TRACE FORMULA TALK

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1 $\mathrm{SL}(2,\mathbb{R})$

Let $G = \mathrm{SL}(2,\mathbb{R})$ and $\Gamma \leq G$ be a compact lattice. Taking for granted that there are no issues with convergence, the trace formula reads: (Recalling that $\cdot_\gamma$ is the centralizer of γ in $\cdot$, and that R^Γ is the right regular action of G on $L^2(\Gamma \setminus G)$) for a test function $f \in C_c^\infty(G)$,

$$\sum_{\pi \in \hat{G}} m_\pi^\Gamma \operatorname{tr} \pi(f) = \operatorname{tr} R^\Gamma(f) = \sum_{\gamma \in \{\gamma\}} \mathrm{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(g^{-1}\gamma g)\, \mathrm{d}g.$$

The aim of this talk is to demonstrate how this formula manifests in the following

Theorem 1. Let $\{\lambda_j\}$ be the eigenvalues of the hyperbolic Laplacian acting on $L^2(\Gamma\backslash\mathfrak{H})$, and $g \in C_c^\infty(\mathbb{R})$. Then

$$\sum \hat{g}(\sqrt{\lambda_j - 1/4}) = \frac{\mathrm{vol}(\Gamma\backslash\mathfrak{H})}{2\pi} \int_0^\infty \hat{g}(r) r \tanh(r)\, \mathrm{d}r + 1/2 \sum_{\gamma \in \{\Gamma\}} \frac{\ell(\gamma_o)}{e^{\ell(\gamma)/2} - e^{-\ell(\gamma)/2}} g(\ell(\gamma))$$

where $\hat{g}$ is the Euclidean Fourier transform of g; where γ_o is the generator of Γ_γ (hence primitive geodesic in the class); and $\ell(\gamma)$ is the translation length of γ.

1.1 Structure of $\mathrm{SL}(2,\mathbb{R})$

Put coordinates on G via the Iwasawa decomposition $G = NAK$:

$$A = \{a(u) = \begin{bmatrix} e^u & 0 \\ 0 & e^{-u} \end{bmatrix} : u \in \mathbb{R}\}$$

$$N = \{n(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbb{R}\}$$

$$K = \{k(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} : \theta \in [0, 2\pi)\}.$$

Classification of conjugacy classes:

(1) $g \in G$ is hyperbolic if
  (a) $\operatorname{tr} g > 2$
  (b) g is conjugate to an element in A
  (c) The action of g on $\overline{\mathfrak{H}}$ has two distinct fixed points on the boundary
(2) $g \in G$ is elliptic if
  (a) $\operatorname{tr} g < 2$
  (b) g is conjugate to an element in K
  (c) The action of g on $\overline{\mathfrak{H}}$ has one fixed point in $\mathfrak{H}$.
(3) $g \in G$ is parabolic if

(a) $\operatorname{tr} g = 2$
(b) g is conjugate to an element in N
(c) The action of g on $\overline{\mathfrak{H}}$ has exactly one point on the boundary

Fact: a discrete subgroup Γ is cocompact iff it doesnt contain any parabolic elements.

1.2 Spectral side

To understand the RHS, we need to compute $\operatorname{tr} \pi(f)$ for those $\pi \in \hat{G}$ with $m_\pi^\gamma \neq 0$. To do this, we relate these π with special functions on $\Gamma\backslash\mathfrak{H}$. Here's the theorem:

Theorem 2. Suppose $m_\pi^\Gamma \neq 0$. Then π^K (that is, the right K invariant vectors in π) is one dimensional, say $\pi^K = \mathbb{C}\varphi$. Then, upon identifying φ with a function on $\Gamma\backslash\mathfrak{H}$, we have

$$\Delta^{\mathfrak{H}}\varphi = \lambda\varphi$$

for $\lambda \in \mathbb{R}_{\geq 0}$ dependent only on the isomorphism class of π.

