F is a local field of characteristic zero, $G = \mathrm{SL}(2)$. Take a cartan subgroup T of G, defined over F. A special role will be played by the group $H^1(F, T)$.

Set $\tilde{G} = \mathrm{GL}_2$, then $\tilde{T} = C_{\tilde{G}}(T)$ is a cartan subgroup of $\tilde{G}$. That $\tilde{G}$ has a single $\tilde{G}(F)$ conjugacy class of cartans, one has $H^1(F, \tilde{T}) = 1$.

Any $h \in \tilde{G}(F) = \mathrm{GL}(2, F)$ is a product $s^{-1}g$ with $g \in G(\overline{F}) = \mathrm{SL}(2, \overline{F})$ and $s \in \tilde{T}(\overline{F})$.

Let L be the centralizer of $T(F)$ in the algebra of 2×2 matrices over F. Then the *set* of determinants of elements in $\tilde{T}(F)$ coincides with the algebra-theoretic norms of elements in $L^\times$.

The map $g \to \det h \mod \mathrm{Nm}_{L/F} L^\times$ yields an identification of $F^\times / \mathrm{Nm}_{L/F}$ with $H^1(F, T)$.

If F is an extension of a field E, consider groups G' defined over E sandwhiched between the restrictions of scalars: $\mathrm{Res}_{L/E}\, G \leq G' \leq \mathrm{Res}_{L/E}\, \tilde{G}$. Then G' is defined by a subgfroup A of $\mathrm{Res}_{L/E}\, \mathbb{G}_m$ and $G'(F) = \{g \in \tilde{G}(F) | \det g \in A(E)\}$

Take $T' = C_{G'}(\mathrm{Res}_{F/E}\, T)$ and set $\mathfrak{D}(T'; E) = F^\times / A(E)\mathrm{Nm}_{L/F} L^\times$. A slight extension is to consider $G' = \{g \in \tilde{G}(F) \det g \in A\}$ for any closed subgroup A of $F^\times$, which may or may not be the set of points of a group rational over some field.

Let $\kappa : X_*(T) \to \mathbb{C}^\times$ which is $\mathrm{Gal}(\overline{F})$ invariant.

The LLC associates to the pair (T, κ) a group H, which must be either G or T.

Fix haar measures on G' and T' and let $\gamma \in T'$ be regular. For $h \in \tilde{G}(F) = \mathrm{GL}(2, F)$, we can transfer the measure on T' to $h^{-1}T'h$.

Def: G' over F is an inner twist if there exists an isomorphism $\psi : G' \to G$ defined over an extension K/F such that $\sigma(\psi)\sigma^{-1}$ is inner for all $\sigma \in \mathrm{Gal}(K/F)$. Langlands' prediction is that there should be an injection of the automorphic representation of $G'(\mathbb{A}_F)$ into those of $G(\mathbb{A}_F)$

Consider $\gamma' \in G'(F)$, semi-simple, then the conjugacy class of $\psi(\gamma')$ is defined[1] over F. Steinberg assures us that the conjugacy class of $\psi(\gamma')$ in $G(\overline{F})$ contains an element $\gamma \in G(F)$. This means that there is an injection of the elliptic (i.e. nonsplit) conjugacy classes of $G'(F)$ into those of $G(F)$. These classes form the indexing set for the respective trace formula for G' and G. One wonders if this injection respects the orbital integrals in some sense.

For $\gamma \in T$ let $\gamma_1, \gamma_2 \in \bar{F}$ be its eigenvalues. When T is split, the function $d(\gamma) = \frac{|(\gamma_1 - \gamma_2)^2|^{1/2}}{|\gamma_1\gamma_2|^{1/2}}$ will play a special role.

[1] what does this mean here? I think this means that the conjugacy class of $\psi(\gamma')$ is stable under $\mathrm{Gal}(K/F)$, perhaps pointwise so

Fix a regular element γ^0 in $\tilde{T}(F))$ and let ψ be a fixed nontrivial additive character of F. Fix an ordering γ^0_1, γ^0_2 on the eigenvalues of γ^0, which in turn determines an order on those for γ.

For a given regular γ, the quotient $\tilde{T}(F)\backslash\tilde{G}(F)$ may be identified with the orbit $\mathcal{O}(\gamma)$ of γ under $\tilde{G}(F)$ conjugation.

