Math 107 Calculus II – Solutions

1. Evaluate the following integrals.

   (a) (7 points) $\int \ln(x)dx$

   Solution: By parts with $u = \ln(x), v' = 1$: We have $u' = \frac{1}{x}, v = x$ and so

$$\int \ln(x)dx = \ln(x)x - \int \frac{1}{x}xdx = \ln(x)x - x + C$$

   (b) (7 points) $\int (x^2 + x)e^x dx$

   Solution: By parts twice:

$$\int (x^2 + x)e^x dx = (x^2 + x)e^x - \int (2x + 1)e^x dx =$$

$$(x^2 + x)e^x - \left(\int (2x + 1)e^x dx\right) = (x^2 + x)e^x - \left((2x + 1)e^x - \int 2e^x dx\right)$$

$$= (x^2 + x)e^x - (2x + 1)e^x + 2e^x + C$$

2. (10 points) Suppose $f(x)$ is a function with values and derivatives as given in the following table:

x	-1	0	1	2	3	4	5
$f(x)$	8	6	2	7	4	9	1
$f'(x)$	7	3	5	-9	5	7	2

Use the table of values given above and an appropriate substitution to find the exact value of $\int_1^2 xf'(x^2 + 1)\,dx$. (Some of the data given in the table might not be needed.) As always, please show your work, and state explicitly what substitution you used to evaluate the integral.

Solution: Let $w = x^2+1$ so $dw = 2xdx$. Then $\int_1^2 xf'(x^2+1)\,dx = (1/2)\int_2^5 f'(w)dw = (1/2)(f(5) - f(2)) = -3$.

3. (12 points) Evaluate $\int \frac{x^2 + x + 2}{x^2 + x}\,dx$. (Hint: Start with long division)

   Solution: Use long division and partial fractions to get
   $\int \frac{x^2 + x + 2}{x^2 + x}\,dx = \int (1 + \frac{2}{x} - \frac{2}{x + 1})\,dx = x + 2\ln|x| - 2\ln(|x + 1|) + C.$

4. (12 points) Evaluate $\int \frac{1}{(1 + y^2)^{3/2}}dy$.

   Solution: Use the trig sub $y = \tan(\theta), dy = \sec^2(\theta)d\theta$ to get

$$\int \frac{1}{(1 + y^2)^{3/2}}dy = \int \frac{1}{(\sec^2(\theta))^{3/2}}\sec^2(\theta)d\theta = \int \frac{1}{\sec(\theta)}d\theta = \int \cos(\theta)d\theta = \sin(\theta) + C$$

Using a triangle, $\sin(\theta) = \frac{y}{\sqrt{1+y^2}}$ and so

$$\int \frac{1}{(1+y^2)^{3/2}} dy = \frac{y}{\sqrt{1+y^2}} + C.$$

5. Let $v(t)$ be the velocity, in feet per second, of a car driving along a straight road. Assume the function $v(t)$ is increasing and concave down. Answer the following questions concerning $\int_0^6 v(t)dt$. Show how you obtain your answers.

t	0	1	2	3	4	5	6	7	8
$v(t)$	5	37	53	61	65	67	68	68.5	68.75

(a) (4 points) Approximate the distance traveled between $t = 0$ and $t = 6$ by finding LEFT(2).

Solution: $LEFT(2) = 3 \cdot (v(0) + v(3)) = 3 \cdot 66 = 198$ feet

(b) (4 points) Is the approximation you found in part (a) an under-estimate or an over-estimate for the exact distance travelled? How do you know?

Solution: It's an under-estemate since the function is increasing.

(c) (4 points) Is MID(2) an under-estimate or an over-estimate for the exact distance travelled from $t = 0$ to $t = 6$? How do you know? (You don't need to find MID(2) to answer this part.)

Solution: It would be an over-estamate since the function is concave down.

6. Evaluate each of the following improper integrals, and either **find its exact value** or **show that it diverges**.

(a) (10 points) $\int_0^\infty \frac{1}{1+x^2} dx$

Solution:

$$\int_0^\infty \frac{1}{1+x^2} dx = \lim_{b\to\infty} (\arctan(x)) |_0^b = \lim_{b\to\infty} (\arctan(b) - \arctan(0)) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

(b) (10 points) $\int_2^9 \frac{1}{7(x-2)^3} dx$

Solution: Use $w = x - 2, dw = dx$:

$$\int_2^9 \frac{1}{7(x-2)^3} dx = \int_0^7 \frac{1}{7w^3} dw = \lim_{a\to 0^+} -\frac{1}{14w^2}\Big|_a^7 = \lim_{a\to 0^+} \left(-\frac{1}{14 \cdot 7^2} + \frac{1}{14a} \right) = \infty$$

So, this integral diverges.

7. Determine if the following improper integrals converge or diverge using any method taught in class. You do **<u>NOT</u>** need to evaluate those that converge. If you use a test from class, be sure to justify its use.

(a) (10 points) $\displaystyle\int_2^\infty \frac{x^2}{x^3-1}dx$

Solution: We have $\frac{x^2}{x^3-1} \geq \frac{x^2}{x^3} = \frac{1}{x}$ (since $x^3 - 1 \leq x^3$). We know $\int_2^\infty \frac{1}{x}dx$ diverges (by the p-test). So, by the Comparison Test, $\displaystyle\int_2^\infty \frac{x^2}{x^3-1}dx$ also diverges.

(b) (10 points) $\displaystyle\int_1^\infty \frac{x+1}{2x^4-1}dx$

Solution: $\frac{x+1}{2x^4-1} \leq \frac{2}{x^3}$ for $x \geq 1$, because $x^3(x+1) \leq 4x^4 - 2$ for $x \geq 1$ (since the latter is equivalent to $x^3 \leq 3x^4 - 2$). We know $\int_1^\infty \frac{2}{x^3}dx$ converges (by the p-test). So $\int_1^\infty \frac{x+1}{2x^4-1}dx$ also converges, by the Comparison Test.

Alternative Solution: $\frac{x+1}{2x^4-1} \leq \frac{1}{x^3}$ for $x \geq 2$, because $x^3(x+1) \leq 2x^4 - 1$ for $x \geq 2$ (since the latter is equivalent to $x^3 \leq x^4 - 1$). Since

$$\int_1^\infty \frac{x+1}{2x^4-1}dx = \int_1^2 \frac{x+1}{2x^4-1}dx + \int_2^\infty \frac{x+1}{2x^4-1}dx$$

and $\int_2^\infty \frac{1}{x^3}dx$ converges (by the p test), we know $\int_1^\infty \frac{x+1}{2x^4-1}dx$ also converges, by the Comparison Test.