Handwritten notes

FEM

tir 2.1 2024

Exercise 1.1

a) Derive the exact expression for the element matrix K_i for the i'th element.

Let $y = (x - x_i)/h_i$ and recall that $N_i(x) = 1 - \frac{x - x_i}{h_i} = 1 - y$ and $N_{i+1}(x) = x - x_i / h_i = y$.

Then

$$k_{r,s}^{(i)} = \int_{x_i}^{x_{i+1}} [(N_r^{(i)})'(N_s^{(i)})' + N_r^{(i)} N_s^{(i)}]\, dx \quad , \quad r,s \in \{1,2\}$$

So

$$k_{1,1}^{(i)} = \int_{x_i}^{x_{i+1}} [(N_1^{(i)})'(N_1^{(i)})' + N_1^{(i)} N_1^{(i)}]\, dx$$

$$= \int_{x_i}^{x_{i+1}} [((1 - \frac{x - x_i}{h_i})')^2 + (1 - \frac{x - x_i}{h_i})^2\, dx$$

$$= \int_{x_i}^{x_{i+1}} (-\frac{1}{h_i})^2 + (1 - \frac{x - x_i}{h_i})^2\, dx$$

$$= \frac{1}{h_i} + \int_{x_i}^{x_{i+1}} (1 - \frac{x - x_i}{h_i})^2\, dx$$

Use integration by substitution with $y = (x - x_i)/h_i$.

$$k_{1,1}^{(i)} = \frac{1}{h_i} + \int_1^0 -h_i \cdot y^2\, dy$$

$$= \frac{1}{h_i} + h_i \int_0^1 y^2\, dy$$

$$= \frac{1}{h_i} + h_i \left[\frac{1}{3} y^3\right]_0^1 = \frac{1}{h_i} + \frac{h_i}{3}$$

and

$$k_{1,2}^{(i)} = \int_{x_i}^{x_{i+1}} [(N_1^{(i)})'(N_2^{(i)})' + N_1^{(i)} N_2^{(i)}]\, dx$$

$$= \int_{x_i}^{x_{i+1}} [(-\frac{1}{h_i}) \cdot (+\frac{1}{h_i}) + (1 - \frac{x - x_i}{h_i}) \frac{x - x_i}{h_i}]\, dx$$

$$= -\frac{1}{h_i} + \int_{x_i}^{x_{i+1}} (\frac{x - x_i}{h_i} - (\frac{x - x_i}{h_i})^2)\, dx$$

$$= -\frac{1}{h_i} + \int_{x_i}^{x_{i+1}} \frac{x - x_i}{h_i}\, dx - \int_{x_i}^{x_{i+1}} (\frac{x - x_i}{h_i})^2\, dx$$

$$= -\frac{1}{h_i} + \int_0^1 h_i y\, dy - \int_0^1 h_i y^2\, dy$$

$$= -\frac{1}{h_i} + h_i \left[\frac{1}{2} y^2\right]_0^1 - h_i \left[\frac{1}{3} y^3\right]_0^1$$

$$= -\frac{1}{h_i} + \frac{h_i}{2} - \frac{h_i}{3}$$

$$= -\frac{1}{h_i} + \frac{h_i}{6}$$

$\frac{1}{2}+\frac{1}{3} = \frac{3}{6}\ \frac{2}{6}$

b) Let $L=2$, $c=1$ ($\hat{u}_1$), $d=e^2$ ($\hat{u}_3$) and $M=3$. Solve for $\hat{u}_i$

Uniform mesh: $h_1 = h_2 = h$

We solve

$$k_{2,1}^{(1)} \hat{u}_1 + (k_{2,2}^{(1)} + k_{1,1}^{(2)})\hat{u}_2 + k_{1,2}^{(2)}\hat{u}_3 = 0$$

$$\Updownarrow$$

$$\left(-\frac{1}{h}+\frac{h}{6}\right)\hat{u}_1 + \left(\frac{1}{h}+\frac{h}{3}+\frac{1}{h}+\frac{h}{3}\right)\hat{u}_2 + \left(-\frac{1}{h}+\frac{h}{6}\right)\hat{u}_3 = 0$$

$$\Updownarrow$$

$$\left(-\frac{1}{h}+\frac{h}{6}\right) + \left(\frac{2}{h}+\frac{2h}{3}\right)\hat{u}_2 + \left(-\frac{1}{h}+\frac{h}{3}\right)e^2 = 0$$

$$\Updownarrow$$

$$\left(-\frac{1}{h}+\frac{h}{6}\right)(1+e^2) + \left(\frac{2}{h}+\frac{2h}{3}\right)\hat{u}_2 = 0$$

