1. Reductive Groups

Let G be a connected algebraic group over an algebraically closed field k. Say that G is semisimple if the only smooth connected solvable normal subgroup of G is trivial, and reductive if the only smooth connected unipotent normal subgroup of G is trivial. Any unipotent group over an algebraically closed field has a composition series in which each quotient is isomorphic to $\mathbb{G}_a$. For reductive G, the inner action of G on itself induces a homomorphism of k-group functors $G \to \mathrm{Aut}(G)$, and automorphisms of G can be differentiated to elements of $\mathrm{Aut}(\mathfrak{g})$: this is the adjoint action of G on $\mathfrak{g}$.

A representation of a torus T on a vectorspace V is tantamount to a grading of V by $X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$. When T is a (maximal) torus in reductive G and $V = \mathfrak{g}$, the decomposition is

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R(T,G)} \mathfrak{g}_\alpha$$

where $R(G,T) \leq X(T)$ are the relative to T, and $\mathfrak{g}_\alpha$ is the subspace on which T acts by α. Each $\mathfrak{g}_\alpha$ (since k is algebraically closed) is one dimensional: hence may be identified with $\mathbb{G}_a$. Pulling back the natural action of $\mathbb{G}_m$ on $\mathbb{G}_a$ by scaling through α, we obtain an action of T on $\mathbb{G}_a$. Up to scalar, there is a unique *root homomorphism* $x_\alpha : \mathbb{G}_a \to \mathfrak{g}$ intertwining the actions of T on $\mathbb{G}_a$ and on $\mathfrak{g}$, inducing an isomorphism $\mathrm{d}x_\alpha : \mathrm{Lie}(\mathbb{G}_a) \approx \mathfrak{g}_\alpha$. Let U_α denote the corresponding subgroup of G.

After normalizing x_α and $x_{-\alpha}$ suitably, there is a unique homomorphism $\varphi_\alpha : \mathrm{SL}_2 \to g$ such that $\varphi_\alpha(\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)) = x_\alpha(a)$ and $\varphi_\alpha(\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right)) = x_{-\alpha}(a)$

The dual coroots $\alpha^v \in \hom(\mathbb{G}_m, T)$ are defined by the relation $\alpha^v(\lambda) = \varphi_\alpha(\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right))$

For each $\alpha \in R$, there is an involution $s_\alpha : X(T) \to X(T)$ defined by $s_\alpha(x) = x - \langle x, \alpha^v, \alpha \rangle$, which restricts to a permutation on R.

The *finite weyl group* associated to the root datum (R, X, R^v, X^v) is the group generated by the s_α for $\alpha \in R$.

The weyl group acts transitively on the choices of simple roots $\sigma \subset R$, and subordinate to any such choice on defines the *positive roots*$R_+ = \{\alpha \in R : \alpha \in \sum_{\sigma \in \Sigma} \mathbb{Z}_{\geq 0} \sigma\}$, *simple reflections* $S_f = \{s_\alpha : \alpha \in \Sigma\}$, and the *dominant weights* $X_+ = \{\lambda \in X : \langle \lambda, \alpha^v \rangle \geq 0, \alpha \in \Sigma\}$. (easymotion-prefix)ll A choice of R_+ yeilds a *Borel subgroup* B^+ containing T such that $B^+ = TU^+$ where U^+ is the subgroup generated by the U_α for $\alpha \in R$

1.1. **Parabolic subgroups: tautological representations from flag variety quotients.**

zo At the level of algebraic groups (and algebraic representations,) every rep of G embeds in some number of copies of $k[G]$. As an affine coordinate ring, $k[G]$ is in many regards too large to deal with on its own. *Parabolic subgroups* P of G are those for which the quotient variety G/P is as small (in the algebro-geometric context) as possible.

When $G = SL_2$, the quotient G/B^+ identfies with $\mathbb{P}^1$ viz. the set of lines in k^2: indeed the action of G on such lines is transitive, and B^+ is the stabilizer of the line spanned by $e_1 = (1,0)$.

More generally, when $G = GL_n$, the quotient G/B^+ identifies with the variety $\mathcal{F}$ of full flags $0 \leq V_1 \leq \ldots \leq V_n = k^n$ where each V_i is i-dimensional.

