Converting QCQPs to SOCPs

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This document outlines how to write a quadratically constrained quadratic programming (QCQP) problem into a second order cone programming (SOCP) problem so that it may be solved using the R package `scs`.

Cone constraints

This document involves three types of cones. The first type is the zero cone,

$$\mathcal{K}_0 \equiv \{x \in \mathbb{R}^n : x = 0\}.$$

The second type is the positive orthant,

$$\mathcal{K}_1 \equiv \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for } i = 1, \ldots, n\}.$$

The third type is the second order cone and it comes in two forms,

$$\mathcal{K}_2 \equiv \left\{x \in \mathbb{R}^n : x_1 \geq \sqrt{x_2^2 + \cdots + x_n^2}\right\},$$
$$\mathcal{K}_2^{\text{rot}} \equiv \left\{x \in \mathbb{R}^n : 2x_1x_2 \geq x_3^2 + \cdots + x_n^2\right\},$$

where $\mathcal{K}_2$ is a quadratic cone and $\mathcal{K}_2^{\text{rot}}$ is its rotated counterpart. That is, $\mathcal{K}_2^{\text{rot}}$ is obtained by rotating $\mathcal{K}_2$ around the last $n-2$ axes in $\mathbb{R}^n$ so that, for $x \in \mathcal{K}_2$ and $\widetilde{x} \in \mathcal{K}_2^{\text{rot}}$, we have $x_i = \widetilde{x}_i$ for $i = 3, \ldots, n$.

The mapping from $\mathcal{K}_2$ to $\mathcal{K}_2^{\text{rot}}$ is conveniently characterized by the rotation matrix T,

$$T \equiv \left[\begin{array}{c:c} T_1 & 0_{2\times(n-2)} \\ \hdashline 0_{(n-2)\times 2} & T_2 \end{array}\right] \tag{1}$$
$$T_1 \equiv \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$
$$T_2 \equiv I_{n-2},$$

where $0_{n\times m}$ denotes an $n \times m$ matrix of zeroes, and I_n denotes an $n \times n$ identity matrix. Since T is a symmetric rotation matrix, we have that $T = T' = T^{-1}$. So for $x \in \mathcal{K}_2$, we have $Tx \in \mathcal{K}_2^{\text{rot}}$. Likewise, for $\widetilde{x} \in \mathcal{K}_2^{\text{rot}}$ we have $T\widetilde{x} \in \mathcal{K}_2$.[1]

[1] This page by Mosek has more details on rotating cones and conic quadratic optimization.

Reformulating QCQP problems as SOCP problems with rotated cone constraints

We want to solve a QCQP problem of the form

$$\begin{array}{llrcl} \max\limits_{x} & c'x & & & \\ \text{s.t.} & & A_1 x & = & b_1 \\ & & A_2 x & \leq & b_2 \\ & & x'Qx + q'x & \leq & b_3, \end{array} \tag{2}$$

where $x \in \mathbb{R}^n$, and $Q \in \mathbb{R}^{n \times n}$ is a positive definite symmetric matrix. Let A_1 be an $h_1 \times n$ matrix A_2 be an $h_2 \times n$ matrix. In total, there are there are h_1 linear equality constraints, h_2 linear inequality constraints, and one quadratic constraint.

A second order cone constraint may generally be written as

$$\|Bx + d\|_2 \leq e'x + f, \tag{3}$$

where $B \in \mathbb{R}^{m \times n}$, $d \in \mathbb{R}^m$, $e \in \mathbb{R}^n$, $f \in \mathbb{R}$, and $\| \cdot \|_2$ denotes the Euclidean norm. Since a quadratic constraint can be rewritten in this form, the QCQP problem in (2) may be treated as an SOCP problem.[2] However, the R package `scs` will not allow cones to be passed in the general form of (3). So this document outlines how all the constraints may be reformulated in a different way to be compatible with `scs`.

Begin by introducing the slack variable z_1 into the quadratic constraint,

$$z_1 + q'x = b_3 \tag{4}$$

$$x'Qx \leq z_1. \tag{5}$$

Since Q is positive definite and symmetric, we can write it as $Q = \Omega'\Omega$. Define $\widetilde{x} \equiv \Omega x$ so that (5) defines a rotated quadratic cone

$$x'\Omega'\Omega x = \widetilde{x}_1^2 + \cdots + \widetilde{x}_n^2 \leq 2 z_1 z_2,$$

where $z_2 = \frac{1}{2}$.

What remains is to rewrite all the other components of (2) in terms of $\widetilde{x}$.

[2] A constraint on the square root of the sum of squared residuals from a linear regression may be written in this form by setting B equal to the design matrix, d equal to the dependent variable, e equal to 0, and f equal to the upper bound.