$$\Updownarrow$$

$$\hat{u}_2 = \frac{-\left(-\frac{1}{h}+\frac{h}{6}\right)(1+e^2)}{\frac{2}{h}+\frac{2h}{3}}$$

$$\frac{2h^2}{3h} + \frac{6}{3h} = \frac{6+2h^2}{3h}$$

$$= \frac{\frac{1}{h}-\frac{h}{6}}{\frac{6+2h^2}{3h}}(1+e^2)$$

$$= \frac{3-\frac{1}{2}h^2}{6+2h^2}(1+e^2)$$

$$= \frac{\frac{5}{2}}{8}(1+e^2) = \frac{5}{16}(1+e^2) \approx 2{,}62$$

Non uniform mesh where $h_1 = 2h_2$ $\quad h_2 = \frac{1}{2}h_1$

$$\left(-\frac{1}{h_1}+\frac{h_1}{6}\right)1 + \left(\frac{1}{h_1}+\frac{h_1}{3}+\frac{1}{h_2}+\frac{h_2}{3}\right)\hat{u}_2 + \left(-\frac{1}{h_2}+\frac{h_2}{6}\right)e^2 = 0$$

$$\Updownarrow$$

$$\left(-\frac{1}{h_1}+\frac{h_1}{6}\right) + \left(\frac{1}{h_1}+\frac{h_1}{3}+\frac{2}{h_1}+\frac{h_1}{6}\right)\hat{u}_2 + \left(-\frac{2}{h_1}+\frac{h_1}{12}\right)e^2 = 0$$

$$\Updownarrow$$

$$\left(\frac{3}{h_1}+\frac{h_1}{3}\right)\hat{u}_2 = \frac{1}{h_1} - \frac{h_1}{6} - \left(-\frac{2}{h_1}+\frac{h_1}{12}\right)e^2$$

$$\Updownarrow$$

$$\hat{u}_2 = \frac{\frac{6-h_1^2}{6h_1} + \left(\frac{2}{h_1}-\frac{h_1}{12}\right)e^2}{\frac{3}{2h_1}+\frac{2h_1}{3}}$$

$$L = h_1 + h_2 = 3h_2 \Leftrightarrow h_2 = \frac{2}{3} \Rightarrow h_1 = \frac{4}{3}$$

$$= \left(\frac{6-\frac{16}{9}}{8} + \left(\frac{6}{4}-\frac{4}{36}\right)e^2\right) / \left(\frac{6}{8}+\frac{4}{3}\right)$$

$$= \left(\frac{6}{8} - \frac{2}{9} + (\ldots\right.$$

c) We find $u(x)$, for the exact solution.

$u(0) = C_1 + C_2 = 1$

$u(2) = C_1 e^2 + C_2 e^{-2} = e^2$

Then $C_1 = 1$ and $C_2 = 0$.

We plot in Maple.

d) Same as c

f) The basic steps for FEM:

1) Compute $K^{(i)}$

2) Combine the $K^{(i)}$'s into a matrix A.

3) Solve $A\vec{u} = b$ for $\vec{u}$

4) Compute $\hat{u}(x)$.

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Exercise 1.5

a) $(-(\epsilon u')' + (4u)')v = fv$

$\Leftrightarrow \int_0^1 -(\epsilon u')' + (4u)' v\,dx = \int_0^1 fv\,dx$

$\Leftrightarrow -\int_0^1 (\epsilon u' + 4u)' v\,dx = \int_0^1 fv\,dx$

$\Leftrightarrow -(u(1)v(1) - u(0)v(0) - \int_0^1 (\epsilon u + 4u) v'\,dx) = \int_0^1 fv\,dx$

$\Leftrightarrow \int_0^1 (\epsilon u' + 4u) v'\,dx = \int_0^1 fv\,dx$

$\Leftrightarrow \int_0^1 \epsilon u'v'\,dx + \int_0^1 4uv'\,dx = \int_0^1 fv\,dx$

c) $v^T A v = a\left(\sum_{i=1}^{M} v_i N_i , \sum_{j=1}^{M} v_j N_j\right)$

$$= \int_0^1 \epsilon\left(\sum_{i=1}^{M} v_i N_i' \sum_{j=1}^{M} v_j N_j'\right) - \psi\left(\sum_{i=1}^{M} v_i N_i \sum_{j=1}^{M} v_j N_j'\right) dx$$

$$= \int_0^1 \sum_{j=1}^{M} v_j N_j' \left(\epsilon \sum_{i=1}^{M} v_i N_i' - \psi \sum_{i=1}^{M} v_i N_i\right) dx$$

