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Notice: This material will be included in a forthcoming (summer 2000) book with the tentative title *Experiencing Geometry in Two- and Three-Dimensional Spaces*. This new book will be an expanded and updated version of *Experiencing Geometry on Plane and Sphere*. This material is in draft form and may not be duplicated or quoted without the author's written permission, except for purposes of review or trying out the material with students. As always comments are welcome and will affect the final draft. Send comments to dwh2@cornell.edu.

Chapter 2

Straightness on Sphere, Cylinder, and Cone

... it will readily be seen how much space lies between the two places themselves on the circumference of the large circle which is drawn through them around the earth. ... [W]e grant that it has been demonstrated by mathematics that the surface of the land and water is in its entirety a sphere, ... and that any plane which passes through the center makes at its surface, that is, at the surface of the earth and of the sky, great circles, and that the angles of the planes, which angles are at the center, cut the circumferences of the circles which they intercept proportionately, ...

— Ptolemy, *Geographia* (ca. 150 AD) Book One, Chapter II

Drawing upon the intuitive ideas about straightness developed in the first chapter, this chapter asks for criteria for straightness on a sphere, cone and cylinder. It is important for you to see that, if you are not building a notion of straightness for yourself (for example, if you are taking ideas from books without thinking deeply about them), then you will have difficulty building a concept of straightness on other surfaces. Only by developing a personal meaning of straightness for oneself does it become part of one's active intuition. We say *active* intuition to emphasize that intuition is in a process of constant change and enrichment, that it is not static.

Problem 2.1. *What Is Straight on a Sphere?*

a. *Imagine yourself to be a bug crawling around on a sphere.* (*This bug can neither fly nor burrow into the sphere.*) *The bug's universe is just the surface*; *it never leaves it. What is "straight" for this bug*? *What will the bug see or experience as straight*? *How can you convince yourself of this*? *Use the properties of straightness* (*such as symmetries*) *which you talked about in Problem* **1.1**.

b. *Show* (i.e., convince yourself, and give an argument to convince others) *that the great circles on a sphere are straight with respect to the sphere, and that no other circles on the sphere are straight with respect to the sphere.*

Suggestions

Great circles are those circles which are the intersection of the sphere with a plane through the center of the sphere. Examples: Any longitude line and the equator are great circles on the earth — consider any pair of opposite points as being the poles and thus the equator and longitudes with respect to any pair of opposite points will be great circles. See examples illustrated in Figure 2.1.

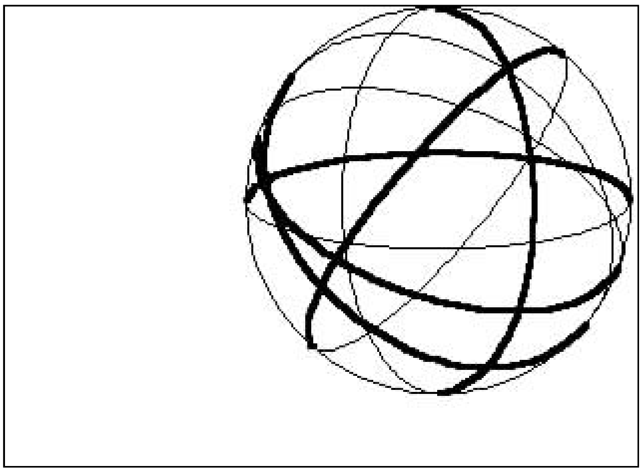

Figure 2.1. **Great Circles.**

The first step to understanding this problem is to convince yourself that great circles are straight lines on a sphere. Think what it is about the great circles that would make the bug experience them as straight. To better visualize what is happening on a sphere (or any other surface, for that matter), **you must use models**. This is a point we cannot stress enough. The use of models will become increasingly important in later problems, especially those involving more than one line. You must make lines on a sphere to fully understand what is straight and why. An orange or an old, worn tennis ball work well as spheres, and rubber bands make good lines. Also, you can use ribbon or strips of paper. Try placing these items on the sphere along different curves to see what happens.

Also look at the symmetries from Problem **1.1** to see if they hold for straight lines on the sphere. The important thing to remember here is to **think in terms of the surface of the sphere, not in 3-space**. Always try to imagine how things would look from the bug's point of view. A good example of how this type of thinking works is to look at an insect called a water strider. The water strider walks on the surface of a pond and has a very 2-dimensional perception of the world around it — to the water strider, there is no up or down; its whole world consists of the 2-dimensional plane of the water. The water strider is very sensitive to motion and vibration on the water's surface, but it can be approached from above or below without its knowledge. Hungry birds and fish take advantage of this fact. For more discussion of water striders and other animals with their own varieties of intrinsic observations, see the delightful book, *The View from the Oak*, by Judith and Herbert Kohl [**Na**: Kohl and Kohl, 1977]. This is the type of thinking needed to adequately visualize properties of straight lines on the sphere.

Lines which are straight on a sphere (or other surfaces) are often called ***geodesics***. This leads us to consider the concept of intrinsic or geodesic curvature versus extrinsic curvature. As an outside observer looking at the sphere in 3-space, all paths on the sphere, even the great circles, are curved — that is, they exhibit *extrinsic* curvature. But relative to the surface of the sphere (*intrinsically*), the lines may be straight. Be sure to understand this difference and to see why all symmetries (such as reflections) must be carried out intrinsically, or from the bug's point of view.

