Exercise 1.1

```
> restart
```

b)

```
> u_2 := K*((K*1/h_1 + h_1/6)*u_1 + (K*1/h_2 + h_2/6)*u_3)/(1/h_1 + h_1/3 + 1/h_2 + h_2/3)
```

$$u_2 := K\,\frac{\left(K\,\frac{1}{h_1}+\frac{h_1}{6}\right)u_1+\left(K\,\frac{1}{h_2}+\frac{h_2}{6}\right)u_3}{\frac{1}{h_1}+\frac{h_1}{3}+\frac{1}{h_2}+\frac{h_2}{3}} \qquad \textbf{(1.1.1)}$$

```
> u_1 := 1
```

$$u_1 := 1 \qquad \textbf{(1.1.2)}$$

```
> u_3 := e^2
```

$$u_3 := e^2 \qquad \textbf{(1.1.3)}$$

```
> h_1 := h_2
```

$$h_1 := h_2 \qquad \textbf{(1.1.4)}$$

```
> simplify(u_2)
```

$$K\,\frac{\left(h_2^{\,2}K\ 6\right)\left(e^2+1\right)}{4\,h_2^{\,2}+12} \qquad \textbf{(1.1.5)}$$

```
> h_1 := 2*h_2
```

$$h_1 := 2\,h_2 \qquad \textbf{(1.1.6)}$$

```
> simplify(u_2)
```

$$\frac{K\,e^2\,h_2^{\,2}K\ 2\,h_2^{\,2}+6\,e^2+3}{6\,h_2^{\,2}+9} \qquad \textbf{(1.1.7)}$$

```
> unassign('h_1')
```

c)

```
> L := h_1 + h_2
```

$$L := h_1 + h_2 \qquad \textbf{(1.2.1)}$$

> $x_1 := 0$

$$x_1 := 0 \tag{1.2.2}$$

> $x_2 := h_1$

$$x_2 := h_1 \tag{1.2.3}$$

> $x_3 := L$

$$x_3 := h_1 + h_2 \tag{1.2.4}$$

> $N_1(x) := piecewise\left(x_1 \le x \le x_2,\ 1 - \frac{x - x_1}{h_1},\ 0\right)$

$$N_1 := x \mapsto \begin{cases} 1 - \frac{x - x_1}{h_1} & x_1 \le x \le x_2 \\ 0 & otherwise \end{cases} \tag{1.2.5}$$

> $N_2(x) := piecewise\left(x_1 \le x \le x_2,\ \frac{x - x_1}{h_1},\ x_2 \le x \le x_3,\ 1 - \frac{x - x_2}{h_2},\ 0\right)$

$$N_2 := x \mapsto \begin{cases} \frac{x - x_1}{h_1} & x_1 \le x \le x_2 \\ 1 - \frac{x - x_2}{h_2} & x_2 \le x \le x_3 \\ 0 & otherwise \end{cases} \tag{1.2.6}$$

> $N_3(x) := piecewise\left(x_2 \le x \le x_3,\ \frac{x - x_2}{h_2},\ 0\right)$

$$N_3 := x \mapsto \begin{cases} \frac{x - x_2}{h_2} & x_2 \le x \le x_3 \\ 0 & otherwise \end{cases} \tag{1.2.7}$$

> $uhat(x) := u_1 \cdot N_1(x) + u_2 \cdot N_2(x) + u_3 \cdot N_3(x)$

$$uhat := x \mapsto u_1 \cdot N_1(x) + u_2 \cdot N_2(x) + u_3 \cdot N_3(x) \tag{1.2.8}$$

> $dsolve\left(\left\{u''(x) - u(x) = 0,\ u(0) = u_1,\ u(L) = u_3\right\}\right)$

$$u(x) = -\frac{\left(e^{h_1 + h_2} - e^2\right) e^{-x}}{-e^{h_1 - h_2} + e^{h_1 + h_2}} + \frac{\left(-e^2 + e^{-h_1 - h_2}\right) e^{x}}{-e^{h_1 - h_2} + e^{h_1 + h_2}} \tag{1.2.9}$$

