Converting QCQPs to SOCPs

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This document outlines how to write a quadratically constrained quadratic programming (QCQP) problem into a second order cone programming (SOCP) problem. However, the SOCP formulation is not compatible with free SOCP solvers in R. The reason is that these R packages only permit non-rotated cones, whereas the reformulation utilizes rotated cones. The only solver that seems to allow for rotated cones is Mosek, which offers free academic licenses.

Reformulating QCQPs as SOCPs

We want to solve a QCQP problem of the form

$$\begin{aligned} \max_x \quad & c'x \\ \text{s.t.} \quad & A_1 x = b_1 \\ & A_2 x \le b_2 \\ & x'Qx + q'x \le s, \end{aligned} \tag{1}$$

or sq root both sides also fine

where $x \in \mathbb{R}^n$, and $Q \in \mathbb{R}^{n \times n}$ is a positive semidefinite symmetric matrix. To keep things simple, let A_1 and A_2 be row vectors, i.e. there is one linear equality constraint, one linear inequality constraint, and one quadratic inequality constraint.

A conic constraint may generally be written as

$$\|Bx + d\|_2 \le e'x + f,$$

where $B \in \mathbb{R}^{m \times n}$, $d \in \mathbb{R}^m$, $e \in \mathbb{R}^n$, $f \in \mathbb{R}$, and $\| \cdot \|_2$ denotes the Euclidean norm.[1] However, the R packages in `scs` and `CLSOCP` only allow cones to be declared in the form of $\|(x_2, \ldots, x_n)\|_2 \le x_1$, i.e., a non-rotated quadratic cone. As shown below, this is problematic as (4) cannot be reformulated as an SOCP problem with a non-rotated quadratic cone (to the best of my knowledge).

The problem stems from the quadratic constraint, which may be written as

slack

$$z_1 + q'x = s \tag{2}$$

[1] A constraint on the square root of the sum of squared residuals from a linear regression may be written in this form by setting B equal to the design matrix, d equal to the dependent variable, e equal to 0, and f equal to the upper bound.

$$x'Qx \le z_1, \tag{3}$$

where z_1 is a new variable in the problem. since Q is positive semidefinite and symmetric, we can write it as $Q = \Omega'\Omega$. Define $\widetilde{x} \equiv \Omega x$ so that (3) defines a *rotated* quadratic cone

$$\|\widetilde{x}\|_2^2 = x'\Omega'\Omega x \le z_1 z_2,$$

?

where $z_2 = 1$.

All that remains is to rewrite the (4) in terms of $\widetilde{x}$.

Ω not invertible unless Q is p.d (not psd) otherwise $\Rightarrow$ $Q^{-1} = (\Omega^{-1})'\Omega^{-1}$ also invertible.

- The objective may be redefined as

$$c'x = \left(c'\Omega^{-1}\right)(\Omega x) = \widetilde{c}'\widetilde{x},$$

  where $\widetilde{c} \equiv (\Omega^{-1})'c$.

- The linear equality constraint may be redefined as

$$A_1 x = \left(A_1\Omega^{-1}\right)(\Omega x) = \widetilde{A}_1\widetilde{x} = b_1,$$

  where $\widetilde{A}_1 \equiv A_1\Omega^{-1}$.

- The linear inequality constraint may be redefined as

$$A_2 x = \left(A_2\Omega^{-1}\right)(\Omega x) = \widetilde{A}_2\widetilde{x} \le b_2,$$

  where $\widetilde{A}_2 \equiv A_2\Omega^{-1}$.

- The quadratic inequality constraint has been replaced by the equality constraint (2), which may be written as

  $z_1 + \tilde{q}'\tilde{x}$?

$$z_1 + q'x = z_1 + \left(q'\Omega^{-1}\right)(\Omega x) = \widetilde{q}'\widetilde{x} = s,$$

  where $\widetilde{q} \equiv (\Omega^{-1})'q$.

Then (4) may be expressed as the following SOCP problem,

$$\begin{array}{lrcl} \max\limits_{\widetilde{x}, z_1, z_2} & \widetilde{c}'\widetilde{x} & & \\ \text{s.t.} & \widetilde{A}_1\widetilde{x} & = & b_1 \\ & \widetilde{A}_2\widetilde{x} & \le & b_2 \\ & z_1 + q'x & = & s \\ & \widetilde{x}'\widetilde{x} & \le & z_1 z_2 \\ & z_2 & = & 1. \end{array} \tag{4}$$

should be $z_1 + \tilde{q}'\tilde{x} = s$?

The value x can be recovered by $x = \Omega^{-1}\widetilde{x}$.

The problem with SOCP solvers

SOCP solvers in R such as `scs` and `CLSOCP` only allow the user to define quadratic cones, i.e., for $x \in \mathbb{R}^n$, the n-dimensional quadratic cone is

$$Q^n \equiv \left\{ x \in \mathbb{R}^n : x_1 \geq \sqrt{x_2^2 + x_3^2 + \cdots + x_n^2} \right\}.$$

A rotated second order cone is instead defined as

$$Q_r^n \equiv \left\{ x \in \mathbb{R}^n : 2x_1x_2 \geq x_3^2 + x_4^2 + \cdots + x_n^2 \right\}.$$

The conic constraint

$$\begin{aligned} \|\widetilde{x}\|_2^2 &= \widetilde{x}'\widetilde{x} \\ &= \widetilde{x}_1^2 + \widetilde{x}_2^2 + \cdots + \widetilde{x}_n^2 \\ &\leq z_1 z_2 \end{aligned}$$

is thus a rotated quadratic cone. Quadratic cones and rotated quadratic cones are isomorphic to each other, so writing the conic constraint in either form is fine.

However, a problem arises when inputting the SOCP problem into R, which only permits non-rotated quadratic cones. The problem stems from the new variable z_1, which appears when converting the quadratic constraint into a conic constraint. Since the software only permits non-rotated cones, it is only possible to define a new variable $\widetilde{z}_1$ satisfying

$$\sqrt{\widetilde{x}_1^2 + \widetilde{x}_2^2 + \cdots + \widetilde{x}_n^2} \leq \widetilde{z}_1.$$

The equality constraint (2) in the SOCP formulation must then be written as

$$\widetilde{z}_1^2 + q'x = s,$$

so that it reverts to a quadratic constraint. Trying to introducing another variable to rewrite $\widetilde{z}_1^2$ as a conic constraint leads to the same problem.

Our problem:

$$\max_\theta \; c'\theta \quad \text{s.t. } \underbrace{\|A\theta - y\|_2^2}_{\text{least squares criterion}} \le \underbrace{\hat{Q}^*}_{\text{constant from first step QP}}$$

forget about linear constraints — your analysis shows how to handle them

Equivalent to:

$$\max_\theta \; c'\theta \quad \text{s.t. } \|A\theta - y\|_2 \le \sqrt{\hat{Q}^*}$$

take square roots to turn into norm

Rewrite as:

$$\max_{x = [\theta', z]'} \; c'\theta \quad \text{s.t. } y = z, \quad \|\tilde{A}x\|_2 \le \sqrt{\hat{Q}^*}$$

combine θ and y notationally

Reformulate as:

$$\max_{x = [\theta', z, w']'} \; c'\theta \quad \text{s.t. } y = z, \quad \tilde{A}x = w, \quad \|w\|_2 \le \sqrt{\hat{Q}^*}$$

get rid of rotation.

Now this is in the right format?