Coursera Notes for Bayesian Statistics: Techniques and Models

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Week 1

Bayesian Modeling

data y, parameter θ
likelihood: $p(y|\theta)$
prior: $p(\theta)$
posterior: $p(\theta|y)$

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{\int p(y|\theta)p(\theta)d\theta} \propto p(y|\theta)p(\theta)$$

If we do not use conjugate priors, or if the models are more complicated, then the posterior distribution may not have a "standard" or well-known form.

Monte Carlo Estimation

Using simulation to determine some properties of a distribution, e.g. mean, variance, probability of an event, quantiles (which all use integration)

Example: Suppose we have $\theta \sim \text{Ga}(a, b)$ and want to know $E[\theta]$

$$E[\theta] = \int_{-\infty}^{\infty} \theta p(\theta) d\theta = \int_0^{\infty} \theta \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta} d\theta = \frac{a}{b}$$

To verify with Monte Carlo, take samples θ_i^* for $i = 1, \ldots, m$ from the Gamma distribution. Estimate sample mean as

$$\overline{\theta^*} = \frac{1}{m} \sum_{i=1}^{m} \theta_i^*$$

Suppose we have some function $h(\theta)$ and we want $E[h(\theta)]$. Can estimate

$$E[h(\theta)] = \int h(\theta) p(\theta) d\theta \approx \frac{1}{m} \sum_{i=1}^{m} h(\theta_i^*)$$

In particular, if $h(\theta)$ is $I_A(\theta)$, i.e. the indicator function for some event A, then we can approximate probabilities as well: $Pr[\theta \in A]$.

Question: How good is this estimate from sampling? By the Central Limit Theorem we know

$$\overline{\theta^*} \dot{\sim} N\Big(E(\theta), \frac{Var(\theta)}{m}\Big)$$

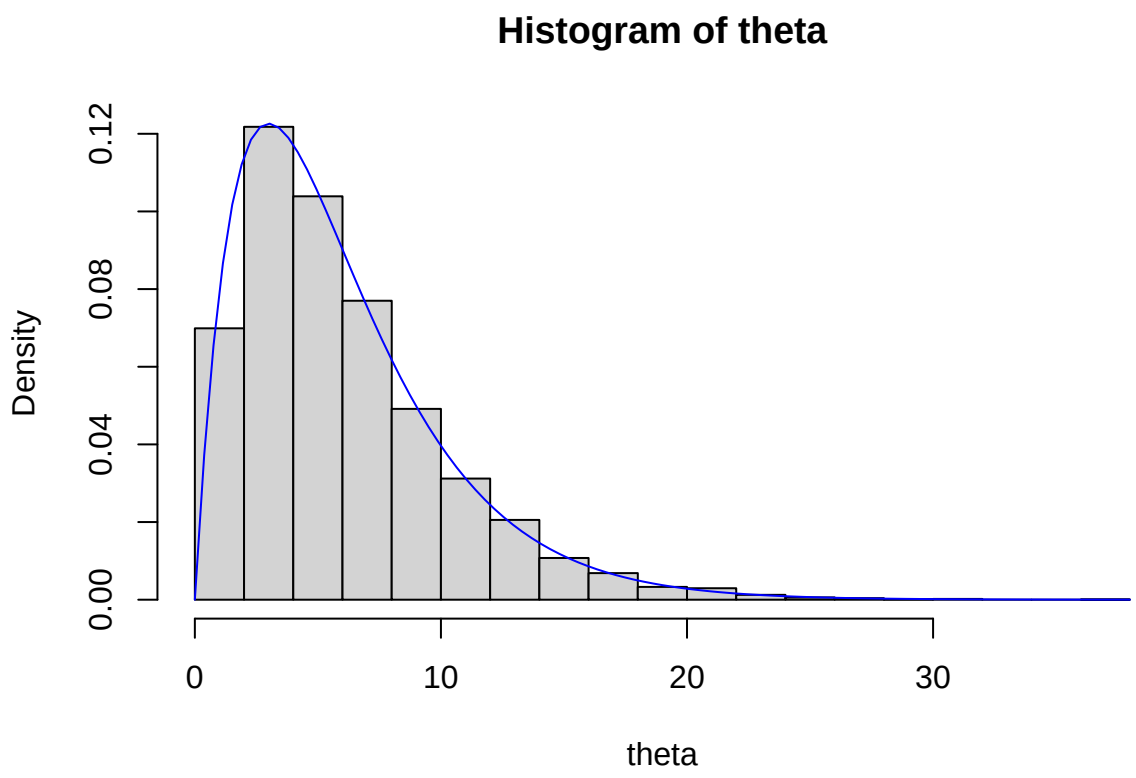
The variance of the estimate is given by