> $assign(\%)$

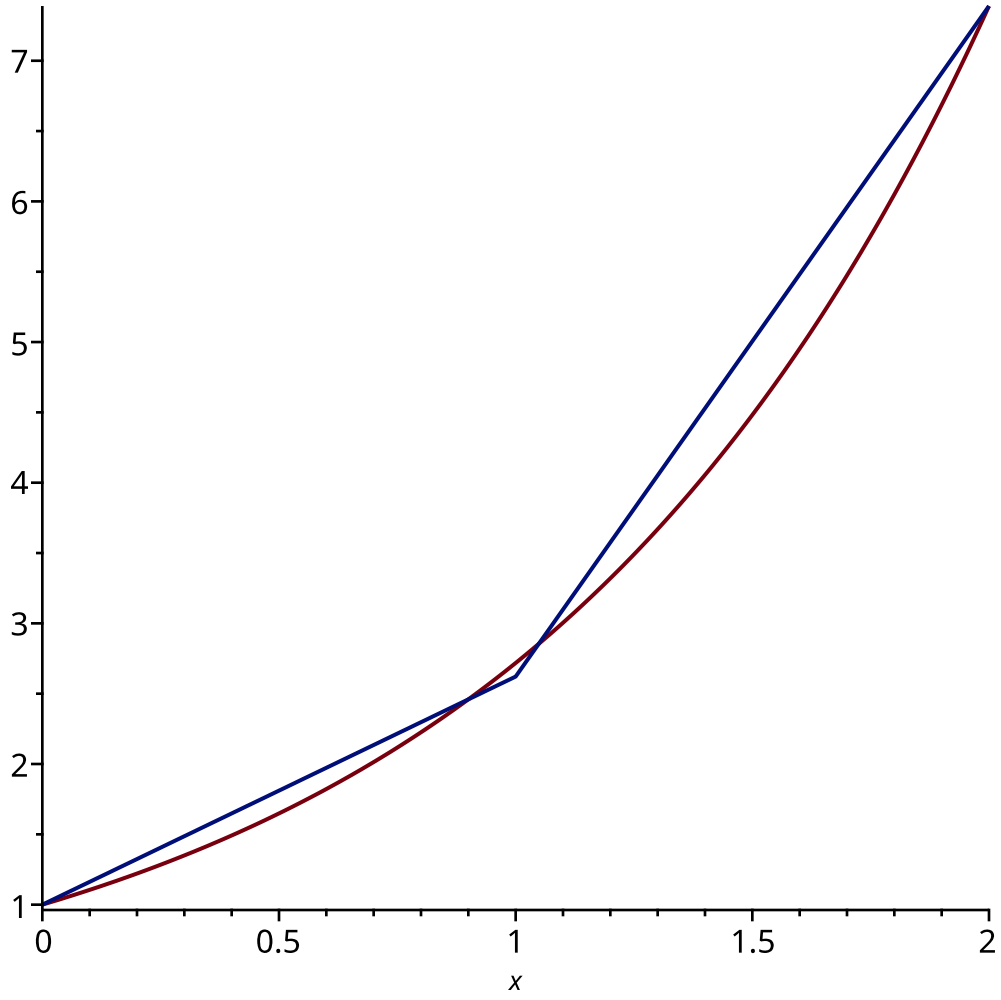
> $h_2 := 1$

$$h_2 := 1 \tag{1.2.10}$$

> $h_1 := h_2$

$$h_1 := 1 \tag{1.2.11}$$

> $plot(\{uhat(x), u(x)\}, x = 0..L)$

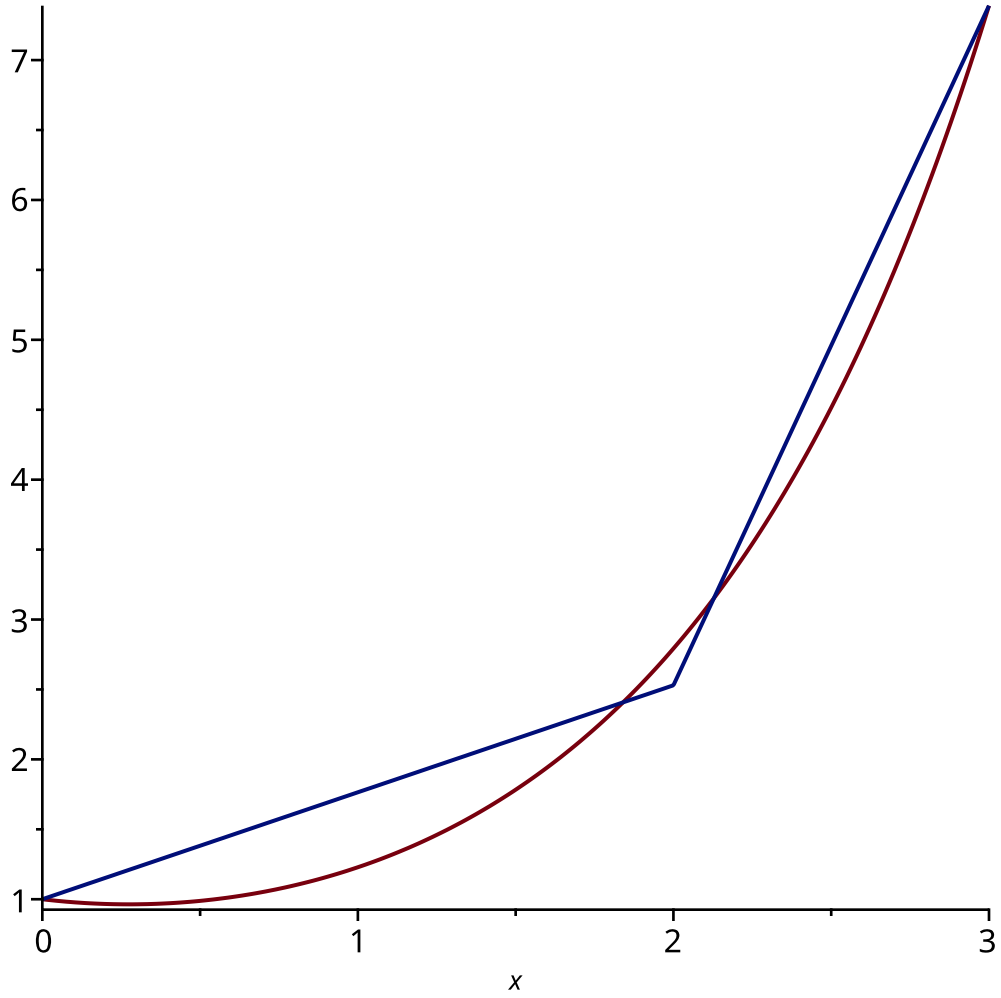


d)

> $h_1 := 2 \cdot h_2$

$$h_1 := 2 \tag{1.3.1}$$

> $plot(\{uhat(x), u(x)\}, x = 0..L)$



>

Exercise 1.6

> *restart*

> $a_1 := \dfrac{u\left(x_i + \dfrac{h}{2}\right) \mathrm{K}\ u(x_i)}{\left(x_i + \dfrac{h}{2}\right) \mathrm{K}\ x_i}$

$$a_1 := \frac{2\left(u\left(x_i + \dfrac{h}{2}\right) \mathrm{K}\ u(x_i)\right)}{h} \tag{2.1}$$

> $a_2 := \dfrac{u(x_i + h) \mathrm{K}\ u\left(x_i + \dfrac{h}{2}\right)}{(x_i + h) \mathrm{K}\ \left(x_i + \dfrac{h}{2}\right)}$

$$a_2 := \frac{2\left(u(x_i+h)\ \mathrm{K}\ u\left(x_i+\frac{h}{2}\right)\right)}{h} \tag{2.2}$$

> $a := \dfrac{u(x_i+h)\ \mathrm{K}\ u(x_i)}{(x_i+h)\ \mathrm{K}\ x_i}$

$$a := \frac{u(x_i+h)\ \mathrm{K}\ u(x_i)}{h} \tag{2.3}$$

