1. PSEUDO-INTRODUCTION

Let M be a compact Riemannian manifold. The eigenvalues of the Laplace operator Δ acting on $L^2(M)$ form a discrete sequence $0 = \lambda_0 \leq \lambda_1 \leq \ldots$. The multiset of eigenvalues $\{\lambda_i\}$ is the **Laplace spectrum** of M. We say that two manifolds are **isospectral** if their Laplace spectra coincide. A manifold is **absolutely spectrally rigid** if the only manifolds to which it is isospectral are in fact isometric.

In this article, we prove the following

Theorem 1. *Let k be a totally real number field with ring of integers R, and A a quaternion algebra over k with type number 1. Further, suppose that there is a unique real place of k at which A is unramified. Let $\mathcal{O}$ be a maximal order in A, $\mathcal{O}^1$ the multiplicative group of norm 1 units. For an ideal I in R, let $\mathcal{O}^1(I) \leq \mathcal{O}^1$ be the kernel of the reduction map. Denote by Γ (resp. $\Gamma(I)$) the image of $\mathcal{O}^1$ (resp. $\mathcal{O}^1(I)$) in $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{Isom}^+(\mathbb{H})$. Suppose that I is not divisible by any prime in R at which A is ramified. Then the Riemann surface $\Gamma(I)\backslash\mathbb{H}$ is absolutely spectrally rigid.*

2. AUDIBLE PROPERTIES

2.1. **Heat invariants.** Let M be a compact Riemannian manifold. Enumerate the spectrum of its Laplace-Beltrami operator as $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$, where each eigenvalue is repeated according to its multiplicity. Pick an $L^2(M)$-orthonormal basis $\{\varphi_j\}$ of corresponding eigenfunctions. The **heat kernel** of M is the function $K : \mathbb{R}_{>0} \times M \times M$ defined by

$$K(t,x,y) = \sum_j e^{-t\lambda_j} \varphi_j(x)\varphi_j(y). \tag{1}$$

2.2. **Scope: Riemannian manifolds.**

Theorem 2. *Volume is an audible property.*

TODO: use asymptotics of Θ_M. □

Theorem 3. *Dimension is an audible property.*

TODO: use asymptotics of Θ_M. □

2.3. **Scope: smooth, compact Riemannian surfaces.**

Theorem 4. *Suppose $\dim M = 2$ and that M is smooth. The homeomorphism class of M is audible.*

TODO: gauss-bonnet. □

Theorem 5. *Suppose $\dim M = 2$, and that M is smooth. The property of having constant curvature is audible.*

TODO: GCB the metric of constant curvature minimizes topological entropy, which is a spectral invariant. □

2.4. **Scope: Riemann surfaces.**

Theorem 6. *Suppose M is a compact Riemann surface. Then arithmeticity is audible.*

TODO: apply Takeuichi. □

2.5. **Scope: Arithmetic Riemann surfaces.**

Theorem 7. *Suppose M is an arithmetic Riemann surface. The commensurability class of M is audible.*

TODO: apply Reid. □

3. The proof

Now let us fix all of data $(k, A, \mathcal{O}, I)$ as in the main theorem. Sequentially applying the theorems of the preceeding section, we may assume that a manifold M isospectral to $\Gamma(I)\backslash\mathbb{H}$ is of the form $M = \Lambda\backslash\mathbb{H}$ for some subgroup Λ of Γ with the same index as $\Gamma(I)$.

To prove the theorem, it suffices to show that Λ is in fact conjugate (in $A^\times$) to $\Gamma(I)$. To this end, we convert the problem into a local one.

[TODO: get to the prime power setting].

4. Arithmetic Subgroups of algebraic groups

A matrix group $G \leq \mathrm{GL}(n, \mathbb{C})$ is said to be **algebraic** if it consists of all invertible matrices whose coefficients annihilate some set of polynomials on $M(n, \mathbb{C})$. If this set of polynomials can be taken with coefficients in some subring $R \leq \mathbb{C}$, then this group is said to be **defined over** R.

4.1. **Base case.**

Let X denote the **Bruhat-Tits tree** for $\mathrm{SL}_2(k)$. The vertices of X are homothety classes of lattices in k^2, and two vertices $x, y \in X$ are adjacent if there exist lattices $L_x \in x$ and $L_y \in y$ so that $L_y \leq L_x$ and $L_y/\pi L_x$ is a $\mathfrak{k}$-line in $L_x/\pi L_x \approx \mathfrak{k}^2$. We write $d : X \times X \to \mathbb{Z}_{\geq 0}$ for the graph-theoretic distance function on X.

For a lattice $L \leq k^2$, we write $[L]$ for the vertex in X corresponding to L. For a subset $A \subset X$ of vertices, write

$$B(A, r) = \{y \in X : d(x, y) \leq r, \forall x \in A\}$$
$$S(A, r) = \{y \in X : d(x, y) = r \, \forall x \in A\}$$

for the ball and sphere about A of radius r, respectively.