- The objective may be redefined as

$$c'x = \left(c'\Omega^{-1}\right)(\Omega x) = \widetilde{c}'\widetilde{x},$$

where $\widetilde{c} \equiv (\Omega^{-1})'c$.

- The linear equality constraints may be redefined as

$$A_1 x = \left(A_1\Omega^{-1}\right)(\Omega x) = \widetilde{A}_1\widetilde{x} = b_1,$$

where $\widetilde{A}_1 \equiv A_1\Omega^{-1}$.

- The linear inequality constraints may be redefined as

$$A_2 x = \left(A_2\Omega^{-1}\right)(\Omega x) = \widetilde{A}_2\widetilde{x} \leq b_2,$$

where $\widetilde{A}_2 \equiv A_2\Omega^{-1}$.

- The quadratic inequality constraint has been replaced by the equality constraint (4), which may be written as

$$z_1 + q'x = z_1 + \left(q'\Omega^{-1}\right)(\Omega x) = z_1 + \widetilde{q}'\widetilde{x} = b_3,$$

where $\widetilde{q} \equiv (\Omega^{-1})'q$.

Define $\widehat{x} \equiv (z_1, z_2, \widetilde{x}')'$. Then (2) may be expressed as the following SOCP problem with a rotated cone constraint,

$$\begin{array}{lrcl}
\max\limits_{\widehat{x}} & \widehat{c}'\widehat{x} & & \\
\text{s.t.} & \widehat{A}_1\widehat{x} & = & b_1 \\
& \widehat{A}_2\widehat{x} & \leq & b_2 \\
& \widehat{A}_3\widehat{x} & = & b_3 \\
& \widehat{A}_4\widehat{x} & = & \frac{1}{2} \\
& \widehat{x}_3^2 + \cdots + \widehat{x}_{n+2}^2 & \leq & 2\widehat{x}_1\widehat{x}_2,
\end{array} \tag{6}$$

where

$$\widehat{A}_1 \equiv \left[\begin{array}{cc} 0_{h_1\times 2} & \widetilde{A}_1 \end{array}\right]$$

$$\widehat{A}_2 \equiv \left[\begin{array}{cc} 0_{h_2\times 2} & \widetilde{A}_2 \end{array}\right]$$

$$\widehat{A}_3 \equiv \left[\begin{array}{ccc} 1 & 0 & \widetilde{q}' \end{array}\right]$$

$$\widehat{A}_4 \equiv e_2',$$

and e_i is the i^{th} standard basis vector in $\mathbb{R}^{n+2}$. The value x can be recovered by

$$x = \Omega^{-1} \begin{bmatrix} \widehat{x}_3 \\ \vdots \\ \widehat{x}_{n+2} \end{bmatrix}.$$

Undoing the rotation of the cone constraint

Unfortunately, the function `scs` does not permit rotated quadratic cones. Nevertheless, we can undo the rotation of the cone constraint in (6) using the rotation matrix T defined in (1).

In order for $\widehat{x}$ to satisfy the rotated cone constraint in (6), it must be that $T^{-1}\widehat{x} = T\widehat{x}$ satisfies a non-rotated cone constraint. So rewrite (6) in terms of $\dot{x} \equiv T\widehat{x}$,

$$\begin{array}{lrcl} \max\limits_{\dot{x}} & \dot{c}'\dot{x} & & \\ \text{s.t.} & \dot{A}_1\dot{x} & = & b_1 \\ & \dot{A}_2\dot{x} & \leq & b_2 \\ & \dot{A}_3\dot{x} & = & b_3 \\ & \dot{A}_4\dot{x} & = & \frac{1}{2} \\ & \sqrt{\dot{x}_2^2 + \dot{x}_3^2 + \cdots + \dot{x}_{n+2}^2} & \leq & \dot{x}_1, \end{array} \tag{7}$$

where

$$\begin{aligned} \dot{A}_1 &\equiv \widehat{A}_1 T = \widehat{A}_1 \\ \dot{A}_2 &\equiv \widehat{A}_2 T = \widehat{A}_2 \\ \dot{A}_3 &\equiv \widehat{A}_3 T \\ \dot{A}_4 &\equiv \widehat{A}_4 T, \end{aligned}$$

with the equalities in the first two lines following from the fact that T is a rotation around the axes for $\dot{x}_3, \ldots, \dot{x}_{n+2}$, i.e. the non-zero columns in $\widehat{A}_1$ and $\widehat{A}_2$ correspond to T_2 in T, which is the identity matrix.

Now we have written the QCQP problem in (2) in the form of a SOCP problem with a non-rotated cone constraint. We can recover x from

$$x = \Omega^{-1} \begin{bmatrix} (T\dot{x})_3 \\ \vdots \\ (T\dot{x})_{n+2} \end{bmatrix} = \Omega^{-1} \begin{bmatrix} \dot{x}_3 \\ \vdots \\ \dot{x}_{n+2} \end{bmatrix},$$

with the second equality again following from the fact that T is a rotation around the axes for $\dot{x}_3, \ldots, \dot{x}_{n+2}$.

