

QGCL CHALLENGE PROGRAM

JK BEATTIE

ABSTRACT. This document is a record of my analysis of the qGCL challenge program as shown by McIver in her lecture of 15 January 2025.

Tuesday 18th November, 2025 at 22:04

1. INTRODUCTION

In her presentation at the annual LMS and BCS-FACS joint seminar, January 2025, McIver included “a challenge program”. The screen grab below shows the program (Figure 1.1 on page 1).

The challenge program takes an argument N , a natural greater than zero, and returns a natural c selected uniformly at random from the range $0 \leq c < N$.

The program uses a single random choice primitive, which selects between two options at equal probability.

The screen grab shows a video player interface. The main slide is titled "A challenge program" in red. It contains a code snippet in a box on the left and a list of questions in a red box on the right. Below the code box is a blue button that says "Set c uniformly between 0 and N-1". The video player controls at the bottom show the video is at 19:22 / 1:29:34. The video title is "Probabilistic Datatypes, Annabelle McIver | LMS BCS-FACS Seminar". The channel is "London Mathematical Society" with 14.3K subscribers. There are buttons for "Subscribe", "9" likes, "Share", "Download", "Clip", "Save", and a menu icon.

A challenge program

```
{ 1/N } # Precondition
var c, v = 0, 1
while(v < N or c ≥ N){
  {Inv × [v < N or c ≥ N]}
  □ (v ≤ N) → v = 2v
    c = 2c 1/2 ⊕ c = 2c+1
  □ (v > N) → v, c = v-N, c-N
  ⊞ Inv }
}
{ [c = i] } # Post condition for any 0 ≤ i < N
```

Set c uniformly between 0 and N-1

- ♦ Why is this fast?
- ♦ Why is this a challenge?
- ♦ What is Inv ?
- ♦ What should we do?

Probabilistic Datatypes, Annabelle McIver | LMS BCS-FACS Seminar

London Mathematical Society
14.3K subscribers

Subscribe

9 Share Download Clip Save ...

Screen grab from Probabilistic Datatypes, Annabelle McIver,
LMS BCS-FACS 15 January 2025

Algorithm 1 McIver's challenge program

```

var c, v = 0, 1
while(V < N or c >= N){
  {Inv x [v < N or C >= N]}
  □ (v <= N) → v = 2v ; c = 2c [½] c = 2c+1
  □ (v > N) → v, c = v-N, c-N
  {Inv}
}

```

The program is efficient in that the random choices which are made in order to get to the result c in the desired range are all effective, none are wasted.

The code is shown in Algorithm 1, based on Dijkstra's guarded command language. Annotations are in braces. Random choice is indicated by $[\frac{1}{2}]$.

The purpose of this paper is to analyse this algorithm, ideally providing loop invariant(s) as appropriate and proving correctness. The algorithm terminates for N a power of 2. The algorithm terminates almost-certainly for N not a power of 2. That is, the probability of that the algorithm will cycle through more loops reduces exponentially with each outer loop, and is zero in the limit. The analysis below explains the meaning of 'outer loop'.

This PDF and the implementation scripts are available on [github](#).

01 May 2025

I have skimmed various papers and the textbook [McIo6] to see if I can express the correctness proof formally. I have tried to use the notation and technology given there to support my proof of correct conditional termination.

2. ANALYSIS

2.1. Restructuring the loop. Referring to Algorithm 1, the two guarded commands inside the while loop above govern two different loop constructs.

So, we convert to nested loops, first using repeat for the outer loop, Algorithm 2. This first try is incomplete, see the next refinement for better.

Now tidy-up to make two while loops and modify the initialisation so that the outer loop can be expressed correctly as a while, Algorithm 3.

Finally, unroll the outer loop once to simplify the initialisation steps, Algorithm 4. Note that the initialisation step now sets $c=0$, $v=1$ which means that the loop invariant is true. It is not true at the top of the outer loop in the previous version. The proposed invariant is defined at the end of the next section. Then the initial run of $I(v, c)$ expands the initial state until $v \geq N$, and then $O(v, c)$ runs if required, including daughter runs of $I(v, c)$.

