts follows from the representation theory of finite groups.

Gassmann Equivalence

Lemma

If G is a finite group, and $G_1, G_2 \leq G$, then the following are equivalent:

- G_1 and G_2 are almost conjugate
- G_1 and G_2 are fixed point equivalent.
- G_1 and G_2 are $\mathbb{Q}$-equivalent.

If these conditions are satisfied, we call (G, G_1, G_2) a Gassmann triple.

- The equivalence of these three concepts follows from the representation theory of finite groups.
- Almost conjugacy gives a relationship between the internal structure of conjugacy classes of G, relative to G_1 and G_2.

Gassmann Equivalence

Lemma

If G is a finite group, and $G_1, G_2 \leq G$, then the following are equivalent:

- G_1 and G_2 are almost conjugate
- G_1 and G_2 are fixed point equivalent.
- G_1 and G_2 are $\mathbb{Q}$-equivalent.

If these conditions are satisfied, we call (G, G_1, G_2) a Gassmann triple.

- The equivalence of these three concepts follows from the representation theory of finite groups.
- Almost conjugacy gives a relationship between the internal structure of conjugacy classes of G, relative to G_1 and G_2.
- Fixed point equivalence gives a relationship between the fixed point sets G_1 and G_2 in representations of G, a notion seemingly external to G.

Gassmann equivalence

- The notion of $\mathbb{Q}$-equivalence has a natural refinement.

Gassmann equivalence

- The notion of $\mathbb{Q}$-equivalence has a natural refinement.

Definition

For a commutative ring R with 1, say that (G, G_1, G_2) are R-equivalent if $R[G/G_1]$ and $R[G/G_2]$ are isomorphic as $R[G]$ modules. When $R = \mathbb{Z}$, we refer to $\mathbb{Z}$-equivalence as **integral equivalence**.

Gassmann equivalence

- The notion of $\mathbb{Q}$-equivalence has a natural refinement.

Definition

For a commutative ring R with 1, say that (G, G_1, G_2) are R-equivalent if $R[G/G_1]$ and $R[G/G_2]$ are isomorphic as $R[G]$ modules. When $R = \mathbb{Z}$, we refer to $\mathbb{Z}$-equivalence as **integral equivalence**.

- Integral equivalence is the key mechanism in our construction.

Riemannian Geometry

- Given a closed Riemannian manifold M, the Laplace-Beltrami operator Δ_M acts on smooth functions, as well as on smooth differential k-forms.

Riemannian Geometry

- Given a closed Riemannian manifold M, the Laplace-Beltrami operator Δ_M acts on smooth functions, as well as on smooth differential k-forms.
- For each $0 \leq k \leq \dim(M)$ let $\mathcal{E}_M^k$ denote the **eigenvalue spectrum** of Δ_M acting on k forms. Here, we think of $\mathcal{E}_M^k$ as a discrete measure on $[0, \infty)$ which assigns to each $\lambda \in [0, \infty)$ the (finite) multiplicity of λ as an eigenvalue for Δ_M acting on k-forms.

Riemannian Geometry

- Given a closed Riemannian manifold M, the Laplace-Beltrami operator Δ_M acts on smooth functions, as well as on smooth differential k-forms.
- For each $0 \leq k \leq \dim(M)$ let $\mathcal{E}_M^k$ denote the **eigenvalue spectrum** of Δ_M acting on k forms. Here, we think of $\mathcal{E}_M^k$ as a discrete measure on $[0, \infty)$ which assigns to each $\lambda \in [0, \infty)$ the (finite) multiplicity of λ as an eigenvalue for Δ_M acting on k-forms.
- When M is negatively curved, let $\mathcal{L}_M^p$ denote the **primitive length spectrum** of M, viewed as a discrete measure on $(0, \infty)$ which assigns to each ℓ the number of primitive geodesics on M of length ℓ.

Riemannian Geometry

- Given a closed Riemannian manifold M, the Laplace-Beltrami operator Δ_M acts on smooth functions, as well as on smooth differential k-forms.
- For each $0 \leq k \leq \dim(M)$ let $\mathcal{E}_M^k$ denote the **eigenvalue spectrum** of Δ_M acting on k forms. Here, we think of $\mathcal{E}_M^k$ as a discrete measure on $[0, \infty)$ which assigns to each $\lambda \in [0, \infty)$ the (finite) multiplicity of λ as an eigenvalue for Δ_M acting on k-forms.
- When M is negatively curved, let $\mathcal{L}_M^p$ denote the **primitive length spectrum** of M, viewed as a discrete measure on $(0, \infty)$ which assigns to each ℓ the number of primitive geodesics on M of length ℓ.
- For a negatively curved manifold, eigenvalue spectrum and the primitive length spectrum are related by the wave trace formula.