Definition 1.1. Suppose G acts on a k-scheme X through $\sigma G \times X \to X$. A G-equivariant sheaf $\mathcal{F}$ on X is a sheaf of $\mathcal{O}_X$ modules together with an isomorphism $\varphi : \sigma^*\mathcal{F} \to p_2^*\mathcal{F}$ of $\mathcal{O}_{G\times X}$ modules, which satisfies the cocycle condition $p_{23}^*\varphi \circ (1_G \times \sigma)^*\varphi = (m \times 1_X)^*\varphi$. The isomorphism φ yeilds a G-equivariant identification of stalks: $\mathcal{F}_{gx} \approx \mathcal{F}_x$ and the cocycle condition ensures that the identifications are compatible: $\mathcal{F}_{ghx} \approx \mathcal{F}_{hx} \approx \mathcal{F}_x$.

For any such sheaf, the k-vectorspace of global sections $\Gamma(X, \mathcal{F})$ admits a natural representation of G. Conversely, for any G module V, G acts on $\mathbb{P}(V^*)$, and the tautological bundle $\mathcal{O}(1)$ is an equivariant line bundle for this action. One recovers the action of G on V from the action on global sections: $\Gamma(\mathbb{P}(V))$

Theorem 1 (Borel fixed point theorem)**.** *Let H be a connected solvable algebraic group acting through regular functions on a nonempty complete variety W over an algebraically closed field. Then there exists a point of W fixed by H.*

Definition 1.2. Let G be a k-group scheme acting freely on a k-scheme X in such a way that X/H is a scheme; let $\pi : X \to X/H$ be the projection map. The **associated sheaf functor** is

$$\mathcal{L} = \mathcal{L}_{X,H} : \{H - \text{modules}\} \to \{\text{vector bundles on } X/H\}$$

defined on objects as follows: if $U \subset X/H$ is open, then

$$\mathcal{L}(M)(U) = \{f \in \mathrm{Hom}_{\text{scheme}}(\pi^{-1}(U), M_a) : f(xh) = h^{-1}f(x)\}.$$

Note: if $\pi^{-1}(U)$ is affine, these sections coincide with $(M \otimes k[\pi^{-1}U])^H$.

For any $\lambda \in X(T) = \mathrm{Hom}(X, \mathbb{G}_m)$, let k_λ be the representation of B pulled back through the projection $B \to B/[B, B] \approx T$, and define the sheaf $\mathcal{O}(\lambda) = \mathcal{L}_{G,B}(k_{-\lambda})$ on G/B.

Given a choice of positive roots R_+ and corresponding Borel B, let $\bar{B}$ be the opposite Borel (corresponding to the choice of $-R_+$ as positive rooots) and $\bar{U}$ its unipotent radical. A consequence of the Bruhat decomposition of G is that the map $\bar{U} \to G/\bar{B}$ sending u to $u\bar{B}/\bar{B}$ is an open inclusion. Furthermore, the (cartesian) product map $(x_\alpha)_{\alpha \in R^+}$ yeilds parametrization of $\bar{U}$ (identiftying the latter with $\mathbb{A}^{|R_+|}$.

2. Witt Vectors

Theorem 2. *Let K be a perfect ring of characteristic p.*

1. *There is a strict p-ring R with residue ring K, unique up to canonical isomorphism.*
2. *There is a unique system of representatives $\tau : K \to R$ (teichmulller representatives) such that $\tau(xy) = \tau(x)\tau(y)$ for $x, y \in K$.*
3. *Every element $x \in R$ can be written uniquely in the form $x = \tau(x_n)p^n$ for $x_n \in K$.*
4. *Formation of R and τ is functorial in K.*

The simplest example: take $R = \mathbb{Z}_p$ and $K = \mathbb{F}_p$, then by Hensel's lemma, each nonzero $x \in \mathbb{F}_p$ has a unique lift $\tau(x)$ to $\mathbb{Z}_p$, and extending τ by 0 to $\mathbb{F}_p$ completes the definition.

A central question: given $x = \sum \tau(x_n)p^n$ and $y = \sum \tau(y_n)p^n$ write $xy = \sum \tau(m_n)p^n$ and $x + y = \sum \tau(s_n)p^n$. How can we determine $\tau(s_n)$ and $\tau(m_n)$ in terms of x and y?