Algorithm 2 Initial re-write, incomplete

```

var v=1
var c=0
repeat {
  ...
  while (v<N or c>=N){
    □ (v <= N ) -> v = 2v; c=2c [½] c=2c+1
  }
  □ (v > N ) → v = v - N ; c = c - N
} until (v>=N and c < N)

```

Algorithm 3 Re-written to be two nested while loops

```

var v=N+1
var c=N
while (c >= N){
  v = v - N ; c = c - N          # Note 1
  while (v<N){                  # Note 2
    v = 2v; c=2c [½] c=2c+1    # Note 3
  }
}

```

- (1) Relative to the previous version Algorithm 2, we don't need the guard, $(v > N) \rightarrow$, since $v > c \geq N$
 - (2) Likewise, we don't need 'or $(c \geq N)$ ', in part because it is false, since $c < v$ and $v < N$ but mostly because it controls the outer while loop, not the inner loop
 - (3) $\frac{1}{2}$ is shorthand for $p : \sigma \rightarrow \frac{1}{2}$, where σ represents the state. See [Kam19, p57 &seq] for the syntax & semantics.
-

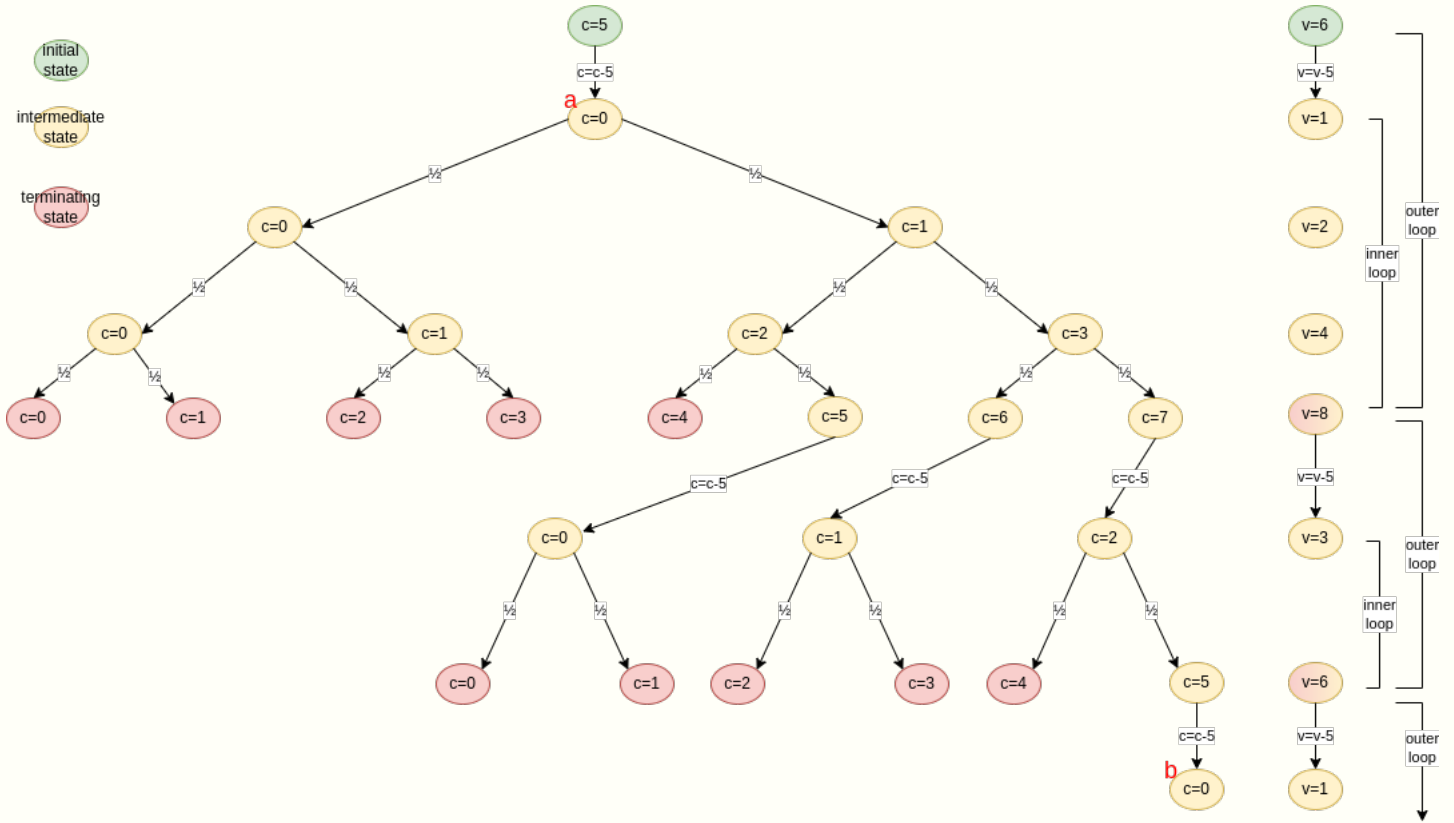
Algorithm 4 Re-written to unroll the outer loop once

```

var v=1
var c=0
while (v<N){
  v = 2v; c=2c [½] c=2c+1
}
while (c >= N){
  v = v - N ; c = c - N
  while (v<N){
    v = 2v; c=2c [½] c=2c+1
  }
}