$$\widehat{Var(\theta)} = \frac{1}{m}\sum_{i=1}^{m}(\theta_i^* - \overline{\theta^*})^2$$

The standard error (SE) is given by

$$\sqrt{\frac{\widehat{Var(\theta)}}{m}}$$

```
set.seed(32)

m=10000
a=2
b=1/3

theta = rgamma(n=m, shape=a, rate=b)

hist(theta, freq=FALSE)
curve(dgamma(x, shape=a, rate=b), col="blue", add=TRUE)
```




```
mean(theta) # Estimated mean
```

```
## [1] 6.022368
```

```
a/b # True mean
```

```
## [1] 6
```

```r
var(theta) # Estimated variance
```

```
## [1] 18.01033
```

```r
a/b^2 # True variance
```

```
## [1] 18
```

```r
ind = theta < 5
mean(ind) # Estimated Prob[theta < 5]
```

```
## [1] 0.4974
```

```r
pgamma(q=5, shape=a, rate=b) # True Prob[theta < 5]
```

```
## [1] 0.4963317
```

```r
quantile(theta, probs=0.9) # Estimated quantile
```

```
##      90%
## 11.74426
```

```r
qgamma(p=0.9, shape=a, rate=b) # True quantile
```

```
## [1] 11.66916
```

```r
se = sd(theta) / sqrt(m) # Standard error of mean
mean(theta) - 2*se # Lower bound CI
```

```
## [1] 5.937491
```

```r
mean(theta) + 2*se # Upper bound CI
```

```
## [1] 6.107245
```

As we can see, Monte Carlo does a pretty good job.

Example: Suppose we have

$$y|\phi \sim Bin(10, \phi)$$
$$\phi \sim Beta(2, 2)$$

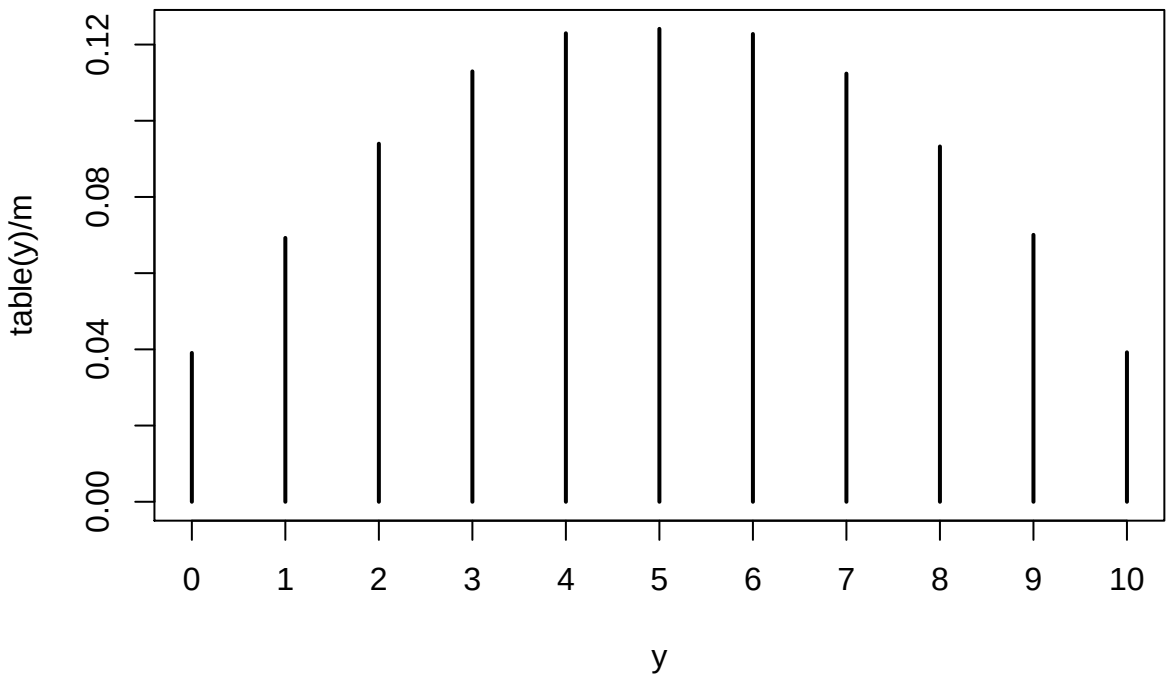
and we want to simulate from marginal distribution of y (which can be difficult to do in general). Can do the following procedure:
1. Draw $\phi_i^* \sim Beta(2, 2)$
2. Given ϕ_i^*, draw $y_i^* \sim Bin(10, \phi_i^*)$

