Coursera Notes for Bayesian Statistics: Techniques and Models

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Week 1

Bayesian Modeling

data y, **parameter** θ

prior $p(\theta)$: Initial "belief" about the parameters

likelihood $p(y|\theta)$: Probability of an outcome given some parameter value. The parameter $\hat{\theta}$ yielding the highest probability is called the *Maximum Likelihood Estimator (MLE)*. Note: we can also use log likelihood since the log function is monotonically increasing.

posterior: $p(\theta|y)$: Updated "belief" about the parameter. Probability of some parameter value given observed data.

$$\begin{aligned} p(\theta|y) &= \frac{p(y|\theta)p(\theta)}{p(y)} \propto p(y|\theta)p(\theta) \\ &= \frac{p(y|\theta)p(\theta)}{\int_\theta p(y,\theta)d\theta} \\ &= \frac{p(y|\theta)p(\theta)}{\int_\theta p(y|\theta)p(\theta)d\theta} \end{aligned}$$

The denominator also called "normalizing constant" since θ does not occur in the expression.

For some choices of prior, the posterior distribution has the same algebraic form as the prior, making the integral easy to solve. These are called *conjugate priors*. For others, or if the models are more complicated, then the posterior distribution may not have a standard form, and numerical integration must be used.

Well-known conjugate priors:

- Beta prior + Binomial/Bernoulli likelihood $\implies$ Beta posterior
- Gamma prior + Poisson likelihood $\implies$ Gamma posterior

Monte Carlo Estimation

Using simulation to determine some properties of a distribution, e.g. mean, variance, probability of an event, quantiles (which all use integration)

Example: Suppose we have $\theta \sim \text{Ga}(a,b)$ and want to know $E[\theta]$

$$E[\theta] = \int_{-\infty}^{\infty} \theta p(\theta)d\theta = \int_0^{\infty} \theta \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}d\theta = \frac{a}{b}$$

To verify with Monte Carlo, take samples θ_i^* for $i = 1, \ldots, m$ from the Gamma distribution. Estimate sample mean as

$$\overline{\theta^*} = \frac{1}{m} \sum_{i=1}^{m} \theta_i^*$$

Suppose we have some function $h(\theta)$ and we want $E[h(\theta)]$. Can estimate

$$E[h(\theta)] = \int h(\theta) p(\theta) d\theta \approx \frac{1}{m} \sum_{i=1}^{m} h(\theta_i^*)$$

In particular, if $h(\theta)$ is $I_A(\theta)$, i.e. the indicator function for some event A, then we can approximate probabilities as well: $Pr[\theta \in A]$.

Question: How good is this estimate from sampling? By the Central Limit Theorem we know

$$\overline{\theta^*} \dot{\sim} N\Big(E(\theta), \frac{Var(\theta)}{m}\Big)$$

The variance of the estimate is given by

$$\widehat{Var(\theta)} = \frac{1}{m} \sum_{i=1}^{m} (\theta_i^* - \overline{\theta^*})^2$$

The standard error (SE) is given by

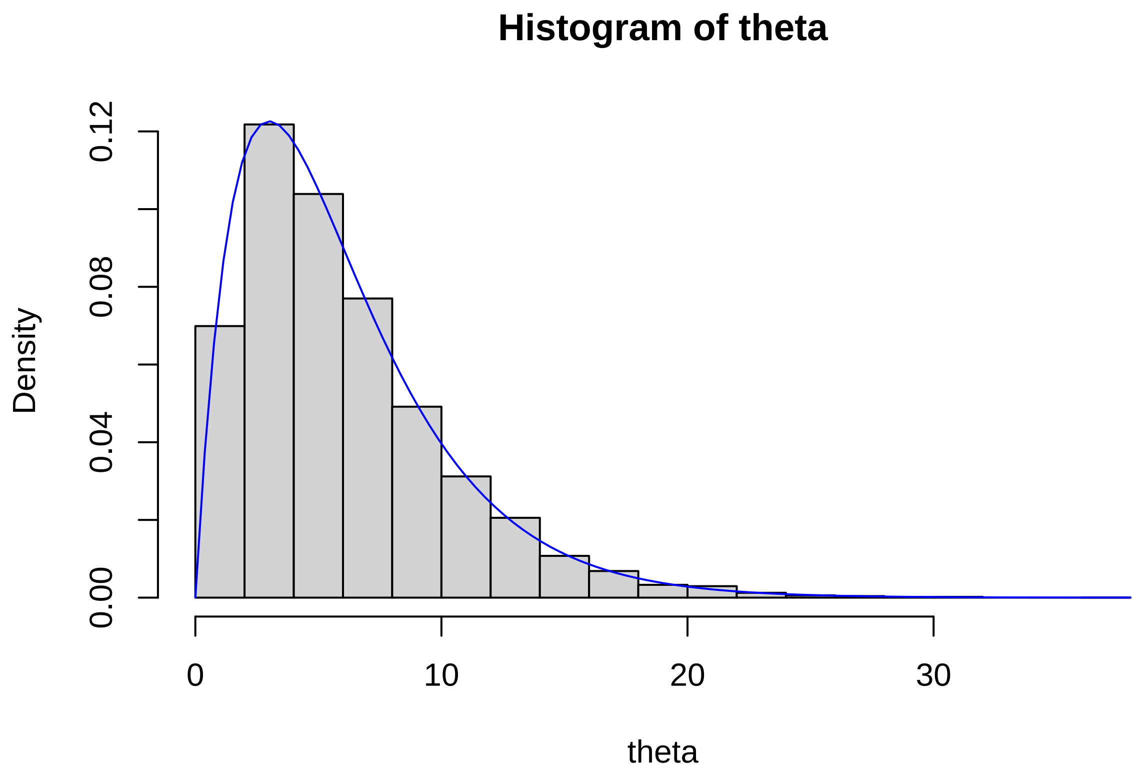
$$\sqrt{\frac{\widehat{Var(\theta)}}{m}}$$

```r
set.seed(32)

m=10000
a=2
b=1/3

theta = rgamma(n=m, shape=a, rate=b)

hist(theta, freq=FALSE)
curve(dgamma(x, shape=a, rate=b), col="blue", add=TRUE)
```




```r
mean(theta) # Estimated mean
```

```
## [1] 6.022368
```

```r
a/b # True mean
```

```
## [1] 6
```

```r
var(theta) # Estimated variance
```

```
## [1] 18.01033
```

```r
a/b^2 # True variance
```

```
## [1] 18
```

```r
ind = theta < 5
mean(ind) # Estimated Prob[theta < 5]
```

```
## [1] 0.4974
```

```r
pgamma(q=5, shape=a, rate=b) # True Prob[theta < 5]
```

```
## [1] 0.4963317
```

```r
quantile(theta, probs=0.9) # Estimated quantile
```

```
##      90%
## 11.74426
```

```r
qgamma(p=0.9, shape=a, rate=b) # True quantile
```

```
## [1] 11.66916
```

```
se = sd(theta) / sqrt(m) # Standard error of mean
mean(theta) - 2*se # Lower bound CI
```

```
## [1] 5.937491
```

```
mean(theta) + 2*se # Upper bound CI
```

```
## [1] 6.107245
```