A **geodesic** is a non-backtracking path in X. The length of a geodesic c is $\sup_{x,y\in c} d(x,y) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. For vertices $x, y \in X$ write $[x, y]$ for the unique geodesic starting at x and finishing at y.

For a subset $A \subset X$ we say that a vertex $x \in A$ is an **interior point** of A and write $x \in A^{\text{int}}$ if $B(x, 1) \subset A$. Otherwise we say x is a **boundary point** of A and write $x \in \partial A$. We say x is an **extremal point** in A if either: $A = \{x\}$ or $\text{Card}(S(x, 1) \cap A) = 1$.

Let $\mathcal{G} = \text{SL}_2$, and write $G = \mathcal{G}(k)$ and $K = \mathcal{G}(R)$. G acts on the set of lattices in k^2, and this action passes to one on X. For a subgroup $H \leq G$, we write X^H for its fixed point set on X. For a subset $A \subset X$, and a subgroup $H \leq G$, we write H_A for its (pointwise) stabilizer in H.

For each vertex x and integer $n \geq 1$, we have a $R/\mathcal{P}^n$-representation $\rho_x : G_x \to \text{SL}(L_x/\pi^n L_x) \approx (R/\mathcal{P}^n)$

Upon picking a basis e_1, e_2 for L_o we may identify G_{x_o} with $\text{SL}_2(R)$. The action of $\text{SL}_2(k)$ on X has exactly two orbits. Letting $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$, the set $\{x_o, \alpha x_o\}$ constitutes a fundamental domain for this action. The respective stabilizers G_{x_o} and $G_{\alpha x_o} = \alpha G_{x_o} \alpha^{-1}$ are representatives of the two conjugacy classes of maximal compact subgroups. The following lemma is standard:

Lemma 1. *For a vertex $x \in X$, there is a G_x equivariant bijection between n-sphere $S(x, n)$ and cyclic $R/\mathcal{P}^n$-submodules of $L_x/\pi^n L_x$. Using a basis for L_x, the latter set is naturally identified with $\mathbb{P}^1(R/\mathcal{P}^n)$*

Lemma 2. *Let $H \leq G$ be a subgroup such that X^H is nonempty.*

1. *For a subgroup $H \leq G$ and a vertex $x \in X^H$, the bijection in lemma 1 restricts to one between H-fixed vertices $X^H \cap S(x, n)$, and H-stable $R/\mathcal{P}^n$-submodules of $L_x/\pi^n L_x$.*
2. *For $x \in X^H$, one has $B(x, n) \leq X^H$ if and only if H acts by scalars under $\rho_{x,n}$.*
3. *A vertex x is a non-extremal boundary point of X^H if and only if $\rho_{x,1}$ decomposes as a direct sum of (distinct) nontrivial characters.*
4. *A vertex x is extremal in X^H if and only if the $\mathfrak{k}$-representation $\rho_{x,1}$ is reducible but indecomposable.*

Proof. item 1 and item 2 follow immediately from the definitions and the equivariance in lemma 1.

item 3: Suppose x is a nonextremal boundary point of X^H. There are then exactly two vertices $y, z \in X^H$ adjacent to x. Under the identification of $S(x, 1)$ with $\mathbb{P}^1(\mathfrak{k})$, we see that H must act as a group of hyperbolic transformations with common fixed points y, z.

item 4 Suppose x is extremal in X^H. Then x has exactly one neighbor in X^H. Under the identification of $S(x, 1)$ with $\mathbb{P}^1(\mathfrak{k})$, we find that H must act as a group of parabolic transformations. $\square$

Proposition 1. *[ref ribetGL2] Let $g \in \text{SL}_2(R)$ and $x \in \partial X^{\langle g \rangle}$. For an integer $n > 0$, the following are equivalent.*

1. *The characteristic polynomial $p_g(T)$ is reducible mod $\mathcal{P}^n$.*
2. *There exists a geodesic of length n based at x and contained in $X^{\langle g \rangle}$.*

Proof. Assuming item 1, pick $\alpha, \beta \in R$ such that $p_g(T) \equiv (T-\alpha)(T-\beta) \mod \mathcal{P}^n$. Then since x is a boundary point, g does not act as a scalar on $L_x/\pi L_x$. Consequently, $L_x/\pi L_x$ is a cyclic $R[g]$ module. Pick a vector $v \in L_x$ such that $\{v, gv\}$ projects to a basis of $L_x/\pi L_x$. By Nakayama's lemma, $\{v, gv\}$ projects to a basis of $L_x/\pi^n L_x$.

