Lemma

MULTIPLICITY ONE FOR FIRST OCCURRENCES

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1 Quaternion Algebras, correspondences, Hecke Operators

Here's the story over for quaternion algebras over $\mathbb{Q}$: Let $a, b \in \mathbb{Z}$ be squarefree, and $a > 0$. Then $A = (a, b/\mathbb{Q})$ is the *indefinite quaternion algebra* defined as follows: There's a basis $\mathbb{Q}$ basis $1, \omega, \Omega, \omega\Omega$ satisfying the relations $\omega^2 = a$, $\Omega^2 = b$, and $\omega\Omega + \Omega\omega = 0$. Define an involution $\bar{\cdot}$ by negation on the basis elements which aren't 1 and extending $\mathbb{Q}$ linearly. Then define trace and norm as one would expect.

Assuming A is a division algebra is the same as assuming that $N(\alpha) = 0$ iff $\alpha = 0$. This is also the same as the corresponding quadratic form is anisotropic/$\mathbb{Q}$.

Fix an embedding φ of A into matrices by

$$\varphi(x_0 + x_1\omega + x_2\Omega + x_3\omega\Omega) = \begin{pmatrix} x_0 + x_1\sqrt{a} & x_2 + x_3\sqrt{a} \\ b(x_2 - x_3\sqrt{a}) & x_0 - x_1\sqrt{a} \end{pmatrix} \tag{1}$$

The image sits in $M_2(F)$, where $F = \mathbb{Q}(\sqrt{a})$ and extends to an isomorphism upon tensoring both sides with $\mathbb{R}$. Note that $\det(\varphi(\alpha)) = N(\alpha)$ and likewise for traces.

An order R in A is a subring containing 1, a free $\mathbb{Z}$ module of rank 4, with all of its elements having trace and norm in $\mathbb{Z}$.

For a fixed order R_o let R be a maximal order containing R. Apparently, since A is *indefinite* (i.e. is unramified at the real places?), R is unique up to conjugation in A. So(?) for a suitable integer D, we have $DR \subset R_o \subset R$.

For each $m \in \mathbb{Z}$ define $R(m)$ to be a elements of R with norm m. Note that $R(1)$ is the group of norm 1 elements of R, and that $R(1)$ naturally acts on $R(m)$ by multiplication (say, on the left). For some reason, for any fixed m, there are only finitely many orbits of $R(1)$ in $R(m)$ (this seems analogous to the action of $SL(n)$ on the level sets of det). Sarnak says there is an integer $q = q(R)$ which behaves something like a conductor: for any m coprime to q, the action of $R(1)$ on $R(m)$ is 'easy to describe'.

Set Γ_R to be the image of $R(1)$ under φ. Viewed as a subgroup of $SL_2(R)$ it is a cocompact lattice, hence gives rise to a compact surface X_R.

We define the 'modular correspondences' $\widetilde{T}_m : X_R \to X_R$ by $z \mapsto \varphi(R(1)\backslash R(m))z$.

This gives rise to an action on $L^2(X_R)$:

$$\sum_{\alpha \in R(1)\backslash R(m)} f(\varphi(\alpha)z)$$

Say an element $\alpha \in R$ is primitive if there is no *rational integer* t such that $\alpha/t \in R$. Then the set $R^{pr}(m)$ of primitive elements in $R(m)$ is invariant under $R(1)$, and so we can consider the correspondence C_m which is like T_m but with the union only over primitive orbits. We also use C_m for the corresponding operators. They are related to the T_m operators by

K has class number 1 this is probably not an issue. integers of K. Again, if K has class number one, this should be no problem.

- Maybe a better approach is through the commensurator. Is the commensurator of $R(1)$ parametrized by the $R(M)$?
- Set $\mathrm{Com}(\Gamma))$ = the set of $g \in SL(2,\mathbb{R})$ so that $g\Gamma g^-1 \cap \Gamma$ has finite index in $\Gamma and g\Gamma g^-1$. Then the integral Hecke ring $H_{\mathbb{Z}}(G,\Gamma)$ is the free abelian group on the double Γ cosets by the elements of the commensurator. Multiplication is nasty. If we tensor up by $\mathbb{C}$ to get $H(G,\Gamma)$ then it turns out to be isomorphic to compactly supported functions on $\Gamma\backslash \mathrm{Com}(\Gamma)/\Gamma$ with multiplication given by convolution. A natural basis is given by the characteristic functions of the double cosets.
- If ζ normalizes Γ, then the associated hecke operator (convolution w/r/t the characteristic function of $\Gamma\zeta\Gamma$) is actually *unitary*. Does this characterize normalizers?
- Solenoids seem like they could be an interesting avenue. Judge has (had?) a student who is thinking about laplacians on solenoids. Two interesting lemmata:1) fix a characteristic series $\Gamma_n \leq \Gamma$. Then $\alpha \in \mathrm{SL}(2,\mathbb{R})$ lives in the commensurator of Γ if and only if for each Γ_n there is a finite index subgroup K_n such that $\alpha K_n \alpha^{-1}$ is finite index in Γ_n. 2) For any α in the commensurator, there is an increasing subsequence of Γ_n 'preserved' under conjugation by α.
- This should have been obvious. The multiplicity of the λ eigenspace is bounded above by the volume of M times the ∞ norm of a unit eigenfunction.
- When γ normalizes Γ, the double coset $\Gamma\gamma\Gamma$ is really a single coset $\gamma^-1\Gamma$. This is the point...the things in the commensurator are close to normalizing. They're 'virtually normalizing?'
- The classical notion of 'multiplicity one' is that an eigenfunction is determined by all of its Hecke eigenvalues.
- A theorem of Ramakrishnan, potentially only in the case of maass forms on $\mathrm{SL}(2,\mathbb{Z})$, but probably generalizable: Let $a_f(p)$ denote the eigenvalue of the hecke operator T_p acting on an eigenfunction f. Then if for two eigenfunctions of the laplacian f,g (with the same eigenvalue) satisfy $a_f(p) = a_g(p)$ for 7/8 of the primes (with natural density), then $f = g$.
- try to remember Sarnak's cute trick for uniformly picking parametrization for isotropy subgroups in compacta.
- Set $\Gamma := \langle r,s,t | r^2 = s^2 = t^2 = (rs)^2 = (st)^3 = (rt)^7 = 1\rangle$, and Γ^+ its orientation preserving subgroup. Note that $\Gamma^+\backslash\mathfrak{H}$ admits a unique reflection, and that its fixed point set is the collection of 3 geodesic segments connecting the fixed points of the three generating rotations.
- A nice argument from Sarnak and pals: Let φ be a Neumann eigenfunction on $\Gamma\backslash\mathfrak{H}$ and β be geodesic segment along which φ vanishes. Then β can't be on the boundary, since its normal derivative already vanishes there. Now suppose β is not strictly contained in the boundary. Let B be extension of a lift of β to a complete geodesic in the upper halfplane, and let R_B denote reflection about that geodesic. Then by reflection principle, upon taking a lift of φ, we have $R_B(\varphi) = -\varphi$. So, in particular, φ^2 is invariant under the group Λ generated by Γ along with R_B. But Γ is a maximal lattice, so Λ is not a discrete group. Same for orientation preserving part. The closure of the orientation part thus contains a one parameter subgroup which fixed φ^2, which by connectedness means it also fixes φ. Thus *phi* is (up to a conjugate) invariant under $SO(2)$ or the diagonal. In either case, the quotient is one dimensional, so φ satisfies an ODE, and thus can't be Γ invariant.

