SECRET SEMINAR TALK: REAL QUADRATIC ZETAS AND PERIODS OF EISENSTEIN SERIES

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1 Arithmetic side

- Let D be square free, for convenience $D = 2, 3 \mod 4$. Let $k = \mathbb{Q}(\sqrt{D})$ and $\mathcal{O}_k = \mathbb{Z}[\sqrt{D}]$.
- The norm map $N : k \to \mathbb{Q}$ is given by $N(a + b\sqrt{D}) = (a + b\sqrt{D})(a - b\sqrt{D}) = a^2 - Db^2$.
- Let I be the collection of integral ideals of $\mathcal{O}_k$ and define

$$\zeta_k(s) = \sum_{\mathfrak{a} \in I/\{0\}} |N(\mathfrak{a})|^{-s} \qquad \mathrm{Re}\, s > 1$$

- Let C_k be the class group (which is finite), and let $\mathfrak{a}_1, .., \mathfrak{a}_h$ be integral representatives of each ideal class.
- ζ_k breaks up

$$\zeta_k(s) = \sum_{i=1}^{h} \sum_{\mathfrak{b} \in [\mathfrak{a}_i]} |N(\mathfrak{b})|^{-s}.$$

- $\mathfrak{b} \in [\mathfrak{a}_i]$ if and only if $\mathfrak{b} = \theta\mathfrak{a}_i$ for some $\theta \in \mathfrak{a}_i^{-1} = \{\theta : \theta\mathfrak{a}_i \subset \mathcal{O}_k\}$, so

$$\zeta_k(s) = \sum_{i=1}^{h} \sum_{\theta \in \mathfrak{a}_i^{-1}} |N(\theta\mathfrak{a}_i)|^{-s}.$$

- Pick a $\mathbb{Z}$ basis e_1, e_2 for $\mathfrak{a}_i^{-1}$

$$\zeta_k(s) = \sum_{i=1}^{h} \sum_{(m,n) \in \mathbb{Z}^2/\{0\}} |N((me_1 + ne_2)\mathfrak{a}_i)|^{-s}.$$

- By algebraic number theory, there are integers A_i, B_i, C_i with $\mathrm{disc}\, Q_i = B_i^2 - 4A_iC_i = d$ and

$$N((me_1 + ne_2)\mathfrak{a}_i) = A_i m^2 + B_i mn + C_i n^2 = Q_i(m, n).$$

- Thus,

$$\zeta_k(s) = \sum_{i=1}^{h} \sum_{(m,n) \in \mathbb{Z}^2/\{0\}} |Q_i(m, n)|^{-s}$$

- Changing basis in the lattice of $\mathfrak{a}_i^{-1}$ amounts to the action of $\mathrm{SL}_2(\mathbb{Z})$ on Q_i, and by theory of quadratic forms, merely permutes the terms in the sum.

2 Geometro-analytic side

- To analyze the Dedekind zeta, we bring the geometry of the moduli space of lattices to bear on it.
- Define a group and its subgroup, using convention that subscripts denote domain of entries

$$G = \mathrm{SL}_2, \quad P = \left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right].$$

- $G_{\mathbb{R}}$ and its subgroups act on the complex upper half plane $\mathfrak{H} = \{z : \operatorname{Im} z > 0\}$ by linear fractional transformations

$$\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \tau = \frac{a\tau + b}{c\tau + d}$$

- (Aux. useful computation)

$$\operatorname{Im} \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \tau = \frac{\operatorname{Im} \tau}{|c\tau + d|^2}$$

- Aside: If $\mathfrak{H}$ is endowed with the hyperbolic metric, $G_{\mathbb{R}}$ acts by isometries.
- Action is transitive, isotropy of i is $K = \mathrm{SO}(2)$:

$$\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] i = i \iff ai + b = -c + id \iff a = d, \quad b = -c, \quad a^2 + b^2 = 1$$