An important

Lemma 1. *Let A be a ring, and $x, y \in A$ such that $x = y \mod pA$. Then for all $i \geq 0$ we have $x^{p^i} = y^{p^i} \mod p^{i+1}A$.*

Note the two maps in play: there is the teichmuller lift $\tau : K \to R$, and an infinite sequence of maps $\pi_n = (\cdot)_n : R \to K$ such that the mapping $\cdot \mapsto \sum \tau((\cdot)_n)p^n$ is the identity on R. A preliminary goal is to understand the compositions $(x, y) \mapsto \pi_n(x + y)$ and $(x, y) \mapsto \pi_n(xy)$.

The answer is as follows:

$$s_1(x, y) = x_1 + y_1 - \sum_{n=1}^{p-1} (p/n) \binom{p}{n} x_0^{n/p} y_0^{(p-n)/p}$$

Definition 2.1. A set P of natural numbers is **divisor-stable if it is nonempty and for all $n \in P$, all divisors of n are also in P. For a divisor stable set P let $\sqrt{P}$ be the set of prime numbers in P. Let $P_p = \{p^n : n \geq 0\}$ and $P_{p(n)} = \{p^j : 0 \leq j \leq n\}$ (these are both divisor stable).**

Definition 2.2. Let $n \in \mathbb{N}$, define the **n-th witt polynomial** as

$$w_n = \sum_{d|n} d x_d^{n/d} \in \mathbb{Z}[\{X_d : d|n\}].$$

For any divisor stable P and any ring A, define

$$W_P(A) = \prod_{n \in P} A.$$

And for $x \in W_P(A)$ write $\pi_n(x) = x_n \in A$ for the projection to the n-th factor. For $P = \mathbb{N}$ write $W(A)$ for $W_P(A)$ and if $P = P_p$m, write $W_p(A)$ for $W_P(A)$.

The witt polynomials w_n are then (set theoretic) maps $w_n : W_P(A) \to A$. Write w_* for the cartesian product of these maps. For $x \in W_P(A)$, the values $w_n(x)$ are called the **ghost components of x.**

Theorem 3. *Let P be a divisor stable set. There is a unique covariant functor $W_P : \mathrm{Alg}_{\mathbb{Z}} \to \mathrm{Alg}_{\mathbb{Z}}$, such that for any ring A,*

(1) *$W_P(A) = \prod_{n \in P} A = A^P$ as sets, and for a ring hom $f : A \to B$, one has*
$$W_P(f)((a_n)_{n \in P}) = (f(a_n))_{n \in P}.$$
(2) *The maps $W_P(A) \to A$ are ring homomorphisms for all $n \in P$.*

Furthermore the zero element is $(0, 0, \ldots)$ and the unit element is $(1, 0, \ldots)$.

A remark: If A is a K algebra, then $W_P(A)$ need not be a K algebra. For example, when $A = \mathbb{F}_p$ and $P = \{p^{\mathbb{N}}\}$, then $W_P(\mathbb{F}_p) = \mathbb{Z}_p$ but the latter is not an algebra over $\mathbb{F}_p$. Nonetheless, W_P sends K-algebras to $\mathbb{Z}$-algebras.

For a ring A, let $\Lambda(A) = 1 + tA[[t]]$ (a multiplicative abelian group). Then for any element $f = 1 + \sum_{n=1}^{\infty} x_n t^n \in \Lambda(A)$, there is a unique expression $f = \prod(1 - y_n t^n)$ for $y_n \in A$. Furthermore, there exist polynomials $Y_n \in \mathbb{Z}[X_1, ..., X_n]$ and $X_n' \in \mathbb{Z}[Y_n', ..., Y_n']$ independent of A such that $y_n = Y_n(x_1, ..., x_n)$ and $x_n = X_n'(y_1, ..., y_n)$.

Consequently: for any ring A the map $x \mapsto f_x : W(A) \to \Lambda(A)$ defined by

$$f_x(t) = \prod(1 - x_n t^n), \tag{1}$$

where $x = (x_1, ...)$ is a bijection.

For any $\mathbb{Q}$-algebra A, the mercator series defines a bijection $\log : \Lambda(A) \to tA[[t]]$ with inverse given by the exponential series $\exp : tA[[t]] \to \Lambda(A)$. In fact, $\log$ is a homomorphism of abelian groups (the former being multiplicative and the latter additive). The map $f \mapsto -t\,\mathrm{d}f/\,\mathrm{d}t$ is an automorphism of $tA[[t]]$ (additive), and its inverse is $\int -t^{-1}(\cdot)\,\mathrm{d}t$. Let $D = -t\frac{\mathrm{d}}{\mathrm{d}t}\log(\cdot) : \Lambda(A) \to tA[[t]]$.