Results in a list of independent pairs (y_i^*, ϕ_i^*) drawn from the joint distribution. Discarding the ϕ_i^*s effectively results in a sample from the marginal distribution of y.

```r
m = 1e5

phi = rbeta(m, shape1=2, shape2=2)
y = rbinom(m, size=10, prob=phi)

table(y) / m
```

```
## y
##       0       1       2       3       4       5       6       7       8       9
## 0.03906 0.06925 0.09398 0.11296 0.12296 0.12412 0.12277 0.11238 0.09325 0.07005
##      10
## 0.03922
```

```r
plot(table(y) / m) # Estimated marginal distribution of y
```




```r
mean(y) # Estimate mean of y
```

```
## [1] 5.00046
```

Week 2

Metropolis-Hastings

Allows us to sample from generic distribution (whose normalizing constant may not be known). To accomplish this, we effectively construct a Markov Chain whose stationary distribution is the target distribution.

Say we want to know $p(\theta)$ but we only know $g(\theta)$ where $p(\theta) \propto q(\theta)$.

Algorithm:

1. Select initial value θ_0
2. for $i = 1, \ldots, m$ repeat:
   a. Draw candidate $\theta^* \sim q(\theta^*|\theta_{i-1})$
   b. Define $\alpha = \frac{g(\theta^*)/q(\theta^*|\theta_{i-1})}{g(\theta_{i-1})/q(\theta_{i-1}|\theta^*)} = \frac{g(\theta^*)}{g(\theta_{i-1})} \frac{q(\theta_{i-1}|\theta^*)}{q(\theta^*|\theta_{i-1}))}$
      i. if $\alpha \geq 1$:
      accept θ^* and set $\theta_i \leftarrow \theta^*$
      ii. $0 < \alpha < 1$:
      with prob α: accept θ^* and set $\theta_i \leftarrow \theta^*$
      with prob $1 - \alpha$: reject θ^* and set $\theta_i \leftarrow \theta_{i-1}$

Where q here is the candidate generating distribution which may or may not depend on θ_{i-1}.

One choice is to make q the same distribution regardless of the value θ_{i-1}. If we take this option, we want $q(\theta)$ to be similar to $p(\theta)$ to best approximate it. A high acceptance rate is a good sign here but still may want q to have a larger variance then p to assure we are exploring the space well.

Another choice – one which *does* depend on θ_{i-1} – is to choose a distribution q that is centered on θ_{i-1}. In any symmetric case, we have the property $q(a|b) = q(b|a)$, so step 2 in the algorithm above reduces to

$$\alpha = \frac{g(\theta^*)}{g(\theta_{i-1})}$$

A common choice for such a distribution is $N(\theta_{i-1}, 1)$, or in order words, a Gaussian random walk: $\theta^* = \theta_{i-1} + N(0, 1)$ In this particular case, we have

$$q(\theta^*|\theta_{i-1}) = \frac{1}{\sqrt{2\pi}} \exp\left[-0.5(\theta^* - \theta_{i-1})^2\right] = q(\theta_{i-1}|\theta^*)$$

The "size" of the random walk step can affect acceptance (and thus convergence) rate. A high acceptance rate is not a good sign here. If random walk is taking too small of steps, it will accept candidate more often but will take a long time to fully explore the space. If it is taking too large of steps, many proposals will have low probabilities which leads to a low acceptance rate. This amounts to "wasted" samples. Ideally, a random walk sampler should have an acceptance rate between 23% and 50%.

