Coursera: Time Series Analysis

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1 Stationarity

Suppose we have a time series $X \equiv \{X_1, X_2, \ldots, X_n\}$ and the following notation:

$$\begin{aligned} \mu(t) &\equiv \mu_t \equiv \mathbb{E}[X_t] \\ \sigma^2(t) &\equiv \sigma_t^2 \equiv \mathrm{Var}[X_t] \\ \gamma(s,t) &\equiv \mathrm{Cov}[X_s, X_t] \equiv \mathbb{E}[(X_s - \mu_s)(X_t - \mu_t)] \end{aligned}$$

where γ is known as the *autocovariance function*.

Definition (Strict Stationarity)**.** X is strictly stationary if the joint distribution of $\{X_1, X_2, \ldots, X_k\}$ equals that of $\{X_{1+\tau}, X_{2+\tau} \ldots, X_{k+\tau}\}$.

If X is strictly stationary, then the X_i are identically distributed (though not necessarily independent). This implies the functions μ_t and σ_t^2 are both constant. Can also show the autocovariance function only depends on the *lag* spacing. That is, $\gamma(s,t)$ depends only on $\tau \equiv |s-t|$. Can write $\gamma(\tau)$ by slight abuse of notation.

Definition (Weak Stationarity)**.** X is weakly stationary if the following conditions hold.

1. μ_t is constant
2. σ_t^2 is constant
3. $\gamma(s,t)$ depends only on $\tau \equiv |s-t|$

Thus, strict stationarity implies weak stationarity, but we will only consider weak stationarity.

Example. Some common stochastic processes:

- White noise **is** stationary: $X_t \sim \text{iid}(0, \sigma^2)$
- Random walks are **not** stationary: $X_t = \sum_{i=1}^{t} Z_t$, where $Z_t \sim \text{iid}(\mu, \sigma^2)$, since $\mu_t = \mu t$ and $\sigma_t^2 = \sigma^2 t$
- MA(q) process **is** stationary (which we will get to next)

2 MA(q) processes

Intuitively, an MA(q) process is a weighted average of current and prior "white noise" movements. Formally,

$$X_t = \beta_0 Z_t + \dots \beta_q Z_{t-q}$$

for $Z_t \sim \text{iid}(0, \sigma^2)$, constants β_i.

Proof of stationarity: By linearity μ_t is constant (and zero) and by independence σ_t^2 is constant.

$$\begin{aligned} \text{Cov}[X_s, X_t] &= \mathbb{E}[X_s X_t] - \mathbb{E}[X_s]\,\mathbb{E}[X_t] \\ &= \mathbb{E}[X_s X_t] \\ \implies \text{Cov}[X_t, X_{t+k}] &= \sigma^2 \sum_{i=0}^{q-k} \beta_i \beta_{i+k} \end{aligned}$$

which depends only on k.

3 AR(p) processes

Intuitively, an AR(p) process is a weighted average of its own previous values plus some additional noise. Formally,

$$X_t = Z_t + \phi_1 X_{t-1} + \dots + \phi_p X_{t-p}$$

for $Z_t \sim \text{iid}(0, \sigma^2)$, constants ϕ_i.

A random walk is a (trivial) example of an AR(1) process. Thus, not all AR processes are stationary.

4 Backshift Operator

For sake of convenient notation, we define the backshift operator B so that

$$\mathrm{B}\,X_t = X_{t-1}$$
$$\mathrm{B}^k\,X_t = X_{t-k}$$

MA(q) process:

$$\boxed{X_t = \boldsymbol{\beta}(\mathrm{B})Z_t} \qquad \text{where } \boldsymbol{\beta}(\mathrm{B}) \equiv \beta_0 + \beta_1\,\mathrm{B} + \cdots + \beta_q\,\mathrm{B}^q.$$

AR(p) process:

$$\boxed{\boldsymbol{\Phi}(\mathrm{B})X_t = Z_t} \qquad \text{where } \boldsymbol{\Phi}(\mathrm{B}) \equiv 1 - \phi_1\,\mathrm{B} - \cdots - \phi_p\,\mathrm{B}^p$$

5 Relating MA(q) and AR(p) processes

Theorem. *AR(p) processes are MA(∞) processes.*

Proof. Given an AR(p) process we can write

$$X_t = \frac{1}{\boldsymbol{\Phi}(\mathrm{B})} Z_t$$

Since $\frac{1}{\boldsymbol{\Phi}(\mathrm{B})}$ is of the form $\frac{1}{1-\boldsymbol{f}(\mathrm{B})}$ for some operator $\boldsymbol{f}$, by Taylor expansion, this fraction is equal to $1 + \theta_1\,\mathrm{B} + \theta_2\,\mathrm{B}^2 + \dots$ for some constants θ_i.

Thus, we have $X_t = (1 + \theta_1\,\mathrm{B} + \theta_2\,\mathrm{B}^2 + \dots)Z_t \in MA(\infty)$ □

Corollary. *For an AR(p) process, the following hold true.*

$\mu_t = 0$

$\sigma_t^2 = \sigma^2 \sum_{i=0}^{\infty} \theta_i^2$

$\gamma(\tau) = \sigma^2 \sum_{i=0}^{\infty} \theta_i \theta_{i+\tau}$

Theorem. *An AR(p) process is stationary if all of the roots of* $\boldsymbol{\Phi}(\mathrm{B})$ *lie outside the unit circle.*

One can introduce a similar idea of *invertibility* that transforms an MA process to an AR process, and one can similarly show that an MA process is invertible if the roots of $\boldsymbol{\beta}(\mathrm{B})$ lie outside the unit circle.

6 Estimating Parameters and Judging Models

Given a time series, how can we determine the *order* of an MA(q) or AR(p) process?

For an MA(q) process, the ACF cuts off after q lags.

For an AR(p) process, the PACF cuts off after p lags.

We can use the *Yule-Walker* equations to determine a closed form of the ACF (and thus the coefficients ϕ_i). Note, however, that this assumes a stationary process. Yule-Walker can be solved for using matrix algebra.

We can judge the quality of a fitted model using Akaike Information (AIC).