Let $w = (g-\beta)v$. Then, compute modulo $\mathcal{P}^n$:

$$\begin{aligned} gw &\equiv (g^2 - g\beta)v \\ &\equiv ((\alpha+\beta)g - \alpha\beta g - g\beta)v \\ &\equiv \alpha(g-\beta)v \\ &\equiv \alpha w \end{aligned}$$

Thus, the cyclic $R/\mathcal{P}^n$-submodule of $L_x/\pi^n L_x$ spanned by w is g stable. The corresponding vertex lies in $X^{\langle g \rangle}$ and is at distance n from x as desired.

Now assume that $y \in X^{\langle g \rangle}$ has $d(x,y) = n$. Pick a lattice L_y representing y with $\pi^n L_x < L_y < L_x$. Then L_y projects to a g-stable free rank 1 $R/\mathcal{P}^n$ submodule of $L_x/\pi^n L_y$ on which g must act by an element $\alpha \in (R/\mathcal{P}^n)^\times$. It follows that $(t-\alpha)$ divides $p_g(t)$ modulo $\mathcal{P}^n$, proving reducibility. □

Proposition 2. *Let $g \in \mathrm{SL}_2(R)$. For an integer $n > 0$, the following are equivalent:*

(1) $\mathrm{tr}(g) = \pm 2 \mod \mathcal{P}^{2n}$
(2) *There is a point $x \in X^g$ such that $B(x,n) \subset X^g$.*

Proof. The argument is by induction on n.

Base case: suppose $\mathrm{tr}(g) = 2 \pm 2 \mod \mathcal{P}^2$. By proposition 1 there is a geodesic γ of length 2 contained in X^g. Conjugating by an element of $\mathrm{GL}_2(k)$ if needed, we may assume that γ takes the form (y, x_o, z). Since g has two fixed points in the 1 neighborhood of x_o, it acts semisimply on $L_{x_o}/\pi L_{x_o}$ say with eigenvalues $\alpha, \beta \in R/\mathcal{P}$. These eigenvalues must satisfy $\alpha + \beta = \pm 2 \mod \mathcal{P}$ and $\alpha\beta = 1 \mod \mathcal{P}$, but then g acts as $\pm\,\mathrm{id}$ and the claim follows.

Suppose now that for all $k < n$ that $\mathrm{tr}(g) = \pm 2 \mod \mathcal{P}^{2k}$ iff and only if g fixes some k ball in X. Let $\mathrm{tr}(g) = \pm 2 \mod \mathcal{P}^{2n}$. Applying proposition 1 as before, there is some geodesic γ passing through x_o of length $2n$ which is fixed pointwise by g.

□

4.2. **Some projective geometry.** Let F be a field, let V be a two dimensional vectorspace over F. The **projective line** over F is the set of F-lines in V. There are several useful coordinate systems that one can put on $\mathbb{P}^1(F)$.

The first system requires no choices: the association of a nonzero vector v with the F-line that it spans yeilds a surjection $V \setminus \{0\} \to \mathbb{P}^1(F)$. The multiplicative group $F^\times$ acts transitively on the fibers, yeilding our first identification: $\mathbb{P}^1(F) = F^\times/(V - \{0\})$

A bijection $\mathbb{P}^1(F) \to \mathbb{P}^1(F)$ is a **projective transformation** if it lifts to a linear automorphism of V. Write G' for the group of projective transformations.

Lemma 3. *A nonidentity projective transformation can have at most* 2 *fixed points on* $\mathbb{P}^1(F)$

Proof. The points in $\mathbb{P}^1(F)$ fixed by g correspond to eigenlines of its lifts to $\mathrm{GL}(V)$. Since V is two dimensional, it supports at most two linearly independent lines. $\square$

We classify nonidentity transformations accordingly:

- say g is **hyperbolic** if it fixes two points on $\mathbb{P}^1(F)$
- say g is **parabolic** if it fixes exactly one point on $\mathbb{P}^1(F)$
- say g is **elliptic** if it fixes no points in $\mathbb{P}^1(F)$.

The following lemma demonstrates how to determine the category of a projective transformation in terms of its lifts. For a linear transformation $\tilde{g} \in \mathrm{GL}(V)$ write $p_{\tilde{g}}(T) = \det(\tilde{g} - T\,\mathrm{id}) \in F[T]$ for its characteristic polynomial and $\delta(\tilde{g}) = \mathrm{tr}(g)^2 - 4\det(g)$ for its discriminant.

Lemma 4. *Let g be a nonidentity projective transformation.*

- *g is hyperbolic if and only if any lift $\tilde{g}$ is semisimple and diagonalizable over F, for any lift $\tilde{g}$. This is so if and only if $\delta(\tilde{g}) \in (F^\times)^2$.*
- *g is parabolic if and only if $\tilde{g}$ is not semisimple for any lift $\tilde{g}$. This is so if and only if $\delta(\tilde{g}) = 0$.*
- *g is parabolic if and only if $\tilde{g}$ is not semisimple for any lift $\tilde{g}$. This is so if and only if $\delta(\tilde{g}) \in F^\times \setminus (F^\times)^2$.*