It is natural for you to have some difficulty experiencing straight on surfaces other than the plane and that consequently you will start to look at the properties of spheres and at the curves on spheres as 3-D objects. Imagining that you are a 2-dimensional bug walking on a sphere emphasizes the importance of experiencing straightness and will help you to shed your limiting extrinsic 3-D vision of the curves on a sphere. Ask yourself:

- What does the bug have to do, when walking on a non-planar surface, in order to walk in a straight line?
- How can the bug check if it is going straight?

Experimentation with models plays an important role here. Working with models that *you create* helps you to experience that great circles are, in fact, the only straight lines on the surface of a sphere. Convincing yourself of this notion will involve recognizing that straightness on the plane and straightness on a sphere have common elements. When you are comfortable with "great-circle-straightness," you will be ready to transfer the symmetries of straight lines on the plane to great circles on a sphere and, later, to geodesics on other surfaces. Here are some activities that you can try, or visualize, to help you experience great circles and their intrinsic straightness on a sphere. However, it is better for you to come up with your own experiences.

- Stretch something elastic on a sphere. It will stay in place on a great circle, but it will not stay on a small circle if the sphere is slippery. Here, the elastic follows a path that is approximately the shortest since a stretched elastic always moves so that it will be shorter. Using the shortest distance criterion directly is not a good way to check for straightness because one cannot possibly measure all paths. But, it serves a good purpose here.
- Roll a ball on a straight chalk line (or straight on a freshly painted floor!). The chalk (or paint) will mark the line of contact on the sphere and it will be a great circle.
- Take a stiff ribbon or strip of paper that does not stretch, and lay it "flat" on a sphere. It will only lie properly along a great circle. Do you see how this property is related to local symmetry? This is sometimes called the *Ribbon Test*. (For further discussion of the Ribbon Test, see the Appendix of this book and Problems **3.4** and **7.6** of [**DG**: Henderson (1998)].)
- The feeling of turning and "non-turning" comes up. Why is it that on a great circle there is no turning and on a latitude line there is turning? Physically, in order to avoid turning, the bug has to move its left feet the same distance as its right feet. On a non-great circle (for example, a latitude line that is not the equator), the bug has to walk faster with the legs that are on the side closest to the equator. This same idea can be experienced by taking a small toy car with its wheels fixed so that, on a plane, it rolls along a straight line. Then on the sphere the car will roll around a great circle but it will not roll around other curves.
- Also notice that, on a sphere, straight lines are circles (points on the surface a fixed distance away from a given point) — special circles whose circumferences are straight! Note that the equator is a circle with two centers: the north pole and the south pole. In fact, any circle on a sphere has two centers.

These activities will provide you with an opportunity to investigate the relationships between a sphere and the geodesics of that sphere. Along the way, your experiences should help you to discover how great circles on a sphere have most of the same symmetries as straight lines on a plane.

You should pause and not read further until you have expressed your thinking and ideas about this problem.

Symmetries of Great Circles

Reflection-thru-itself symmetry: We can see this globally by placing a hemisphere on a flat mirror. The image in the mirror exactly recreates the hemisphere. Figure 2.2 shows a reflection through the great circle *g*.

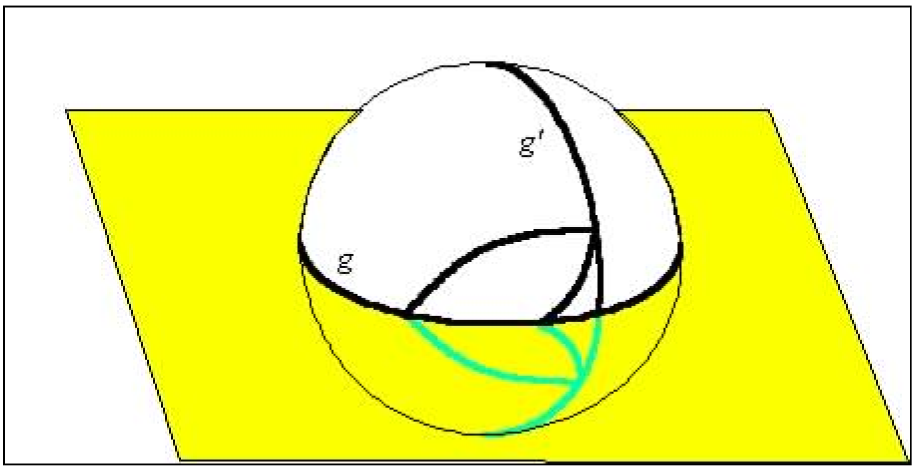

Figure 2.2. **Reflection-thru-itself symmetry.**

Reflection-perpendicular-to-itself symmetry: A reflection through any great circle will take any great circle (for example, *g*′ in Figure 2.2) which is perpendicular to the original great circle onto itself.

Half-turn symmetry: A rotation through half of a full revolution about any point *p* on a great circle interchanges the part of the great circle on one side of *p* with the part on the other side of *p*.