Riemannian Geometry: continued

- For the purposes of this talk, say M_1 and M_2 are **Laplace isospectral** if $\mathcal{E}^k_{M_1} = \mathcal{E}^k_{M_2}$ as measures, for all k.

Riemannian Geometry: continued

- For the purposes of this talk, say M_1 and M_2 are **Laplace isospectral** if $\mathcal{E}^k_{M_1} = \mathcal{E}^k_{M_2}$ as measures, for all k.
- Say M_1 and M_2 are **length isospectral** if $\mathcal{L}^p_{M_1} = \mathcal{L}^p_{M_2}$ as measures.

Riemannian Geometry: continued

Theorem (Sunada 1985)

If M is a closed Riemannian manifold, and $\rho : \Gamma = \pi_1(M) \to G$ is a surjective homomorphism to a finite group such that $G_1, G_2 \leq G$ are subgroups with (G, G_1, G_2) forming a Gassmann triple, then the covers M_1 and M_2 corresponding to $\rho^{-1}(G_1)$ and $\rho^{-1}(G_2)$ are Laplace isospectral and length isospectral.

Riemannian Geometry: continued

Theorem (Sunada 1985)

If M is a closed Riemannian manifold, and $\rho : \Gamma = \pi_1(M) \to G$ is a surjective homomorphism to a finite group such that $G_1, G_2 \leq G$ are subgroups with (G, G_1, G_2) forming a Gassmann triple, then the covers M_1 and M_2 corresponding to $\rho^{-1}(G_1)$ and $\rho^{-1}(G_2)$ are Laplace isospectral and length isospectral.

- Proving M_1 and M_2 are Laplace isospectral is easily seen through the fixed point equivalence.

Riemannian Geometry: continued

Theorem (Sunada 1985)

If M is a closed Riemannian manifold, and $\rho : \Gamma = \pi_1(M) \to G$ is a surjective homomorphism to a finite group such that $G_1, G_2 \leq G$ are subgroups with (G, G_1, G_2) forming a Gassmann triple, then the covers M_1 and M_2 corresponding to $\rho^{-1}(G_1)$ and $\rho^{-1}(G_2)$ are Laplace isospectral and length isospectral.

- Proving M_1 and M_2 are Laplace isospectral is easily seen through the fixed point equivalence.
- This is a manifestation of function spaces being *contravariant*.

Riemannian Geometry: continued

Theorem (Sunada 1985)

If M is a closed Riemannian manifold, and $\rho : \Gamma = \pi_1(M) \to G$ is a surjective homomorphism to a finite group such that $G_1, G_2 \leq G$ are subgroups with (G, G_1, G_2) forming a Gassmann triple, then the covers M_1 and M_2 corresponding to $\rho^{-1}(G_1)$ and $\rho^{-1}(G_2)$ are Laplace isospectral and length isospectral.

- Proving M_1 and M_2 are Laplace isospectral is easily seen through the fixed point equivalence.
- This is a manifestation of function spaces being *contravariant*.
- By contrast, length isospectrality is best seen through almost conjugacy, since this describes how conjugacy classes in $\pi_1(M)$ factor in the covers associated to $\rho^{-1}(G_j)$.

Riemannian Geometry: continued

Theorem (Sunada 1985)

If M is a closed Riemannian manifold, and $\rho : \Gamma = \pi_1(M) \to G$ is a surjective homomorphism to a finite group such that $G_1, G_2 \leq G$ are subgroups with (G, G_1, G_2) forming a Gassmann triple, then the covers M_1 and M_2 corresponding to $\rho^{-1}(G_1)$ and $\rho^{-1}(G_2)$ are Laplace isospectral and length isospectral.

- Proving M_1 and M_2 are Laplace isospectral is easily seen through the fixed point equivalence.
- This is a manifestation of function spaces being *contravariant*.
- By contrast, length isospectrality is best seen through almost conjugacy, since this describes how conjugacy classes in $\pi_1(M)$ factor in the covers associated to $\rho^{-1}(G_j)$.
- This is a manifestation of length sets being *covariant*.

Riemannian Geometry: continued

Theorem (Sunada 1985)

If M is a closed Riemannian manifold, and $\rho : \Gamma = \pi_1(M) \to G$ is a surjective homomorphism to a finite group such that $G_1, G_2 \leq G$ are subgroups with (G, G_1, G_2) forming a Gassmann triple, then the covers M_1 and M_2 corresponding to $\rho^{-1}(G_1)$ and $\rho^{-1}(G_2)$ are Laplace isospectral and length isospectral.