Essence of proof:

- The convolution algebra of left-and-right K invariant functions on G is commutative: use Cartan decomposition $G = KAK$ to show first that functions on $K\backslash G/K$ are invariant under transpose, then note that transpose induces an anti-automorphism of $C_c^\infty(K\backslash G/K)$, which is the identity.
- Irreducible unitary reps of G appearing in $L^2(\Gamma\backslash G)$ are admissible (meaning the restriction of the rep to K is completely reducible, with irreducible constituents having finite multiplicity).
- For an irreducible admissible rep π, the subspace π^K affords a representation of $C_c^\infty(K\backslash G/K)$ which is irreducible. Irreducible reps of a commutative algebra are at most one dimensional.
- Under the identification of functions on $\Gamma\backslash\mathfrak{H}$ with right K invariant, left Γ invariant functions on G, the hyperbolic Laplacian agrees with the Casimir operator of G up to a constant.
  - Quick and dirty on Casimir:
    The action of G on $C^\infty(G)$ induces an action of its Lie algebra via the formula
    $$R(X)f(g) = \frac{d}{dt}|_{t=0}R(\exp(tX))f(g) = \frac{d}{dt}|_{t=0}f(g\exp(e^{tX})).$$
    Thus, we can view R as a Lie algebra homomorphism $\mathfrak{g}$ into the associative algebra of differential operators on G such that $R([X,Y]) = R(X)R(Y) - R(Y)R(X)$. The universal object for such maps is $\mathcal{U}(\mathfrak{g})$, the universal enveloping algebra: this is an associative algebra and a map $\mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ such that any Lie algebra map of $\mathfrak{g}$ into an associative algebra (turning Lie bracket into commutant) factors through a unique associative algebra map from $\mathcal{U}(\mathfrak{g})$. *This is all to say that $\mathcal{U}(\mathfrak{g})$ is the smallest place in which it is meaningful to take products of Lie algebra elements.*
  - The center of $\mathcal{U}(\mathfrak{g})$ is manifestly the algebra of differential operators on G which commute with the action of G. The Casimir is a distinguished element of the center, constructed as follows:
    * We'll contrive a nonzero algebra map $\operatorname{End}\mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ and take Casimir to be the image of the identity (which is certainly in the center):
      First, use the canonical map $\operatorname{End}\mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}^*$, then use a nondegenerate bilinear form on $\mathfrak{g}$ to pin down an isomorphism $\mathfrak{g} \to \mathfrak{g}^*$ to obtain a map $\operatorname{End}\mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}$.

Then use the canonical inclusion $\mathfrak{g}\otimes\mathfrak{g}$ into the tensor algebra $\bigotimes\mathfrak{g}$, and quotient by brackets to get a map to $\mathcal{U}(\mathfrak{g})$.

- With the basis of $\mathfrak{g}$

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

  and using the trace form to make the identification of $\mathfrak{g}$ with its dual, we have

$$h^* = h \quad e^* = f/2 \quad f^* = e/2.$$

  so Casimir is $h^2 + 2ef + 2fe$.
- We want to compute the action of this differential operator on left K invariant functions on G. To do this, first note that $e-f$ exponentiates into K, so will act by zero on left K invariant functions. So we'll compute h^2+4e^2.
- Exponentials are

$$\exp(th) = a(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$$
$$\exp(te) = n(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

  so compute, for right K invariant f on G, using the fact that A normalizes N via $a(-u)n(x)a(a) = n(e^u x)$ so $n(x)a(u) = a(u)a(-u)n(x)a(u) = a(u)n(e^u x)$

$$R(e)f(n(x)a(u)) = \frac{d}{dt}|_{t=0} f(n(x)a(u)n(t)) = \frac{d}{dt}|_{t=0} f(n(x+e^{ut})a(u)) = e^u \frac{\partial}{\partial_x} f(n(x)a(u))$$
$$R(h)f(n(x)a(u)) = \frac{d}{dt}|_{t=0} f(n(x)a(u)a(t)) = \frac{d}{dt}|_{t=0} f(n(x)a(u+t)) = \frac{\partial}{\partial u} f(n(x)a(u))$$

- Because casimir commutes with G, it provides an G-invariant automorphism of any irreducible to itself. Thus, by schur, it acts as a scalar on that irreducible. I.e. the one dimensional subspace π^K of π appearing in $L^2(\Gamma\backslash G)$ is in fact an eigenspace of casimir.
- Since laplacian and casimir agree up to scale on left K invariant functions on G, this shows that the function in π^K is in fact a laplacian eigenfunction.

So now we look at the irreps π of G such that π^K is one dimensional. These are called *spherical* representations.

