**SPECTRAL RIGIDITY OF HURWITZ SURFACES**

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For an element $\gamma \in \mathrm{PSL}(2,\mathbb{R})$, there is a unique lift $\widetilde{\gamma} \in \mathrm{SL}(2,\mathbb{R})$ such that $\mathrm{Tr}(\widetilde{\gamma}) \geq 0$. Define $\mathrm{Tr}(\gamma) := \mathrm{Tr}(\widetilde{\gamma})$. We say that an element $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ is hyperbolic if $\mathrm{Tr}(\gamma) > 2$. In this case, there is a unique real number $N(\gamma) > 1$ such that $\widetilde{\gamma}$ is conjugate in $\mathrm{SL}(2,\mathbb{R})$ to $\left[\begin{smallmatrix} N(\gamma) & 0 \\ 0 & N(\gamma)^{-1} \end{smallmatrix}\right]$. We call $N(\gamma)$ the *norm* of $\gamma$. The norm and trace are related by the equation $\mathrm{Tr}(\gamma) = N(\gamma) + N(\gamma)^{-1}$. In particular, they uniquely determine one another.

Let $\Gamma \leq \mathrm{PSL}(2,\mathbb{R})$ be a cofinite Fuchsian group. We say that a hyperbolic element $\gamma$ is primitive if it is not a proper power. Let $[\Gamma]_{\mathrm{prim}}$ denote the collection of conjugacy classes of primitive hyperbolic elements of $\Gamma$. Since trace and norm are both conjugation invariant, we may extend their definition to $[\Gamma]_{\mathrm{prim}}$.

Let $\rho : \Gamma \to \mathrm{GL}(V)$ be a representation of $\Gamma$ on a finite dimensional vectorspace $V$. Define, for $\mathrm{Re}(s) \gg 0$ the (twisted) Selberg zeta function

$$Z_\gamma(s,\rho) := \prod_{[\gamma]\in[\Gamma]_p} \prod_{k=0}^{\infty} \det(1-\rho(\gamma)N(\gamma)^{-(s+k)})$$

#### 0.0.0.1 Multiplicative independence

Set $\Lambda_\Gamma(s,\rho) = \log Z_\Gamma(s,\rho)$. Compute for $\mathrm{Re}(s) \gg 0$,

$$\begin{aligned}
\Lambda_\Gamma(s,\rho) &= \sum_{[\gamma]\in[\Gamma]_p} \sum_{k=0}^{\infty} \log\det(1-\rho(\gamma)N(\gamma)^{-(s+k)}) \\
&= \sum_{[\gamma]\in[\Gamma]_p} \sum_{k=0}^{\infty} \mathrm{Tr}\log(1-\rho(\gamma)N(\gamma)^{-(s+k)}) \\
&= -\sum_{[\gamma]\in[\Gamma]_p} \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \frac{\chi_\rho(\gamma^m)}{m} N(\gamma)^{-m(s+k)},
\end{aligned}$$

since $\log(1-A) = -\sum_{m=1}^{\infty} \frac{A^m}{m}$ for any matrix $A$ with sufficiently small entries. In the last line, $\chi_\rho(\gamma) = \mathrm{Tr}(\rho(\gamma))$ is the character of $\rho$.

The expression in the last line of the previous display is sensible for $\chi_\rho$ for any class function on $\Gamma$. For $\chi$ a class function on $\Gamma$, we take this as a definition of $\Lambda_\Gamma(s,\chi)$ and thereby $Z_\Gamma(s,\chi)$. For a class function $\chi$, suppose there are finitely many nonzero rational numbers $a_\rho$, for $\rho \in \hat{\Gamma}$ such that $\chi = \sum_{\rho\in\hat{\Gamma}} a_\rho \chi_\rho$. We call such a $\chi$ a *virtual character*, and write $\chi \in \mathrm{Vchar}(\Gamma)$. For such a $\chi$,

$$\Lambda_\Gamma(s,\chi) = \sum_{\rho\in\hat{\Gamma}} a_\rho \Lambda_\Gamma(s,\rho)$$

and

$$Z_\Gamma(s,\chi) = \prod_{\rho\in\hat{\Gamma}} Z_\Gamma(s,\rho)^{a_\rho}$$

Suppose there is some multiplicative dependency among the functions $Z_\Gamma(s,\rho_1), ..., Z_\Gamma(s,\rho_n)$, i.e. there are rational numbers $a_{\rho_i}$ so that $\prod_{i=1}^n Z_\Gamma(s,\rho_i)^{a_{\rho_i}} = 1$. Then this may be expressed concisely as $Z_\Gamma(s,\chi) = 1$, where $\chi = \sum_{i=1}^n a_{\rho_i}\chi_{\rho_i}$.