> $b_1 := u(x_i)\ \mathrm{K}\ a_1 \cdot x_i$

$$b_1 := u(x_i)\ \mathrm{K}\ \frac{2\left(u\left(x_i+\frac{h}{2}\right)\ \mathrm{K}\ u(x_i)\right)x_i}{h} \tag{2.4}$$

> $b_2 := u(x_i+h)\ \mathrm{K}\ a_2 \cdot (x_i+h)$

$$b_2 := u(x_i+h)\ \mathrm{K}\ \frac{2\left(u(x_i+h)\ \mathrm{K}\ u\left(x_i+\frac{h}{2}\right)\right)(x_i+h)}{h} \tag{2.5}$$

> $b := u(x_i)\ \mathrm{K}\ a \cdot x_i$

$$b := u(x_i)\ \mathrm{K}\ \frac{(u(x_i+h)\ \mathrm{K}\ u(x_i))\,x_i}{h} \tag{2.6}$$

> $simplify\left((a_1 \cdot x + b_1\ \mathrm{K}\ (a \cdot x + b))^2\right)$

$$\frac{\left(u(x_i)+u(x_i+h)\ \mathrm{K}\ 2u\left(x_i+\frac{h}{2}\right)\right)^2 (x\ \mathrm{K}\ x_i)^2}{h^2} \tag{2.7}$$

> $simplify\left((a_2 \cdot x + b_2\ \mathrm{K}\ (a \cdot x + b))^2\right)$

$$\frac{\left(u(x_i)+u(x_i+h)\ \mathrm{K}\ 2u\left(x_i+\frac{h}{2}\right)\right)^2 (h\ \mathrm{K}\ x+x_i)^2}{h^2} \tag{2.8}$$