- By orbit stabilizer theorem

$$G_{\mathbb{R}}/K \approx \mathfrak{H} \quad \text{via} \quad gK \mapsto gi.$$

- The moduli space of lattices is given by $G_{\mathbb{Z}} \setminus \mathfrak{H} \approx G_{\mathbb{Z}} \setminus G_{\mathbb{R}}/K$
- The Real analytic Eisenstein series are functions either on $\mathfrak{H}$ or $G_{\mathbb{R}}$ designed to descend to these quotients:

$$\begin{aligned} E_s(g) &= \sum_{\gamma \in P_{\mathbb{Z}} \setminus G_{\mathbb{Z}}} (\operatorname{Im}(\gamma g i))^s \qquad \operatorname{Re}(s) > 1 \\ \text{or } E_s(z) &= \sum_{\gamma \in P_{\mathbb{Z}} \setminus G_{\mathbb{Z}}} (\operatorname{Im}(\gamma z))^s \\ &= \sum_{\left[\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right] \in G_{\mathbb{Z}}} \frac{(\operatorname{Im} z)^s}{2|cz + d|^{2s}} \end{aligned}$$

- (Note, being in $G_{\mathbb{Z}}$ forces $(c, d) = 1$)
- Convenient modification: a sum over all $(c, d) \neq 0$, can be written as $\sum_{e=1}^{\infty} \sum_{(c,d)=e}$. Upon multiplication by $\zeta(2s)$ we thus have

$$\zeta(2s) E_s(z) = \sum_{(c,d) \neq 0} \frac{(\operatorname{Im} z)^s}{2|cz + d|^{2s}}$$

3 $D < 0$ easy

- For imaginary quadratic, $D < 0$ we've just shown that the Dedekind zeta breaks into a sum of a definite quadratic form at lattice points.
- Erstwhile,

$$\begin{aligned} |cz + d|^2 &= (c \operatorname{Re}(z) + d)^2 + \operatorname{Im}(z)^2 c^2 \\ &= |z|^2 c^2 + 2 \operatorname{Re}(z) cd + d^2 \end{aligned}$$

is a quadratic form of discriminant $4(\operatorname{Re} z)^2 - 4|z|^2 = -4(\operatorname{Im} z)^2$.

- By reduction theory, for each imaginary quadratic ideal class, with corresponding quadratic form Q_i there is a unique z_i in the standard fundamental domain $G_{\mathbb{Z}} \setminus \mathfrak{H}$ with $\frac{\operatorname{Im} z}{|mz+n|^2} = \frac{\sqrt{D}/2}{Q_i(c,d)}$.
- This gives

$$\frac{(\sqrt{D}/2)^s}{\zeta(2s)}\zeta_k(s) = \sum_{i=1}^{h} E_s(z_i).$$

4 $D > 0$ preparations

- For real quadratic, $D > 0$ the quadratic forms Q_i are *indefinite*. There is no hope of finding z_i to make the above work.
- Instead, approach from the geometric side. There is a canonical embedding $\iota : k^* \to \mathrm{GL}_{\mathbb{Q}}(k)$ via multiplication.
- With basis $1, \sqrt{D}$, the embedding is

$$a + b\sqrt{D} \mapsto \begin{bmatrix} a & bD \\ b & a \end{bmatrix}.$$

- Dirichlet's theorem on units shows that the intersection $H_{\mathbb{Z}} = \iota(k^*) \cap G_{\mathbb{Z}}$ is nontrivial. This subgroup corresponds to $\iota(\mathcal{O}_{k\ >0}^{\times})$, positive normed units.
- $H_{\mathbb{Z}}$ is a subgroup of $H_{\mathbb{R}} = H_{\mathbb{Z}} \otimes \mathbb{R}$, a one parameter subgroup which is easy to describe

$$H_{\mathbb{R}} = \{\begin{bmatrix} a & bD \\ b & a \end{bmatrix} : a^2 - b^2D = 1\} = \{h(t) = \begin{bmatrix} \cosh t & \sinh t\sqrt{D} \\ \sinh t/\sqrt{D} & \cosh t \end{bmatrix} : t \in \mathbb{R}\}$$

- For any nonidentity $h = \left[\begin{smallmatrix} a & bD \\ b & a \end{smallmatrix}\right] \in H_{\mathbb{Z}}$, $\mathrm{tr}(h) = 2a > 2$ (since $a^2 = 1 + b^2D$ if $b^2 \neq 0$ then $a^2 \geq 1 + D$ and one can check for $D = 2, 3$ that always $2a > 2$.)
- That is, nontrivial units in $\mathcal{O}_{k\ >0}^{\times}$ correspond to hyperbolic transformations.
- Hyperbolic transformations have 2 distinct real fixed points, one acting as a sink, the other as a source.
- For $a + b\sqrt{D} \mapsto hH_{\mathbb{Z}}$, the fixed points of h are its eigenvalues, which are the conjugates $a \pm b\sqrt{D}$.
- The group $H_{\mathbb{R}}$ acts freely and transitively on the geodesic from $a \pm b\sqrt{D}$, and by the existence of nontrivial units $H_{\mathbb{Z}} \setminus H_{\mathbb{R}}$ is closed (hence compact).