As we can see, Monte Carlo does a pretty good job.

Example: Suppose we have

$$y|\phi \sim Bin(10, \phi)$$

$$\phi \sim Beta(2, 2)$$

and we want to simulate from marginal distribution of y (which can be difficult to do in general). Can do the following procedure:

1. Draw $\phi_i^* \sim Beta(2, 2)$
2. Given ϕ_i^*, draw $y_i^* \sim Bin(10, \phi_i^*)$

Results in a list of independent pairs (y_i^*, ϕ_i^*) drawn from the joint distribution. Discarding the ϕ_i^*s effectively results in a sample from the marginal distribution of y.

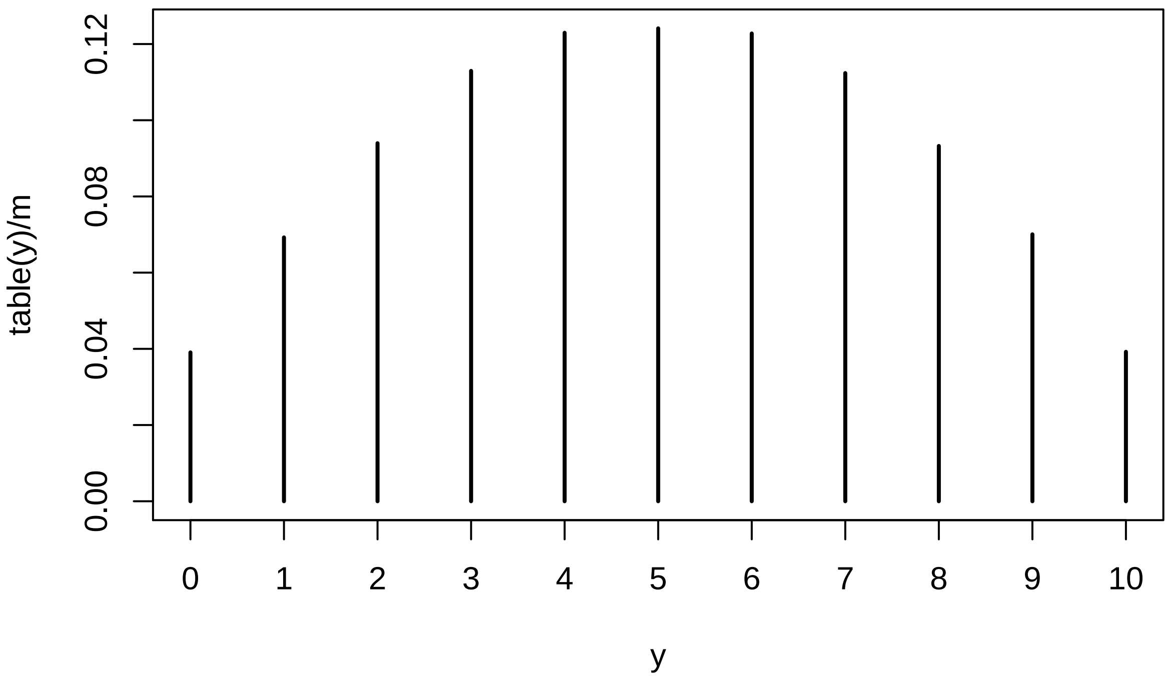
```
m = 1e5

phi = rbeta(m, shape1=2, shape2=2)
y = rbinom(m, size=10, prob=phi)

table(y) / m
```

```
## y
##       0       1       2       3       4       5       6       7       8       9
## 0.03906 0.06925 0.09398 0.11296 0.12296 0.12412 0.12277 0.11238 0.09325 0.07005
##      10
## 0.03922
```

```
plot(table(y) / m) # Estimated marginal distribution of y
```




```r
mean(y) # Estimate mean of y
```

```
## [1] 5.00046
```