Let $\Gamma' \leq \Gamma$ be a finite index normal subgroup, and set $G = \Gamma'\backslash\Gamma$. For each class function $\chi$ of $G$, let $\widetilde{\chi}$ be its inflation to $\Gamma$. Now suppose $\chi$ is a virtual character of $G$ such that $Z_\Gamma(s,\widetilde{\chi}) = 1$. This is equivalent to

$$\Lambda_\Gamma(s,\widetilde{\chi}) = -\sum_{[\gamma]\in[\Gamma]_p}\sum_{k=0}^\infty\sum_{m=1}^\infty \frac{\widetilde{\chi}(\gamma^m)}{m} N(\gamma)^{-m(s+k)} = 0$$

Recall (find ref) the uniqueness principle for generalized Dirichlet series: let $\nu_1, \nu_2, ...$ be a sequence of distinct positive real numbers and $a(\nu_1), a(\nu_2), ...$ a sequence of complex numbers such that the series

$$\sum_{i=1}^\infty a(\nu_i)\nu_i^{-s}$$

converges absolutely to an analytic function $f(s)$ for $\mathrm{Re}\, s \gg 0$. Then $f(s) = 0$ identically if and only if $a(\nu_i) = 0$ for all $\nu_i$.

In order to apply the uniqueness principle for (generalized) Dirichlet series, we must collect summands according to their base. To this end, let $\mathcal{N}_\Gamma$ denote the *primitive norm spectrum of* $\Gamma$. That is, $\mathcal{N}_\Gamma$ is the (multi-)set of values taken by the norm map $N$ at primitive hyperbolic conjugacy classes in $\Gamma$. For every real number $\nu$, let $\mathcal{N}_\Gamma(\nu)$ denote the collection of classes $[\gamma] \in [\Gamma]_p$ with $N(\gamma) = \nu$. By definition, $\mathcal{N}_\Gamma(\nu)$ is empty unless $\nu \in \mathcal{N}_\Gamma$. Now we have

$$\Lambda_\Gamma(s,\widetilde{\chi}) = -\sum_{\nu\in\mathcal{N}_\Gamma}\sum_{[\gamma]\in\mathcal{N}_\Gamma(\nu)}\sum_{k=0}^\infty\sum_{m=1}^\infty \frac{\widetilde{\chi}(\gamma^m)}{m}\nu^{-m(s+k)} = 0.$$

The sum over $k$ is a geometric series with base $\nu^{-m}$, so

$$\Lambda_\Gamma(s,\widetilde{\chi}) = -\sum_{\nu\in\mathcal{N}_\Gamma}\sum_{[\gamma]\in\mathcal{N}_\Gamma(\nu)}\sum_{m=1}^\infty \frac{\widetilde{\chi}(\gamma^m)}{m(1-\nu^{-m})}\nu^{-ms} = 0$$

The set consisting of all powers of a fixed primitive norm and the set consisting of a fixed power of all primitive norms may intersect nontrivially. Indeed, by McReynolds-Lafont, arithmetic non-compact hyperbolic surfaces admit arbitrarily long arithmetic progressions in their length spectrum.

For each primitive norm $\nu$, and each integer $m > 0$, let $A_m(\nu)$ denote the set of primitive hyperbolic conjugacy classes $[\gamma]$ in $\Gamma$, such that $N(\gamma^m) = \nu$. For a fixed $\nu \in \mathcal{N}_\Gamma$, only finitely many $A_m(\nu)$ are nonempty. Let $m(\nu)$ be the largest $m$ such that $A_m(\nu)$ is nonempty. Then we have a partition $\mathcal{N}_\Gamma(\nu) = A_1(\nu) \sqcup ... \sqcup A_{m(\nu)}(\nu)$. Thus, for each $\nu \in \mathcal{N}_\Gamma$, the coefficient $b(\nu,\chi)$ of $\nu^{-s}$ in $\Lambda_\Gamma(s,\chi)$ is

$$b(\nu,\chi) := -\sum_{m=1}^{m(\nu)} \frac{1}{m(1-\nu^{-m})}\sum_{[\gamma]\in A_m(\nu)} \widetilde{\chi}(\gamma^m).$$

which we conclude is zero for every primitive length $\nu$. This does not immediately imply that the class function $\widetilde{\chi}$ is zero.