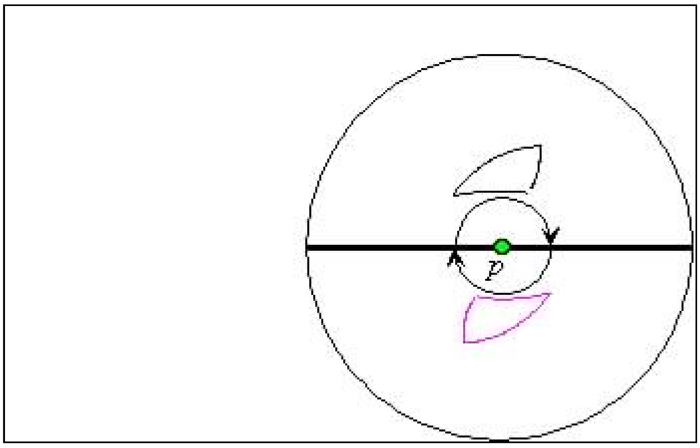

Figure 2.3. Half-turn symmetry.

Rigid-motion-along-itself symmetry: For great circles on a sphere we call this a translation along the great circle and a rotation around the poles of that great circle. This property of being able to move rigidly along itself is not unique to great circles since any circle on the sphere whose center is a pole of the great circle will also have the same symmetry.

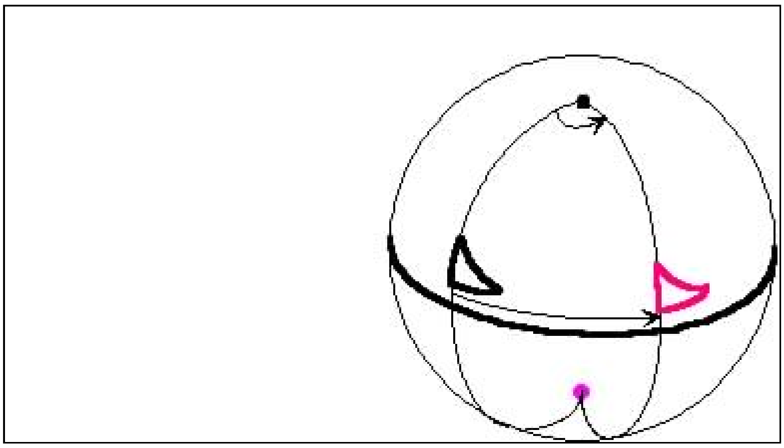

Figure 2.4. Rigid-motion-along-itself symmetry.

Central symmetry, or point symmetry: Viewed intrinsically (that is, from the 2-dimensional bug's point-of-view), central symmetry through a point *P* on the sphere sends any point *A* to the point at the same great circle distance from *P* but on the opposite side.

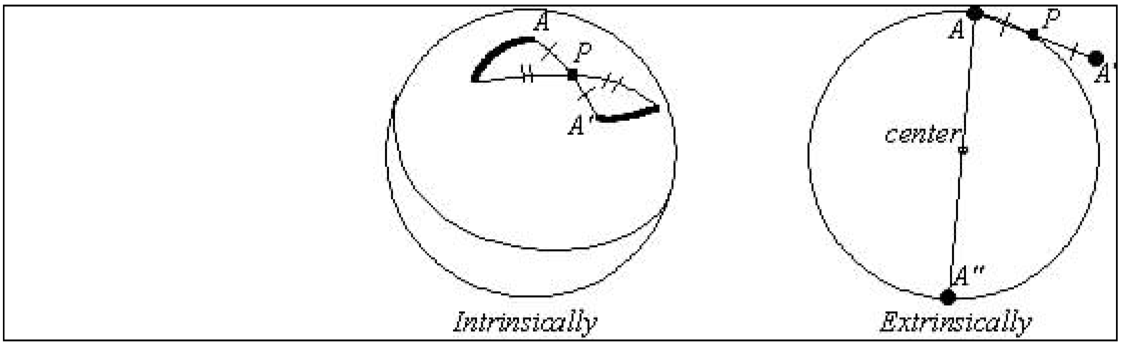

Figure 2.5. Central symmetry.

Extrinsically (i.e., from our 3-dimensional point-of-view), the only central symmetry of the sphere (and the only one for great circles on the sphere) is through the center of the sphere. (See Figure 2.5.) The transformation that is intrinsically central symmetry is extrinsically half-turn symmetry (about the diameter through *P*). Intrinsically, as on a plane, central symmetry does not differ from half-turn symmetry with respect to the end result. This distinction between intrinsic and extrinsic is important to experience at this point.

<u>*3-dimensional rotation symmetry*</u>: **This symmetry does not hold for great circles in 3-space**; however, it does hold for great circles in a 3-sphere. See Chapter 12.

<u>*Similarity or self-similarity*</u>: **This symmetry does not hold on spheres**, as we shall see in Problem **9.4**.

You will probably notice that other objects on the sphere, besides great circles, have some of the symmetries mentioned here. It is important for you to construct such examples, and to attempt to find an object that has all of the symmetries mentioned here but is not a great circle. This will help you to realize that straightness and the five symmetries discussed here are intimately related.