2 forms as points or closed geodesics

Given a binary quadratic form $M(x, y) = ax^2 + bxy + cy^2$ with coefficients in $\mathbb{R}$, we associate the discriminant $d(M) = ac - b^2/4$ (this is the determinant of the symmetric matrix corresponding to M). For any M with $d(M) < 0$, there is a unique $z(M)$ in the upper half plane satisfying $M(z, 1) = 0$. The map $M \mapsto z(M)$ is an $\mathrm{SL}(2, \mathbb{R})$ equivariant map, where it acts by $M \mapsto g^\top Mg$ on matrices and $z \mapsto g^-1z$ on points. The correspondence $M \mapsto z(M)$ is ambiguous up to scaling M, but other than that is a bijection.

If $d(M) > 0$, then $z(M, 1)$ has two real roots. Consder $\gamma(M)$ the geodesic in $\mathfrak{H}$ connecting those roots. Then the map $M \mapsto \gamma(M)$ induces a g equivariant bijection between such forms (up to scale) and geodesics in $\mathfrak{H}$.

3 a *very* brief summary of the relationship between eigenfunctions on $\Gamma\backslash\mathfrak{H}$ and the decomp of $L^2(\Gamma\backslash G)$ (a la Lubotsky)

Here's the story over $\mathbb{R}$. Set $G = PSL(2, \mathbb{R})$ and $\Gamma \leq G$ a cocompact lattice. Then, as with any unitary rep of G, $L^2(\Gamma\backslash G)$ decomposes as a direct integral relative to a Plancharel-like-measure over its whole unitary dual. The unitary dual has all the familiar parts,the trivial rep, the principal series, the discrete series, the limit of discrete series, and the complementary series. Now the reps in occuring in $L^2(\Gamma\backslash G)$ will give rise to functions on the surface $\Gamma\backslash G/K$ precisely when they have a K invariant vector. It turns out that the only such reps are the principal and complementary series.

Aside: The discrete series do not have K fixed vectors, but *do* have vectors which transform according to a character of K i.e. sections of a line bundle on the quotient by K. These are the analogues of holomorphic modular forms on $\Gamma\backslash G/K$.

The principal series and complementary series have similar models: let P be the upper triangulars in G and $\chi_s : p \to \mathbb{C}^\times$ be a character defined by $\chi_s(\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]) = |ad^-1|^s$. Then induce this character up to G to get ρ_s, which acts on functions on G, left equivariant by P transforming according to χ_s. Relative to the L^2 norm on functions on G, this is unitary when $s = it$ is pure imaginary (note, different folks use different normalizations of induction to get different lines of unitarity). These are the principal series reps. Now, when $s \in (-1/2, 1/2)$, ρ_s is unitarizable, but one needs to take a different norm (not just the one on $L^2(G)$).

So, the ρ_{it} for $t \in \mathbb{R}$ and ρ_s for $s \in (-1/2, 1/2)$ are the only candidates for reps with K fixed vectors which occur in $L^2(\Gamma\backslash G)$. Most of them don't though. Since Γ is cocompact, the decomposition is discrete, so there is a discrete collection of t's and s's which actually do occur, and when they do, they occur with finite multiplicity. This verbiage should sound quite familiar.

The t's and s's which do occur are closely related to the eigenvalues of the Laplacian on $L^2(\Gamma\backslash\mathfrak{H})$. Here's the correspondence: for now, I'm not going to distinguish between principal series and complementary; I'll just write them both as ρ_s. Turns out that each ρ_s actually has a unique K invariant vector. (Prove that the convolution algebra of K bi-invariant functions on G a) acts on the space of K fixed vectors in ρ_s, b) commutes with ρ_s there, c) is itself commutative, and d)that ρ_s^K is irreducible for the action. From these, conclude that it's one dimensional) Call it u. Then form the matrix coefficient function $\varphi_s(g) = \langle \rho_s(g)u, u\rangle$. (note here that the inner product changes depending on where s is.) Now φ_s is a K bi-invariant function on G, so in particular descends to the quotient $G/K = \mathfrak{H}$. The Casimir operator, on the one hand, distinguishes irreducibles by its eigenvalues (which is $1/4 - s^2$ for both families of irreps), and on the other is G invaraiant and upon restricting to K invariant functions, acts as the Laplacian on the quotient. IOU...

4 more interesting, more unfamiliar: the p-adic analog

So most of the first three paragraphs carry thru verbatim to the case of $G = PGL_2(\mathbb{Q}_p)$. Some modifications, $K = PGL_2(\mathbb{Z}_p)$, and when we define the character on P, we need to use the p-adic norm. The principal series are the same, but the complementary series occur along the shifted intervals: $s + n\pi ni/\log p$ for $s \in (-1/2, 1/2)$ and $n \in \mathbb{Z}$.

As before, these are the only reps with K fixed vectors. Now let's look at $G/K = PGL_2(\mathbb{Q}_p)/PGL_2(\mathbb{Z}_p)$. Lubotsky says that this is a $p+1$ regular tree. Here's why: we're going to make G act transitively on a $p+1$ regular tree such that K is the stabilizer of a vertex.

Let V be a 2dim'l vs over $\mathbb{Q}_p$. A lattice in V is a rank 2 $\mathbb{Z}_p$ submodule which contains a $\mathbb{Q}_p$ basis (i.e. it's not a rank 2 submodule in a dumb way). Say two lattices L, L' are equivalent if there is an $\alpha \in \mathbb{Q}_p$ such that $L = \alpha L'$ i.e. L and L' are homothetic. Our vertices will be equivalence classes of lattices. Say two classes are adjacent if, for a representative of one (say) L, there is a represntative of the other (say) L' such that L has index p in L'. Call the resulting graph X

Now $GL_2(\mathbb{Q}_p)$ acts transitively on bases of V, hence on lattices. Its center preserves classes, so $PGL_2(\mathbb{Q}_p)$ acts on the vertices. Essentially by definition, the stabilzier of any particular lattice is $PGL_2(\mathbb{Z}_p)$, so we're good on the vertices. Lubotsky says that preservation of adjacency is obvious, and I'm inclined to believe him.