Indeed, for each fixed $\nu$, as we vary $\chi$ among the virtual characters, the function $\lambda_\nu : \chi \mapsto b(\nu, \chi)$ is actually a linear functional on the vectorspace $\mathrm{Vchar}(\Gamma' \backslash \Gamma)$. In order to conclude that $\chi = 0$, we must demonstrate that the (infinite!) set of functionals $\{\lambda_\nu : \nu \in \mathcal{N}_\Gamma\}$ spans the (finite dimensional!) dual space to $\mathrm{Vchar}(\Gamma' \backslash \Gamma)$.

#### 0.0.0.2 Conjugacy classes in $\mathrm{GL}(2, q)$ and $\mathrm{PGL}(2, q)$

Let $q$ be an odd prime power and set $G = \mathrm{GL}(2, q)$ and $\bar{G} = \mathrm{PGL}(2, q) = G/Z$ where $Z$ is the center of $G$. For an element $g \in G$, define the *characteristic polynomial* $p_g := x^2 - \mathrm{tr}(g)x + \det(g) \in F_q[x]$. Among non central conjugacy classes, the characteristic polynomial of an element completely determines its class:

**Lemma 1.** Suppose $g$ and $h$ are non central. Then $g$ and $h$ are conjugate in $G$ if and only if $p_g = p_h$

*Proof.* This follows from the theory of Jordan normal forms. □

Conjugation in $\bar{G}$ can be described in terms of conjugation in $G$:

**Lemma 2.** Elements $gZ$ and $hZ$ in $\bar{G}$ are conjugate if and only if $g$ is conjugate to $\lambda h$ in $G$ for some $\lambda \in Z$.

For an element $gZ$ of $\bar{G}$, let $\bar{p}_{gZ}$ denote the collection of characteristic polynomials of lifts of $gZ$ to $G$. That is, $\bar{p}_{gZ} = \{p_{\lambda g} : \lambda \in Z\}$. Combining the preceding lemmas, we obtain a characterization of nonidentity conjugacy classes in $\bar{G}$:

**Lemma 3.** Nonidentity elements $gZ$ and $hZ$ are conjugate in $\bar{G}$ if and only if $\bar{p}_{gZ} = \bar{p}_{hZ}$.

#### 0.0.0.3 Multiplicative independence for principal congruence covers of semi-arithmetic surfaces

Now suppose $\Gamma$ is a subgroup of $\mathrm{PGL}(2, \mathrm{O}_K)$ where $K$ is a totally real number field, and $\mathrm{O}_K$ is its ring of integers. We further suppose that under some embedding $K \to \mathbb{R}$, the image of $\Gamma$ is a lattice in $\mathrm{PGL}(2, \mathbb{R})$. For any prime ideal $\mathfrak{p}$ of $\mathrm{O}_K$, the reduction mod $\mathfrak{p}$ map $\mathrm{O}_K \to \mathbb{F}_\mathfrak{p} := \mathrm{O}_K / \mathfrak{p}\, \mathrm{O}_K$ induces a map $\mathrm{PGL}(2, O_K) \to G(\mathfrak{p}) := \mathrm{PGL}(2, \mathbb{F}_\mathfrak{p})$. Let $\pi_\mathfrak{p}$ be its restriction to $\Gamma$. For each $\mathfrak{p}$, define the *principal congruence subgroup* $\Gamma(\mathfrak{p}) := \ker \pi_\mathfrak{p}$ *of level* $\mathfrak{p}$. Let $S$ be the set of primes $\mathfrak{p}$ such that the map $\pi_\mathfrak{p}$ is surjective. Thus, when $\mathfrak{p} \in S$, $\pi_\mathfrak{p}$ induces an isomorphism of $\Gamma(\mathfrak{p}) \backslash \Gamma$ with $G(\mathfrak{p}) = \mathrm{PGL}(2, O_K/\mathfrak{p}O_K)$. If $H$ is one of the groups $\Gamma, \Gamma(\mathfrak{p})$ or $G(\mathfrak{p})$, let $\widetilde{H}$ denote its lift to $\mathrm{GL}(2, \mathbb{F}_\mathfrak{p})$.

**Theorem 1.** Suppose $\gamma \in \Gamma - \Gamma(\mathfrak{p})$ is hyperbolic and that the characteristic polynomial $p_\gamma$ is irreducible over $K$. If $\lambda \in \Gamma$ is such that $N(\gamma) = N(\lambda)$, then $\pi_\mathfrak{p}(\gamma)$ and $\pi_\mathfrak{p}(\lambda)$ are conjugate in $G(\mathfrak{p})$.