4.1 Computation for the quadratic form $a^2 - Db^2$

- The quadratic form corresponding to the trivial class is just the norm $a^2 - Db^2$. Dehomogenizing, we have the quadratic polynomial $a^2 - D = 0$, with roots $a = \pm\sqrt{D}$. Let C_1 be the geodesic joining $\pm\sqrt{D}$.
- $i\sqrt{D}$ is on this geodesic, so

$$C_1 = \{h(t) \cdot i\sqrt{D} = \sqrt{D}\frac{i\cosh t + \sinh t}{i\sinh t + \cosh t} : t \in \mathbb{R}\}.$$

- The integral to compute is then $\int_{\mathbb{R}} E_s(\sqrt{D}\frac{i\cosh t+\sinh t}{i\sinh t+\cosh t})\,\mathrm{d}t$

$$
\begin{aligned}
&= \int_{H_{\mathbb{Z}}\backslash H_{\mathbb{R}}} E_s(hi\sqrt{D})\,\mathrm{d}h \\
&= \int_{H_{\mathbb{Z}}\backslash H_{\mathbb{R}}} \sum_{\gamma\in P_{\mathbb{Z}}\backslash G_{\mathbb{Z}}} \mathrm{Im}(\gamma hi\sqrt{D})^s\,\mathrm{d}h \\
&= \int_{H_{\mathbb{Z}}\backslash H_{\mathbb{R}}} \sum_{a\in P_{\mathbb{Z}}\backslash G_{\mathbb{Z}}/H_{\mathbb{Z}}} \sum_{b\in aP_{\mathbb{Z}}a^{-1}\cap H_z\backslash H_{\mathbb{Z}}} \mathrm{Im}(abhi\sqrt{D})^s\,\mathrm{d}h \\
&= \sum_{a\in P_{\mathbb{Z}}\backslash G_{\mathbb{Z}}/H_{\mathbb{Z}}} \int_{H_{\mathbb{Z}}\backslash H_{\mathbb{R}}} \sum_{b\in\pm1\backslash H_{\mathbb{Z}}} \mathrm{Im}(abhi\sqrt{D})^s\,\mathrm{d}h \\
&= \sum_{a\in P_{\mathbb{Z}}\backslash G_{\mathbb{Z}}/H_{\mathbb{Z}}} \int_{\pm1\backslash H_{\mathbb{R}}} \mathrm{Im}(ahi\sqrt{D})^s\,\mathrm{d}h \\
&= \frac{1}{\zeta(2s)} \sum_{(c,d)\in\mathbb{Z}^2-\{0\}/H_{\mathbb{Z}}} \int_{\pm1\backslash H_{\mathbb{R}}} \left(\frac{\mathrm{Im}(hi\sqrt{D})}{|chi\sqrt{D}+d|^2}\right)^s\,\mathrm{d}h \\
&= \frac{1}{\zeta(2s)} \sum_{(c,d)\in\mathbb{Z}^2-\{0\}/H_{\mathbb{Z}}} \int_{\mathbb{R}} \left(\frac{\sqrt{D}}{|c\sqrt{D}(i\cosh t+\sinh t)+d(i\sinh t+\cosh t)|^2}\right)^s\,\mathrm{d}t \\
&= \frac{1}{\zeta(2s)} \sum_{(c,d)\in\mathbb{Z}^2-\{0\}/H_{\mathbb{Z}}} \int_{\mathbb{R}} \left(\frac{\sqrt{D}}{(c^2D+d^2+2cd\sqrt{D})e^{2t}+(c^2D+d^2-2cd\sqrt{D})e^{-2t}}\right)^s\,\mathrm{d}t
\end{aligned}
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