$$= \sum_{j=1}^{M} \int_0^1 v_j N_j' \left(\epsilon \sum_{i=1}^{M} v_i N_i' - \psi \sum_{i=1}^{M} v_i N_i\right) dx$$

$$= \sum_{j=1}^{M} \int_0^1 v_j N_j' \sum_{i=1}^{M} (\epsilon v_i N_i' - \psi v_i N_i) dx$$

$$= \sum_{j=1}^{M} \sum_{i=1}^{M} \int_0^1 \epsilon v_i v_j N_i' N_j' - \psi v_i v_j N_i N_j' dx \qquad = \int_0^1 f \sum_{j=1} v_j N_j dx$$

Let $\hat{v}' = \sum_{j=1}^{M} v_j N_j'$ and $\hat{v} = \sum_{j=1}^{M} v_j N_j$ then

$$v^T A v = \int_0^1 \epsilon \hat{v}' \hat{v}' - \psi \hat{v} \hat{v}' dx$$

$$= \int_0^1 \epsilon \hat{v}' \hat{v}' - \psi \int_0^1 \hat{v} \hat{v}' dx$$

$$= \int_0^1 \epsilon (\hat{v}')^2 dx - \frac{\psi}{2} \int_0^1 (\hat{v}^2)' dx$$

$$= \int_0^1 \epsilon (\hat{v}')^2 dx - \frac{\psi}{2} [\hat{v}^2]_0^1$$

$$= \int_0^1 \epsilon (\hat{v}')^2 dx - \psi/2 (\hat{v}(1)^2 - \hat{v}(0)^2)$$

$$= \int_0^1 \epsilon (\hat{v}')^2 dx \geq 0$$

Since $\epsilon > 0$ and $(\hat{v}')^2 \geq 0$

If $v^T A v = 0$ then

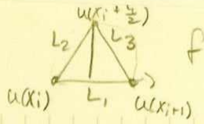
$$\int_0^1 \epsilon (\hat{v}')^2 = 0 \Rightarrow \int_0^1 (\hat{v}')^2 = 0 \Rightarrow (\hat{v}')^2 = 0$$

Then

$$\left(\sum_{j=1}^{M} v_j N_j'\right)^2 = 0 \Rightarrow v_j = 0 \quad \forall j \in \{1, \dots, M\}$$

because $N_j' \neq 0$ when $x \in]x_{j-1}, x_{j+1}[\ \forall j \in \{1, \dots, M\}$ so if any $v_j > 0$, then $\sum v_j N_j' > 0$.

$L_1 = ax+b$
$L_2 = cx+d$
$L_3 = ex+f$



$f = \begin{cases} L_2 - L_1 & x \in [x_i, x_{i+\frac{h}{2}}] \\ L_3 - L_1 & x \in [x_{i+\frac{h}{2}}, x_{i+1}] \end{cases}$

Exercise 1.6

a) $\| \Pi_{h/2,i} u - \Pi_{h,i} u \|_{L^2(e^i)} = \left(\int_{x_i}^{x_{i+1}} | \Pi_{\frac{h}{2},i} u - \Pi_{h,i} u_2 |^2 \, dx \right)^{\frac{1}{2}}$

$$= \left(\int_{x_i}^{x_i+\frac{h}{2}} |f|^2 dx + \int_{x_i+\frac{h}{2}}^{x_{i+1}} |f|^2 dx \right)^{\frac{1}{2}}$$

$$= \left(\int_{x_i}^{x_i+\frac{h}{2}} |cx+d-(ax+b)|^2 dx + \int_{x_i+\frac{h}{2}}^{x_{i+1}} |ex+f-(ax+b)|^2 dx \right)^{\frac{1}{2}}$$

$$= \left(\int_{x_i}^{x_i+\frac{h}{2}} |(c-a)x+(d-b)|^2 dx + \int_{x_i+\frac{h}{2}}^{x_{i+1}} |(e-a)x+(f-b)|^2 dx \right)^{\frac{1}{2}}$$

$$= \left(\int_{x_i}^{x_i+\frac{h}{2}} (c-a)^2 x^2 + ((c-a)(d-b)x + (d-b)^2 dx + \int_{x_i+\frac{h}{2}}^{x_{i+1}} (e-a)^2 x^2 + (e-a)(f-b)x + (f-b)^2 dx \right)$$