Every Geodesic is a Great Circle

But notice that you were not asked to proof that *every geodesic on the sphere is a great circle*. This is true but more difficult to prove. Many texts simply *define* the great circles to the straight lines (geodesics) on the sphere. We have not taken that approach. We have shown that the great circles are geodesics and it is clear that two points on the sphere are always joined by a great circle arc which shows that there are sufficient great circle geodesics to do the geometer which we wish.

To show that great circles are the only geodesics involve some notions from Differential Geometry. In Problem **3.2b** of [**DG**: Henderson (1998)] this is proved using special properties of plane curves. More generally, a geodesics satisfies a differential equation with the initial condition being a point on the geodesic and the direction of the geodesic at that point (see Problem **8.4b** of [**DG**: Henderson (1998)]); and thus it follows from the analysis theorem on the *Existence and Uniqueness of Differential Equations* that:

At every point on a smooth surface there is a unique geodesic going in every direction from that point.

From this it follows that all geodesics on a sphere are great circles. Do you see why?

Intrinsic Curvature

You have tried wrapping the sphere with a ribbon and noticed that the ribbon will only lie flat along a great circle. (If you haven't experienced this yet, then do it now before you go on.) Arcs of great circles are the only paths of a sphere's surface that are tangent to a straight line on a piece of paper wrapped around the sphere.

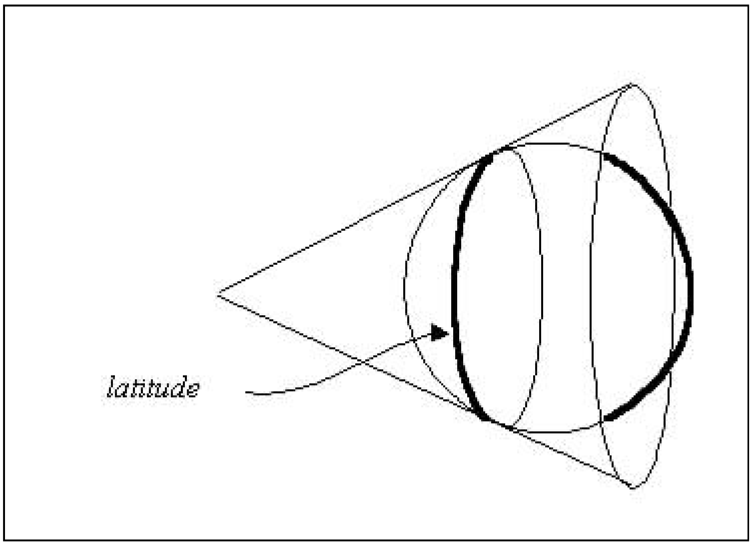

Figure 2.6. **Finding the intrinsic curvature.**

If you wrap a piece of paper tangent to the sphere around a latitude circle (see Figure 2.6), then, extrinsically, the paper will form a portion of a cone and the curve on the paper will be an arc of a circle. The *intrinsic curvature* of a path on the surface of a sphere can be defined as the curvature that one gets when one "unwraps" the path onto a plane. For related information see the Appendix, "A Geometric Introduction to Differential Geometry".

Now we continue with straightness, but now the goal is to think intrinsically. By the end of Problem **2.2**, you should be comfortable with straightness as a *local intrinsic notion* — this is the bug's view. This notion of straightness is the basis for the notion of *geodesics* in differential geometry.

Problem 2.2. *Straightness on Cylinders and Cones*

When looking at great circles on the surface of a sphere, we were able (except in the case of central symmetry) to see all the symmetries of straight lines from global extrinsic points of view. For example, a great circle extrinsically divides a sphere into two hemispheres which are mirror images of each other. When looking at straightness on a sphere, it is a natural tendency to use the more familiar and comfortable extrinsic lens instead of taking the bug's local and intrinsic point of view. However, on a cone and cylinder you must use the local, intrinsic point of view because there is no extrinsic view that will work.

a. *What lines are straight with respect to the surface of a cone or a cylinder*? *Why*? *Why not*?

Suggestions

In Problem **2.1**, you were asked to consider straightness on a sphere from the bug's point of view. Problem **2.2** also encourages you to look at straightness from the bug's point of view. The questions in Problem **2.2** are similar to those in Problem **2.1**, but this time the surfaces are the cylinder and a cone.

Make paper models, but consider the cone or cylinder as continuing indefinitely with no top or bottom (except, of course, at the cone point). Again, imagine yourself as a bug whose whole universe is a cone or cylinder. As the bug crawls around on one of these surfaces, what will the bug experience as straight? As we mentioned before, paths which are straight with respect to a surface are often called the "geodesics" for the surface.