Thus X has a vertex transitive automorphism group, so is k regular for some k. To figure out k, we need to know how many neighbors a class has. For a fixed lattice L, the neighbors come from picking a line in $L/pL = F_p \times F_p$ of which there are $p+1$.

Here's how to parametrize the neighbors (to a given lattice) by said lines (equivalently by points in projective space). For each finite point, set $A_i = \left[\begin{smallmatrix} p & i \\ 0 & i \end{smallmatrix}\right]$ and at infinity $A_\infty = \left[\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right]$. Then the neighbors to L are $A_1 L, ..., A_\infty L$.

Stepping back for a moment, lets compare our picture of $\mathfrak{H}$ as G/K with this new picture of X as G/K. In the upper half plane, K is the stabilizer of i, and it spins the plane about i; acting freely and transitively on the geodesic circles about i. Thus, the orbits 'laminate' the plane as concentric geodesic circles about i. In particular, a function on G/K which is *also* left K invariant can only care about the distance of a point from i. That is, it's a function of $d(z, i)$ alone. The picture for X is actually pretty similar: K is the stabilizer of a point x in the tree, and acts freely and transitively on the set of points exactly n edges away from x, i.e. the geodesic circle about x. Likewise, a K bi-invariant function on X is thus a function of $d(z, x)$ alone.

Really, a double coset should be read as the K orbit of a point zK in X, i.e. the geodesic circle of radius $d(z, x)$ from x.

Easy examples of such functions: characteristic functions of geodesic circles about x. e.g. $\overline{\delta}$ the characteristic function of $K \left[\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right] K$. Note that $A_0 x = \left[\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right] x$ is 1 away from x for any x.

Convolving a function f on X with $\overline{\delta}$ gives $\delta(f)(y) = \int_G f(yg^-1)\overline{\delta}(g)\,\mathrm{d}g$ then, using right K invariance of f and δ, this is $\int_{G/K} f(yg^-1K)\overline{\delta}(gK)\,\mathrm{d}gK$ and $\overline{delta}(gK)$ is zero unless gK is a neighbor to X so the integral is the finite sum $\sum f(y)$ along the $p+1$ neighbors to x.

5 notes on Jacquet-Langlands

- F global field
- M' quaternion algebra over F
- G' multiplicative group of M'
- M two by two matrices
- $G = GL_2$
- Local Hecke algebras:
  - Let k be a nonarchimedean local field, G a group.
  - $\mathcal{H}_k$ is convolution algebra of measures coming from locally constant compactly supported functions on $G(k)$.
  - Here's how to make a idempotent element of $\mathcal{H}_k$. Given a finite set of inequivalent f.d. irreps of $G(O_k)$, say $\pi_1, ...\pi_r$. Set $\xi_i(g) = \dim(\pi_i)\operatorname{tr}(\pi_i)(g^-1)$ for $g \in G(O_k)$ and extend to a function on $G(k)$ by zero. Then $\xi = \sum \xi_i$ is idempotent. Such ξ are called elementary idempotent. Seems like they are projections onto the $\oplus\pi_i$ isotypic component?
  - IOU archimedean
- Global Hecke algebras:
  - Let F be global, $\mathbb{A}$ the adeles of F.
  - For each place v of F, set $M'_v = M'_{F_v} = M' \otimes_F F_v$.
  - We say M'_v is split if there is an isomorphism $\theta_v : M'_v \to M(2, F_v)$. All but finitely many M'_v are split.
  - Fix a basis B of M over F and let L_v be the O_v submodule of M_v generated by B. We fix a θ_v for each v over which M' is split. We may chose these θ_v so that $\theta_v(L_v) = M(2, O_v)$ (possible subtlty here). Such choices are determined up to a globally inner automorphism.
  - Over each split place v define a maximal compact subgroup $K'_v \leq G'_v$ as the preimage of $K_v = G(O_v)$ under θ_v.
  - Over the not split places, set $K'_v = \{x \in M'_v | |v(x)|_v = 1\}$. It's compact.
  - Over the split places, push $\mathcal{H}_v$ forward through θ_v to define $\mathcal{H}'_v$.
  - Over the nonarchimidean not split places, define $\mathcal{H}'_v$ as the (convolution) algebra of measures defined by locally constant compactly supported functions on G'_v. Over the archimidean nonsplit places, define $\mathcal{H}_v$ as the (direct?) sum of (algebra of measures defined by left-and-right K'_v finite smooth compactly supported functions on G'_v) and (the algebra of measures defined by functions on K'_v spanned by matrix coefficient functions of f.d. reps of K'_v).
  - Let ε_v and ε'_v be normalized Haar measures on K_v and K'_v. (viewed as functions, these are indicators on K_v and K'_v). Then ε_v is an elementary idempotent of $\mathcal{H}_v$ and likewise with apostrophes. Define $\mathcal{H}$ and $\mathcal{H}'$ as the restricted direct product of their respective local components.
  - Let S be the set of places at which M'_v does not split. Define $\mathcal{H}_S$ as the product over nonsplit components, and $\hat{\mathcal{H}}_S$ as the product over the split components. Likewise for $\mathcal{H}'S$ and $\hat{\mathcal{H}}'_S$. Clearly $\hat{\mathcal{H}}_S$ and $\hat{\mathcal{H}}'_S$ are isomorphic (with isomorphism given by the θ_v).
  - Make same definitions for G, G'.