1.3 Spherical representations of $\mathrm{SL}(2,\mathbb{R})$

. Consider the subgroup

$$B = AN = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}.$$

Parametrize the multiplicative characters on B by

$$\chi_s(\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}) = |a|^s,$$

and induce from χ_s the representation

$$V_s = \{\varphi : G \to \mathbb{C} : \varphi(bg) = \chi_{s+1}(b)\varphi(g)\}.$$

(the $+1$ is a normalization that we'll benefit from later). Since $G = NAK$, a function in V_s is determined by its values on K. Denote by π_s the $\varphi \in V_s$ such that $\varphi|_K \in L^2(K)$. Let G act on π_s by right translation. Then

Theorem 3. When s is not an odd integer, π_s is irreducible with $\dim \pi_s^K = 1$. When s is an odd integer, π_s has a unique irreducible subquotient which we'll renotate to π_s. Every irreducible spherical representation of G arises as a π_s and pairs $\pi_s, \pi_{s'}$ are distinct unless $s = \pm s'$.

In order for $\mathfrak{m}_{\pi_s}^K \neq 0$ we need π_s to be unitary. The pairing on $\pi_{s_1} \times \pi_{s_2}$

$$\langle \varphi_1, \varphi_2 \rangle = \int_K \varphi_1(k)\overline{\varphi_2(k)}\, \mathrm{d}k$$

is G invariant precisely when $s_1 = -\overline{s}_2$ and thus gives a G invariant innerproduct on π_s when $s \in i\mathbb{R}$.

When $s \in (0,1)$, the map $M(s) : \varphi \mapsto \int_{-\infty}^{\infty} f(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \cdot)\, \mathrm{d}x$ is an intertwining map $\pi_s \to \pi_{-s}$, which induces an inner product on π_s iff $s \in (-1,1)$.

Theorem 4. These irreducible unitary spherical reps of G are the π_s with $s \in i\mathbb{R} \cup (-1,1)$.

Since each of these π_s are irreducible, casimir acts on it by a scalar. To compute this scalar, it suffices to compute its value on a $\varphi \in \pi_s^K$ which we normalize to take the value 1 on K. Then identifying φ with a function on $\mathfrak{H}$, the condition

$$\varphi(bg) = \chi_{s+1}(b)\varphi(g)$$

translates to

$$\varphi(x+iy) = y^{(s+1)/2}$$

and since casimir is laplacian, the casimir eigenvalue is obtained by

$$\Delta^{\mathfrak{H}}\varphi(x+iy) = y^2(\frac{\partial^2 l}{\partial x^2} + \frac{\partial^2 l}{\partial y^2})y^{(s+1)/2} = \frac{(s+1)(s-1)}{4}y^{(s+1)/2}.$$

Casmir acts on π_s by $(s^2-1)/4$.
From all of this, we conclude:

$$m_{\pi_s}^{\Gamma} = \dim\{\varphi : \mathfrak{H} \to \mathbb{C} | \Delta^{\mathfrak{H}}\varphi = \frac{s^2-1}{4}\varphi\}$$

To finish our analysis of the LHS, we'll compute the traces $\operatorname{tr} \pi_s(f)$ for $f \in C_c^\infty(K\backslash G/K)$. Recalling that this algebra is commutative, it acts by a scalar on irreducibles, and its trace is that scalar. To compute that scalar, check it on the unique function $\varphi \in \pi_s^K$ such that $\varphi(1) = 1$: compute

$$\begin{aligned}
(\pi_s(f)\varphi)(1) &= \int_G f(g)R(g)\varphi(1))\,\mathrm{d}g \\
&= \int_G f(g)\varphi(g)\,\mathrm{d}g \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{[0.2\pi)} f(n(x)a(u)k(\theta))\varphi(n(x)a(u)k(\theta))\,\mathrm{d}\theta\,\mathrm{d}u\,\mathrm{d}x \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} f(n(x)a(u))\varphi(n(x)a(u)k(\theta))\,\mathrm{d}u\,\mathrm{d}x \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} f(\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}\begin{bmatrix}e^u & 0\\ 0 & e^{-u}\end{bmatrix})\chi_{s+1}(\begin{bmatrix}e^u & 0\\ 0 & e^{-u}\end{bmatrix})\,\mathrm{d}u\,\mathrm{d}x \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} f(\begin{bmatrix}e^{u/2} & e^{u/2}x\\ 0 & e^{-u/2}\end{bmatrix})e^{\frac{u(1+s)}{2}}\,\mathrm{d}u\,\mathrm{d}x \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} f(\begin{bmatrix}e^{u/2} & x\\ 0 & e^{-u/2}\end{bmatrix})e^{us/2}\,\mathrm{d}u\,\mathrm{d}x
\end{aligned}$$