$$= \left(\left[\frac{(c-a)^2}{3} x^3 + \frac{(c-a)(d-b)}{2} x^2 + (d-b)^2 x \right]_{x_i}^{x_i+\frac{h}{2}} + \left[\frac{(e-a)^2}{3} x^3 + \frac{(e-a)(f-b)}{2} x^2 + (f-b)^2 x \right]_{x_i+\frac{h}{2}}^{x_{i+1}} \right)^{\frac{1}{2}}$$

$$= \left(\frac{(c-a)^2}{3} \left((x_i+\tfrac{h}{2})^3 - x_i^3 \right) + \frac{(c-a)(d-b)}{2} \left((x_i+\tfrac{h}{2})^2 - x_i^2 \right) + (d-b)^2 \frac{h}{2} + \frac{(e-a)^2}{3} \left(x_{i+1}^3 - (x_i+\tfrac{h}{2})^3 \right) + \frac{(e-a)(f-b)}{2} \left(x_{i+1}^2 - (x_i+\tfrac{h}{2})^2 \right) + (f-b)^2 \frac{h}{2} \right)^{\frac{1}{2}}$$

we define a, b, c, d and e.

$$a = \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i} = \frac{u(x_{i+1}) - u(x_i)}{h} \quad b =$$

$$c = \frac{u(x_i+\frac{h}{2}) - u(x_i)}{x_i + \frac{h}{2} - x_i} = \frac{2(u(x_i+\frac{h}{2}) - u(x_i))}{h}$$

$$e = \frac{u(x_{i+1}) - u(x_i+\frac{h}{2})}{x_{i+1} - (x_i+\frac{h}{2})} = \frac{2(u(x_{i+1}) - u(x_i+\frac{h}{2}))}{2}$$

Then

$$c - a = \frac{2u(x_i+\frac{h}{2}) - 2u(x_i)}{h} - \frac{u(x_{i+1}) - u(x_i)}{h} = \frac{2u(x_i+\frac{h}{2}) - (u(x_{i+1}) + u(x_i))}{h}$$

$$e - a = \frac{2u(x_{i+1}) - 2u(x_i+\frac{h}{2})}{h} - \frac{u(x_{i+1}) - u(x_i)}{h} = \frac{u(x_{i+1}) + u(x_i) - 2u(x_i+\frac{h}{2})}{h}$$

$3h - 12 - \frac{4}{h}$

Because N_i only overlaps with N_{i-1}, N_i and N_{i+1} we can simplify the system to

$$\sum_{j=1}^{M} a_{ij} \cdot u_j = a_{i,i-1} u_{i-1} + a_{i,i} u_i + a_{i,i+1} u_{i+1} = \int_0^1 f N_i(x) dx$$

We then define $k_{r,q}^{(i)} = \int_{x_i}^{x_{i+1}} N_s'(x) (\epsilon N_r'(x) - \psi N_r(x)) dx$ and compute:

$$k_{2,1}^{(i)} = \int_{x_i}^{x_{i+1}} N_1'(x) (\epsilon N_2'(x) - \psi N_{2}(x)) dx$$

$$= \int_{x_{i-1}}^{x_{i+1}} \frac{1}{h_i} (\epsilon (-\frac{1}{h_i}) - \psi (1 - \frac{x-x_i}{h_i}) dx$$

$$= \frac{\epsilon}{-h_i} - \psi \int_{x_i}^{x_{i+1}} \frac{x-x_i}{h_i} dx \qquad y = 1 - \frac{x-x_i}{h_i}$$

$$= -\frac{\epsilon}{h_i} - \psi \int_0^1 y \, dx$$

$$= -\frac{\epsilon}{h_i} + \psi [\tfrac{1}{2} y^2]_0^1 = -\frac{\epsilon}{h_i} - \frac{\psi}{2}$$

$$\cancel{= -\frac{\epsilon}{h_i} + \psi [\tfrac{1}{2} y^2]_0^1 = -\frac{\epsilon}{h_i} + \frac{\psi}{2} = \frac{\psi}{2} - \frac{\epsilon}{h_i}}$$

$$k_{1,1} = \int_{x_i}^{x_{i+1}} N_1'(x) (\epsilon N_1'(x) - \psi N_1(x)) dx$$

$$= \int_{x_i}^{x_{i+1}} \frac{1}{h_i} (\epsilon \frac{1}{h_i} - \psi (1 - \frac{x-x_i}{h_i}) dx$$

$$= \frac{\epsilon}{h_i} - \int_{x_i}^{x_{i+1}} \psi (1 - \frac{x-x_i}{h_i}) dx$$

