Math 107 Calculus II – Spring 2018 – Exam 2 – Solutions

1. Consider the series

$$1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \frac{z^4}{16} + \cdots .$$

   (a) (5 points) Find the sum of the series.
   Solution: $\frac{1}{1-\frac{z}{2}}$ or $\frac{2}{2-z}$

   (b) (5 points) For what values of z does the series converge to the sum you found in (a)? (Give an interval for your answer.)
   Solution: It converges when z is in $(-2, 2)$ or, equivalently, when $-2 < z < 2$.

2. (10 points) By setting up and evaluating a definite integral, demonstrate that the volume of a cone of radius 7 and height 11 is $\frac{1}{3} \cdot \pi \cdot 7^2 \cdot 11$. Specify precisely the name and the meaning of the variable you use in the blanks below, and indicate in the picture how you are slicing the cone.

Name of variable: h

Meaning of variable: distance down from the top of the cone

Solution: We slice it horizontally, giving circular cross section. At h the radius is $\frac{7}{11}h$ and so the volume of an infinitesimal slice is $\pi\left(\frac{7}{11}h\right)^2 dh$. Total volume is thus

$$\int_0^{11} \pi \frac{7^2}{11^2} h^2\, dh = \pi \frac{7^2}{11^2}\frac{h^3}{3}\Big|_0^{11} = \pi \frac{7^2}{11^2}\frac{11^3}{3} = \pi \cdot 7^2 \cdot 11 \cdot \frac{1}{3}.$$

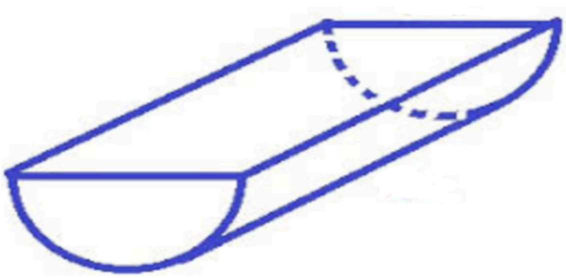
One could also let y be the distance up from the bottom (i.e., $y = 11 - h$). This leads to a cross-sectional area of $\pi(7 - \frac{7}{11}y)^2$ and

$$\int_0^{11} \pi(7 - \frac{7}{11}y)^2\, dy = \int_0^{11} \pi(7^2 - 2\frac{7}{11}y + \frac{7^2}{11^2}y^2)\, dy$$

$$= \pi(7^2 y - \frac{7^2}{11}y^2 + \frac{1}{3}\frac{7^2}{11^2}y^3)\Big|_0^{11} = \pi(7^2 11 - 7^2 \cdot 11 + \frac{1}{3} \cdot 7^2 \cdot 11) = \pi \cdot 7^2 \cdot 11 \cdot \frac{1}{3}.$$

3. The tank pictured below is a half-cyinder of radius 5 feet and length 16 feet, and its rectangular top is at gound level. It is filled with a fluid that weighs 70 pounds/ft^3.

   (a) (6 points) Give a formula for the approximate weight of the layer of fluid located h feet down from the top of tank having very small thickness Δh.



Solution: The cross section h feet down is a rectangle of length 16 and width $w(h) = 2\sqrt{25-h^2}$. So the volume of the layer is $32 \cdot \sqrt{25-h^2}\,\Delta h$ and its weight is

$$70 \cdot 32 \cdot \sqrt{25-h^2}\,\Delta h = 2240 \cdot \sqrt{25-h^2}\,\Delta h \text{ pounds}$$

(b) (6 points) Set up, but do *not* evaluate a definite integral that give the total work needed to pump all of the fluid to ground level.

Solution: The layer of the previous part wieghs $2240 \cdot \sqrt{25-h^2}\,\Delta h$ pounds and needs to be pumped upward by h feet, and thus the work needed for this layer is

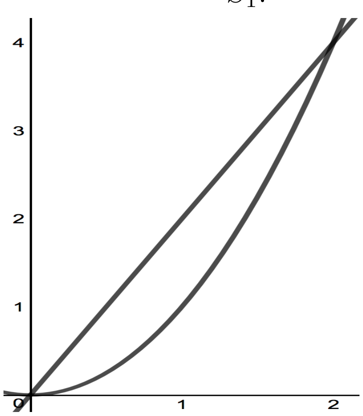
$$h \cdot 2240 \cdot \sqrt{25-h^2}\,\Delta h \text{ pounds}$$

The total work is thus

$$\int_0^5 h \cdot 2240 \cdot \sqrt{25-h^2}\,dh \text{ foot-pounds}$$

4. Let R be the region bounded by the curves $y = 2x$ and $y = x^2$. See the picture below.

(a) (6 points) Let S_1 be the solid of revolution obtained by rotating R about the vertical line $x = 3$. Set up, but do *not* evaluate, a definite integral that gives the volume of S_1.