We arrange for the measure on $\tilde{T}(F)\backslash\tilde{G}))$ is of the ofrm $|\omega_\gamma|/|\gamma_1 - \gamma_2|$ for a certain form ω_γ.

For $a \in F^\times$, set $\gamma(a) = a\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. If $\tilde{T}$ is split and $a \in Z(T')$, set $\Phi^{T'}(a, f) = |a|^{-1} \int_{\mathcal{O}(\gamma(a))} f(h)\,\mathrm{d}h$.

For a quadratic extension L/F, regard $\tilde{G}(F)$ as the group of invertible F linear transformations of L. Then $\tilde{T}(F) = L^\times$ acting on L by multiplication.

Pick an F basis $\{1, \tau\}$ for L. Then there are $u, v \in F$ so that $\tau^2 = u\tau + v$.

If $\gamma = a + b\tau \in \tilde{T}(F)$ or $L^\times$, its eigenvalues are of the form $\gamma_1 = a + b\tau$ and $\gamma_2 = a + b\bar{\tau}$ so that $\gamma_1 - \gamma_2 = b(\tau - \bar{\tau})$.

In these coordinates, γ corresponds to $\left(\begin{smallmatrix} a & bv \\ b & a+bu \end{smallmatrix}\right)$

For $g = \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)$, then

$$\gamma^g = \begin{pmatrix} * & -b\mathrm{Nm}_{L/F}(b_1+d_1v)/\det g \\ b\mathrm{Nm}_{L/F}(a_1+c_1v)/\det g & * \end{pmatrix}$$

Let $\tilde{G}(\mathcal{O}_F)$ be the stabilizer of $\mathcal{O}_L$ in $\tilde{G}(F)$. After averaging a function f on $\tilde{G}(F)$ over $\tilde{G}(\mathcal{O}_F)$, we can assume that $f(g^k) = \kappa'(\det(k))f(g)$.

If π is a uniformizer for $\mathcal{O}_F$, then every double coset in $\tilde{T}(F)\backslash\tilde{G}(F)/\tilde{G}(\mathcal{O}_F)$ contains a g so that $g\mathcal{O}_L = \mathcal{O}_F + \pi^m\mathcal{O}_F\tau$, for some $m \geq 0$.

Equivalently, in coordinates, it contains a representative of the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \pi^m \end{smallmatrix}\right)$ for $m \geq 0$.

Every $g \in G'$ can be written as $g = nak$ with $k \in K' = G' \cap \tilde{G}(O_F)$. Set $\beta(g) = \|\alpha/\beta\|$ if $g = nak$ with $a = \left(\begin{smallmatrix} \alpha & \\ & \beta \end{smallmatrix}\right)$ and $\lambda(g) = \beta(g) + \beta(wg)$, so that $\lambda(g) = \beta(wn)$ when $g = nak$. For $\gamma \in A'$ regular, set $\Delta(\gamma) = \|\alpha - \beta\|/\|\alpha\beta\|^{1/2}$

Define distributions

$$F(\gamma, f) = \Delta(\gamma) \int_{A\backslash G} f(g^{-1}\gamma g)\,\mathrm{d}g$$

and

$$A_1(\gamma, f) = \Delta(\gamma) \int_{A\backslash G} f(g^{-1}\gamma g) \ln \lambda(g)\,\mathrm{d}g$$

Given a character η of A, consider the representation $g \mapsto \rho(g, \eta)$ of G acting by (right) translation on the space of smooth left $N(F)$ invariant functions on G satisfying

$$\varphi(ag) = \eta(a)\beta(a)^{1/2}\varphi(g)$$

for all $a \in A$. We can regard the space of $\rho(\eta)$ as a space of functions on $G(O_F)$. The space of functions is the same for η as it is for $\eta_s : a \mapsto \eta(a)\beta(a)^s$

The kernel for $\rho(f, \eta)$ is given by

$$K_\eta(k_1, k_2) = \int_A \int_{N(F)} f(k_1^{-1} a n k_2)\lambda(a)^{1/2}\,\mathrm{d}a\,\mathrm{d}n$$

with the measure on K' chosen so that

$$\int_G f(g)\,\mathrm{d}g = \int_A \int_{N(F)} \int_K f(ank)\,\mathrm{d}a\,\mathrm{d}n\,\mathrm{d}k$$

[1]

References

[1] *Singular Homology*, pages 1–52. WORLD SCIENTIFIC.