As you begin to explore these questions, it is likely that many other related geometric ideas will arise. Do not let seemingly irrelevant excess geometric baggage worry you. Often, you will find yourself getting lost in a tangential idea, and that's understandable. Ultimately, however, the exploration of related ideas will give you a richer understanding of the scope and depth of the problem. In order to work through possible confusion on this problem, try some of the following suggestions which others have found helpful. Each suggestion involves constructing or using models of cones and cylinders:

- If we make a cone or cylinder by rolling up a sheet of paper, will "straight" stay the same for the bug when we unroll it? Conversely, if we have a straight line drawn on a sheet of paper and roll it up, will it continue to be experienced as straight for the bug crawling on the paper?
- Lay a stiff ribbon or straight strip of paper on a cylinder or cone. Convince yourself that it will follow a straight line with respect to the surface. Also, convince yourself that straight lines on the cylinder or cone, when looked at locally and intrinsically, have the same symmetries as on the plane.
- If you intersect a cylinder by a flat plane and unroll it, what kind of curve do you get? Is it ever straight?
- On a cylinder or cone, can a geodesic ever intersect itself? How many times? This question is explored in more detail in Problem **11.1**, which the interested reader may turn to now.
- Can there be more than one geodesic joining two points on a cylinder or cone? How many? Is there always at least one? Again this question is explored in more detail in Problem **11.1**.
- You may find it helpful to explore cylinders first before beginning to explore cones. This problem has many aspects, but focusing at first on the cylinder will simplify some things.

There are several important things to keep in mind while working on this problem. First, **you absolutely must make models**. If you attempt to visualize lines on a cone or cylinder, you are bound to make claims that you would easily see are mistaken if you investigated them on an actual cone or cylinder. Many students find it helpful to make models using transparencies.

Second, as with the sphere, you must think about lines and triangles on the cone and cylinder in an intrinsic way — always look at things from a bug's point of view. We are not interested in what's happening in 3-

space; only what you would see and experience if you were restricted to the surface of a cone or cylinder.

And last, but certainly not least, you must look at cones of different shapes, that is, cones with varying cone angles.

Cones with Varying Cone Angles

Geodesics behave differently on differently shaped cones. So an important variable is the cone angle. The ***cone angle*** is generally defined as the angle measured around the point of the cone on the surface. Notice that this is an intrinsic description of angle. The bug could measure a cone angle, first, by making a model of a one-degree angle and, then, by determining how many of them it would take to go around the cone point. We can determine the cone angle extrinsically in the following way: If we cut the cone along a generator and flatten it, then the measure of the cone angle is the number of degrees in the planar sector.

For example, if we take a piece of paper and bend it so that half of one side meets up with the other half of the same side, we will have a 180-degree cone:

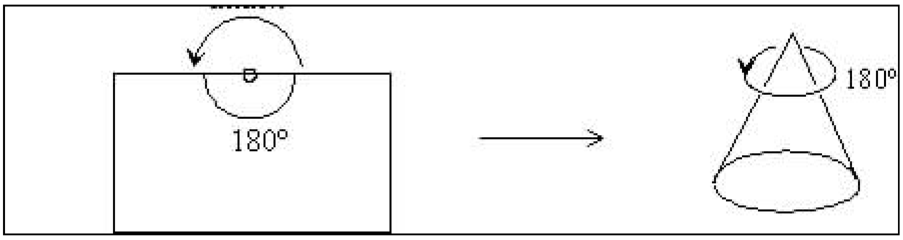

Figure 2.7. Making a 180° cone.

A 90º cone is also easy to make — just use the corner of a sheet of paper and bring one side around to meet with the adjacent side. Also be sure to look at larger cones. One convenient way to do this is to make a cone with a variable cone angle. This can be accomplished by taking a sheet of paper and cutting (or tearing) a slit from one edge to the center. (See Figure 2.8.) A rectangular sheet will work but a circular sheet is easier to picture. Note that it is not necessary that the slit be straight!

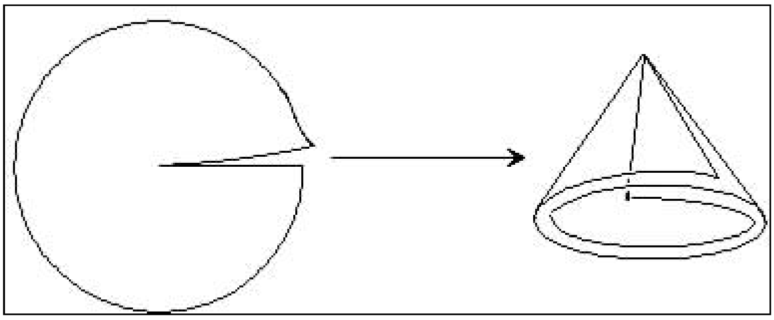

Figure 2.8. A cone with variable cone angle (0 - 360°).

You've already looked at a 360º cone in some detail — it's just a plane. The cone angle can also be larger than 360º. A common larger cone is the 450º cone. You probably have a cone like this somewhere on the walls, floor, and ceiling of your room. You can easily make one by cutting a slit in a piece of paper and inserting a 90º slice (360º + 90º = 450º):

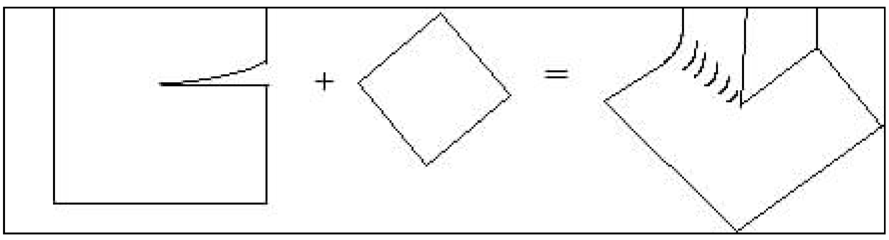

Figure 2.9. How to make a 450 º cone.