> $simplify\left(\sqrt{\int_{x_i}^{x_i+\frac{h}{2}} (a_1 \cdot x + b_1\ \mathrm{K}\ (a \cdot x + b))^2\,dx + \int_{x_i+\frac{h}{2}}^{x_i+h} (a_2 \cdot x + b_2\ \mathrm{K}\ (a \cdot x + b))^2\,dx}\right)$

$$\frac{\sqrt{3}\sqrt{h\left(u(x_i)+u(x_i+h)\ \mathrm{K}\ 2u\left(x_i+\frac{h}{2}\right)\right)^2}}{6} \tag{2.9}$$

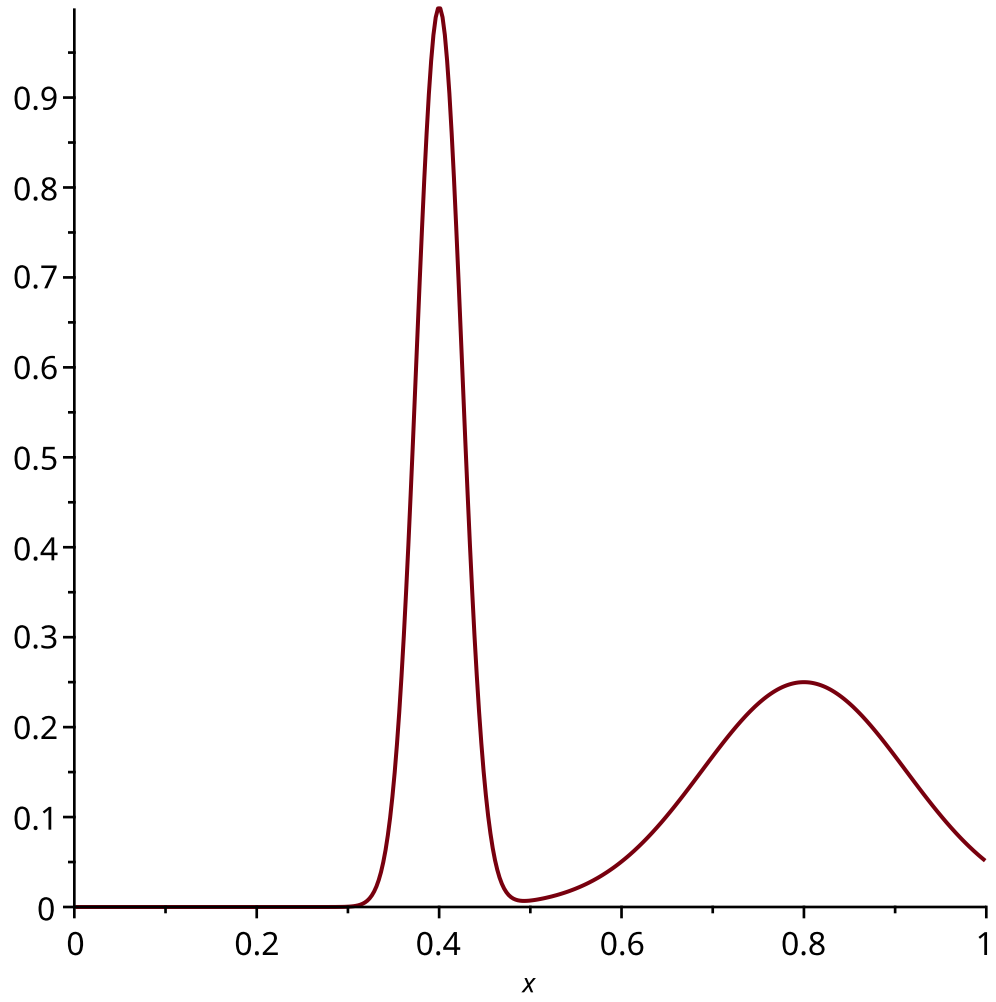
> $err(h, x_i) := \dfrac{\sqrt{3}\sqrt{h\left(u(x_i) + u(x_i + h)\ \mathrm{K}\ 2\,u\left(x_i + \frac{h}{2}\right)\right)^2}}{6}$

$$err := (h, x_i) \mapsto \frac{\sqrt{3}\cdot\sqrt{h\cdot\left(u(x_i) + u(h + x_i)\ \mathrm{K}\ 2\cdot u\left(x_i + \frac{h}{2}\right)\right)^2}}{6} \qquad \textbf{(2.10)}$$

> $u(x) := e^{\mathrm{K}\ 800\cdot(x\,\mathrm{K}\ 0.4)^2} + 0.25\cdot e^{\mathrm{K}\ 40\cdot(x\,\mathrm{K}\ 0.8)^2}$

$$u := x \mapsto e^{\mathrm{K}\ 800\cdot(x\,\mathrm{K}\ 0.4)^2} + 0.25\cdot e^{\mathrm{K}\ 40\cdot(x\,\mathrm{K}\ 0.8)^2} \qquad \textbf{(2.11)}$$