```

Figure 2.1 on page 4 shows the evolution of the state space for $N=5$. The arrows show successive transitions of the two state variables, v, c . They transition in tandem, going down the diagram. At $v = 8$ the probability range has been doubled three times and is now larger than $N = 5$. Each

Evolution of state space for $N=5$

possible value of c has equal probability of being chosen. Hence if the program has selected a value of c in the desired range $0 \leq c < N$, then this value has an equal probability with any other value in the desired range.

If the program has selected a value in the complement of the desired range, then it reduces the state c, v such that $c < v$ & $v < N$ and then cycles to expand the range represented by v .

Finally, it is worth noting that the state space repeats indefinitely, but at a smaller scale in each repeat, and is thus a fractal. In Figure 2.1 on page 4 the state at a is repeated at b.

There is a relevant relationship with the binary expansion of $1/N$. For $N = 5$, $1/5 = 0.0011_2$, as shown in the long division in Figure 2.2 on page 5, which we discuss briefly, so as to connect it with the evolution of the state space.

Long division by hand involves terms such as dividend and trial value. These reference fairly obvious numbers in the diagram.

Bringing down a zero from the dividend means appending a zero to the right-hand end of the current trial value. Ignoring the binary point and treating the number just as a string of bits, appending zero at the right works in the same way as multiplying by two. Zeroes are brought down

$$\begin{array}{r}
 00110011 \\
 101 \overline{) 1.000000000000\dots} \\
 \underline{101} \\
 110 \\
 \underline{101} \\
 1000 \\
 \underline{101} \\
 110 \\
 \underline{101} \\
 1\dots
 \end{array}$$

Long division of $1/5$ in binary

until the trial value is bigger than N , which in the example means 1000_2 or 8. Then we subtract 101_2 , leaving remainder 11_2 . This is equivalent to the stage $v = 8$, Figure 2.1 on page 4. The desired range for c is $0 \leq c < 5$, size 5, and the complement is size 3. If $c \geq 5$, i.e. c is in the complement range, then the outer cycle loops. v is reduced to 3 giving a fresh range which starts at zero, i.e. $0,1,2$. It is less than N , as it must be as it is the remainder $v\%N$. c is reduced so that $c < v$ is maintained.

Doubling v gives $6 > 5$ so 1 is appended to the quotient. In the state space, the inner loop terminates immediately and the outer loop tests for $c < N$ for termination.

The long division also shows the obvious repetition for the expansion of a rational in any base. This matches the fractal repeat of the state space.

Another interpretation for the binary expansion of $1/N$ is that the algorithm is consuming increasing quantities of the probability mass: $1 = 5/8 + 5/16 + 5/128 + 5/256 + \dots$. That is, on the first cycle of the outer loop, the sub-distribution at termination has mass $5/8$, after the second, the sub-distribution has mass $5/16$ and so on.