~~Int~~ Integration by substitution

$$= \frac{\epsilon}{h_i} + \frac{\psi}{2}$$

$$k_{2,2} = \int_{x_i}^{x_{i+1}} N_2'(x) (\epsilon N_2'(x) - \psi N_2'(x)) dx \qquad y = \frac{x-x_i}{h_i}$$

$$= \frac{\epsilon}{h_i} - \psi \int_{x_i}^{x_{i+1}} \frac{x-x_i}{h_i} dx$$

$$= \frac{\epsilon}{h_i} - \psi \int_0^1 y \, dy$$

$$= \frac{\epsilon}{h_i} - \frac{\psi}{2}$$

$$k_{1,2} = \int_{x_i}^{x_{i+1}} N_2'(x) (\epsilon N_1'(x) - \psi N_1(x)) dx$$

$$= -\frac{\epsilon}{h_i} - \psi \int_{x_i}^{x_{i+1}} (1 - \frac{x-x_i}{h_i}) dx \qquad y = 1 - \frac{x-x_i}{h_i}$$

$$= -\frac{\epsilon}{h_i} + \psi \int_0^1 y^2 dy$$

$$= -\frac{\epsilon}{h_i} + \frac{\psi}{2}$$

b) DE: $(\epsilon u')' + (\psi u)' = f$ (1), $u(0) = 0 = u(1)$, ie: $L = 1$

From (a) we know that we can reformulate the problem to:

$$\int_0^1 \epsilon u' v' dx + \int_0^1 \psi u v' dx = \int_0^1 f v \, dx \qquad (2)$$

and that if u solves (2), then v also solves (1).

We use the finite element method on the weak formulation on a general mesh with M nodes so we approximate $u(x)$ with:

$$\hat{u}(x) = \sum_{j=1}^{M} \hat{u}_j N_j(x)$$

We note that $N_j(0) = \begin{cases} 1 & j=1 \\ 0 & \text{otherwise} \end{cases}$ and $N_j(1) = \begin{cases} 1 & j=M \\ 0 & \text{otherwize} \end{cases}$ hence to satisfy the boundary conditions we set

$$\hat{u}(0) = \hat{u}_1 = 0 \quad \text{and} \quad \hat{u}(1) = \hat{u}_M = 0$$

We substitute $\hat{u}$ for $u(x)$ and put $v(x) = N_i(x)$ to find $\hat{u}_i$.

~~Then~~

$$\int_{x_i}^{x_{i+1}} (\epsilon \hat{u}' N_i(x)' dx + \int_{x_i}^{x_{i+1}} \psi \hat{u} N_i(x) dx = \int_{x_i}^{x_{i+1}} f N_i(x) \, dx$$

$$\int_{x_i}^{x_{i+1}} (\epsilon (\sum_{j=1}^{M} \hat{u}_j N_j(x))' N_i(x)' dx + \int_{x_i}^{x_{i+1}} \psi \sum_{j=1}^{M} \hat{u}_j N_j(x) N_i(x)' dx = \int_{x_i}^{x_{i+1}} f N_i(x) dx$$

$$\int_{x_i}^{x_{i+1}} (\epsilon (\hat{u}_{i+1} N_{i+1}(x) + \hat{u}_i N_i(x))' N_i(x)') dx + \int_{x_i}^{x_{i+1}} \psi (\hat{u}_{i+1} N_{i+1}(x) + N_i(x) \hat{u}_i) N_i(x)' dx = -\text{''}-$$

$$\int_{x_i}^{x_{i+1}} (\epsilon \hat{u}_{i+1} N_{i+1}'(x) + \hat{u}_i N_i'(x)) N_i'(x) dx + \int_{x_i}^{x_{i+1}} \psi (\hat{u}_{i+1} N_{i+1}(x) + N_i(x) \hat{u}_i) N_i'(x) dx = -\text{''}-$$

$$\int_{x_i}^{x_{i+1}} (\epsilon (\hat{u}_{i+1} \tfrac{1}{h_i}(x) + \hat{u}_i (-\tfrac{1}{h_i})) (-\tfrac{1}{h_i}) dx + \int_{x_i}^{x_{i+1}} \psi (\hat{u}_{i+1} N_{i+1}(x) + N_i(x) \hat{u}_i) (-\tfrac{1}{h_i}) dx = -\text{''}-$$

$$\int_{x_i}^{x_{i+1}} (\epsilon (u_{i+1}$$

Then

$$\int_0^1 (\epsilon \hat{u}' N_i(x)' dx - \int_0^1 \psi \hat{u} N_i(x)' dx = \int_0^1 f v \, dx$$