You may have trouble believing that this is a cone, but remember that just because it can't hold ice cream, doesn't mean it's not a cone. If the folds and creases bother you, they can be taken out — the cone will look ruffled instead. It is important to realize that when you change the shape of the cone like this (i.e., by ruffling), you are only changing its extrinsic appearance. Intrinsically (from the bug's point of view) there is no difference. You can even ruffle the cone so that it will hold ice cream if you like, although changing the extrinsic shape in this way is not useful to a study of its intrinsic behavior.

You can also make a cone with variable angle of more than 180° by taking two sheets of paper and slitting them together to their centers as in Figure 2.10. Then tape the left side of the top slit to the right side of the bottom slit as pictured.

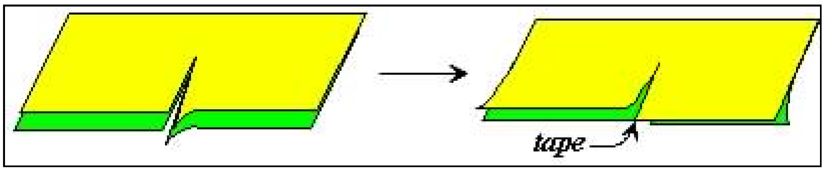

Figure 2.10. Variable cone angle larger than 360°.

It may be helpful for you to discuss some definitions of a cone. The following is one definition: *Take any simple (non-intersecting) closed curve **a** on a sphere and consider a point **P** at the center of the sphere. A **cone** is the union of the rays that start at **P** and go through each point on **a**.* The cone angle is then equal to (length of ***a***)/(radius of sphere), in radians. Do you see why?

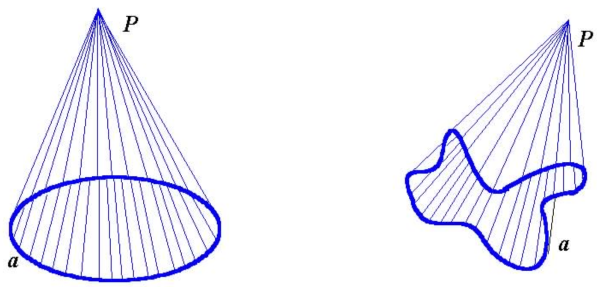

Figure 2.11. Cones.

Experiment by making out of paper examples of cones like those shown above. What happens to the triangles and lines on a 450 ° cone? Is the shortest path always straight? Does every pair of points determine a straight line?

Finally, also consider line symmetries on the cone and cylinder. Check to see if the symmetries you found on the plane will work on these surfaces, and remember to think intrinsically and locally. A special class of geodesics on the cone and cylinder are the generators. These are the straight lines that go through the cone point on the cone or go parallel to the axis of the cylinder. These lines have some extrinsic symmetries (*can you see which ones*?), but in general, geodesics have only local, intrinsic symmetries. Also, on the cone, think about the region near the cone point — what is happening there that makes it different from the rest of the cone?

It is best if you experiment with paper models to find out what geodesics look like on the cone and cylinder before going on to the next page. You will benefit much more from thinking about this problem on your own first, even if you make mistakes.

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Geodesics on Cylinders

Let's first look at the three classes of straight lines on a cylinder. When walking on the surface of a cylinder, a bug might walk along a vertical generator.

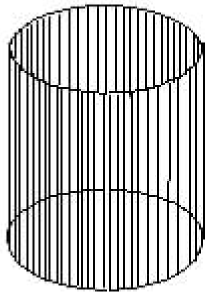

Figure 2.12. **Vertical generators are straight.**

It might walk along an intersection of a horizontal plane with the cylinder, what we will call a *great circle* or a *generator circle*.

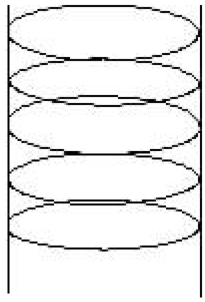

Figure 2.13. **Generating circles are intrinsically straight.**

Or, the bug might walk along a spiral or helix of constant slope around the cylinder.

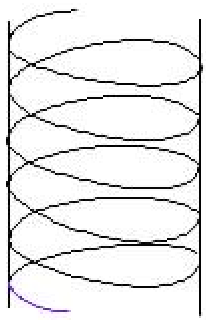

Figure 2.14. **Helixes are intrinsically straight.**

Why are these geodesics? How can you convince yourself? And why are these the only geodesics?

Geodesics on Cones

Now let's look at the classes of straight lines on a cone.

Walking along a generator: When looking at straight paths on a cone, you will be forced to consider straightness at the cone point. You might decide that there is no way the bug can go straight once it reaches the cone point, and thus a straight path leading up to the cone point ends there. Or you might decide that the bug can find a continuing path that has at least some of the symmetries of a straight line. Do you see which path this is?