Example: Suppose $y_i|\mu \overset{iid}{\sim} N(\mu, 1)$ for $i = 1, \ldots, n$ and $\mu \sim t(0, 1, 1)$. We want to sample from the posterior distribution $p(\mu|y_1, \ldots, y_n)$, which we can show is proportional to $\frac{\exp[n(\overline{y}\mu - \mu^2/2)]}{1+\mu^2}$

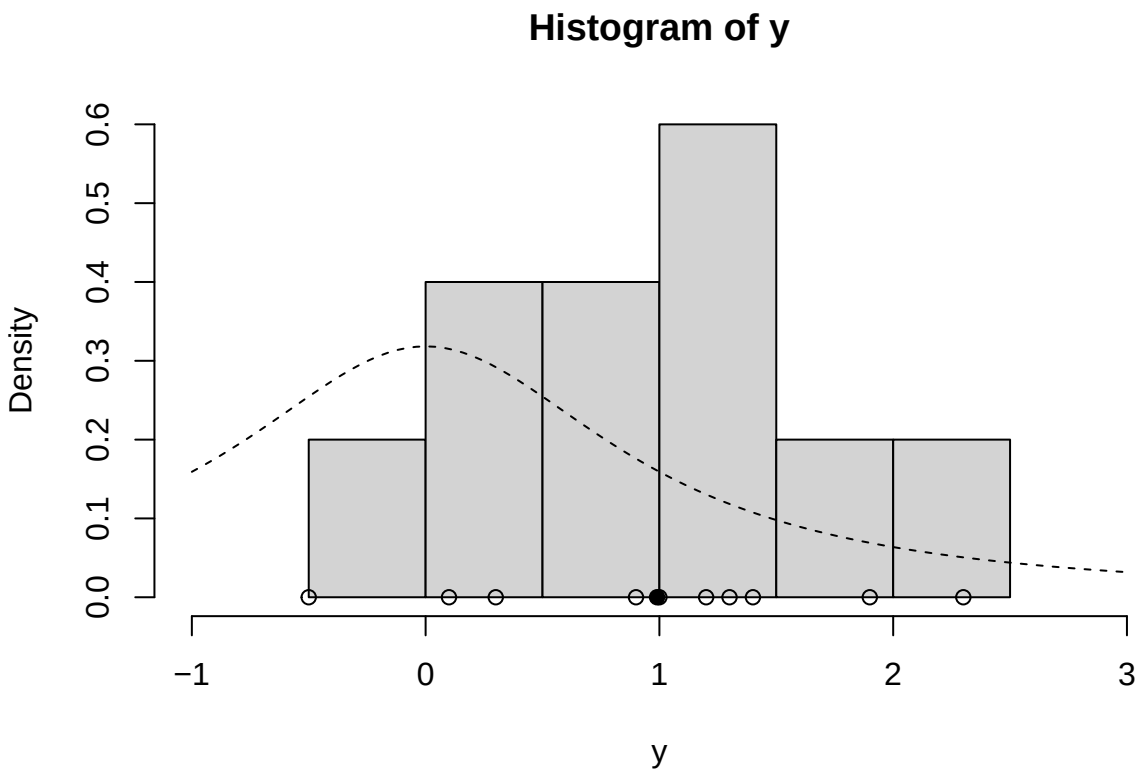
```
# using log(g(x)) instead of g(x) for numerical stability
log_g = function(mu, n, ybar) {
  n * (ybar * mu - mu^2 / 2) - log(1 + mu^2)
}
```

```
metropolis_hastings = function(n, ybar, n_iter, mu_init, cand_sd) {
  # Random-Walk Metropolis-Hastings algorithm

  # step 1
  mu_out = numeric(n_iter)
```

```
  n_accept = 0
  mu_now = mu_init
  lg_now = log_g(mu=mu_now, n=n, ybar=ybar)

  for (i in 1:n_iter) {
    # step 2a
    mu_cand = rnorm(1, mean=mu_now, sd=cand_sd) # draw candidate

    # step 2b
    lg_cand = log_g(mu=mu_cand, n=n, ybar=ybar)
    lg_alpha = lg_cand - lg_now
    alpha = exp(lg_alpha)

    u = runif(1) # less than alpha with prob min(1, alpha)
    if (u < alpha) { # accept candidate
      mu_now = mu_cand
      n_accept = n_accept + 1
      lg_now = lg_cand
    }

    mu_out[i] = mu_now
  }

  list(mu=mu_out, accept_rate=n_accept/n_iter)
}
```