- Proving M_1 and M_2 are Laplace isospectral is easily seen through the fixed point equivalence.
- This is a manifestation of function spaces being *contravariant*.
- By contrast, length isospectrality is best seen through almost conjugacy, since this describes how conjugacy classes in $\pi_1(M)$ factor in the covers associated to $\rho^{-1}(G_j)$.
- This is a manifestation of length sets being *covariant*.
- Provided M has a sufficiently rich covering space theory, $\pi_1(M)$ will surject any finite group. The natural requirement is **largeness**.

A cautionary note

- Even if G_1 and G_2 are non-conjugate almost conjugate subgroups of G, it is possible that the manifolds M_1 and M_2 constructed in Sunada's theorem are actually isometric.

A cautionary note

- Even if G_1 and G_2 are non-conjugate almost conjugate subgroups of G, it is possible that the manifolds M_1 and M_2 constructed in Sunada's theorem are actually isometric.
- Provided M satisfies various structural hypotheses (for example, being negatively curved), this can only occur when $\rho^{-1}(G_1)$ and $\rho^{-1}(G_2)$ are conjugate via an element of the commensurator of $\pi_1(M)$ inside the isometry group of the universal cover of M.

A cautionary note

- Even if G_1 and G_2 are non-conjugate almost conjugate subgroups of G, it is possible that the manifolds M_1 and M_2 constructed in Sunada's theorem are actually isometric.
- Provided M satisfies various structural hypotheses (for example, being negatively curved), this can only occur when $\rho^{-1}(G_1)$ and $\rho^{-1}(G_2)$ are conjugate via an element of the commensurator of $\pi_1(M)$ inside the isometry group of the universal cover of M.
- In order to prohibit such accidental isometries, one must add conditions on M to make sure this commensurator is small.

A cautionary note

- Even if G_1 and G_2 are non-conjugate almost conjugate subgroups of G, it is possible that the manifolds M_1 and M_2 constructed in Sunada's theorem are actually isometric.
- Provided M satisfies various structural hypotheses (for example, being negatively curved), this can only occur when $\rho^{-1}(G_1)$ and $\rho^{-1}(G_2)$ are conjugate via an element of the commensurator of $\pi_1(M)$ inside the isometry group of the universal cover of M.
- In order to prohibit such accidental isometries, one must add conditions on M to make sure this commensurator is small.
- When M is hyperbolic, a theorem of Margulis makes **non-arithmeticity** the natural requirement.

How similar are manifolds arising from Sunada's construction?

- Granting that M_1 and M_2 are isospectral but non-isometric, a natural question is "what geometric invariants can be used to distinguish M_1 and M_2?"

How similar are manifolds arising from Sunada's construction?

- Granting that M_1 and M_2 are isospectral but non-isometric, a natural question is "what geometric invariants can be used to distinguish M_1 and M_2?"
- Applying Hodge's theorem to the 0 eigenspace, Laplace isospectrality (from Sunada or otherwise) implies equality of deRham cohomology: $H^k_{dR}(M_1) \approx H^k_{dR}(M_2)$ as real vector spaces.

How similar are manifolds arising from Sunada's construction?

- Granting that M_1 and M_2 are isospectral but non-isometric, a natural question is "what geometric invariants can be used to distinguish M_1 and M_2?"
- Applying Hodge's theorem to the 0 eigenspace, Laplace isospectrality (from Sunada or otherwise) implies equality of deRham cohomology: $H^k_{dR}(M_1) \approx H^k_{dR}(M_2)$ as real vector spaces.
- Equivalently, Laplace isospectral Riemannian manifolds have the same Betti numbers.

How similar are manifolds arising from Sunada's construction?

- Granting that M_1 and M_2 are isospectral but non-isometric, a natural question is "what geometric invariants can be used to distinguish M_1 and M_2?"
- Applying Hodge's theorem to the 0 eigenspace, Laplace isospectrality (from Sunada or otherwise) implies equality of deRham cohomology: $H^k_{dR}(M_1) \approx H^k_{dR}(M_2)$ as real vector spaces.
- Equivalently, Laplace isospectral Riemannian manifolds have the same Betti numbers.
- What can be said of the torsion part of integral homology/cohomology?

How similar are manifolds arising from Sunada's construction?

- In general: nothing (Sunada or otherwise).

How similar are manifolds arising from Sunada's construction?

