Math 107 Calculus II – Spring 2018 – Exam 2 – Solutions

Question 1.

Consider the following power series:

$$f(x) = \sum_{n=1}^{\infty} \frac{(x+1)^n}{\sqrt{n}\,(1.1)^n}.$$

(a) (3 points) Fill in the answer to the following:
This is a power series about $x = a$ where $a =$ ____________
Solution: $a = -1$

(b) (8 points) What is the radius of convergence of this power series? Justify your answer.
Solution: Applying the ratio test:

$$\lim_{n\to\infty} \frac{|x+1|^{n+1}}{\sqrt{n+1}(1.1)^{n+1}} \frac{\sqrt{n}(1.1)^n}{|x+1|^n} = \frac{|x+1|}{1.1} < 1$$

if and only if $|x+1| < 1.1$. So, $R = 1.1$.

Question 2. (10 points)

Consider the following power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n5^n}.$$

It has radius of convergence 5. (**You do not need to check this fact.**). Determine the interval of convergence.

Solution: Since it is centered at 2 and $R = 5$, we need to check the points -3 and 7. Plugging in $x = -3$ gives $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, the altenating harmonic series, which is known to converge (Alternating Series Theorem). Plugging in $x = 7$ gives $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, which is known to diverge (the p test).

So, the interval of convergence is $[-3, 7)$ or, in other words, it coverges precisely when $-3 \le x < 7$.

Question 3.

For each statement below say whether the claim is true or false, and provide brief justification.

(a) (3 points) True or False: If the power series $\sum_{n=1}^{\infty} c_n(x-4)^n$ converges at $x = 10$ then it converges at $x = 0$.
Solution: True. It it converges at 10 the radius of convergence is at least 6, and $|0-4| = 4 < 6$.

(b) (3 points) True or False: If $\sum_{n=1}^{\infty} a_n$ converges and $a_n \ge 0$ for all n then $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ must also converge.
Solution: True. This is the Absolute Convergence Theorem.

(c) (3 points) True or False: If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ must converge.
Solution: False. The n-term test can only be used to show divergence. The harmonic series is a counter example.

Question 4. (10 points)

Consider a function $f(x)$ having the following properties:

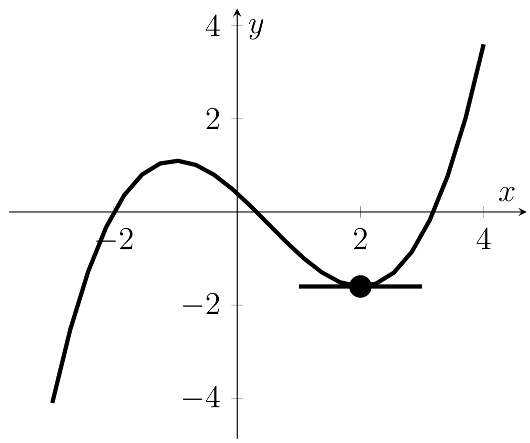
$$f(1) = 2 \qquad f'(1) = -1 \qquad f''(1) = -\frac{3}{4} \qquad f^{(3)}(1) = 6 \qquad f^{(4)}(1) = -3$$

What is the Taylor polynomial of degree 3 for $f(x)$ near $x = 1$?

Solution: $2 - (x-1) - \frac{3}{8}(x-1)^2 + (x-1)^3$

Question 5. (9 points)

Consider the graph of $y = f(x)$ below.



The dot is at x-coordinate 2. Suppose we approximate $f(x)$ near $x = 2$ by the second degree Taylor polynomial

$$P_2(x) = a + b(x-2) + c(x-2)^2.$$

In each of the blanks below write either "**positive**", "**negative**", or "**zero**". Then in each case give a one sentence justification for your answer.

The value of a is ____________.

Solution: Negative. a is the value of $f(x)$ at 2, which is clearly negative.

The value of b is ____________.