Or, it is using increasingly accurate binary approximations of the desired probability, $1/N$. In the example case, $1/5 = 1/8 + 1/16 + 1/128 + 1/256 + \dots$. Where the algorithm terminates after the first cycle of the outer loop,

the probability of the value c is $1/8$. If it terminates after the second, the probability is $1/16$. Visually, instead of dividing the unit interval into five equal slots, we divide it first into eight equal slots and assign five at $1/8$ each. We then subdivide the remaining three slots to make six and assign five at $1/16$ each. We then start again with the remaining $1/16$ slot and divide that into eight, repeating the process, but at one-sixteenth of the overall mass. Overall, the unit interval is subdivided into a fractal, assigning increasingly accurate approximations of one-fifth of the interval to each possible result.

Note that, when N is a power of 2, the algorithm definitely terminates after one cycle of the outer loop. This corresponds to the binary expansion of $\frac{1}{2^m}$, which has a single bit set to 1 and all the rest are zero.

The algorithm does not definitely-terminate for N not a power of 2; it may, in principle, continue indefinitely. However, the probability of non-termination decreases exponentially.

Hence these two cases are different and must both be discussed.

2.2. qGCL. The programs in Algorithm 5 comprise the algorithm. $C(v, c)$ is the overall program.

Definition 1. Define a version of *Unif* ([Kam19, p60]) where the total probability mass can be other than 1. This is a uniform distribution labelled by integers $0 \leq i < n$, total weight w .

$$Unif(w, n) = \sum_{i=0}^{n-1} \frac{w}{n} |i\rangle \quad (2.1)$$

$$(n \in \mathbb{N}, w \in \mathbb{Q}, 0 < w \leq 1) \quad (2.2)$$

Definition 2. The following equations define the semantics of the algorithm, where $N \in \mathbb{N}$ and $w \in [0, 1] \cap \mathbb{Q}$ are auxilliary variables. N is an external parameter of the program, i.e. the input argument. w is the probability mass of the sub-distribution at termination [Kam19, definition 3.8].

$$Vars = \{v, c\} \quad (2.3)$$

$$Vals = \mathbb{N} \quad (2.4)$$

$$\sigma : Vars \rightarrow \mathbb{N} \quad (2.5)$$

$$F(\sigma, w, N) : \llbracket C(v, c) \rrbracket_\sigma = Unif(w, N) = \sum_{0 \leq i < N} \frac{w}{N} |\sigma[c \mapsto i]\rangle \quad (2.6)$$

$F(\sigma, w, N)$ is the predicate for the desired post-condition. It asserts of the sub-distribution at termination, $\llbracket C(v, c) \rrbracket_\sigma$, that c is distributed uniformly

Algorithm 5 Program components of the algorithm. Note that $C(v, c)$ is the overall program.

```

var v, c, w, N
Init(v, c, w) = {v=1; c=0; w=1}
I(v, c) = while (v < N) {      # I for Inner
    v=2v;
    {c=2c [1/2] c=2c+1}
}
O(v, c, w) = while (c >= N) { # O for Outer
    w=w*(v-N)/v
    v=v-N; c=c-N;
    I(v, c);
}
T(v, w) = {w=w*N/v}
C(v, c) = Init(v, c, w); I(v, c); O(v, c); T(v, w)

```

in the range $0 \leq c < N$. Note that this is a sub-distribution. The total weight is $w \leq 1$.

We want to prove correctness on conditional termination:

$$\{N > 1 \ \& \ (\nexists m. N = 2^m)\} \text{Init}(v, c, w); I(v, c); O(v, c, w); T(v, w) \{F(\sigma, w', N)\}$$

where F is the assertion above. The special cases excluded by the precondition are discussed in Section 3 below.

The invariant for both loops is

$$\text{Inv} : c < v \ \& \ \mu(\sigma) = \text{Unif}(w, v)$$

$\mu(\sigma)$ is the distribution of the random variables in the state at state σ . That is, the distribution of c . v is a definite variable, not a probabilistically distributed variable.