$$\Leftrightarrow \int_0^1 (\epsilon (\sum_{j=1}^{M} \hat{u}_j N_j'(x) N_i'(x) - \psi \sum_{j=1}^{M} \hat{u}_j N_j(x) N_i'(x) dx = \int_0^1 f v dx$$

$$\Leftrightarrow \int_0^1 (\sum_{j=1}^{M} (\epsilon \hat{u}_j N_j'(x) N_i'(x) - \psi \hat{u}_j N_j(x) N_i'(x)) dx = \int_0^1 f N_i(x) dx$$

$$\Leftrightarrow \sum_{j=1}^{M} u_j \underbrace{\int_0^1 (\epsilon N_j'(x) N_i'(x) - \psi N_j(x) N_i'(x)) dx}_{a_{ij}} = \int_0^1 f N_i(x) dx$$

$\parallel N_i'(\epsilon N_j'(x) - \psi N_j(x))$

$$\Leftrightarrow \sum_{j=1}^{M} u_j a_{ij} = \int_0^1 f N_i(x) \, dx$$

Note that $a_{i,i-1} = k^{(i-1)}_{2,1}$, $a_{ii} = k^{(i-1)}_{22} + k^{(i)}_{1,1}$, $a_{i,i+1} = k^{(i)}_{1,2}$
hence we can substitute so we get

$$k^{(i-1)}_{2,1}\hat{u}_{i-1} + (k^{(i-1)}_{2,2} + k^{(i)}_{1,1})\hat{u}_i + k^{(i)}_{1,2}\hat{u}_{i+1} = \int_0^1 f N_i(x)\,dx$$

We then approximate f by

$$\hat{f}(x) = \sum_{j=1}^{m} \hat{f}_j N_j \qquad \hat{f}_j = f(x_j)$$

and substitute f with $\hat{f}$ so

$$b_i = \int_0^1 \hat{f} N_i(x)\,dx = \int_0^1 \sum_{j=1}^{m} \hat{f}_j N_j N_i\,dx = \sum_{j=1}^{m}\left(\int_0^1 \hat{f}_j N_j N_i\,dx\right) = \sum_{j=1}^{m} \hat{f}_j \underbrace{\int_0^1 N_j N_i\,dx}_{C_{i,j}} = \sum_{j=1}^{m} \hat{f}_j C_{i,j}$$

N_i only overlaps with N_{i-1}, N_i and N_{i+1} and consequently

$$b_i = \hat{f}_{i-1}\int_0^1 N_{i-1}N_i\,dx + \hat{f}_i\int_0^1 N_i^2\,dx + \hat{f}_{i+1}\int_0^1 N_{i+1}N_i\,dx$$

$$= \hat{f}_{i-1}\int_{x_{i-1}}^{x_i} N_{i-1}(x)N_i(x)\,dx + \hat{f}_i\int_{x_{i-1}}^{x_{i+1}} N_i(x)^2 + \hat{f}_{i+1}\int_{x_i}^{x_{i+1}} N_{i+1}(x)N_i(x)\,dx$$

$$= \hat{f}_{i-1}\int_{x_{i-1}}^{x_i}\left(1-\frac{x-x_{i-1}}{h_{i-1}}\right)\left(\frac{x-x_{i-1}}{h_{i-1}}\right)dx + \hat{f}_i\left(\int_{x_{i-1}}^{x_i}\left(\frac{x-x_{i-1}}{h_{i-1}}\right)^2 dx + \int_{x_i}^{x_{i+1}}\left(1-\frac{x-x_i}{h_i}\right)^2 dx\right) + \hat{f}_{i+1}\int_{x_i}^{x_{i+1}}\left(\frac{x-x_i}{h_i}\right)\left(1-\frac{x-x_i}{h_i}\right)dx$$

$$= \hat{f}_{i-1}\int_0^1 (1-y)y\,h_{i-1}\,dy + \hat{f}_i\int_0^1 h_{i-1}y^2\,dy + \int_0^1 h_i w^2\,dw + \hat{f}_{i+1}\int_0^1 h_i z(1-z)\,dz$$

$$= \hat{f}_{i-1}h_{i-1}\left[\tfrac{1}{2}y^2 - \tfrac{1}{3}y^3\right]_0^1 + \hat{f}_i h_{i-1}\left[\tfrac{1}{3}y^3\right]_0^1 + \hat{f}_i h_i\left[\tfrac{1}{3}z^3\right]_0^1 + \hat{f}_{i+1}h_i\left[\tfrac{1}{2}z^2 - \tfrac{1}{3}z^3\right]_0^1$$