Problem set up:

```
y = c(1.2, 1.4, -0.5, 0.3, 0.9, 2.3, 1.0, 0.1, 1.3, 1.9) # data
ybar = mean(y) # sample mean
n = length(y)

hist(y, freq=FALSE, xlim=c(-1, 3)) # histogram of data
curve(dt(x=x, df=1), lty=2, add=TRUE) # prior for mu
points(y, rep(0,n), pch=1) # individual data points
points(ybar, 0, pch=19) # sample mean
```



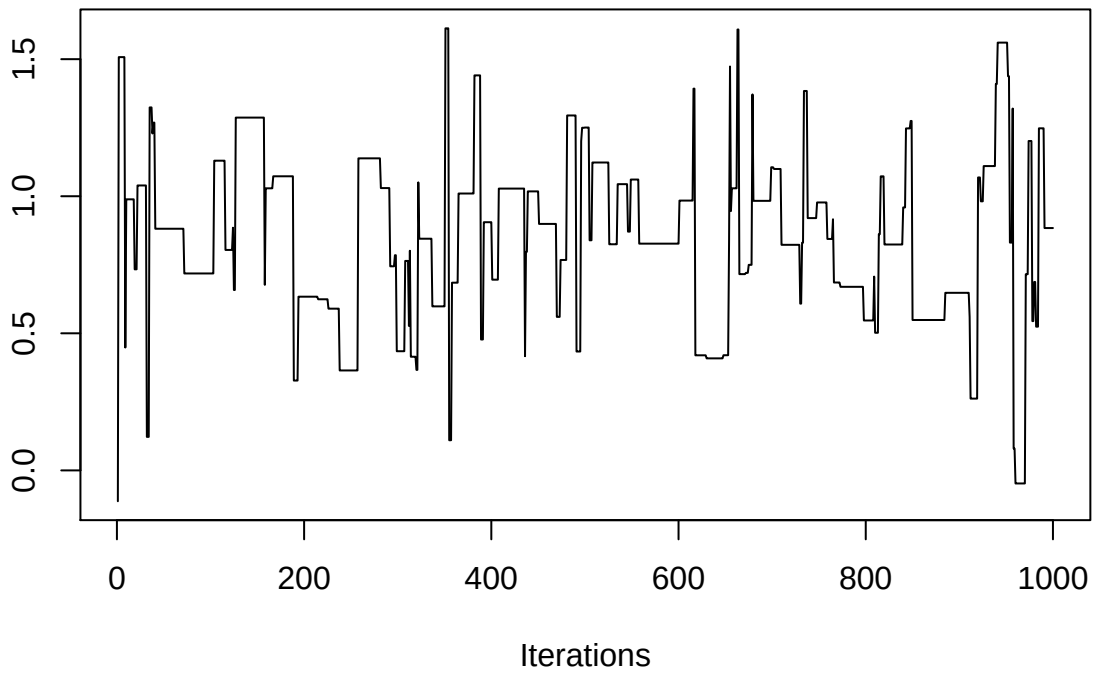

```
set.seed(43) # for reproducibility
library("coda") # traceplot --> helpful to determine convergence
```