Note that the $I(v, c)$ code is executed once before the $O(v, c, w)$ code but the outer loop then includes calls to the inner loop. Thus the first run of I can be eliminated, so that O is called immediately after Init . However, this comes at a cost: the variables v, c, w would have to be initialised to $N+1$, N and $N+1$, respectively. These are strange initial values and, worse, the invariant wouldn't be valid just before the call to O .

Initialising to sensible values and executing I once after initialisation makes the code clearer overall and ensures validity of the invariant. It also helps the analysis where N is a power of 2, see below.

At termination, $\llbracket C(v, c) \rrbracket_\sigma = \mu(\sigma)$, [Kam19, definition 3.8]. At termination, $c < N \leq v$. Together with the invariant, it follows that c is in the distribution $\text{Unif}(w, v)$ and $c < N$, i.e. c is an integer uniformly at random in the range $0 \leq c < N$.

The probability of non-termination goes down exponentially, as shown in Section A & (A.6).

3. ADDITIONAL

3.1. Special cases of N.

$N=1$.

$$\{N=1\} \text{ Init}(v, c, w) ; 0(v, c) \{F(\sigma, 1, 1)\}$$

is also true, since $\{N=1\}\text{Init}(v, c, w)\{F(\sigma, 1, 1)\}$ is true without executing the inner loop, as v is already $1=N$ and c is already $0 < 1$. However, this all follows different reasoning than the case(s) $N > 1$.

$$\begin{aligned} F(\sigma, w, 1) &= \sum_{0 \leq i < N} \frac{w}{N} |\sigma[c \mapsto i]\rangle \\ &= \sum_{0 \leq i < 1} \frac{w}{1} |\sigma[c \mapsto i]\rangle \\ &= |\sigma[c \mapsto 0]\rangle \end{aligned}$$

N a power of 2.

$$\begin{aligned} &\{N > 1 \text{ \& } N \text{ is a power of } 2\} \\ &\text{Init}(v, c, w) ; I(v, c) \\ &\{F(\sigma, 1, N)\} \end{aligned}$$

is also true, since $\{N=2^m\} \text{Init}(v, c, w) ; I(v, c) \{F(\sigma, 1, N)\}$ is true (i.e. the first call of I makes F true. And then the outer cycle, 0 is skipped because $c < N$ already. At termination $v = N$:

$$\begin{aligned} &\{N = 2^m\} \\ &\text{Init}(v, c, w) \\ &I(v, c) ; w = w * N / v \\ &\{v = N \text{ \& } w = 1 \text{ \& } 0 \leq c < N \text{ \& } F(\sigma, w, N)\} \end{aligned}$$

3.2. Script implementations. The algorithm has been implemented as bash and python scripts, see the files nested in the bash and python folders in the github project.

3.3. Generalisation. The algorithm can be generalised to other bases. That is, instead of random choice of 0,1 with probability $1/2$, we generalise to allow random choice in $\{0..B-1\}$ with probability $1/B$ ($B \in \mathbb{N}$). This amounts to selecting a base B for the expansion of $1/N$ as a fraction..

See the nestedB script in the python folder.

One key difference is that the range $0 \leq c < v$ is expanded by multiplying by B , which can be larger than 2. Then $v > N$ may in fact be v greater than several multiples of N . Hence the code treats the range $[0..N..2N..mN..v-1]$ by folding the complete sections of length N . That is, any $c < mN$ is taken modulo N and is a valid result: a selection in the desired range, of equal probability.

If N is a power of B or of one of the factors of B (consider $N = 5, B = 10$) then the algorithm will terminate as in the case of N a power 2 discussed above.

3.4. Other formulations of the basic algorithm. The algorithm as presented by McIver is a single loop containing two guarded commands. The analysis above started by reconfiguring the code to be two loops, one nested inside the other. There are alternatives, two are mentioned below.

Make the outer loop into a guarded command within the inner.

```

E(v,c) = while (v < N) {    # E for expand
    v=2v;
    c=2c [1/2] c=2c+1
    S(v,c)
}
S(v,c) = if(c>=N) {        # S for setup
    v=v-N; c=c-N;
    w=w*(v-N)/v
}
Term(w) = {w=w*N/v}        # sub-distribution weight
C"(v,c) = Init(v,c,w); E(v,c); Term(w)

```

This avoids unrolling the outer loop and allows correct initialisation of c, v, w . However, it means that the guard in S is tested on every loop of E , which often is unnecessary work.