Passing the SOCP problem into the R function scs

The function `scs` only admits problems of the form

$$\begin{array}{rrcl} \max\limits_{x} & c'x & & \\ \text{s.t.} & Ax + s & = & b \\ & s & \in & \mathcal{K}, \end{array} \tag{8}$$

where $\mathcal{K}$ is a Cartesian product of cones. The set $\mathcal{K}$ includes zero cones, the positive orthant, and second order cones.[3] The user may declare which components of s constitute a cone, and of what type. The number of equality constraints must equal the dimension of s, with each equality constraint containing only one component of s.

To adhere to this form, it is actually easier to rewrite (7) in terms of $\widetilde{x}$ and slack variables s (as opposed to combining $\widetilde{x}$ and the slack variables into a single vector, as previously done with $\widehat{x}$ and $\dot{x}$). Let $\mathcal{K}_i^j$ denote a cone $\mathcal{K}_i \subset \mathbb{R}^j$. Let $s_1 \in \mathcal{K}_0^{h_1} \subset \mathbb{R}^{h_1}$. Then s_1 may be used to write the equality constraints defined by $\widetilde{A}_1$ that do not involve conic variables,

$$\widetilde{A}_1\widetilde{x} + s_1 = b_1.$$

To write the inequality constraints defined by $\widetilde{A}_2$, introduce the vector of slack variables $s_2 \in \mathcal{K}_1^{h_2}$,

$$\widetilde{A}_2\widetilde{x} + s_2 = b_2.$$

The non-rotated quadratic constraint is characterized by the final three constraints in (7). The first two of the constraints imply the following,

$$\dot{A}_3\dot{x} = b_3 \Rightarrow z_1 = b_3 - \widetilde{q}'\widetilde{x},$$

[3] It may include several other types of cones that will not be useful for our purpose.

$$\dot{A}_4\dot{x} = \frac{1}{2} \Rightarrow z_2 = \frac{1}{2}.$$

The rotation matrix T only affects z_1 and z_2 through T_1, and does not affect $\widetilde{x}$. So the effect it has on the SOCP problem is restricted to how T_1 transforms z_1 and z_2,

$$\begin{aligned} T_1 \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= T_1 \begin{bmatrix} b_3 - \widetilde{q}'\widetilde{x} \\ \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}}(b_3 - \widetilde{q}'\widetilde{x}) + \frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}}(b_3 - \widetilde{q}'\widetilde{x}) - \frac{1}{2\sqrt{2}} \end{bmatrix}. \end{aligned}$$

Then by defining the slack variables $s_3, s_4 \in \mathbb{R}$ as

$$\begin{bmatrix} s_3 \\ s_4 \end{bmatrix} \equiv T_1 \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},$$

we have that

$$\begin{aligned} \frac{1}{\sqrt{2}}\widetilde{q}'\widetilde{x} + s_3 &= \frac{1}{\sqrt{2}}b_3 + \frac{1}{2\sqrt{2}}, \\ \frac{1}{\sqrt{2}}\widetilde{q}'\widetilde{x} + s_4 &= \frac{1}{\sqrt{2}}b_3 - \frac{1}{2\sqrt{2}}. \end{aligned}$$

To complete the quadratic constraint, define the final vector of slack variables $s_5 \in \mathbb{R}^n$ as

$$-I_n + s_5 = 0.$$

The vector $(s_3, s_4, s_5')'$ belongs to the non-rotated quadratic cone $\mathcal{K}_2^{n+2}$.

So all together, we have

$$\begin{array}{lrcccl} \max\limits_{\widetilde{x}} & c'\widetilde{x} & & & & \\ \text{s.t.} & \widetilde{A}_1\widetilde{x} & + & s_1 & = & b_1 \\ & \widetilde{A}_2\widetilde{x} & + & s_2 & = & b_2 \\ & \frac{1}{\sqrt{2}}\widetilde{q}'\widetilde{x} & + & s_3 & = & \frac{1}{\sqrt{2}}b_3 + \frac{1}{2\sqrt{2}} \\ & \frac{1}{\sqrt{2}}\widetilde{q}'\widetilde{x} & + & s_4 & = & \frac{1}{\sqrt{2}}b_3 - \frac{1}{2\sqrt{2}} \\ & -I_n & + & s_5 & = & 0 \\ & & & s_1 & \in & \mathcal{K}_0^{h_1} \\ & & & s_2 & \in & \mathcal{K}_1^{h_2} \\ & & & (s_3, s_4, s_5')' & \in & \mathcal{K}_2^{n+2}. \end{array} \tag{9}$$

The SOCP problem in (9) is equivalent to the QCQP problem in (2) and may be passed to `scs`.