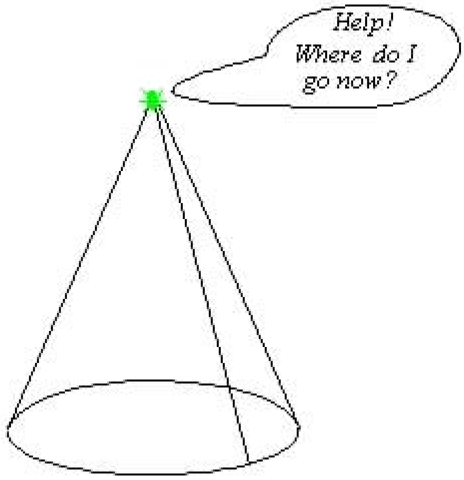

Figure 2.15. **Bug walking straight over the cone point.**

Walking straight and around: If you use a ribbon on a 90º cone, then you can see that this cone has a geodesic like the one depicted in Figure 2.16. This particular geodesic intersects itself. However, check to see that this property depends on the cone angle. In particular, if the cone angle is more than 180°, then geodesics do not intersect themselves. And if the cone angle is less than 90°, then geodesics (except for generators) intersect at least two times. Try it out! Later in this chapter we will describe a tool which will help you determine how the number of self-intersections depends on the cone angle.

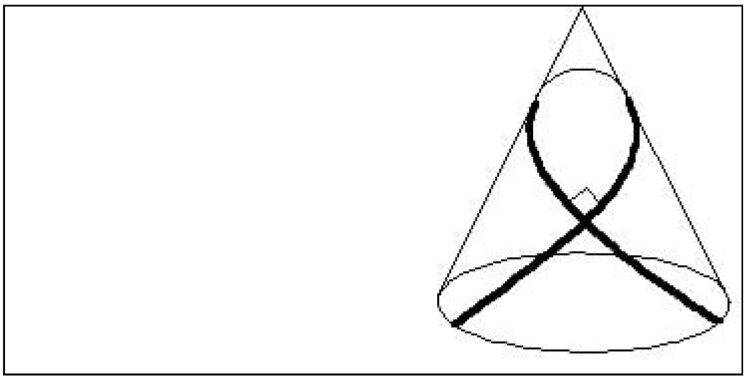

Figure 2.16. **A geodesic intersecting itself on a 90° cone.**

Locally Isometric

By now you should realize that when a piece of paper is rolled or bent into a cylinder or cone, the bug's local and intrinsic experience of the surface does not change except at the cone point. Extrinsically, the piece of paper and the cone are different, but in terms of the local geometry intrinsic to the surface they differ only at the cone point.

Two geometric spaces, **G** and **H**, are said to be ***locally isometric*** at points *G* in **G** and *H* in **H** if the local intrinsic experience at *G* is the same as the experience at *H*. That is, there are neighborhoods of *G* and *H* that are identical in terms of intrinsic geometric properties. A cylinder and the plane are locally isometric (at every point) and the plane and a cone are locally isometric except at the cone point. Two cones are locally isometric at their cone points only if the cone angles are the same.

Since cones and cylinders are locally isometric with the plane it means that locally they have the same geometric properties. We look at this more in Chapter 11. Later, we will show that a sphere is not locally isometric with the plane — *be on the lookout for a result that will imply this.*

Is "Shortest" Always "Straight"?

We are often told that "a straight line is the shortest distance between two points," but, is this really true?

As we have already seen on a sphere, two points which are not opposite each other are connected by two straight paths (one going one way around a great circle and one going the other way). Only one of these paths is shortest. The other is also straight, but not the shortest straight path.

Consider a model of a cone with angle 450°. Notice that such cones appear commonly in buildings as so-called "outside corners" (see Figure 4.20). It is best, however, for you to have a paper model that can be flattened. Use your model to investigate which points on the cone can be joined by straight lines. In particular, look at points like those labeled *A* and *B* in Figure 2.17 below. There is no single straight line on the cone going from *A* to *B*, and thus for these points the shortest path is not straight. Convince yourself that in this case this shortest path is not straight.

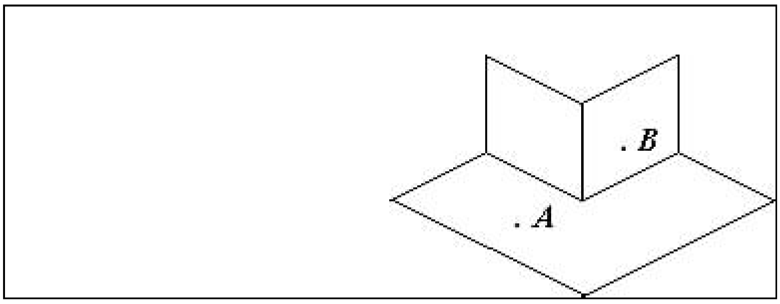

Figure 2.17. **There is no straight path from *A* to *B*.**

Here is another example: Think of a bug crawling on a plane with a tall box sitting on that plane (refer to Figure 2.18). This combination surface — the plane with the box sticking out of it — has eight cone points. The four at the top of the box have 270° cone angles, and the four at the bottom of the box have 450° cone angles (180° on the box and 270° on the plane). What is the shortest path from points *X* and *Y*, points which are on opposite sides of the box? Is the straight path the shortest? Is the shortest path straight? To check that the shortest path is not straight, try to see that at the bottom corners of the box the two sides of the path have different angular measures. (In particular, if *X* and *Y* are close to the box, then the angle on the box side of the path measures a little more than 180° and the angle on the other side measures almost 270°.)