Solution: 0. b is the slope of the graph at 2, which is 0.

The value of c is ____________.

Solution: The sign of c gives the concavity of the graph at 2, which is positive (concave up).

Question 6. (16 points)

Circle your response to the following question and then justify your answer. The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1+\frac{1}{n})}{n^{2/3}}$$

Converges absolutely **Converges conditionally** **Diverges**

Solution: It converges conditionally.

The series itself converges by the Alternating Series Theorem (or Test), since the terms are alternating in sign, decreasing in magnitude, and $\lim_{n\to\infty} a_n = 0$.

Taking absolute values of the terms gives the series $\sum_{n=1}^{\infty} \frac{1+\frac{1}{n}}{n^{2/3}}$. Since $\frac{1+\frac{1}{n}}{n^{2/3}} > \frac{1}{n^{2/3}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges (by the p-test), the series $\sum_{n=1}^{\infty} \frac{1+\frac{1}{n}}{n^{2/3}}$ diverges (by direct comparison test).

Question 7. (10 points)

Find the first three non-zero terms of the Taylor series for $f(x) = \sin(x)\cos(x)$ around $x = 0$. [Hint: since you already know the Taylor series for $\sin(x)$ and $\cos(x)$ it isn't necessary to compute all the derivatives of $f(x)$.]

Solution: We have $f(x) = (x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots)(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots)$. The expansion of this will involve only odd powers of x, so we are looking for the x, x^3, x^5 terms. The expansion is

$$\begin{aligned} &x - \frac{x^3}{2} + \frac{x^5}{4!} - \cdots \\ &-\frac{x^3}{3!} + \frac{x^5}{3! \cdot 2} - \cdots \\ &\frac{x^5}{5!} - \cdots \end{aligned}$$

and, upon combining like terms, the first 3 non-zero terms are

$$x + (-\frac{1}{3!} - \frac{1}{2})x^3 + (\frac{1}{4!} + \frac{1}{3! \cdot 2} + \frac{1}{5!})x^5$$

which simplifies to

$$x - \frac{2}{3}x^3 + \frac{2}{15}x^5.$$

Question 8. (10 points)

By using the Taylor series about $x = 0$ for these functions, order them from largest to smallest for values of x very close to 0.

$$f(x) = 1 - e^{-x} \qquad g(x) = \frac{x}{1+x} \qquad h(x) = \sin(x).$$

Solution: The Taylor series are

$$\begin{aligned} 1 - e^{-x} &= x - \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots \\ \frac{x}{1+x} &= x - x^2 + x^3 - \cdots \\ \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \end{aligned}$$

The P_2 approximiations are $x - \frac{x^2}{2}$, $x - x^2$ and x, and for small values of x we may ignore higher order terms. Since $-x^2 < \frac{x^2}{2} < 0$ (for x non-zero), the correct answer is

$$\sin(x) > 1 - e^{-x} > \frac{x}{1+x}.$$

Question 9.

All parts of this question concern the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{3^{2n}}{(2n)!} = -\frac{3^2}{2!} + \frac{3^4}{4!} - \frac{3^6}{6!} + \cdots$$

(a) (6 points) Use the ratio test to show that this series converges.
Solution: We have

$$\lim_{n\to\infty} \frac{3^{2n+2}}{(2n+2)!} \frac{(2n)!}{3^{2n}} = \lim_{n\to\infty} \frac{3^2}{(2n+2)(2n+1)} = 0$$

Since this limit is < 1, the series converges by the Ratio Test.

(b) (6 points) This series arises from substituting a value $x = s$ into the Taylor series about $x = 0$ of some function $f(x)$. Fill in the blanks below. You do not need to provide further justification.

i. The function $f(x) =$ ________________________

Solution: $f(x) = \cos(x) - 1$

ii. The value $s =$ ____________

Solution: $s = 3$

(c) (3 points) What is the exact value of the series?
Solution: $f(3) = \cos(3) - 1$.