Solution: The cross section at height y, for $0 \leq y \leq 4$, is an annulus with inner radius

$$r_{inner}(y) = 3 - \sqrt{y}$$

and outer radius

$$r_{outer}(y) = 3 - \frac{1}{2}y$$

The area of this annulus is

$$A(y) = \pi(3 - \frac{1}{2}y)^2 - \pi(3-\sqrt{y})^2$$

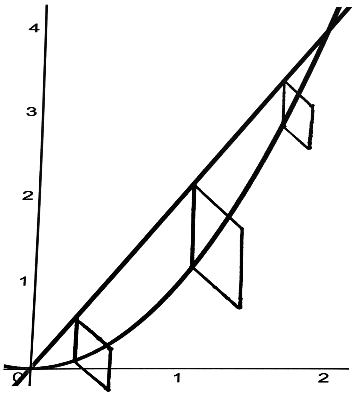
and thus the total volume is

$$\int_0^4 \left(\pi(3 - \frac{1}{2}y)^2 - \pi(3-\sqrt{y})^2\right)\,dy.$$

or equivalently

$$\int_0^4 \pi(3 - \frac{1}{2}y)^2\,dy - \int_0^4 \pi(3-\sqrt{y})^2\,dy.$$

(b) (6 points) Let S_2 be the solid whose base is R and whose cross-sections perpendicular to the x-axis are perfect squares. See the picture on the right. Set up, but do *not* evaluate, a definite integral that gives the volume of S_2.



Solution: The cross section at x, for $0 \le x \le 2$, is a square with side length

$$s(x) = 2x - x^2$$

and thus it has area

$$A(x) = (2x - x^2)^2.$$

The total volume is thus

$$\int_0^2 (2x - x^2)^2 \, dx$$

or equivalently

$$\int_0^2 (4x^2 - 4x^3 + x^4) \, dx.$$

5. (12) Use the integral test to determine whether the series

$$\sum_{n=1}^{\infty} \frac{2n}{e^{n^2}}$$

converges or diverges. Be sure to explain why the test can be used.

Solution: The function $f(x) = \frac{2x}{e^{x^2}}$ is positive and decreasing for all $x \ge 1$. (The latter could be checked carefully by computing the derivative.) So, we may apply the integral test. The improper integral $\int_1^\infty \frac{2x}{e^{x^2}} \, dx$ converegs since

$$\int_1^\infty \frac{2x}{e^{x^2}} \, dx = \lim_{b \to \infty} e^{-x^2}|_1^b = e^{-1}.$$

So the series does too.

6. (10 points) Consider the series

$$\sum_{n=1}^{\infty} \frac{\sin(n) + 4}{n^2}$$

It is a fact that this series converges. Circle ALL of the following statements that are valid justifications of this fact:

(a) The series converges because $\lim_{n\to\infty} \frac{\sin(n)+4}{n^2} = 0$.

(b) The series converges by the integral test.

(c) The series converges by the (direct) comparison test, using $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n^2}$.

(d) The series converges by the limit comparison test, using $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n^2}$.

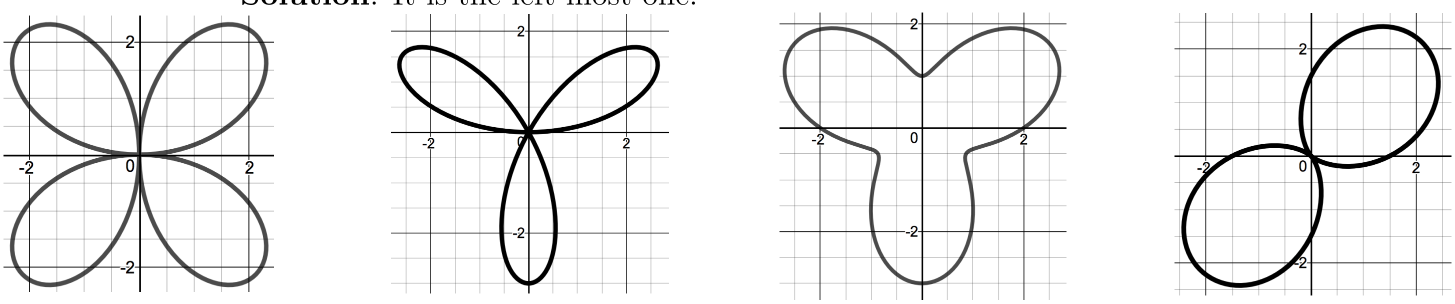
(e) The series converges since it is a geometric series.

