

## Converting QCQPs to SOCPs

January 19, 2021

This document outlines how to write a quadratically constrained quadratic programming (QCQP) problem into a second order cone programming (SOCP) problem. However, the SOCP formulation is not compatible with free SOCP solvers in R. The reason is that these R packages only permit non-rotated cones, whereas the reformulation utilizes rotated cones. The only solver that seems to allow for rotated cones is [Mosek](#), which offers free academic licenses.

### Reformulating QCQPs as SOCPs

We want to solve a QCQP problem of the form

$$\begin{aligned} \max_x \quad & c'x \\ \text{s.t.} \quad & A_1x = b_1 \\ & A_2x \leq b_2 \\ & x'Qx + q'x \leq s, \end{aligned} \tag{1}$$

where  $x \in \mathbb{R}^n$ , and  $Q \in \mathbb{R}^{n \times n}$  is a positive semidefinite symmetric matrix. To keep things simple, let  $A_1$  and  $A_2$  be row vectors, i.e. there is one linear equality constraint, one linear inequality constraint, and one quadratic inequality constraint.

A conic constraint may generally be written as

$$\|Bx + d\|_2 \leq e'x + f,$$

where  $B \in \mathbb{R}^{m \times n}$ ,  $d \in \mathbb{R}^m$ ,  $e \in \mathbb{R}^n$ ,  $f \in \mathbb{R}$ , and  $\|\cdot\|_2$  denotes the Euclidean norm.<sup>1</sup> However, the R packages in `sos` and `CLSOCP` only allow cones to be declared in the form of  $\|(x_2, \dots, x_n)\|_2 \leq x_1$ , i.e., a non-rotated quadratic cone. As shown below, this is problematic as (4) cannot be reformulated as an SOCP problem with a non-rotated quadratic cone (to the best of my knowledge).

The problem stems from the quadratic constraint, which may be written as

$$z_1 + q'x = s \tag{2}$$

---

<sup>1</sup> A constraint on the square root of the sum of squared residuals from a linear regression may be written in this form by setting  $B$  equal to the design matrix,  $d$  equal to the dependent variable,  $e$  equal to 0, and  $f$  equal to the upper bound.

$$x'Qx \leq z_1, \quad (3)$$

where  $z_1$  is a new variable in the problem. since  $Q$  is positive semidefinite and symmetric, we can write it as  $Q = \Omega'\Omega$ . Define  $\tilde{x} \equiv \Omega x$  so that (3) defines a *rotated* quadratic cone

$$\|\tilde{x}\|_2^2 = x'\Omega'\Omega x \leq z_1 z_2,$$

where  $z_2 = 1$ .

All that remains is to rewrite the (4) in terms of  $\tilde{x}$ .

- The objective may be redefined as

$$c'x = (c'\Omega^{-1})(\Omega x) = \tilde{c}'\tilde{x},$$

where  $\tilde{c} \equiv (\Omega^{-1})'c$ .

- The linear equality constraint may be redefined as

$$A_1 x = (A_1 \Omega^{-1})(\Omega x) = \tilde{A}_1 \tilde{x} = b_1,$$

where  $\tilde{A}_1 \equiv A_1 \Omega^{-1}$ .

- The linear inequality constraint may be redefined as

$$A_2 x = (A_2 \Omega^{-1})(\Omega x) = \tilde{A}_2 \tilde{x} \leq b_2,$$

where  $\tilde{A}_2 \equiv A_2 \Omega^{-1}$ .

- The quadratic inequality constraint has been replaced by the equality constraint (2), which may be written as

$$z_1 + q'x = z_1 + (q'\Omega^{-1})(\Omega x) = \tilde{q}'\tilde{x} = s,$$

where  $\tilde{q} \equiv (\Omega^{-1})'q$ .

Then (4) may be expressed as the following SOCP problem,

$$\begin{aligned} \max_{\tilde{x}, z_1, z_2} \quad & \tilde{c}'\tilde{x} \\ \text{s.t.} \quad & \tilde{A}_1 \tilde{x} = b_1 \\ & \tilde{A}_2 \tilde{x} \leq b_2 \\ & z_1 + \tilde{q}'\tilde{x} = s \\ & \tilde{x}'\tilde{x} \leq z_1 z_2 \\ & z_2 = 1. \end{aligned} \quad (4)$$

The value  $x$  can be recovered by  $x = \Omega^{-1}\tilde{x}$ .

### The problem with SOCP solvers

SOCP solvers in R such as `sos` and `CLSOCP` only allow the user to define quadratic cones, i.e., for  $x \in \mathbb{R}^n$ , the  $n$ -dimensional quadratic cone is

$$Q^n \equiv \left\{ x \in \mathbb{R}^n : x_1 \geq \sqrt{x_2^2 + x_3^2 + \cdots + x_n^2} \right\}.$$

A rotated second order cone is instead defined as

$$Q_r^n \equiv \left\{ x \in \mathbb{R}^n : 2x_1x_2 \geq x_3^2 + x_4^2 + \cdots + x_n^2 \right\}.$$

The conic constraint

$$\begin{aligned} \|\tilde{x}\|_2^2 &= \tilde{x}'\tilde{x} \\ &= \tilde{x}_1^2 + \tilde{x}_2^2 + \cdots + \tilde{x}_n^2 \\ &\leq z_1z_2 \end{aligned}$$

is thus a rotated quadratic cone. Quadratic cones and rotated quadratic cones are isomorphic to each other, so writing the conic constraint in either form is fine.

However, a problem arises when inputting the SOCP problem into R, which only permits non-rotated quadratic cones. The problem stems from the new variable  $z_1$ , which appears when converting the quadratic constraint into a conic constraint. Since the software only permits non-rotated cones, it is only possible to define a new variable  $\tilde{z}_1$  satisfying

$$\sqrt{\tilde{x}_1^2 + \tilde{x}_2^2 + \cdots + \tilde{x}_n^2} \leq \tilde{z}_1.$$

The equality constraint (2) in the SOCP formulation must then be written as

$$\tilde{z}_1^2 + q'x = s,$$

so that it reverts to a quadratic constraint. Trying to introducing another variable to rewrite  $\tilde{z}_1^2$  as a conic constraint leads to the same problem.