> $plot(u(x), x = 0..1)$



> $simplify(err(h, x_i))$

$$0.2886751347\left(h\left(e^{\mathrm{K}\ 32.\left(5.x_i\,\mathrm{K}\ 2.\right)^2} + 0.25\,e^{\mathrm{K}\ 1.6\left(5.x_i\,\mathrm{K}\ 4.\right)^2} + e^{\mathrm{K}\ 32.\left(5.x_i + 5.h\,\mathrm{K}\ 2.\right)^2}\right.\right.$$
$$\left.\left. + 0.25\,e^{\mathrm{K}\ 1.6\left(5.x_i + 5.h\,\mathrm{K}\ 4.\right)^2} \mathrm{K}\ 2.\,e^{\mathrm{K}\ 8.\left(10.x_i + 5.h\,\mathrm{K}\ 4.\right)^2}\right)\right. \qquad \textbf{(2.12)}$$

$$K\ 0.5\, e^{K\ 0.4\left(\left(10.x_i + 5.h K\ 8.\right)^2\right)^2}\Bigg)^{1/2}$$

```
> err(0.2, 0)
```

$$1.027190526 \times 10^{K\ 8}\sqrt{3} \tag{2.13}$$

```
> err(0.2, 0.3)
```

$$0.1485731197\sqrt{3} \tag{2.14}$$

```
>
```

Exercise 1.7

a)

```
> restart
> u(x) := e^(K 800·(x K 0.4)^2) + 0.25·e^(K 40·(x K 0.8)^2)
```

$$u := x \mapsto e^{K\ 800\cdot(x K\ 0.4)^2} + 0.25\cdot e^{K\ 40\cdot(x K\ 0.8)^2} \tag{3.1.1}$$

```
> u''(x)
```

$$K\ 1600\, e^{K\ 800\,(x K\ 0.4)^2} + \left(K\ 1600\, x + 640.0\right)^2 e^{K\ 800\,(x K\ 0.4)^2} K\ 20.00\, e^{K\ 40\,(x K\ 0.8)^2} + 0.25\left(K\ 80\, x + 64.0\right)^2 e^{K\ 40\,(x K\ 0.8)^2} \tag{3.1.2}$$

```
> f(x) := u''(x) K u(x) : f(x)
```

$$K\ 1601\, e^{K\ 800\,(x K\ 0.4)^2} + \left(K\ 1600\, x + 640.0\right)^2 e^{K\ 800\,(x K\ 0.4)^2} K\ 20.25\, e^{K\ 40\,(x K\ 0.8)^2} + 0.25\left(K\ 80\, x + 64.0\right)^2 e^{K\ 40\,(x K\ 0.8)^2} \tag{3.1.3}$$