*Proof.* It suffices to show that $N(\gamma)$ completely determines the characteristic polynomial $p_\gamma$. To this end, observe that the characteristic polynomial $p_\gamma$ of $\gamma$ is also the minimal polynomial for the unit $N_\gamma$ in the quadratic extension $K_\gamma := K(N(\gamma))$ of $K$. Let $\sigma$ denote the nontrivial Galois automorphism of $K_\gamma/K$. Then $p_\gamma = (t - N(\gamma))(t - N(\gamma)^\sigma)$, as claimed. Specifically, the constant term $\det(\gamma)$ of $p_\gamma$ is the Galois norm $N(\gamma)N(\gamma)^\sigma$ of the quadratic unit $N(\gamma)$. □

**Corollary 1.** Suppose $\chi$ is the inflation of a class function from $G(\mathfrak{p})$ to $\Gamma$. Then $\chi$ is constant along each set $\mathcal{N}_\Gamma(\nu)$ such that $\nu \neq 1 \mod \mathfrak{p}$. If $\nu = 1 \mod \mathfrak{p}$, and $\mathcal{N}_\Gamma(\nu) \cap \Gamma(\mathfrak{p}) = \emptyset$, then $\chi$ is constant on $\mathcal{N}_\Gamma(\nu)$.

**Theorem 2.** Suppose $\Gamma$ is as above, and $\mathfrak{p} \in S$. Let $\chi$ be a class function on $\Gamma$ inflated from one on $G(\mathfrak{p})$. Then $\Lambda_\Gamma(s, \chi) = 0$ if and only if $\chi = 0$.

*Proof.* If $\Lambda_\Gamma(s, \chi) = 0$, then for all $\nu \in \mathcal{N}_\Gamma$, we have $b(\nu, \chi) = 0$.

First, suppose $\nu \neq 1 \mod \mathfrak{p}$. Then along the set $\mathcal{N}_\Gamma(\nu)$ of primitive hyperbolic $\gamma \in \Gamma$ with norm $\nu$, the function $\chi$ takes a common value which we denote $\chi(\nu)$. Let $a_m(\nu)$ denote the cardinality of $A_m(\nu)$, the set of primitive hyperbolic $\gamma \in \Gamma$ such that $N(\gamma^m) = \nu$. Then

$$b(\nu, \chi) = -\sum_{m=1}^{m(\nu)} \frac{a_m(\nu)}{m(1-\nu^{-m})}\chi(\nu) = 0$$

To conclude that $\chi(\nu) = 0$, it suffices to demonstrate the existence of a primitive hyperbolic $\gamma \in \Gamma$ such that $N(\gamma) = \nu \mod \mathfrak{p}$. This follows from the Cheboratev density theorem of (cite Sarnak):

**Theorem 3.** Let $C$ be a conjugacy class in $G(\mathfrak{p})$ and $F(T, C)$ denote the collection of primitive hyperbolic $\gamma$ in $\Gamma$ with $N(\gamma) < T$ such that $\pi_\mathfrak{p}(\gamma) \in C$. Then as $T \to \infty$,

$$F(T, C) = \frac{|C|}{|G(\mathfrak{p})|}\frac{T}{\log T} + o(1).$$

Thus, for any primitive hyperbolic $\gamma \in \Gamma$ with $N(\gamma) \neq 1 \mod \mathfrak{p}$, we have $\chi(\gamma) = 0$.

Now suppose $\nu = 1 \mod \mathfrak{p}$. Then if $\gamma \in \mathcal{N}_\Gamma(\nu)$, either $\pi_\mathfrak{p}(\gamma) \in I$ the identity conjugacy class, or $\pi_\mathfrak{p}(\gamma) \in P$ the unique parabolic conjugacy class in $G(\mathfrak{p})$. Note that $|I| = 1$, and $|P| = q^2 - 1$, where $q = |O_K/\mathfrak{p}O_K|$. By Sarnak's version of Chebotarev's density theorem, there are infinitely $\nu \in \mathcal{N}_\Gamma$ with $\nu = 1 \mod \mathfrak{p}$ such that $\mathcal{N}_\Gamma(\nu) \cap \Gamma(\mathfrak{p}) = \emptyset$. For any such $\nu$, $\chi$ is constant along $\mathcal{N}_\Gamma(\nu)$. As above, for such $\nu$, $b(\nu, \chi) = 0$ implies $\chi(\nu) = 0$.

□

## References