Posterior sampling:

```
post = metropolis_hastings(n=n, ybar=ybar, n_iter=1e3, mu_init=0, cand_sd=3)
str(post)
```

```
## List of 2
##  $ mu         : num [1:1000] -0.113 1.507 1.507 1.507 1.507 ...
##  $ accept_rate: num 0.122
```

```
traceplot(as.mcmc(post$mu))
```

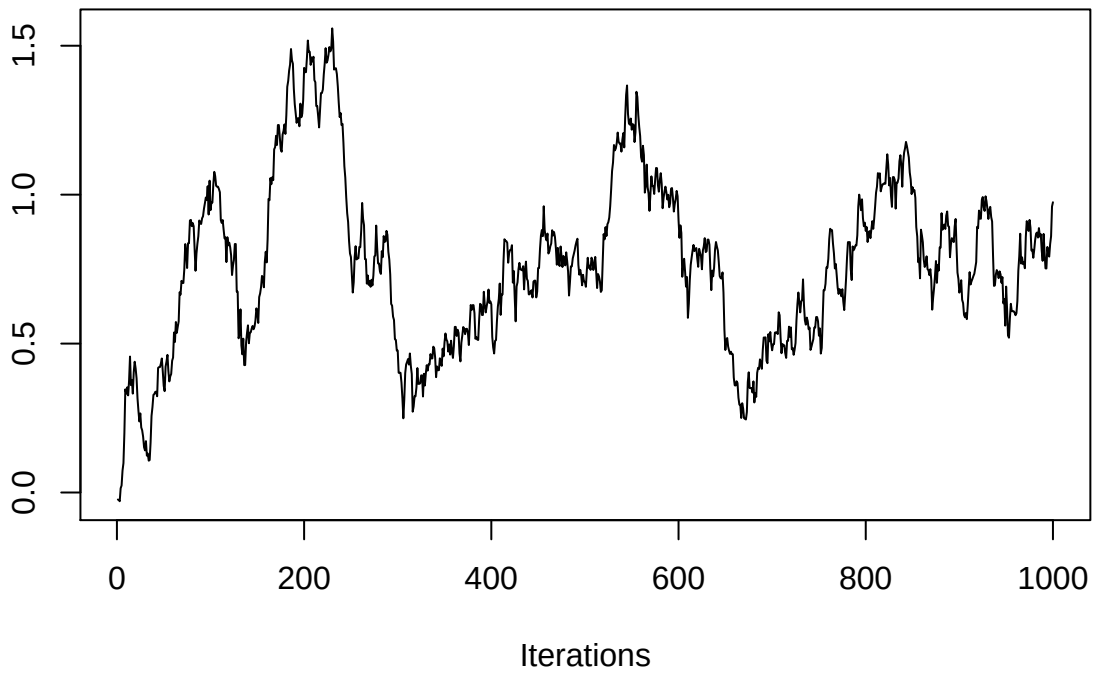


Step size too large (low acceptance rate). Let's try another.

```
post = metropolis_hastings(n=n, ybar=ybar, n_iter=1e3, mu_init=0, cand_sd=0.05)
str(post)
```

```
## List of 2
##  $ mu         : num [1:1000] -0.0236 -0.0247 -0.0293 0.0142 0.0235 ...
##  $ accept_rate: num 0.946
```

```
traceplot(as.mcmc(post$mu))
```