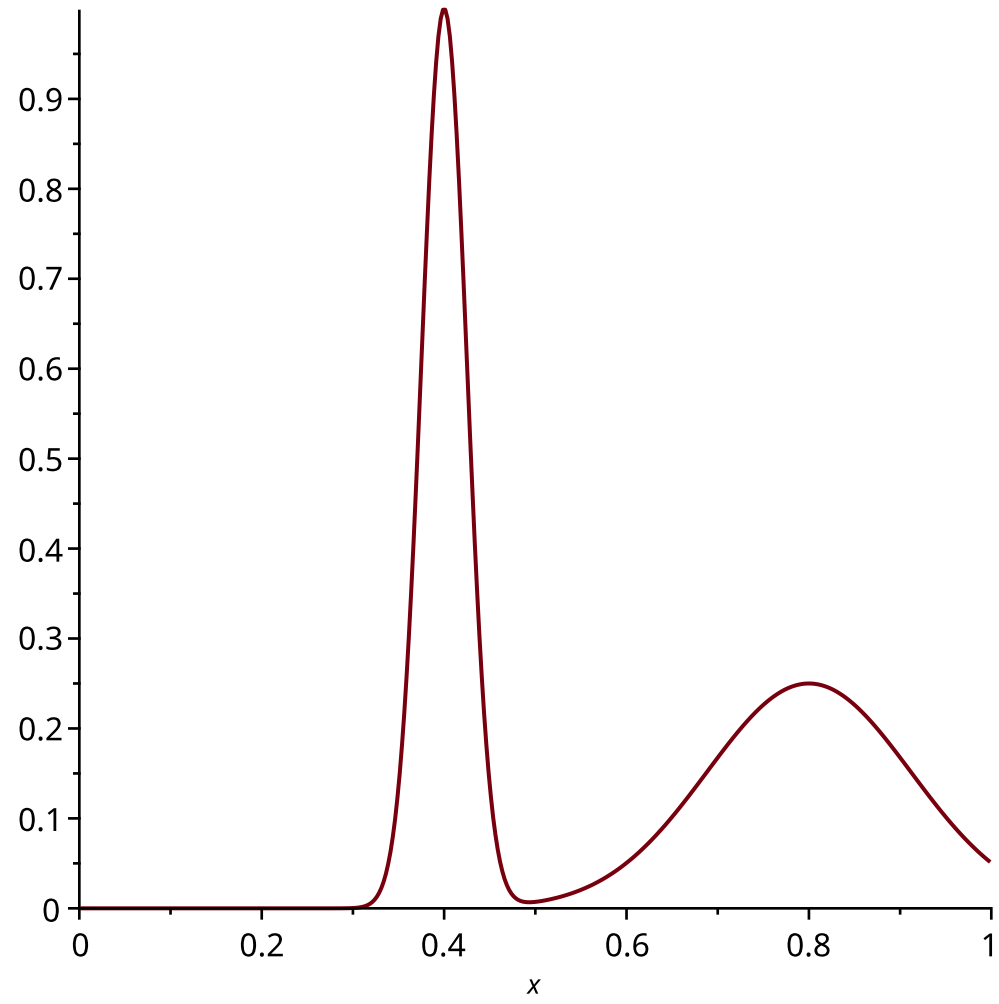
```
> c := u(0)
```

$$c := 1.905466299 \times 10^{K\ 12} \tag{3.1.4}$$

```
> d := u(1)
```

$$d := 0.05047412950 \tag{3.1.5}$$

```
> plot(u(x), x = 0..1)
```



b)

```
> restart
```

$$> a_1 := \frac{u_{new}\left(x_i + \frac{h}{2}\right) \mathrm{K}\ u_{new}(x_i)}{\left(x_i + \frac{h}{2}\right) \mathrm{K}\ x_i}$$

$$a_1 := \frac{2\left(u_{new}\left(x_i + \frac{h}{2}\right) \mathrm{K}\ u_{new}(x_i)\right)}{h} \qquad \textbf{(3.2.1)}$$

$$> a_2 := \frac{u_{new}(x_i + h) \mathrm{K}\ u_{new}\left(x_i + \frac{h}{2}\right)}{(x_i + h) \mathrm{K}\ \left(x_i + \frac{h}{2}\right)}$$

$$a_2 := \frac{2\left(u_{new}(x_i + h) \mathrm{K}\ u_{new}\left(x_i + \frac{h}{2}\right)\right)}{h} \qquad \textbf{(3.2.2)}$$

> $a := \dfrac{u_{old}(x_i + h) \text{ K } u_{old}(x_i)}{(x_i + h) \text{ K } x_i}$

$$a := \frac{u_{old}(x_i + h) \text{ K } u_{old}(x_i)}{h} \tag{3.2.3}$$

> $b_1 := u_{new}(x_i) \text{ K } a_1 \cdot x_i$

$$b_1 := u_{new}(x_i) \text{ K } \frac{2\left(u_{new}\left(x_i + \frac{h}{2}\right) \text{ K } u_{new}(x_i)\right) x_i}{h} \tag{3.2.4}$$

> $b_2 := u_{new}(x_i + h) \text{ K } a_2 \cdot (x_i + h)$

$$b_2 := u_{new}(x_i + h) \text{ K } \frac{2\left(u_{new}(x_i + h) \text{ K } u_{new}\left(x_i + \frac{h}{2}\right)\right)(x_i + h)}{h} \tag{3.2.5}$$

> $b := u_{old}(x_i) \text{ K } a \cdot x_i$

$$b := u_{old}(x_i) \text{ K } \frac{(u_{old}(x_i + h) \text{ K } u_{old}(x_i))\, x_i}{h} \tag{3.2.6}$$

> $simplify\left((a_1 \cdot x + b_1 \text{ K } (a \cdot x + b))^2\right)$

$$\frac{1}{h^2}\left((2x \text{K } 2x_i)\, u_{new}\left(x_i + \frac{h}{2}\right) + (\text{K } x + x_i)\, u_{old}(x_i + h) + (h \text{K } 2x + 2x_i)\, u_{new}(x_i) \text{ K } u_{old}(x_i)\, (h \text{K } x + x_i)\right)^2 \tag{3.2.7}$$

> $simplify\left((a_2 \cdot x + b_2 \text{ K } (a \cdot x + b))^2\right)$

$$\frac{1}{h^2}\left((\text{K } 2h + 2x \text{K } 2x_i)\, u_{new}\left(x_i + \frac{h}{2}\right) + (h \text{K } 2x + 2x_i)\, u_{new}(x_i + h) + (x \text{ K } x_i)\, u_{old}(x_i + h) + u_{old}(x_i)\, (h \text{K } x + x_i)\right)^2 \tag{3.2.8}$$