  - Set $\theta = \otimes\theta_v$, an isomorphism $\hat{G'_S} \to \hat{G_S}$.
  - Elements of $\mathcal{H}$ (resp $\mathcal{H}'$) can be thought of as measures on $G_{\mathbb{A}}$ (resp $G'_{\mathbb{A}}$). That is, elements of $\mathcal{H}$ are finite linear combinations of product measures. Let $\mathcal{H}_1$ be the subalgebra of measures coming from functions.
  - Admissibility: Consider a rep π of $\mathcal{H}$ on V. Say π is admissible if
    * every $w \in V$ is a linear combination of the form $\sum \pi(f_i)w_i$ for some f_i's in $\mathcal{H}_1$. (find a way to interpret this)
    * If ξ is elementary idempotent, then range of $\pi(\xi)$ is finite dimensional (seems like one should think of idempotents, or at least elementary idempotents, as projection operators. So this requirement is that they project onto f.d. subspaces)
    * Let v_o be archimedean, and let ξ be an elementary idempotent which is ε_v at almost all of its factors. Then the map $f_{v_o} \to \pi(f_{v_o} \otimes \bigotimes_{v \neq v_o} \xi_v)w$ is continuous.
  - Given π_v admissible reps of $\mathcal{H}_v$ on V_v such that $\pi_v(\varepsilon_v)$ has nonzero range for all but finitely many v (i.e. V has K_v fixed vectors for all but finitely many v), pick for each such v a vector e_v fixed by $\pi(\varepsilon_v)$. Relative to these e_v, form the restricted product $\otimes V_v$ and rep $\pi = \otimes\pi_v$. Then π is admissible. Reps equivalent to ones of this form are called factorizable.
  - Admissible irreps are factorizable, with factors unique up to equivalence.
- ...
- A continuous function on $G'_F \backslash G'_{\mathbb{A}}$ is an automorphic form if for every elementary idempotent ξ in $\mathcal{H}'$, the space $\{\rho(\xi f)\varphi | f \in \mathcal{H}'\}$ is finite dimensional. That is, the ξ isotypic component of the representation $\rho(\mathcal{H}')\varphi$ generated by φ is finite dimensional.
- Let A' be the space of automorphic forms on $G'_{\mathbb{A}}$.
- For η a not-necessarily-unitary (aka quasi) character of $F^\times \backslash \mathbb{I}$, let $A'(\eta)$ be the η isotypic component (under the action of $Z'_{\mathbb{A}}$) of A'.
- Some facts:
- 1) if an admissible irrep π of $\mathcal{H}'$ lives in A', then it lives in some $A'(\eta)$.
- 2) Each $A'(\eta)$ is a direct sum of admissible $\mathcal{H}'$ irreps with finite multiplicity. (compactness is key here)
- (recall) For $\pi = \otimes_v \pi_v$ admirrep of $\mathcal{H}'$, then each π_v is an admirrep of $\mathcal{H}'_v$.

6 things I really ought to have known

- Let D be an indefinite division algebra over F, and π an irreducible automorphic representation of $D^\times(\mathbb{A})$ with unitary central character. Then π is unitary and cuspidal (! there is no constant term along which π needs to vanish.) By multiplicity one, there is a unique subspace of $L^2(D^\times(F) \backslash D^\times(\mathbb{A}))$ and a unique map to it which intertwines π and right translation.
- Multiplicity of a rep ρ can be replaced (recast?) by occurance of a product $\rho \otimes 1_G^{m(\rho)}$.
- what the fuck? The expected gap between consecutive eigenvalues is asymptotically $1/t$
- Let $\rho : G \to GL(W)$ be a rep. Suppose $W = W_1 \oplus ... \oplus W_m$ is a decomposition into subspaces which are permuted transitively by $\rho(G)$. Then $W \approx \mathrm{Ind}_H^G W_1$ where H is the stabilizer of W_1.
- If W_1 is one dimensional, then it is spanned by a vector with orbit of size $\dim(W)$ under the action of $\rho(G)$. Going backwards, if we have a one dimensional rep of a subgroup occuring in W, can we find a basis of W containing it, such that $\rho(G)$ permutes the basis?
- (this is not quite right) Let A be any abelian subgroup of G, and ρ an irrep of G on V. Then *rho* restricts to a rep of A, which decomposes as a direct sum of character's. Equivalently,

there is a basis of V consisting of common eigenvectors of A. Picking any one, say v, of those eigenvectors gives rise to a character on A. Since ρ is irreducible, $\mathbb{C}[G]v = V$. Setting $W_1 = \mathbb{C}v$ and $W_i = \mathbb{C}\rho(g_i)v$ for a collection of g_i's such that the collection of $\rho(g_i)v$ forms a basis, we are in the situation of the prop. Note, the subgroup H from which we induce may be bigger than the A that we started with, for example if A is properly contained in its centralizer. This seems to be the only possible obstruction, meaning for any irreducible ρ of G and any abelian A with centralizer D, there is a character χ on D such that $\rho = \mathrm{Ind}_D^G \chi$. Note that maximal abelian subgroups are self centralizing.

- This theorem emphatically does *not* say that any χ on D will induce an irrep on G. The χ is dictated by A and ρ.
  - This could be very good for us. For a (full) hurwitz surface with isometry group G generated by r, s, t with irrep ρ on V. Since r, s commute, there's a basis of common eigenvectors. Pick a v from that basis. The the span of the $\rho(G)$ orbit of v is V, since ρ is irreducible. Thus, setting $W_1 = \mathbb{C}v$ puts us in the scenario of the above proposition. That is, with $H = \{g \in G : \rho(g)v \in \mathbb{C}v\}$ and $\chi_v : H \to \mathbb{C}^\times$ the representation given by the action of H on v, we have $\rho = \mathrm{Ind}_H^G(\chi_v)$. Note that, by our choice of v, at the very least $\langle r, s\rangle \leq H$.
- Interesting thm in Mackey, attributed to Wigner: Say a finite group G is 'simply reducible' if a) every conjugacy class is 'self inverse' (?? maybe meaning every element is conjugate to its inverse?) and b) for every two irreducibles ρ and σ of G, the tensor product $\rho \otimes \sigma$ is multiplicity free. Here's another characterization:
  - For each $x \in G$, let $v(x)$ denote the number of elements of G which commute with x, and $\zeta(x)$ denote the number of solutions of $y^2 = x$.
  - Then G is simply reducible if and only if $\sum v(x)^2 = \sum \zeta(x)^3$.
- More generally, for any subgroup $H \leq G$, the corresponding permutation representation is multiplicity free (+ with intertwining operators between irr components π, π^* symmetric) iff every H double coset in G is self inverse.
- For M a quaternion algebra over F, with r_1 split real places, r_2 ramified real places, and c complex places, $M \otimes_{\mathbb{Q}} \mathbb{R} = M_2(\mathbb{R})^{r_1} \times H^{r_2} \times M_2(\mathbb{C})^c$ where H is the unique division algebra (hamiltons' quaternions) over $\mathbb{R}$.
- Here's the relationship between lattices in G_∞ and $G_{\mathbb{A}}$. Let $K = K_\infty \times K_f \leq G_{\mathbb{A}}$ with K_f open compact in $G_{\mathbb{A}_f}$ and K compact in G_∞. Define Γ_K as the projection of $G_\infty K \cap G(F)$ to G_∞. Then Γ_K is a lattice in G_∞. Question: given a lattice Γ in G_∞, under what condition does there exist such a compact K so that $\Gamma = \Gamma_K$? Is this arithmeticity? Commensurability to $\Gamma_{K_{max}}$?
- My situation: G is group of norm 1 elts of Hurwitz quaternion algebra (the unique division algebra over $F = \mathbb{Q}(\eta)$, the cubic totally real subfield of 7th cyclotomic which is ramified at exactly two infinite place). Then $G_\infty = G(F \otimes \mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ (I think), and $K_{max} = \mathrm{SO}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \prod_{v<\infty} SL_2(O_v) = K_\infty \times K_f$ and $\Gamma_{K_{max}}$ is the $(2, 3, 7)$ triangle group. Also $\Gamma_{K_{max}}$ is the norm 1 elements in the unique maximal order of G.
- Apparently, the fact that this setup gives a cocompact lattice is attributed to Kate Hey (1929)
- Kinda neat: let H, H' be finite index in G, and $\mathbb{R}$ a commutative ring. The map $\Phi(H, H')$: $R[H\backslash G/H'] \to \hom_{\mathbb{R}[G]}(R[G/H], R[G/H'])$ defined by $f \mapsto (Hg \mapsto \sum_{H'g' \in H'\backslash G} f(Hgg'^-1H')H'g')$ is an R module isomorphism.
- Here's how Bruhat-Tits tells you how to make a lattice. (Notation: let k be local nonarch with uniformizer π_k, define the distance $d(R_1, R_2)$ between two maximal orders of $M(2, k)$ as the integer n such that $R_1/R_1 \cap R_2 = O_k/\pi_k^n O_k$. Then the tree T_k has vertices maximal orders in $M(2, k)$, with two vertices adjacent if they are distance 1 from eachother.) Now $M^\times/F^\times$ acts on $T_\nu = T_{F_v}$ via conjugation. Now pick a finite set S of finite primes of F