Express the algorithm using coroutines. See the coroutine scripts in the bash and python folders. One benefit of the coroutine formulation is to make it easier to see how to modify the loops as shown in the alternatives above.

3.5. Web pages. See also [stackoverflow](https://en.wikibooks.org/wiki/Fractals/Mathematics/binary) which discusses manual algorithms corresponding to long division.

<https://en.wikibooks.org/wiki/Fractals/Mathematics/binary>

<https://www.omnicalculator.com/math/binary-fraction>

APPENDIX A. ESTIMATES OF PROBABILITY OF TERMINATION, BASE 2

This appendix gives detailed calculations of the estimates of termination for the basic algorithm, i.e. where the primitive random choice is $\left[\frac{1}{2}\right]$ and $\frac{1}{N}$ is expanded in binary, base 2.

Let the bits in the expansion of $\frac{1}{N}$ be $\{b(i)\}_{i=1}^{\infty}$:

$$\frac{1}{N} = \sum_{i=1}^{\infty} \frac{b(i)}{2^i} \quad (\text{A.1})$$

Let the repeat length for the expansion be l and let B be defined as

$$B = \sum_{i=1}^l \frac{b(i)}{2^i}$$

So B comprises the bits in the first cycle. Hence

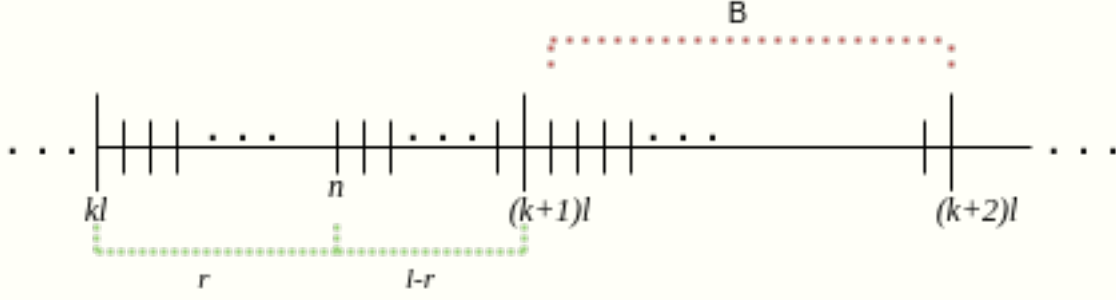
$$\frac{1}{N} = \sum_{j=0}^{\infty} \frac{B}{(2^l)^j} = B \sum_{j=0}^{\infty} \left(\frac{1}{2^l}\right)^j = B \frac{1}{1 - \frac{1}{2^l}} = B \frac{2^l}{2^l - 1} \quad (\text{A.2})$$

$$NB = \frac{2^l - 1}{2^l} \quad (\text{A.3})$$

Note that the sum starts from $j = 0$.

Let $S(b, n)$ be the n^{th} partial sum of [A.1](#). This is an approximation to $\frac{1}{N}$. The probability that the algorithm terminates at or before the n^{th} cycle of the inner loop $I(v, c)$ is $NS(b, n)$, on the following argument. Each I-cycle extends the fraction expansion by one place. Let that be the i^{th} place. The additional weight of the probability space which is added to the weight so far is $N \frac{b(i)}{2^i}$. So zero is added when $b(i) = 0$. The cumulative weight is the probability of termination at or before i .