> $simplify\left(\sqrt{\int_{x_i}^{x_i + \frac{h}{2}} (a_1 \cdot x + b_1 \text{K } (a \cdot x + b))^2 \, dx + \int_{x_i + \frac{h}{2}}^{x_i + h} (a_2 \cdot x + b_2 \text{K } (a \cdot x + b))^2 \, dx}\right)$

$$\frac{1}{6}\left(\sqrt{3}\,\sqrt{2}\left(h\left(2\,u_{new}\left(x_i + \frac{h}{2}\right)^2 + (u_{new}(x_i) \text{ K } 3\,u_{old}(x_i) + u_{new}(x_i + h) \text{ K } 3\,u_{old}(x_i + h))\right) u_{new}\left(x_i + \frac{h}{2}\right) + 2\,u_{old}(x_i + h)^2 + \left(\text{K } \frac{u_{new}(x_i)}{2} + 2\,u_{old}(x_i)\right)\right.\right. \tag{3.2.9}$$

$$\mathrm{K}\ \frac{5\,u_{new}(x_i+h)}{2}\Bigg)\,u_{old}(x_i+h)+u_{new}(x_i)^2\,\mathrm{K}\ \frac{5\,u_{old}(x_i)\,u_{new}(x_i)}{2}+2\,u_{old}(x_i)^2$$

$$\mathrm{K}\ \frac{u_{old}(x_i)\,u_{new}(x_i+h)}{2}+u_{new}(x_i+h)^2\Bigg)\Bigg)^{1/2}\Bigg)$$

> $err(h, x_i) :=$

$$\frac{1}{6}\Bigg(\sqrt{3}\,\sqrt{2}\,\Bigg(h\,\Bigg(2\,u_{new}\Big(x_i+\frac{h}{2}\Big)^2+\big(u_{new}(x_i)\ \mathrm{K}\ 3\,u_{old}(x_i)+u_{new}(x_i+h)\ \mathrm{K}\ 3\,u_{old}(x_i+h)\big)\,u_{new}\Big(x_i+\frac{h}{2}\Big)+2\,u_{old}(x_i+h)^2+\Bigg(\mathrm{K}\ \frac{u_{new}(x_i)}{2}+2\,u_{old}(x_i)\ \mathrm{K}\ \frac{5\,u_{new}(x_i+h)}{2}\Bigg)\,u_{old}(x_i+h)+u_{new}(x_i)^2\,\mathrm{K}\ \frac{5\,u_{old}(x_i)\,u_{new}(x_i)}{2}+2\,u_{old}(x_i)^2\ \mathrm{K}\ \frac{u_{old}(x_i)\,u_{new}(x_i+h)}{2}+u_{new}(x_i+h)^2\Bigg)\Bigg)^{1/2}\Bigg)$$

$$err := (h, x_i) \mapsto \frac{1}{6}\Bigg(\sqrt{3}\cdot\sqrt{2}\cdot\Bigg(h\cdot\Bigg(2\cdot u_{new}\Big(x_i+\frac{h}{2}\Big)^2+\big(u_{new}(x_i)\ \mathrm{K}\ 3\cdot u_{old}(x_i)+u_{new}(h+x_i)\ \mathrm{K}\ 3\cdot u_{old}(h+x_i)\big)\cdot u_{new}\Big(x_i+\frac{h}{2}\Big)+2\cdot u_{old}(h+x_i)^2+\Bigg(\mathrm{K}\ \frac{u_{new}(x_i)}{2}+2\cdot u_{old}(x_i)\ \mathrm{K}\ \frac{5\cdot u_{new}(h+x_i)}{2}\Bigg)\cdot u_{old}(h+x_i)+u_{new}(x_i)^2\,\mathrm{K}\ \frac{5\cdot u_{old}(x_i)\cdot u_{new}(x_i)}{2}+2\cdot u_{old}(x_i)^2\,\mathrm{K}\ \frac{u_{old}(x_i)\cdot u_{new}(h+x_i)}{2}+u_{new}(h+x_i)^2\Bigg)\Bigg)^{1/2}\Bigg) \qquad \textbf{(3.2.10)}$$