- In general: nothing (Sunada or otherwise).
- In one direction, Bartel–Page (2016) proved: for any finite set S of primes, there exist (Sunada) isospectral hyperbolic 3-manifolds which have have non-isomorphic cohomology rings with coefficients in F_p for $p \in S$, and isomorphic cohomology rings with coefficients in F_p for p outside S.

How similar are manifolds arising from Sunada's construction?

- In general: nothing (Sunada or otherwise).
- In one direction, Bartel–Page (2016) proved: for any finite set S of primes, there exist (Sunada) isospectral hyperbolic 3-manifolds which have have non-isomorphic cohomology rings with coefficients in F_p for $p \in S$, and isomorphic cohomology rings with coefficients in F_p for p outside S.
- Such a cosntruction is possible is due to the ubiquity of $\mathbb{Q}$ equivalent triples (G, G_1, G_2) which are not $\mathbb{Z}$ equivalent.

How similar are manifolds arising from Sunada's construction?

- In general: nothing (Sunada or otherwise).
- In one direction, Bartel–Page (2016) proved: for any finite set S of primes, there exist (Sunada) isospectral hyperbolic 3-manifolds which have have non-isomorphic cohomology rings with coefficients in F_p for $p \in S$, and isomorphic cohomology rings with coefficients in F_p for p outside S.
- Such a cosntruction is possible is due to the ubiquity of $\mathbb{Q}$ equivalent triples (G, G_1, G_2) which are not $\mathbb{Z}$ equivalent.

How similar are manifolds arising from Sunada's construction?

- We proved the other direction:

How similar are manifolds arising from Sunada's construction?

- We proved the other direction:

Theorem (Arapura, Katz, Mcreynolds, Solapurkar 2017

Let M be a closed hyperbolic n manifold that is **large** and **non-arithmetic**. Then for each integer $j > 0$, there exist pairwise nonisometric Riemannian covers $M_1,\ldots,M_j$ of M such that the following hold:

How similar are manifolds arising from Sunada's construction?

- We proved the other direction:

Theorem (Arapura, Katz, Mcreynolds, Solapurkar 2017

Let M be a closed hyperbolic n manifold that is **large** and **non-arithmetic**. Then for each integer $j > 0$, there exist pairwise nonisometric Riemannian covers $M_1,\ldots,M_j$ of M such that the following hold:

- (Laplace isospectrality) For each k, i, i', one has $\mathcal{E}^k_{M_i} = \mathcal{E}^k_{M_{i'}}$

How similar are manifolds arising from Sunada's construction?

- We proved the other direction:

Theorem (Arapura, Katz, Mcreynolds, Solapurkar 2017

Let M be a closed hyperbolic n manifold that is **large** and **non-arithmetic**. Then for each integer $j > 0$, there exist pairwise nonisometric Riemannian covers $M_1,\ldots,M_j$ of M such that the following hold:

- (Laplace isospectrality) For each k, i, i', one has $\mathcal{E}^k_{M_i} = \mathcal{E}^k_{M_{i'}}$
- (Length isospectrality) For each i, i', $\mathcal{L}^p_{M_i} = \mathcal{L}^p_{M_{i'}}$.

How similar are manifolds arising from Sunada's construction?

- We proved the other direction:

Theorem (Arapura, Katz, Mcreynolds, Solapurkar 2017

Let M be a closed hyperbolic n manifold that is **large** and **non-arithmetic**. Then for each integer $j > 0$, there exist pairwise nonisometric Riemannian covers M_1,...,M_j of M such that the following hold:

- (Laplace isospectrality) For each k, i, i', one has $\mathcal{E}^k_{M_i} = \mathcal{E}^k_{M_{i'}}$
- (Length isospectrality) For each i, i', $\mathcal{L}^p_{M_i} = \mathcal{L}^p_{M_{i'}}$.
- (Compatible isomorphism of integral cohomology) For each k, i, i' there exist isomorphisms $\psi_k : H^k(M_i, \mathbb{Z}) \to H^k(M_{i'}, \mathbb{Z})$ which commute with restriction and corestriction relative to the covering maps $M_i \to M$ and $M_{i'} \to M$.

The critical difficulty: a lack of examples

- Supposing one has access to a collection of finite groups G containing arbitrarily large families of nonconjugate integral equivalent subgroups $G_1,\ldots,G_j$, the theorem is a straightforward application of Sunada's method plus Shapiro's lemma.