$$= \hat{f}_{i-1}\left(\tfrac{1}{2}-\tfrac{1}{3}\right) + \hat{f}_i\,\tfrac{1}{3} + \hat{f}_{i+1}\left(\tfrac{1}{2}-\tfrac{1}{3}\right)$$

$$= \left(\frac{\hat{f}_{i-1}h_{i-1}}{6} + \frac{\hat{f}_i h_{i-1}}{3} + \frac{\hat{f}_i h_i}{3} + \frac{\hat{f}_{i+1}h_i}{6}\right) \qquad i \in \{2,\dots,M-1\}$$

$$b_1 = \int_0^1 \hat{f} N_1(x)\,dx = \int_0^1 \sum_{j=1}^{M} \hat{f}_j N_j N_1\,dx = \int_0^1 \hat{f}_1 N_1^2 + \hat{f}_2 N_2 N_1\,dx$$

$$= \hat{f}_1\int_{x_1}^{x_2}\left(1-\frac{x-x_1}{h_1}\right)^2 dx + \hat{f}_2\int_{x_1}^{x_2}\left(\frac{x-x_1}{h_1}\right)\left(1-\frac{x-x_1}{h_1}\right)dx$$

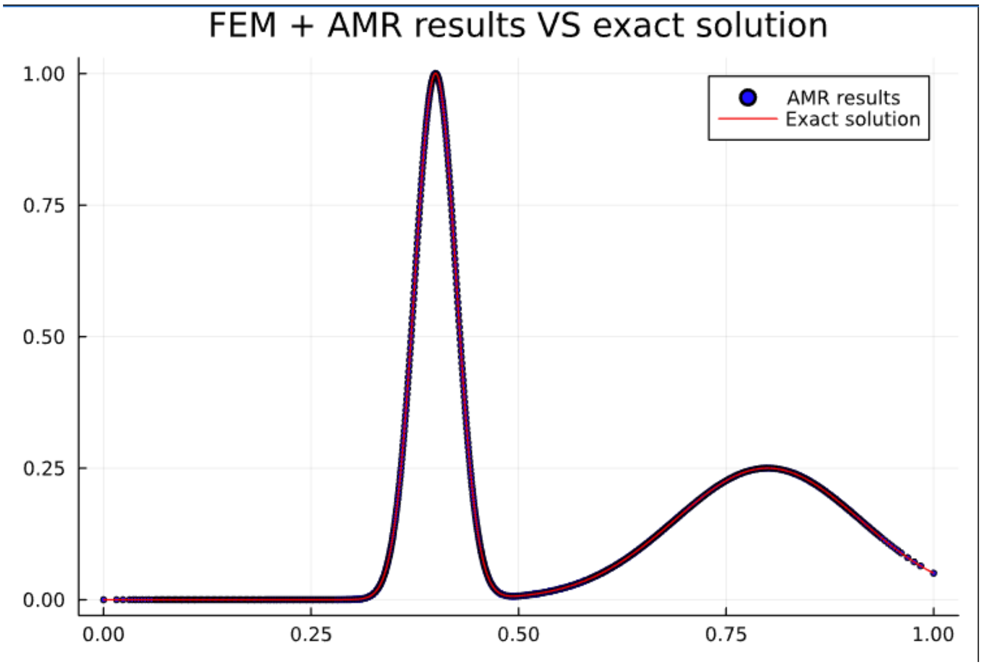
$$= \hat{f}_1\int_0^1 h_1(1-y)^2\,dy + \hat{f}_2\int_0^1 h_1\,y(1-y)\,dy$$