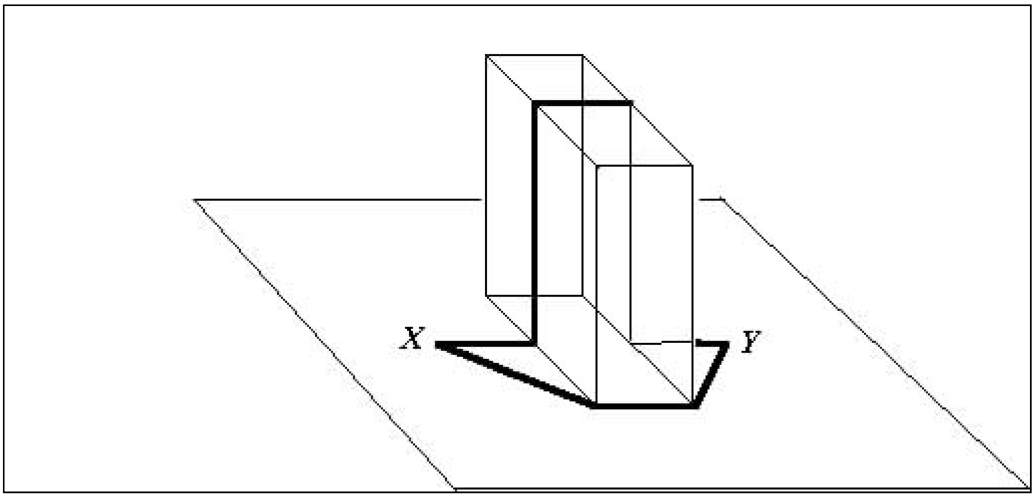

Figure 2.18. **The shortest path is not straight.**

Euclid's Postulates and Differential Geometry

So, we see that sometimes a straight path is not shortest and the shortest path is not straight. So, does it makes sense to say (as most books do) that in Euclidean geometry a straight line is the shortest distance between two points? In differential geometry, on "smooth" surfaces, "straight" and "shortest" are more nearly the same. A *smooth* surface is essentially what it sounds like. More precisely, a surface is smooth at a point if, when you zoom in on the point, the surface becomes indistinguishable from a flat plane. (For details of this definition, see Problem **3.1** in [**DG**: Henderson]. Note that a cone is not smooth at the cone point, but a sphere and a cylinder are both smooth at every point. The following is a theorem from differential geometry:

Theorem: *If a surface is smooth then a straight line on the surface is always the shortest path between "nearby" points. If the surface is smooth and complete* (*every geodesic on it can be extended indefinitely*), *then the shortest path between any two points is always straight.* See [**DG**: Henderson], Problem **7.4b** and **7.4d**.

Consider a planar surface with a hole removed. Check that for points near opposite sides of the hole, the shortest path (on the plane surface with hole removed) is not straight because the shortest path must go around the hole.

We encourage the reader to discuss how each of the previous examples and problems is in harmony with this theorem.

Note that that statement "every geodesic on it can be extended indefinitely" is a reasonable interpretation of Euclid's Second Postulate: *Every line can be extended indefinitely*. In addition, Euclid defines a right angle as follows: *When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right*** [**A**: Euclid's *Elements*]. Note that if you use this definition, then right angles at a cone point are not equal to right angles at points which are locally isometric to the plane. And Euclid goes on to state as his Fourth Postulate: *All right angles are equal to one another.* Thus, Euclid's Fourth Postulate rules out cones and any surface with isolated cones points. What is further ruled out by Euclid's Fourth Postulate would depend on formulating more precisely just what it says. It is not clear (at least to the author!) whether there is something we would want to call a surface which could be said to satisfy Euclid's Fourth Postulate and not be a smooth surface. However, it is clear that Euclid's postulate at least gives part of the meaning of "smooth surface", because it rules out isolated cone points.

Differential geometers often talk about intrinsically straight paths (geodesics) in terms of the velocity vector of the motion as one travels at a constant speed along that path. (The velocity vector is tangent to the curve along which the bug walks.) For example, as you walk along a great circle, the velocity vector to the circle changes direction, extrinsically, in 3-space where the change in direction is toward the center of the sphere. "Toward the center" is not a direction that makes sense to a 2-dimensional bug whose whole universe is the surface of the sphere. Thus, the bug does not experience the velocity vectors at points along the great circle as changing direction. In differential geometry, the rate of change, from the bug's point of view, is called the *covariant* (or *intrinsic*) *derivative*. As the bug traverses a geodesic, the covariant derivative of the velocity vector is zero. This can also be expressed in terms of *parallel transport* which is discussed in Chapters 7, 8, and 10. See also the Appendix and [**DG**: Henderson], Chapters 3, 5, and 8.