away from the ramified primes of M and pick a maximal order R of $M^\times$. Then define $\Gamma_{R,S}$ to be those $x \in M^\times/F^\times$ such that x fixes R_v away from S and fixes an edge of the tree T_v when $v \in S$. Then this is a maximal arithmetic subgroup of $M^\times/F^\times$, and any maximal arithmetic subgroup arises this way.

- (from Farb and Weinberger) If M is compact locally symmetric of noncompact type, w/ no torus factors, with locally symmetric metric h_{loc}. Then for any other metric h on M, isom(M,h) is isomorphic to a subgroup of isom(M,h_{loc})). NOTE: this does not say, nor is it true, that the action of the former is topologically that of the latter. This says nothing about the action, just the group structure. Still, this could be very useful for perturbation.
- Farb has a good notion of hidden symmetry: say an isometry of a cover of M is a hidden symmetry, if it's not a lift of an isometry of M. If we have a covering $M' \to M$ with corresponding inclusion of fundamental groups $\Gamma' \mapsto \Gamma$, let Λ' and Λ denote their respective normalizers in the isometry group of the universal cover of M. Set $G = \Gamma/\Gamma'$, a finite group, which acts on M' via deck transformations of the cover $M' \to M$. I think this is all about whether Λ/Λ' can be bigger than G. In case Γ' is a maximal lattice it is self normalizing, i.e. $\Lambda' = \Gamma'$. So hidden symmetries come from Γ having atypically large normalizer.
- Solid philosophy: "Uniqueness implies invariance"
- Fucking piss, farb above needs $n > 2$:()
- L-indistinguishability: Let π_1 and π_2 be two reps of $G = \mathrm{SL}_n(F)$, F local. Then π_1 and π_2 are L indistinguishable if $\pi_1^g = \pi_2$ for some $g \in \widetilde{G} = \mathrm{GL}_n(F)$. Here $\pi^g(x) = \pi(x^g)$. Equivalently, π_1 and π_2 both occur in the restriction of a rep from $\widetilde{G}$ to G.
- $\mathrm{PSL}(2,q)$ is a hurwitz group when $q = 7$, when q is a prime ± 1 mod 7, and when $q = p^3$ with p prime ± 2 or ± 3 mod 7.

7 Notes on snaith's proof of Brauer's induction theorem

- G compact not nec connected. X is a closed manifold on which G acts smoothly.
- For a closed subgroup H of G let (H) denote its conjugacy class. Then set $X_{(H)}$ as the set of points with stabilizer in (H).
- Set $M = G\backslash X$. Then $M_{(H)}$ are the points with orbit isomorphic to G/H (which is dependent only on (H))
- The structure of isotopy type gives rise to a stratification of a CW-complex on M. There is an ordering on conjugacy classes (H) of subgrps given by containment (of closure) of $M_{(H)}$. (note, the containment is reversed: $\overline{M_{(H)}} \subseteq \overline{M_{(J)}}$ iff $(J) \leq (H)$.
- Set $\chi^{\#}_{(J)}$ the Euler characteristic of $M_{(J)}$ (with respect to compactly supported cohomology with $\mathbb{C}$ coefficients.) Then $\chi^{\#}_{(H)} = \chi(\overline{M_{(H)}}) - \chi(\overline{M_{(H)}} - \chi(M_{(H)})$, where the RHS is singular cohomology.
- Topological interpretation of $\chi^{\#}_{(H)}$: Suppose $f : M \to M$ is a map homotopic to the identity by a homotopy which preserves the isotropy stratification (so called isovariant) with isolated fixed points. Then the restriction of f to $M_{(H)}$ is homotopic to the identity so $\chi^{\#}_{(H)}$ is given by the sum of the local fixed point indices of f.
- Such an f always exists.
- We can lift the homotopy from identity to f on M to homotopy on X from identity to a G map $F : X \to X$. Then above any point on $M_{(H)}$ fixed by f, there is an orbit (isomorphic to G/H) in X which is pointwise fixed under F.
- Nonstandard terminology: Say $S \leq G$ is a cartan subgroup of G if it is topologically cyclic and finite index in its normalizer. Then S is a product of a torus and finite cyclic group.