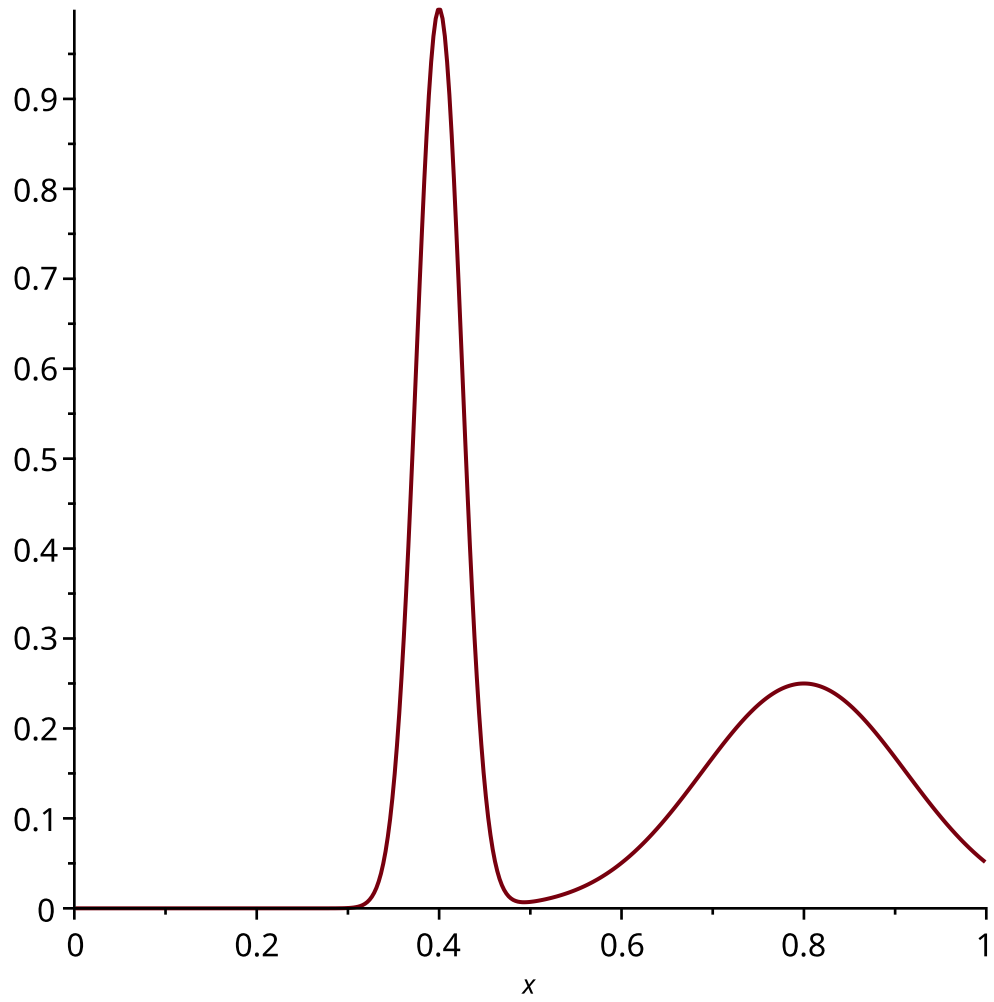
> $u_{old}(x) := \mathrm{e}^{\mathrm{K}\,800\cdot(x\,\mathrm{K}\ 0.4)^2}+0.25\cdot\mathrm{e}^{\mathrm{K}\,40\cdot(x\,\mathrm{K}\ 0.8)^2}$

$$u := x \mapsto \mathrm{e}^{\mathrm{K}\,800\cdot(x\,\mathrm{K}\ 0.4)^2}+0.25\cdot\mathrm{e}^{\mathrm{K}\,40\cdot(x\,\mathrm{K}\ 0.8)^2} \qquad \textbf{(3.2.11)}$$

> $u_{new}(x) := u_{old}(x)$

$$u_{new} \qquad \textbf{(3.2.12)}$$

> $plot(u(x), x = 0..1)$



> $simplify(err(h, x_i))$

$$0.2886751347\left(h\left(e^{K\,32.\left(5.x_i K\,2.\right)^2}+0.25\,e^{K\,1.6\left(5.x_i K\,4.\right)^2}+e^{K\,32.\left(5.x_i+5.h K\,2.\right)^2}+0.25\,e^{K\,1.6\left(5.x_i+5.h K\,4.\right)^2}\,K\,2.\,e^{K\,8.\left(10.x_i+5.h K\,4.\right)^2}\,K\,0.5\,e^{K\,0.4\left(10.x_i+5.h K\,8.\right)^2}\right)^2\right)^{1/2} \quad \textbf{(3.2.13)}$$

> $err(0.2, 0)$

$$1.027190526\times 10^{K\,8}\sqrt{3} \quad \textbf{(3.2.14)}$$

> $err(0.2, 0.3)$

$$0.1485731197\sqrt{3} \quad \textbf{(3.2.15)}$$

>

>

>

>

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