ON LIE GROUPS WITH APPLICATIONS IN QUANTUM MECHANICS

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1. Introduction

Lie groups are a special type of object that connects the fields of Differential Calculus and Abstract Algebra together. This allows them to be considered as a smooth transformation, while allowing them to retain the power of group theory. This makes Lie Groups essential for anything to do with Continuous Symmetries, which consequently makes them a pivotal part of algebra and geometry while also being a indispensable piece of equipment for particle physics.

2. Lie Groups

All Lie Theory starts with the basic element, Lie Groups. A Lie Group is a set G with the two properties that G is a group and is a smooth and real manifold, meaning the group operations of multiplication and inversion are smooth maps.

Definition 2.1 (Lie Group)**.** A *Lie Group* G is a differentiable manifold which is also endowed with a group structure such that the multiplication map

$$G \times G \to G$$

and the inverse map

$$\iota : G \to G,\ \iota(x) = x^{-1}$$

are both smooth.

Putting it another more informal way by John Baez:

$$\text{Lie Groups} = \underset{\text{(Groups)}}{\text{Symmetry}} + \underset{\text{(Manifolds)}}{\text{Calculus}}$$

Where the group operation is smooth. Smooth here can actually be interpreted in many ways, C^1, C^∞, analytic, yet as a result of Hilbert's 5th problem they are all equivalent: every C^0 lie group has a unique analytic structure. We will assume here that "smooth" = C^∞ though.

Example 2.1. Here are a couple of examples of Lie Groups:

- $\mathbb{R}^n$, $+$
- $\mathbb{R}^*$, $\times$
- $\mathrm{GL}(n, \mathbb{R})$
- $\mathrm{SL}(n, \mathbb{R})$
- $\mathrm{O}(n)$
- $\mathrm{SO}(n)$, $\mathrm{SO}(2)$, $\mathrm{SO}(3)$
- $\mathrm{SU}(2)$

and many more.

We will go into more depth for $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{SO}(n)$, $\mathrm{SO}(2)$, $\mathrm{SO}(3)$, and $\mathrm{SU}(2)$.

Let us show that some of these are Lie Groups.

One convenient fact is that most of the Lie groups that matter are subspaces of some $\mathbb{R}^n$.

3. Lie Matricies

Definition 3.1 (Lie Matrix Group)**.** A Lie Group is defined as a continuous subgroup of all non-singular $n \times n$ matrices over a field $\mathbb{F}$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.

This is also known as $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$. Taking a page out of Axler, $\mathbb{F}$ stands for either $\mathbb{R}$ or $\mathbb{C}$, depending on the context.

Theorem 3.2 (Closed Subgroup Theorem)**.** *Let G be a Lie Group and H be a closed subgroup of G, then H is a Lie Subgroup of G.*

Although this is a highly non trivial result, the proof can be broken down into 4 steps. We must first show that

Proposition 3.3. *If $\{u_n\}_n$ is a sequence in T_eG such that $\frac{u_n}{||u_n||} \to v \in T_eG$, $||u_n|| \to 0$, and $expu_n \in H$ for all n, then $exptv \in H$ for all $t \in \mathbb{R}$.*

Then we must show that

Proposition 3.4. *For all $v_1, v_2 \in V = \{v | exp(tv) \in H\}$, $exp(t(v_1 + v_2)) \in H$.*

Then we can use that to show that

Proposition 3.5. *$exp(V)$ is a neighborhood of $e \in H$*

in order to find the inclusion map to prove a smooth embedding, hence satisfying the result.

Since many of the most important Lie Groups are closed subgroups of $\mathrm{GL}(n, \mathbb{F})$, this suffices to show they are a Lie Group

Proof that the General Linear Group is a Lie Group. Since $u : \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is smooth and $\mathrm{GL}(n, \mathbb{R}) \subset R^{n\times n} \implies u|_{\mathrm{GL}(n,\mathbb{R})\times\mathrm{GL}(n,\mathbb{R})}$ is smooth.

4. Lie Algebras

Lie Algebras are the tangent spaces at the identity of a Lie Algebra. They can be thought of as infinitesimal symmetry motions. This will feed later into why they matter for Quantum Mechanics.

To start, lets define the *Lie Bracket*. The Lie Bracket on vector fields is an operator that assigns to any two fields X and Y on a smooth manifold M a third vector field denoted $[X, Y]$. There are multiple proper definitions for the lie bracket, but we will define it as followed.

Definition 4.1 (Lie Bracket)**.** The commutator $\delta_1 \circ \delta_2 - \delta_3 \circ \delta_1$ of any two derivations δ_1, δ_2 is a derivation, where $\circ$ is a composition of operators. Then the Lie Bracket is

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \quad \forall f \in C^\infty(M).$$

Note most problems can just describe the Lie Bracket for each group, so full understanding of the definition is not necessarily needed to understand Lie Algebras to some degree as long as you understand that whatever operation give must satisfy the identity.