- Say g is regular if it generates a cartan (i.e. if the closure of the cyclic group it generates is a cartan as above)
- Regular elements are dense, and for their action on any homogeneous space G/H, their fixed points are isolated.
- Set $R_{.} = \hom(R(G), \mathbb{Z})$, where $R(G)$ is the complex representation ring of G, which in turn is the ring generated by characters of irreps. So an element of $R_{.}(G)$ assigns an integer to each character.
- Any inclusion $i : H \to G$ induces an inclusion $(i_H^G)_* : R_{.}(H) \to R_{.}(G)$.
- T is the complexification of the tangent space of G/H at the identity coset. Since H fixes the identity coset, T affords a representation of H. Set $\lambda_{G/H} = \sum(-1)^i \Lambda^i T$, a representation of H, equivalently an element of $R(H)$.
- Identify $R(G)$ with functions in $R_{.}(G)$ with finite support (we must be thinking of elements of $R_{.}(G)$ as picking out which reps occur in an element of $R(G)$.)
- Then $(i_H^G)_*(R(H)\lambda_{G/H})$ lies in $R(G)$ (i.e. is a finitely supported function in $R_{.}(G)$ (i.e. only finitely many irreps occur in it)).
- Define $\mathrm{Ind}_H^G : R(H) \to R(G)$ so that $\mathrm{Ind}_H^G(M) = (i_H^G)_*(M\lambda_{G/H})$ for $M \in R(H)$.

7.1 Alternative definition using elliptic complex on G/H:

- Let Ω^i denote the ($\mathbb{C}$ vectorspace) of i-fields(?) on G/H. Then exterior differentiation makes the complex $0 \to \Omega^0 \to \Omega^1 \to ...$ into a chain complex with ith homology $H^i(G/H : \mathbb{C})$.
- Define the bundle $E_M = G \times_H M \to G/H$ and choose a G-invariant connection on it (??).
- Let $\Gamma(\cdot)$ denote the differentiable sections of a bundle $\cdot$.
- Apparently, the connection is what provides the maps $d_M : \Gamma(E_M \otimes \Lambda^p T^*) \to \Gamma(E_M \otimes \Lambda^{p+1} T^*)$.
- Apparently choosing a G invariant metric on G/H is what allows us to form the adjoint d_M^*.
- Set $D_M = d_M + d_M^* : \bigoplus \Gamma(E_M \otimes \Lambda^{2k} T^*) \to \Gamma(E_M \otimes \Lambda^{2k+1} T^*)$. It's a G invariant elliptic operator, and apparently (!!) $\mathrm{Ind}_H^G(M)$ is the representation of G on the space $\ker(D_M) - coker(D_M)$.

7.2 properties of induction

- Ind_H^G is an $R(G)$ module homomorphism.
- $\mathrm{Ind}_H^G(1) = \sum_j (-1)^i H^j(G/H, \mathbb{C}) \in R(G)$ (!!?!) by the de Rham theorem
- The value of the character of $\mathrm{Ind}_H^G(M)$ at a regular element g is given by $\sum M(x^-1gx)$ where the sum is over xH such that $gxH = xH$. Since $M \in R(G)$, it is identified as function with finite support, so this sum is finite. Note that regular elements are dense, so this formula determines Ind_H^G for all $g \in G$.

7.3 the main theorem

- Here it is: Let G be a compact lie group, with closed subgroups J and H. Let J act on G/H by left translation, and set $M = J\backslash G/H$. Then the composition $R(H) \to R(G) \to R(J)$ obtained by inducing from H to G then restricting to J is given by the formula

$$\mathrm{Res}_J^G(\mathrm{Ind}_H^G(M)) = \sum_{x \in M_(V)} \chi_{(V)}^{\#} \mathrm{Ind}_{J\cap xHx^{-1}}^J(x^* \mathrm{Res}_{x^{-1}Jx\cap H}^H(M)))$$

where (V) runs over conjugacy classes of subgroups of J and for each (V), x is a representative of its stratum.

- Let X_n denote any one of the simple $n \times n$ matrix groups over $\mathbb{C}$. Let Y_n denote the normalizer of the diagonals in X_n, so Y_n is the wreath product of the diagonal along with a permutation group. For a compact G with homomorphism $vG \to X_n$, let G act on $X = X_n/Y_n$ via v. Let M denote the orbit space. Then:
  (1) Then in $R(G)$,
  $$1 = \sum \chi^{\#}_{(V)} \operatorname{Ind}_H^G(1)$$
  where (V) runs over conjugacy claasses of subgroups of G. Here, H is the collection of $h \in G$ with $\pi v(h)(1) = 1$, where π is the projection of Y_n onto the permutation part,
  (2)

8 notes on L-indistingusihability, by langlands and labesse

- General philosophy: there should be a (rough) correspondence between conjugacy classes of a group G and irreducible representations of G. (There is an explicit bijection for symmetric groups. For finite groups, they at least have the same) cardinality.
- When G is a reductive algebraic group over a field F, say two regular semisimple elements in $G(F)$ are stably conjugate if there is a $g \in G(\overline{F})$ which conjugates one to the other. Note that honest $G(F)$ conjugate elements are stably conjugate.
- Stable conjugacy gives rise to a partition of the set of conjugacy classes of $G(F)$. Finite sets? In any case, passing this notion through the general philosophy in the first bullet, there should be a corresponding partition of the set of irreducible reps of $G(F)$. These are L packets. Turns out they have the same L functions.
- Apparently, if G is $\mathrm{GL}(n)$ then conjugacy and stable conjugacy coincide.
- Say two elements of $\mathrm{SL}(2, F)$ are stably conjugate if they lie in the same orbit under the action of $\mathrm{GL}(2, F)$ (acting by conjugation $h \mapsto h^g = g^{-1}hg$). (does this coincide with the discussion above?)
- Say two irreps of $\mathrm{SL}(2, F)$ are L-indistinguishable if they lie in the same $\mathrm{GL}(2, F)$ orbit (acting by pulling back along conjugation: $\pi \mapsto \pi^g$ where $\pi^g(h) = \pi(h^g)$). L.L. emphasize that this really should be stated in terms of reps of the Hecke algebra.
- Here's their motivating question: Suppose F is global, and $\pi = \otimes \pi_v$ is an automorphic rep. For a finite set of places, pick a π'_v which is L-indistinguishable from π_v and not equivalent to it. Then form the rep $\otimes_v \pi_v \otimes \otimes \pi'_v$. Is this rep automorphic? The answer is yes, with the only possible exception being the following: if π comes from the characters of the group of ideles of norm one in a quadratic extension.
- If G is $\mathrm{SL}(2)$ or a twisted form thereof, the ${}^LG$ is $PGL(2, \mathbb{C})$.

