MATH460 Module 2 Artifact

Abstract

Let $\mathcal{C}(m, r)$ be an open ball of radius r centered at $m \in M$, a smooth manifold, and consider $N(m, c, \varepsilon)$, the number of disjoint congruent balls of radius ε that can be drawn on M within $\mathcal{C}(m, c\varepsilon)$ for $c \geq 3$. We investigate N with theoretical methods suggesting that N will approach a finite upper bound as $\varepsilon \to 0$.

1. Introduction

   First, we may look at Euclidean space of familiar dimension (namely 2 and 3) as a basis for investigating other smooth manifolds such as the sphere or torus. There, this question of counting the number of ε-balls that can cover a region on the manifold can be magnified. Since any open patch of M^2 is homeomorphic to $\mathbb{E}^2$, we may consider the open $c\varepsilon$-ball for $c \geq 3$ as specified in the problem statement a distorted patch of the Euclidean plane, which has familiar tilings. Notice how, because we can create necessarily invertible charts from M to $\mathbb{E}^n$ and because smooth manifolds possess complete Riemannian metrics, the Euclidean analogy is not unhelpful. To motivate it further, recall that a packing (of congruent objects) is fully described by its associated set of center points. That is, a circle with fixed radius ε is uniquely identified by the location of its center. The only constraint when generating such a (non-overlapping) packing from a set of centers is that the radius cannot exceed half of the minimum distance between any two points. In $\mathbb{E}^n$, however, there are many ways to form a lattice such that every point is necessarily equidistant from all its neighbors. Furthermore, when fully tiling the space with other shapes, one can simply inscribe a ball within that shape, sharing a center point. Consider $\mathbb{R}$, on which we can pack disks by choosing our set of centers to be $\mathbb{Z}$. Thus, the line is packed with ε-balls (intervals) with $\varepsilon = 0.5$. In $\mathbb{R}^2$, we can select our set of centers to be every lattice point of the Cartesian grid, allowing us to tile the plane with squares before inscribing circles. Alternatively, we could tile the plane with hexagons (side length 1), still allowing us to pack ε-balls of radius 0.5. Respectively, the densities of these packings are $\frac{\pi}{4}$ and $\frac{\pi}{\sqrt{12}}$ (in two dimensions, the hexagonal lattice does provide the densest circle packing).

2. Stereographic Projection

   Considering this question as raised with $M = S^2$, we can imagine a stereographic projection of the sphere onto the Euclidean plane, which is a conformal mapping that will therefore preserve (locally) shapes. If S^2 is the unit sphere centered at the origin and with pole $p = (0, 0, 1)$, then the projective map may be represented by

$$\chi : S^2 \backslash \{p\} \to \mathbb{R}^2$$

   and defined with

$$\chi(m) = \left(\frac{x}{1 - z}, \frac{y}{1 - z} \right).$$

Its inverse χ^{-1} is then given by

$$\chi^{-1}(x,y) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right),$$

allowing us to project from the plane onto the sphere. If we are to project a patch of S^2 in the shape of an open $c\varepsilon$-ball onto $\mathbb{R}^2$, create a lattice Λ on that patch which optimizes $N(m,c,\varepsilon)$, then return it to the sphere via χ^{-1}, there is now a deformed patch of the plane that contains already a packing of disks with maximal density.

3. Contact Graphs

Perhaps to avoid charts altogether, we may construct a different type of translation. Consider, within $\mathcal{C}(m, c\varepsilon)$, the number and orientation of disjoint, congruent ε-balls in a packing that maximizes N. The simple graph whose vertices are these balls and whose edges represent the contact of the vertices they connect is called the contact graph of the packing. As $\varepsilon \to 0$, the number of vertices on this graph will increase, but not without bound. Just as $\mathcal{C}(m, c\varepsilon)$ restricts the balls to scale with its own radius, the number of vertices of the graph is restricted by the number of edges. Every time a new vertex is added, it must be connected with at least one edge (even in low-density packings this number is still much larger than 1).