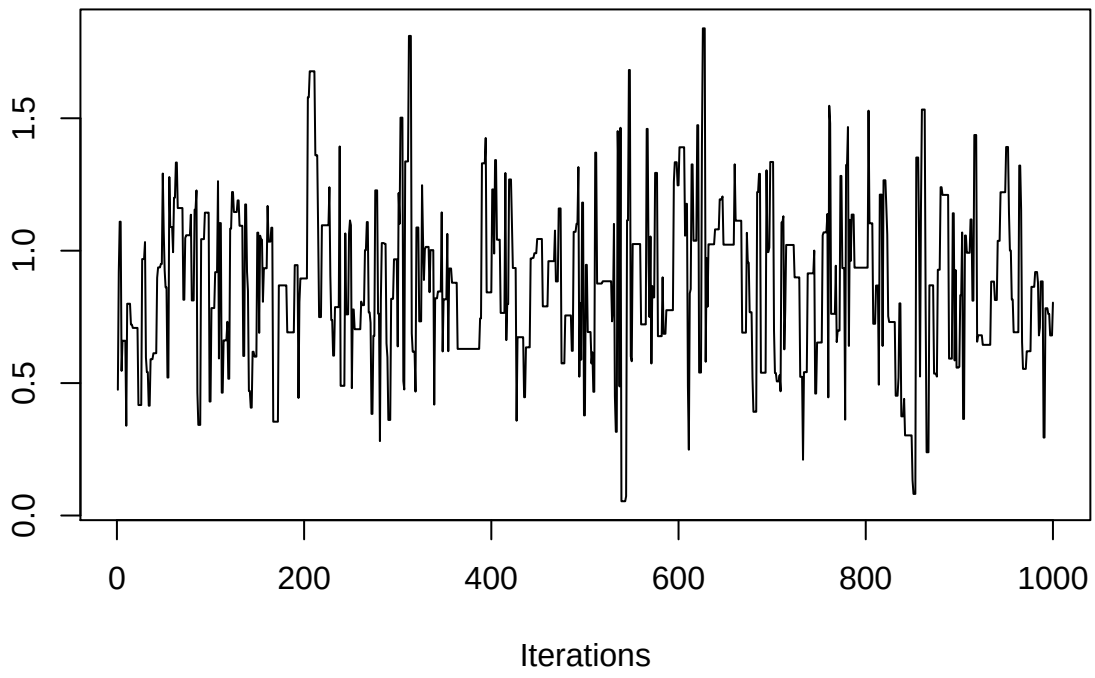


Step size too small (high acceptance rate). Let's try another.

```
post = metropolis_hastings(n=n, ybar=ybar, n_iter=1e3, mu_init=0, cand_sd=0.9)
str(post)
```

```
## List of 2
##  $ mu         : num [1:1000] 0.475 0.92 1.109 1.109 0.546 ...
##  $ accept_rate: num 0.38
```

```
traceplot(as.mcmc(post$mu))
```

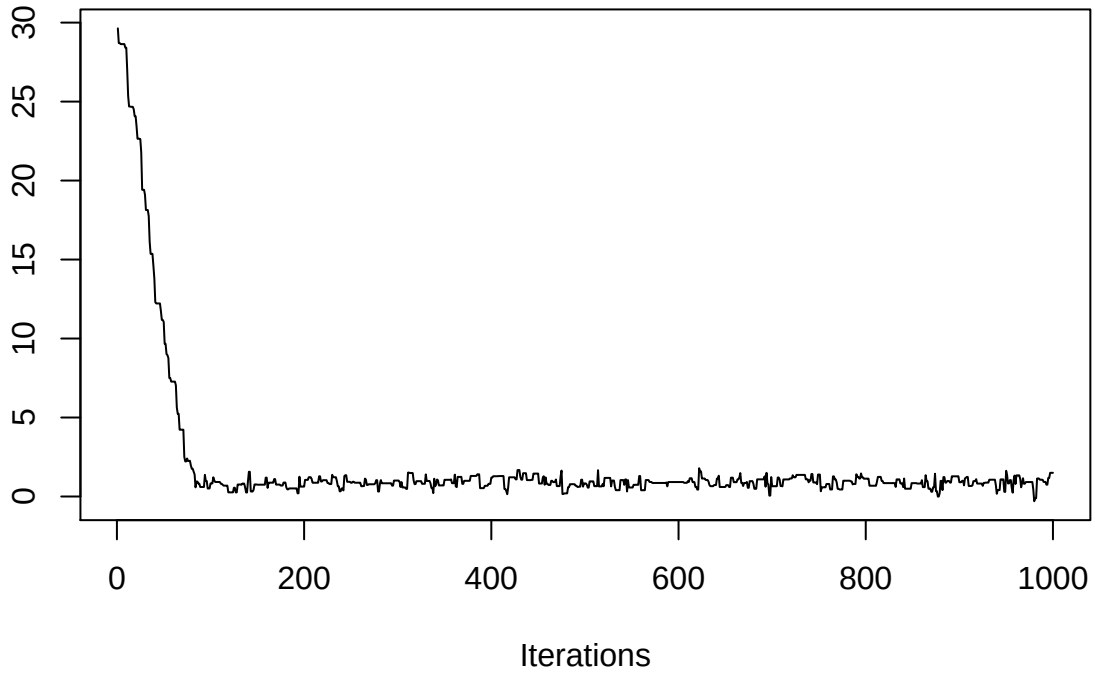


Looks good. Experimenting with different initial value:

