Line–sphere intersection

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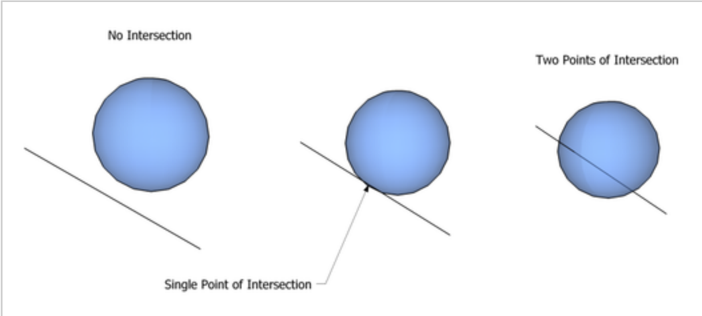
In analytic geometry, a line and a sphere can intersect in three ways: no intersection at all, at exactly one point, or in two points. Methods for distinguishing these cases, and determining equations for the points in the latter cases, are useful in a number of circumstances. For example, this is a common calculation to perform during ray tracing (Eberly 2006:698).



The three possible line-sphere intersections:
1. No intersection.
2. Point intersection.
3. Two point intersection.

Calculation using vectors in 3D

In vector notation, the equations are as follows:

Equation for a sphere

$$\|\mathbf{x}-\mathbf{c}\|^2 = r^2$$

- $\mathbf{c}$ - center point
- r - radius
- $\mathbf{x}$ - points on the sphere

Equation for a line starting at $\mathbf{o}$

$$\mathbf{x} = \mathbf{o} + d\mathbf{l}$$

- d - distance along line from starting point
- $\mathbf{l}$ - direction of line (a unit vector)
- $\mathbf{o}$ - origin of the line
- $\mathbf{x}$ - points on the line

Searching for points that are on the line and on the sphere means combining the equations and solving for d:

Equations combined

$$\|\mathbf{o}+d\mathbf{l}-\mathbf{c}\|^2 = r^2 \Leftrightarrow (\mathbf{o}+d\mathbf{l}-\mathbf{c})\cdot(\mathbf{o}+d\mathbf{l}-\mathbf{c}) = r^2$$

Expanded

$$d^2(\mathbf{l}\cdot\mathbf{l}) + 2d(\mathbf{l}\cdot(\mathbf{o}-\mathbf{c})) + (\mathbf{o}-\mathbf{c})\cdot(\mathbf{o}-\mathbf{c}) = r^2$$

Rearranged

$$d^2(\mathbf{l}\cdot\mathbf{l}) + 2d(\mathbf{l}\cdot(\mathbf{o}-\mathbf{c})) + (\mathbf{o}-\mathbf{c})\cdot(\mathbf{o}-\mathbf{c}) - r^2 = 0$$

The form of a quadratic formula is now observable. (This quadratic equation is an example of Joachimsthal's Equation [1] (http://mathworld.wolfram.com/JoachimsthalsEquation.html).)

$$ad^2 + bd + c = 0$$

where

- $a = \mathbf{l}\cdot\mathbf{l} = \|\mathbf{l}\|^2$

- $b = 2(\mathbf{l} \cdot (\mathbf{o} - \mathbf{c}))$
- $c = (\mathbf{o} - \mathbf{c}) \cdot (\mathbf{o} - \mathbf{c}) - r^2 = \|\mathbf{o} - \mathbf{c}\|^2 - r^2$

Simplified

$$d = \frac{-(\mathbf{l} \cdot (\mathbf{o} - \mathbf{c})) \pm \sqrt{(\mathbf{l} \cdot (\mathbf{o} - \mathbf{c}))^2 - \|\mathbf{l}\|^2 (\|\mathbf{o} - \mathbf{c}\|^2 - r^2)}}{\|\mathbf{l}\|^2}$$

Note that $\mathbf{l}$ is a unit vector, and thus $\|\mathbf{l}\|^2 = 1$. Thus, we can simplify this further to

$$d = -(\mathbf{l} \cdot (\mathbf{o} - \mathbf{c})) \pm \sqrt{(\mathbf{l} \cdot (\mathbf{o} - \mathbf{c}))^2 - \|\mathbf{o} - \mathbf{c}\|^2 + r^2}$$

- If the value under the square-root ($(\mathbf{l} \cdot (\mathbf{o} - \mathbf{c}))^2 - \|\mathbf{o} - \mathbf{c}\|^2 + r^2$) is less than zero, then it is clear that no solutions exist, i.e. the line does not intersect the sphere (case 1).
- If it is zero, then exactly one solution exists, i.e. the line just touches the sphere in one point (case 2).
- If it is greater than zero, two solutions exist, and thus the line touches the sphere in two points (case 3).

See also

- Analytic geometry
- Line-plane intersection
- Line of intersection between two planes

References

- David H. Eberly (2006), *3D game engine design: a practical approach to real-time computer graphics*, 2nd edition, Morgan Kaufmann. ISBN 0-12-229063-1