8.1 local theory

- F is a local field of char zero, and $G = \mathrm{SL}(2)$. T is a cartan (meaning?) of G defined over F.
- Some definitions:
- $\mathfrak{A}(T)$ is the set of g in $G(\overline{F})$ for which $g^{-1}Tg$ and the map $t \mapsto g^{-1}tg$ are both defined over F. Equivalently $g \in \mathfrak{A}(T)$ iff $a_\sigma = \sigma(g)g^{-1}$ lives in $T(\overline{F})$ for all $\sigma \in Gal(\overline{F}/F)$. As σ runs over the galois group, the a_σ form a cohomology class in $H^1(F, T)$.
- Set $\mathfrak{D}(T) = T(\overline{F}) \backslash \mathfrak{A}(T) / G(F)$.
- Then the map $g \mapsto \{a_\sigma\}$ (the cohomology class defined above) gives an inclusion $\mathfrak{D}(T) \to H^1(F, T)$ (remember, H^1 measures whether things 'with norm 1' must come from things of the form $x/\sigma x$. Norm in this case must mean the product over all galois conjugates.).

- For some reason, its image is the kernel (meaning?) of $H^1(F,T)$. The g in $\mathfrak{A}(T)$ are the things which give rise to stable conjugacy.
- $\mathfrak{D}(T)$ parametrizes the $G(F)$ conjugacy classes within the stable conjugacy class of T (which is a union of $G(F)$ conjugacy classes.(in what sense?)
- G is simply connected, so $H^1(F,G)$ is trivial (i.e. if $g \in G(F)$ satisfies $\prod_\sigma g^\sigma = 1$ then there is some $h \in G(F)$ such that $g = h * (h^-1)^\sigma$).
- In our scenario: $\mathfrak{D}(T) = H^1(F,T)$. (so the $G(F)$ conjugacy classes of $T(F)$ that make up the stable conjugacy class of $T(F)$ are parametrized by $H^1(F,T)$.)
- Let $\widetilde{G}$ be GL(2).
- The centralizer $\widetilde{T}$ of T in $\widetilde{G}$ is a cartan there. By (maybe?)hilbert 90 $H^1(F,\widetilde{T}) = 1$.
- So (why?) any $g \in \mathfrak{A}$ is of the form sh with $s \in \widetilde{T}(\overline{F})$ and $h \in \widetilde{G}(F)$. The point is that any g which does the stable conjugation of T can have all of its non-torus-ness factored out into an F-point of an extended group (i.e. $GL(2)$) and an element of $\widetilde{T}(\overline{F})$.
-

9 questions

- Let G be a reductive group over a number field F with adeles $\mathbb{A}$. Under what conditions can we guarantee that $G(\mathbb{Q})$ is self normalizing (i.e. is its own normalizer) in $G(\mathbb{A})$?
- With notation from tirothk, if $\Gamma \triangleright \Gamma'$, and for some K, K' we have $\Gamma = \Gamma_K$ and $\Gamma' = \Gamma_{K'}$ then do we have $K \triangleright K'$? If so, do we have $K'/K = \Gamma'/\Gamma$?

10 Notes on Analytic continuation of representations and estimates of automorphic forms Bernstein and Reznikov

- G a lie group, (π, G, V) a continuous rep of G on a TVS V. Say a vector v in V is analytic if the function $\xi_v : g \mapsto \pi(g)v$ is an analytic function on G (taking values in V). The complexification $G_\mathbb{C}$ of G is designed to do the following: for such an analytic function ξ_v, there is a neighborhood $U = U_v$ of G in $G_\mathbb{C}$ such that ξ_v extends to a holomorphic function on U. This allows us to make sense of $\pi(g)v$ for g in the neighborhood U of G. *Note, in case the rep π arises as the action of g on (say) $L^2(X)$ (or one of its subspaces) for some space X, this gives rise to operators which do not come merely from translation.*
- Now set $G = \mathrm{SL}_2(\mathbb{R})$ so $G_\mathbb{C} = \mathrm{SL}_2(\mathbb{C})$. We'll deal with principal series reps (so called 'typical'), and use the following model: For $\lambda \in \mathbb{C}$ define D_λ as the space of smooth homogeneous functions of degree $\lambda - 1$ on $\mathbb{R}^2 \setminus 0$. That is, $\varphi \in D_\lambda$ satisfies $\varphi(ax, ay) = |a|^{\lambda-1}\varphi(x,y)$. Then G acts on D_λ through its action on $\mathbb{R}^2$.
- Restriction to the unit circle identifies D_λ with even smooth functions on S^1. Through this identification, we have a 'basis' of vectors $e_k = \exp(2ik\theta)$.
- When $\lambda = it$, then π_λ is unitary with the norm coming from that on $L^2(S^1)$.
- Let $v \in D_\lambda$ be the unique function in D_λ which is identically 1 on S^1. Thus, v is represented by the function $(x^2+y^2)^{\frac{\lambda-1}{2}}$.
- For $a > 0$, let g_a be $\mathrm{diag}(a, a^-1)$. Then $\xi_v(g_a) = \pi_\lambda(g_a)v = (a^2x^2 + a^{-2}y^2)^{\frac{\lambda-1}{2}}$. The RHS makes sense for any complex a with $|\arg(a)| < \pi/4$. Define $I \subset G_\mathbb{C}$ as those such g_a. Then set $U = \mathrm{SL}_2(\mathbb{R}) \cdot I \cdot K_\mathbb{C}$ where $K_\mathbb{C}$ is $\mathrm{SO}_2(\mathbb{C}) = C^\times$. Then ξ_v extends analytically to U.

10.1 reznikov bernstein perspective on eigenfunctions and reps

- $\Gamma \leq G$ is a lattice, $X = \Gamma\backslash G$ and $Y = \Gamma\backslash G/K = X/K$.
- For an eigenfunction φ, let $L_\varphi \subset L^2(X)$ be the rep of G generated by φ (i.e. the closure of the span of right translates of φ viewed as a function on G.) Then $(\pi, L) = (\pi_\varphi, L_\varphi)$ is a unitary irrep of G.
- Conversely, for a irreducible unitary (π, L) of G with K fixed unit vector $v_o \in L$, any G map $\nu : L \to L^2(X)$ defines an eigenfunction $\varphi = \nu(v_o)$.
- Thus, eigenfunctions are in bijective correspondence with tuples (π, L, v_o, ν).