The critical difficulty: a lack of examples

- Supposing one has access to a collection of finite groups G containing arbitrarily large families of nonconjugate integral equivalent subgroups $G_1, \ldots, G_j$, the theorem is a straightforward application of Sunada's method plus Shapiro's lemma.
- Unlike Q equivalent configurations, which are fairly abundant, all known examples of nontrivial $\mathbb{Z}$ equivalent triples (G, G_1, G_2) are essentially derived from a single example:

The critical difficulty: a lack of examples

- Supposing one has access to a collection of finite groups G containing arbitrarily large families of nonconjugate integral equivalent subgroups $G_1,\ldots,G_j$, the theorem is a straightforward application of Sunada's method plus Shapiro's lemma.
- Unlike Q equivalent configurations, which are fairly abundant, all known examples of nontrivial $\mathbb{Z}$ equivalent triples (G, G_1, G_2) are essentially derived from a single example:
- In 1993 Scott proved that there exists a nonconjugate pair of subgroups G_1 and G_2, both isomorphic to A_5, of $G = \mathrm{PSL}(2, F_{29})$ such that the (G, G_1, G_2) forms a $\mathbb{Z}$-equivalent triple.

The critical difficulty: a lack of examples

- Supposing one has access to a collection of finite groups G containing arbitrarily large families of nonconjugate integral equivalent subgroups $G_1,\ldots,G_j$, the theorem is a straightforward application of Sunada's method plus Shapiro's lemma.
- Unlike Q equivalent configurations, which are fairly abundant, all known examples of nontrivial $\mathbb{Z}$ equivalent triples (G, G_1, G_2) are essentially derived from a single example:
- In 1993 Scott proved that there exists a nonconjugate pair of subgroups G_1 and G_2, both isomorphic to A_5, of $G = \mathrm{PSL}(2, F_{29})$ such that the (G, G_1, G_2) forms a $\mathbb{Z}$-equivalent triple.
- The novelty of our result lies in the production of arbitrarily large nontrivial such families using this single example.

Rough sketch of construction

- Starting with a single nontrivial integrally equivalent triple (G, G_1, G_2): subgroups of G^m consisting of products $\prod_{n=1}^m G_{\nu_n}$ where $\nu_n \in \{1, 2\}$ give rise 2^m pairwise nonconjugate $\mathbb{Z}$ equivalent subgroups of G^m.

Rough sketch of construction

- Starting with a single nontrivial integrally equivalent triple (G, G_1, G_2): subgroups of G^m consisting of products $\prod_{n=1}^{m} G_{\nu_n}$ where $\nu_n \in \{1, 2\}$ give rise 2^m pairwise nonconjugate $\mathbb{Z}$ equivalent subgroups of G^m.
- Since M is large, there are plenty of surjections $\pi_1(M) \to G^m$, for each m. Pulling back any of the Z equivalent subgroups above produces lots of $\mathbb{Z}$ equivalent subgroups of $\pi_1(M)$.

Rough sketch of construction

- Starting with a single nontrivial integrally equivalent triple (G, G_1, G_2): subgroups of G^m consisting of products $\prod_{n=1}^{m} G_{\nu_n}$ where $\nu_n \in \{1, 2\}$ give rise 2^m pairwise nonconjugate $\mathbb{Z}$ equivalent subgroups of G^m.
- Since M is large, there are plenty of surjections $\pi_1(M) \to G^m$, for each m. Pulling back any of the Z equivalent subgroups above produces lots of $\mathbb{Z}$ equivalent subgroups of $\pi_1(M)$.
- That said, many of these subgroups are actually conjugate in $\pi_1(M)$. In order to produce non-conjugate $\mathbb{Z}$ equivalence subgroups, one looks at the growth of the hom spaces $\pi_1(M) \to G^m$ as a function of m.

Rough sketch of construction

- Starting with a single nontrivial integrally equivalent triple (G, G_1, G_2): subgroups of G^m consisting of products $\prod_{n=1}^m G_{\nu_n}$ where $\nu_n \in \{1, 2\}$ give rise 2^m pairwise nonconjugate $\mathbb{Z}$ equivalent subgroups of G^m.
- Since M is large, there are plenty of surjections $\pi_1(M) \to G^m$, for each m. Pulling back any of the Z equivalent subgroups above produces lots of $\mathbb{Z}$ equivalent subgroups of $\pi_1(M)$.
- That said, many of these subgroups are actually conjugate in $\pi_1(M)$. In order to produce non-conjugate $\mathbb{Z}$ equivalence subgroups, one looks at the growth of the hom spaces $\pi_1(M) \to G^m$ as a function of m.
- Standard estimates for asymptotics of hom spaces of nonabelian free groups to finite groups, along with the assumption of non-arithmeticity on $\pi_1(M)$ shows that there are so many $\mathbb{Z}$-equivalent collections arising from this construction that they cannot all be accounted for by conjugacy. QED