Let $T(B, m)$ be the m^{th} partial sum of [A.2](#). Then the probability that the algorithm terminates on or before $n = ml$ cycles of the inner loop $I(v, c)$ is $NT(B, m)$. The probability of non-termination on or before $n = ml$ I-cycles is



The vertical ticks represent successive places in the binary expansion of N , i.e. at the i -th tick the term of the infinite sum is $b(i)/2^i$

$$\begin{aligned}
 1 - NT(B, m) &= N \sum_{j=0}^{\infty} \frac{B}{(2^l)^j} - N \sum_{j=0}^{m-1} \frac{B}{(2^l)^j} \\
 &= N \sum_{j=m}^{\infty} \frac{B}{(2^l)^j} \quad \{i = j - m\} \\
 &= N \sum_{i=0}^{\infty} \frac{B}{(2^l)^{i+m}} \\
 &= \frac{N}{(2^l)^m} \sum_{i=0}^{\infty} \frac{B}{(2^l)^i} \\
 &= \frac{N}{(2^l)^m} \frac{1}{N} = \frac{1}{2^{ml}} = \frac{1}{2^n}
 \end{aligned} \tag{A.4}$$

Likewise for the general case, non-termination at or before n I-cycles, where $n = kl + r, 0 \leq r < l$:

$$\begin{aligned}
 1 - NS(b.n) &= N \left(\sum_{i=1}^{\infty} \frac{b(i)}{2^i} - \sum_{i=1}^n \frac{b(i)}{2^i} \right) \\
 &= N \sum_{i=n+1}^{\infty} \frac{b(i)}{2^i} = N \left(\sum_{j=1}^{\infty} \frac{b(j+n)}{2^{j+n}} \right) \quad \{j = i - n\} \\
 &= N \left(\sum_{j=1}^{\infty} \frac{b(j+kl+r)}{2^{j+kl+r}} \right) = \frac{N}{2^{kl}} \left(\sum_{j=1}^{\infty} \frac{b(j+r)}{2^{j+r}} \right) \quad \{n = kl + r; \forall p, k : b(kl + p) = b(p)\} \\
 &= \frac{N}{2^{kl}} \left(\sum_{j=1}^{l-r} \frac{b(j+r)}{2^{j+r}} + \sum_{j=l-r+1}^{\infty} \frac{b(j+r)}{2^{j+r}} \right) \quad \{i = j + r\} \\
 &= \frac{N}{2^{kl}} \left(\sum_{i=r+1}^{l} \frac{b(i)}{2^i} + \sum_{i=l+1}^{\infty} \frac{b(i)}{2^i} \right) \\
 &= \frac{N}{2^{kl}} \left(\sum_{i=r+1}^{l} \frac{b(i)}{2^i} + \sum_{j=1}^{\infty} \frac{b(j+l)}{2^{j+l}} \right) \quad \{j = i - l\} \\
 &= \frac{N}{2^{kl}} \left(\sum_{i=r+1}^{l} \frac{b(i)}{2^i} + \frac{1}{2^l} \sum_{j=1}^{\infty} \frac{b(j)}{2^j} \right) \\
 &= \frac{N}{2^{kl}} \left(\sum_{i=r+1}^{l} \frac{b(i)}{2^i} \right) + \frac{N}{2^{(k+1)l}} \frac{1}{N} \\
 &\leq \frac{NB}{2^{kl}} + \frac{1}{2^{(k+1)l}} \\
 &= \frac{(2^l - 1)}{2^{(k+1)l}} + \frac{1}{2^{(k+1)l}} = \frac{1}{2^{kl}}
 \end{aligned} \tag{A.5}$$

$$\lim_{n \rightarrow \infty} \frac{1}{2^{kl}} = 0 \tag{A.6}$$

Note that kl is the next multiple of l below n .

REFERENCES

- [Kam19] Benjamin Lucien Kaminski. “Erweiterte wp-Kalküle für Probabilistische Programme”. en. PhD thesis. RWTH Aachen University, 2019, p. 2019. DOI: [10.18154/RWTH-2019-01829](https://doi.org/10.18154/RWTH-2019-01829). URL: <https://publications.rwth-aachen.de/record/755408/files/755408.pdf> (visited on 02/25/2025) (cited pp 3, 6, 7).
- [McIo6] Annabelle McIver. *Abstraction, Refinement and Proof for Probabilistic Systems*. Ed. by Charles C. Morgan. Monographs in computer science. Dordrecht: Springer-Verlag New York Inc, 2006. 397 pp. ISBN: 9780387401157 (cited p 2).

INDEX