Definition 4.2 (Lie Algebra)**.** A *Lie Algebra* is a real vector space V with a bi-linear operation

$$[\cdot,\cdot] : V \times V \to V$$

such that for all $X, Y, Z \in V$

(1) $[X < Y] = -[Y, X]$
(2) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Where (1) is the Skew-symmetry and (2) is the Jacobi identity.

Example. For example in $\mathbb{R}^3$,

$$[x, y] = x \times y.$$

Satisfies the Jacobi Identify

$$x \times (y \times z) = (x \times y) \times z + y \times (x \times z).$$

Where $\times$ is the cross product in $\mathbb{R}^3$, making it a Lie Algebra.

Definition 4.3 (Exponential Map)**.** The Exponential Map is a map from the Lie Algebra $\mathfrak{g}$ to the Lie Group G denoted as $\exp : \mathfrak{g} \to G$

Suppose G is a Lie Group and H is a subgroup of G implies that H is topologically closed. Since H is a submanifold of G by definition, we know that it is locally closed. Because it is locally closed there exists an U of $e \in G$ such that $U \cap H = U \cap \bar{H}$. Now since for all $h \in G$, hU is open implies that $hU \cap H \neq \varnothing$. $h' \in hU \cap H$ implies that $h^{-1}h' \in U$. But since $h \in \hat{H}$ then there exists $h_n \in H$ that converges to h. Since we also know that $h_n^{-1}h' \in H$ converges to $h^{-1}h'$ then $h^{-1}h' \in U \cap \bar{H} = U \cap H \implies \bar{H} \subset H$.

5. O(n) Groups

Definition 5.1. *Orthogonal Groups* of dimension n is the set of all distance preserving transformations of a Euclidean Space of n, denoted $\mathrm{O}(n)$.

This also means that it is the group of $n \times n$ orthogonal matrices under matrix multiplication.

Since orthogonality implies $A^tA = 1 \implies \det(A)^2 = 1 \implies \det(A) = \pm 1$. This effectively creates a group that implies its transformations are all reflections and rotations.

6. SO(2) and SO(3) Groups

We can restrict the $\mathrm{O}(n)$ group by defining the $\mathrm{SO}(n)$ subgroup as the subgroup of the group $O(n)$ containing all matrices with determinant 1. This adds a main property to the group, mainly that all matrix transformations now only represent rotations in their respective places.

Definition 6.1. A rotation is a smooth map $A : \mathbb{R} \to \mathbb{R}$ such that A preserves the origin, angles, distances, and orientations.

The $\mathrm{SO}(n)$ groups are in general called the special orthogonal groups. They consist of all orthogonal matrices with determinant 1. This gives the $\mathrm{SO}(n)$ groups a special name: the *Rotation Group*. This is because any transformation done by the $\mathrm{SO}(n)$ groups satisfy the property of rotation.

It is not hard to see why a rotation group for $\mathbb{R}^2$ and $\mathbb{R}^3$ might be incredibly useful. We can define SO(2) first.

6.1. **SO**(2).

Definition 6.2. The SO(2) group is the set of all matrices

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

such that $a^2 + b^2 = 1$

6.2. **SO**(3). The SO(3) group is very similar to the SO(2) group in that it represents rotation, just in $\mathbb{R}^3$ instead of $\mathbb{R}^2$. We can represent this group as followed. The group SO(3) is generated by all elementary rotations $R_x(\alpha)$, $R_y(\beta)$, and $R_z(\gamma)$ given by

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix},\ R_y = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix},\ R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

7. $SU(2)$ Group

SU(n) is the complex analogue to SO(n).

Definition 7.1 (SU(2)). SU(2) is the set of all 2×2 matrices of det$= 1$ that satisfies $A^\dagger A$,

$$A = \begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{bmatrix},\ \text{where } |\alpha|^2 + |\beta|^2 = 1$$

Theorem 7.2. *There is also a two-to-one homomorphic map from $SU(2)$ onto $SO(3)$, making both groups nearly identical.*

8. Application to Quantum Mechanics

Quantum mechanics is one of the many places that Lie Groups shine. This is because due to the fact that certain Lie Groups are representative of rotations, they can be used to describe the switching of states in particles.

Definition 8.1. The Quantum State of a physical system is specified by a non-zero vector in Hilbert space over the complex numbers.

For an electron, we can represent its spin as a linear combination of two numbers

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \alpha, \beta \in \mathbb{C}$$

But since the probability of both states must add to 1, $|\alpha|^2 + |\beta|^2 = 1$ must be an additional applied constraint.

Now suppose that we rotate the coordinate axis and the spinor starts to change. In order to see how they might change, we can write the following equation.

$$\begin{bmatrix} \alpha' \\ \beta' \end{bmatrix} = U(\theta) \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

where

$$U(\theta) = e^{\frac{-i(\theta\sigma)}{2}}.$$

Now since this matrix is unitary and has determinant of 1, it is represented under $SU(2)$! Thus we can define transformations of an electron's spin using Lie Groups.

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