Solution: Only (c) is correct. (Note that (b) is wrong since the evident function is not eventually decreasing.)

7. Three questions on $r = 3\sin(2\theta)$.

(a) (3 points) Pictured below are four polar curves. Which one is the graph of $r = 3\sin(2\theta)$? Circle the correct one.

Solution: It is the left-most one.



(b) (3 points) Find both points in the first quadrant where the curve $r = 3\sin(2\theta)$ meets the circle of radius $\frac{3}{\sqrt{2}}$ centered at the origin. Express both points in *polar coordinates*.

Solution: If $\frac{3}{\sqrt{2}} = 3\sin(2\theta)$ then $\sin(2\theta) = \frac{1}{\sqrt{2}}$, and this occurs when $2\theta = \pi/4$ or $3\pi/4$ or .... So, the two points are

$$\left(\frac{3}{\sqrt{2}}, \frac{\pi}{8}\right) \text{ and } \left(\frac{3}{\sqrt{2}}, \frac{3\pi}{8}\right).$$

(c) (6 points) Set up, but do not evaluate, a definite integral (or a difference of definite integrals) that gives the area of the region in the first quadrant that is *inside the curve* $r = 3\sin(2\theta)$ and *outside the curve* $r = \frac{3}{\sqrt{2}}$.

Solution: Either

$$\int_{\pi/8}^{3\pi/8} \frac{(3\sin(2\theta))^2}{2}\, d\theta - \int_{\pi/8}^{3\pi/8} \frac{(\frac{3}{\sqrt{2}})^2}{2}\, d\theta$$

or

$$\int_{\pi/8}^{3\pi/8} \frac{(3\sin(2\theta))^2}{2} - \frac{(\frac{3}{\sqrt{2}})^2}{2}\, d\theta$$

8. (12 points) An off-shore oil rig is leaking, causing an oil slick on the surface of the ocean. The density of the oil is given by $\delta(r) = 1000 - \dfrac{r^2}{1000}$ kg/m^2 where r is the distance, in meters, from the center. If the oil slick is a perfect circle, extending from $r = 0$ to $r = 1000$ meters, find the total mass of the oil in the slick.

   Solution: The annulus of radius r of small thickness Δr has approximate area $2\pi r\Delta r$ and the oil in such an annulus has approximate mass $\delta(t)2\pi r\Delta r$. So the total mass is

$$\int_0^{1000} (1000 - \frac{r^2}{1000})2\pi r\, dr = 2\pi(1000\frac{r^2}{2} - \frac{r^4}{4 \cdot 1000})|_0^{1000} =$$

$$2\pi(\frac{1000^3}{2} - \frac{1000^3}{4}) = \frac{1000^3}{2}\pi = 500,000,000\pi \text{ kilograms.}$$

9. (10 points) Use a comparison test to determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges or diverges. Be sure to indicate which test you are using and to explain why that test can be used.

   Solution: The series converges

   We use the Limit Comparison Test with comparison series $\sum_{n=1}^{\infty} \frac{1}{2^n}$. The latter converges (to 1, in fact) since it is a geometric seires with ratio $\frac{1}{2}$. We have

$$\lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - 2^{-n}} = 1.$$

   Since this limit exists and is > 0, and since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, by the Limit Comparison Test, the original series also converges.

   We can also do this using the Direct Comparison Test, with comparison series $\sum_{n=1}^{\infty} \frac{10}{2^n}$. The latter converges (to 10, in fact) since it is a geometric seires with ratio $\frac{1}{2}$. We have

$$\frac{1}{2^n - 1} \le \frac{10}{2^n} \iff 2^n \le 10 \cdot 2^n - 10 \iff 10 \le 9 \cdot 2^n$$

   which holds for all $n \ge 1$. So, by the Direct Comparison test, since $\sum_{n=1}^{\infty} \frac{10}{2^n}$ converges, so does the original series.