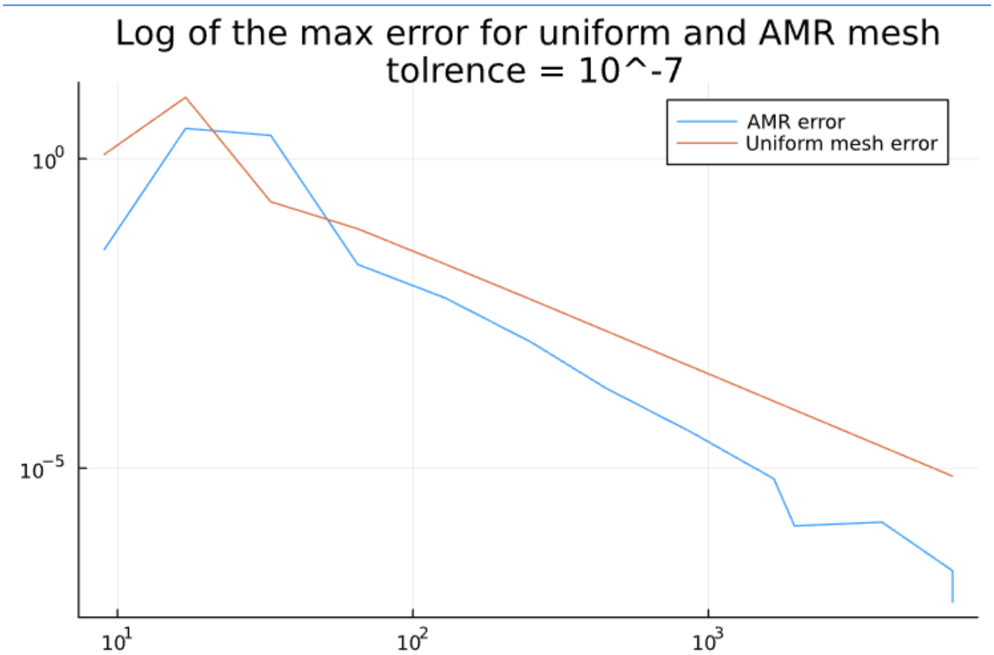
$$= \hat{f}_1 h_1\left[\tfrac{1}{3}y^3 - y^2 + y\right]_0^1 + \hat{f}_2 h_1\left[\tfrac{1}{2}y^2 - \tfrac{1}{3}y^3\right]_0^1$$

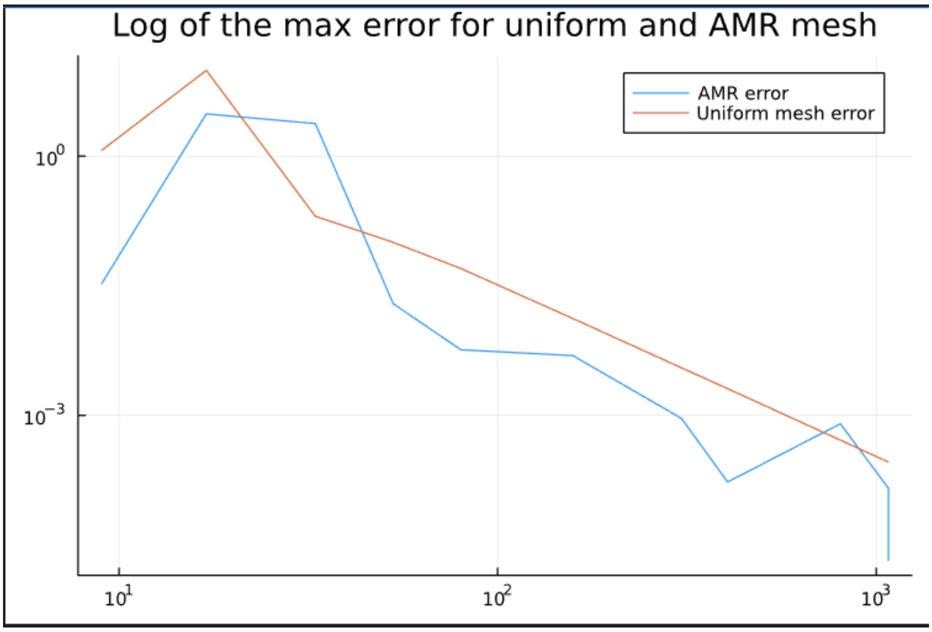
$$= \frac{\hat{f}_1 h_1}{3} + \frac{\hat{f}_2 h_1}{6} \qquad = \frac{h_1}{2}$$

$$= f_i\left(\frac{h_{i-1}+h_i}{3}\right) + \frac{f_{i-1}h_{i-1}}{6} + \frac{f_{i+1}h_i}{h}$$

1) The CPU time is: 0.004477s
2) AMR iteration count: 10
3) Degrees of freedom for final mesh: 724
4) Relevant plots:







5) We used a computer with 8 logical processors with arch linux as its operating system. We wrote the code in julia 1.10 and computed that our CO2 consumption is 0.000069953125g.
6) CO2 consumption formula: 15 what/8 *(cpu time / 3600)h * 30g/kwh
7) We used Error tolerance (1e-4) and initial mesh configuration used (x=[0.0, 0.5, 1.0], M=3)
8) For each iteration of our AMR we create a fine mesh, where we have subdivided each element. We then compute the estimated errors for each old element. The elements where the error is larger than our tolerance we keep, and the rest we do not update. We keep iterating until there are no elements where the estimated error is larger than the tolerance.
9) We used this website to find the CO2 calculation formula. https://devblogs.microsoft.com/sustainable-software/how-can-i-calculate-co2eq-emissions-for-my-azure-vm/