11 notes on jacquet gelbart

11.1 division algebras

- F is a number field, H a division algebra of degree p^2 over F, G its units viewed as a group over F, Z its center which is isomorphic to $F^\times$. Set $\overline{G} = G/Z$.
- For a character ω of $F^\times\backslash\mathbb{A}^\times$ let $L^2(\omega, G)$ denote the ω isotypic component of $L^2(G(F)\backslash G(\mathbb{A}))$ viewed as a rep of $Z(\mathbb{A})$. Let ρ_ω denote the action of $G(\mathbb{A})$ on the former.
- Pick an F basis $\{c_1, ..., c_{p^2}\}$ for H. Then for ae v, the R_v (ring of integers of F_v) span of this basis is a maximal order O_v, hence for such v, $O_v^\times$ is a maximal compact subgroup K_v of $H_v^\times = G(F_v)$.
- Suppose φ is a complex valued function on $G(\mathbb{A})$ (not nec automorphic) which transforms according to ω^{-1}. Further, suppose that it's of the form $\varphi(g) = \prod \varphi_v(g_v)$ where for each v, φ_v is smooth compactly supported satisfying $\varphi_v(z_v g_v) = \omega_v^{-1}(z)\varphi_v(g_v)$ for $z_v \in Z(F_v)$. Assume further that for almost all v, φ_v is supported on $K_v Z_v$. Then φ defines an integral operator $\rho_\omega(\varphi) := \int_{G(\mathbb{A})} \varphi(g)\rho_\omega(g)\,\mathrm{d}g$ on $L^2(\overline{G}(F)\backslash\overline{G}(\mathbb{A}))$. Its kernel is $K(x, y) = \varphi(x^{-1}\gamma y)$ where the sum runs over $\overline{G}(F)$.
- Thm: $\rho_\omega(\varphi)$ is of trace class. Corollary: ρ_ω decomposes as a discrete direct sum of irreps, each occuring with finite multiplicity. Proof: by thm, $\rho_\omega(\varphi)$ is compact for good φ.
- For an irrep π of $G(\mathbb{A})$, let $m(\pi)$ denote its multiplicity in ρ_ω.
- Then $\mathrm{Tr}(\rho_\omega(\varphi)) = \sum m(\pi)\,\mathrm{Tr}(\pi)$. Since each component of ρ_ω has ω as its central character, we only need to sum over such π.
- But also, $\mathrm{Tr}(\rho_\omega(\varphi)) = \int_{\overline{G}(F)\backslash\overline{G(\mathbb{A})}} K(x, x)\,\mathrm{d}x$.
- Since p, the 'degree' (sqrt of dimension) of H over F is prime, any noncentral element of $Z(F)$ is regular (meaning?). Thus $\int_{\overline{G}(F)\backslash\overline{G(\mathbb{A})}} K(x, x)\,\mathrm{d}x = \mathrm{vol}(\overline{G(F)}\backslash\overline{G}(\mathbb{A}))\varphi(e) + \int \sum_{\gamma\neq e} \varphi(x^{-1}\gamma x)\,\mathrm{d}x$.
- Every noncentral element ξ of $G(F)$ generates an extension L of F of degree p in H (???).
- If ξ is conjugate to ξ' in $G(F)$, then the extension L' coming from ξ' is F isomorphic to L via an F isomorphism taing ξ' to ξ.
- Let X be a set of representatives for the isomorphism classes of extensions of degree p of F which imbed in H.Then any nonidentity $\gamma \in \overline{G}(F)$ can be written as $\gamma = \eta^{-1}\xi\eta$ for nonidentity $\xi \in L^\times/F^\times$ for some L in X and η is a representative of $(F^\times\backslash L^\times)\backslash\overline{G}(F) = L^\times\backslash G(F)$. (i.e. any γ in $F^\times\backslash G(F)$ is conjugate to an element of a representative extension in X via an element of $L^\times\backslash G(F)$.)
- γ uniquely determines the representative in X, and the number of ways to $L^\times\backslash G(F)$ conjugate γ to some ξ is g_L, the number of F automorphisms of L.

- Thus

$$K(x,x) = \varphi(e) + \sum_{L\in X} (g_L)^{-1} \sum_{\xi} \sum_{\eta} \varphi(x^{-1}\eta^{-1}\xi\eta x) \tag{2}$$

  for ξ, η as above. Integrating,

$$\int K(x,x)\,\mathrm{d}x = \mathrm{vol}(\overline{G(F)}\backslash\overline{G}(\mathbb{A}))\varphi(e)+\sum_{L\in X}(g_L)^{-1}\,\mathrm{vol}(F^\times(\mathbb{A})L^\times\backslash L^\times(\mathbb{A}))\sum_{\xi\in L^\times - F^\times}\int_{L^\times\backslash\overline{G}(\mathbb{A})}\varphi(x^{-1}\xi x)\,\mathrm{d}x \tag{3}$$

- Suppose the measure on $L^\times(\mathbb{A})\backslash\overline{G}(\mathbb{A})$ is a product measure. Then

$$\int \varphi(x^-1\xi x)\,\mathrm{d}x = \prod_v \int_{L_v^\times\backslash G_v} \varphi_v(\xi_v^{-1}\xi x_v)\,\mathrm{d}x_v \tag{4}$$

- Given $\xi \in L^\times\backslash F^\times$, for almost every v, $\xi \in K_v$. And since almost every φ_v is supported on $K_vF_v^\times$, $\varphi_v(x_v^{-1}\xi x_v) = 0$ unless $x_v \in K_vL_v^\times$. Thus almost all of the factors in the preceding integral are $\mathrm{vol}(L_v^\times\backslash L_v^\times K_v)$.
- To summarize, we've expressed the character $\mathrm{Tr}\,\rho_\omega(\varphi)$ in terms of the local invariant (??) distributions $\varphi_v \mapsto \varphi_v(e)$ and $\varphi_v \mapsto \int_{L_v^\times\backslash G_v}\varphi_v(x_v^{-1}\xi x_v)$.
- ...

11.2 Cuspforms on GL(2)

- Now G is GL(2). Let Z, P, A, N be as usual. We say a function φ on $G(F)\backslash G(\mathbb{A})$ is cuspidal if $\int_{N(F)\backslash N(\mathbb{A})}\varphi(ng)\,\mathrm{d}n$ is identically zero as a function of g.

12 notes on P. Humphries (2017) Spectral Multiplicity for Maaß Newforms of Non-Squarefree Level

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13 list

- Dubi Kelmer
- Chandrasheel Bhagwat
- C.S. Rajan
- Sarnak: The behaviour of eigenstates of arithmetic hyperbolic manifolds
- Jian-Shu Li and Joachim Schwermer On the Cuspidal Cohomology of Arithmetic Groups
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- Sobolev Norms of Automorphic Functionals page 5 very nice recast using frob recip. J. Bernstein and A. Reznikov

- RANKIN-SELBERG WITHOUT UNFOLDING AND BOUNDS FOR SPHERICAL FOURIER COEFFICIENTS OF MAASS FORMS ANDRE REZNIKOV
- Analytic continuation of representations and estimates of automorphic forms Bernstein and Reznikov
- P. Humphries (2017) “Spectral Multiplicity for Maaß Newforms of Non-Squarefree Level,”