ZETA FUNCTIONS OF REAL QUADRATIC FIELDS AS PERIODS OF EISENSTEIN SERIES

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1. Introduction

Set $k = \mathbb{Q}(\sqrt{D})$ and $\mathcal{O}_k$ its the ring of integers. The zeta function attached to k is

$$\zeta_k(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_k} \frac{1}{|N(\mathfrak{a})|^{-s}}$$

where the sum is over nonzero integral ideals in $\mathcal{O}_k$ and $\mathrm{Re}(s) > 1$. A suitable modification of Riemann's argument for the continuation of $\zeta = \zeta_{\mathbb{Q}}$ shows that ζ_k has meromorphic continuation to the entire s-plane.

Let $G = \mathrm{SL}_2(\mathbb{R})$, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and P be the parabolic of upper triangular elements of $\mathrm{SL}_2(\mathbb{R})$. As usual, G acts on the upper half plane $\mathfrak{H}$ by fractional linear transformations. The for complex s with $\mathrm{Re}(s) > 1$, the s^{th} Eisenstein series on the upper half plane is

$$E_s(z) = \sum_{\gamma \in (P \cap \Gamma) \backslash \Gamma} (\mathrm{Im}(\gamma z))^s,$$

which is Γ-invariant by design. For fixed z, the map $s \mapsto E_s(z)$ has meromorphic continuation to the entire s-plane.

This writeup shows that ζ_k are integrals of Eisenstein series over closed geodesics.

2. Backstory

In setting the stage for real quadratic fields, let's quickly rehearse an argument on imaginary quadratic fields. In this section $k = \mathbb{Q}(\sqrt{D})$ with $D < 0$. The sum defining ζ_k decomposes into a double sum, first over classes of ideals, then over representatives within each class,

$$\zeta_k(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_k} \frac{1}{|N(\mathfrak{a})|^{-s}} = \sum_{\text{Classes } \mathfrak{a}} \sum_{\mathfrak{b} \approx \mathfrak{a}} \frac{1}{|N(\mathfrak{b})|^{-s}}.$$

Quadratic fields have finitely many distinct classes of ideals, so the outer sum is finite. The ideal norm is multiplicative....

3. Real quadratic

In this section $k = \mathbb{Q}(\sqrt{D})$ with $D < 0$. As a $\mathbb{Q}$ module, k is $\mathbb{Q}^2$. The multiplicative subgroup $k^\times$ acts transitively on $\mathbb{Q}^2$. Choose the basis $\{\sqrt{D}, 1\}$ and compute for $a+b\sqrt{D} \in k^\times$,

$$(a+b\sqrt{D}) \times \sqrt{D} = a \cdot \sqrt{D} + bD \cdot 1$$
$$(a+b\sqrt{D}) \times 1 = b \cdot \sqrt{D} + a \cdot 1.$$

In coordinates, we have an embedding

$$k^\times \to \mathrm{GL}_2(\mathbb{Q})$$
$$a+b\sqrt{D} \mapsto \begin{pmatrix} a & bD \\ b & a \end{pmatrix}$$

Let G' be the image of $k^\times$ in $\mathrm{GL}_2(\mathbb{Q})$. Note that the determinant of the image of $a+b\sqrt{D}$ is $a^2 - b^2 D = N(a+b\sqrt{D})$. The existence of nontrivial units in $\mathcal{O}_k$ implies that the subgroup $H' = G' \cap G$ is nontrivial. As a subgroup of $G = \mathrm{SL}_2(\mathbb{R})$, the group H' sensibly acts on the upper half-plane. Taking the trace of a generic matrix $Tr \begin{pmatrix} a & bD \\ b & a \end{pmatrix} = 2a$ and recalling that the units of $\mathcal{O}_k$ are integral shows that all nonidentity elements of H_1 are hyperbolic. As such, any nonidentity element of H' fixes two distinct points on $\mathbb{R} \cup \{\infty\}$.

Although H' is discrete in G, it lies in a one parameter subgroup H of G' defined by parameterizing the solutions of $a^2 - b^2 D = 1$ viz

$$H' \subset H = \{\begin{pmatrix} \cosh(t) & \sinh(t)\sqrt{D} \\ \sinh(t)/\sqrt{D} & \cosh(t) \end{pmatrix} : \quad t \in \mathbb{R}\}$$

In fact, $H' = H \cap \Gamma$. Denote an element of H by h_t. One can compute that each h_t fixes $-\sqrt{D}$ and $\sqrt{D}$. Consequently, H fixes the geodesic $mathcalC_{\sqrt{D}}$ running from $-\sqrt{D}$ to $\sqrt{D}$, set-wise. In particular, the radius of the semicircle defining $\mathcal{C}_{\sqrt{D}}$ is $\sqrt{D}$, so the point $i\sqrt{D} \in \mathcal{C}_{\sqrt{D}}$. Pointwise, each h_t translates a point rightward along $\mathcal{C}_{\sqrt{D}}$. The orbit of $i\sqrt{D}$ under the nontrivial discrete subgroup $H' = H \cap \Gamma$ partitions the orbit $H \cdot i\sqrt{D}$ into congruent intervals. The resulting quotient $H' \backslash H \cdot i\sqrt{D}$ is compact.