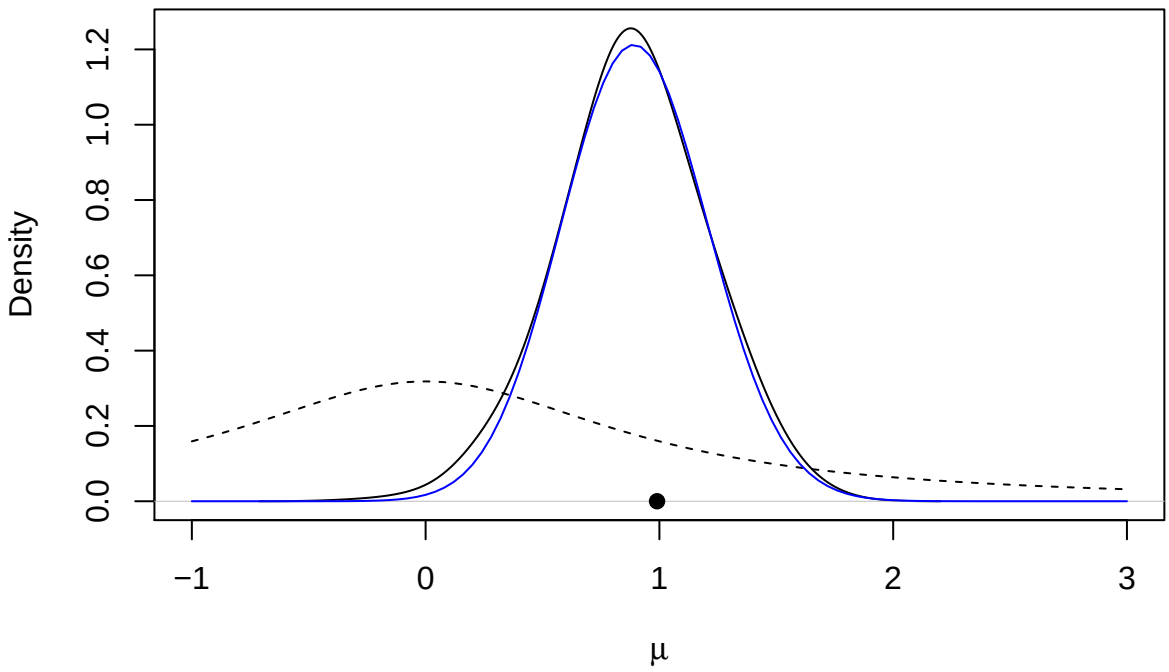
```
post = metropolis_hastings(n=n, ybar=ybar, n_iter=1e3, mu_init=30, cand_sd=0.9)
str(post)
```

```
## List of 2
##  $ mu         : num [1:1000] 29.6 28.7 28.7 28.6 28.6 ...
##  $ accept_rate: num 0.387
```

```
traceplot(as.mcmc(post$mu))
```




```
post$mu_keep = post$mu[-c(1:100)] # discard the first 100 samples
plot(density(post$mu_keep, adjust=2), main="", xlim=c(-1, 3), xlab=expression(mu)) # plot estimated pos
curve(dt(x=x, df=1), lty=2, add=TRUE) # prior for mu
points(ybar, 0, pch=19) # sample mean

curve(0.017*exp(log_g(mu=x, n=n, ybar=ybar)), from=-1, to=3, add=TRUE, col="blue") # approximation to t
```


Density
0.0
0.2
0.4
0.6
0.8
1.0
1.2
−1
0
1
2
3
μ