Exam 2 Review Session

March 12, 2018

Area

Finding the area of geometric shapes or between two curves:
Key Steps:

i) Draw a picture (if one is not provided) – label all parameters
ii) Determine how to “slice” your image
iii) Identify the shape of the slice and write the appropriate formula for the area of that slice
iv) Use various techniques (similar triangles/shapes, pythagorean theorem, etc) to write the area formula using one variable
v) Set up an integral using the bounds for the variable in your area formula (and evaluate as necessary)

Note that the shape of the slices for area between/under curves are rectangles, with lengths given by some combination of the given function values.

Area Example

Use a definite integral to find the are of the region between the curves $y = x^3$ and $y = \sqrt{x}$.

i) Draw a picture!

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Make sure to read the directions to see which way to slice the above image! They may ask you to do one or both directions.

Horizontal Slices:

iii) Shape of slice: rectangle; area of slice: $\ell \Delta y$

iv) Since we are using horizontal slices, we need to solve each of the given equations for y to find the length of our rectangles:$x = y^{\frac{1}{3}}$, $x = y^2$. Then $\ell = y^{\frac{1}{3}} - y^2$.

v) Thus, the area of each slice is $y^{\frac{1}{3}} - y^2 \Delta y$.

vi) The area between the curves is $\displaystyle\int_0^1 (y^{\frac{1}{3}} - y^2)dy$.

Note that the limits of the integral are the y values of the intersection points.

Area Example

If we wanted to evaluate the same area using **vertical slices**, we use methods learned in Calculus 1:

iii) Slices are again rectangles with area $h\Delta x$.

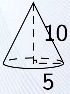
iv) h is given by $\sqrt{x} - x^3$.

v) Area of slice: $\sqrt{x} - x^3\Delta x$

vi) Total area between curves: $\int_0^1 (\sqrt{x} - x^3)dx$

Volume

We use the same steps as we did to find the area of geometric shapes, by are now concerned with the volume of the shape in question.
Consider a cone of height 10 cm and radius 5 cm. Use an integral to find the volume of the cone.

i) 

ii) We will use horizontal slices.

iii) Slices are disks, so the volume of each slice is $\pi r^2 \Delta h$.

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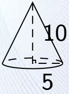
i) 

ii) We will use horizontal slices.

iii) Slices are disks, so the volume of each slice is $\pi r^2 \Delta h$.

iv) We now want to rewrite the area to have only one variable, h. To do this, set up a ratio: $\frac{r}{h} = \frac{5}{10}$. Then $r = \frac{1}{2}h$, so the volume of the slice is $\pi(\frac{h}{2})^2 \Delta h$.

v) Finally, the volume of the cone is $\int_0^{10} \pi(\frac{h}{2})^2 dh$.

Volume (Revolutions)

For solving volumes using revolutions, we apply the same key steps as we used for area and the previous volumes section.

- We will revolve a solid (bounded within a region) about a given line
  - Typically this line is the x or y axis or a line $y = c$ or $x = c$ where c is some constant.
- The axis of revolution determines the direction of the slice:
  - x-axis or a line $y = c$ implies we use vertical slices.
  - y-axis or a line $x = c$ implies we use horizontal slices.
  - Note: when we revolve around a line vertically, we use vertical slices, and vice versa.
- Make sure the limits of integration correspond to the possible values of the variable used in the integral.

Polar Coordinates

Converting between Polar and Cartesian

To convert from Cartesian coordinates to Polar use the formulas

$$r = \sqrt{x^2 + y^2},$$
$$\tan(\theta) = \frac{y}{x}.$$

Remember to draw the picture!

To convert from Polar to Cartesian use the formulas

$$x = r\cos(\theta),$$
$$y = r\sin(\theta).$$

Areas in Polar Coordinates

To find the area enclosed by a polar curve, use the formula

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} [f(\theta)]^2 d\theta$$

To find the area enclosed by two polar coordinates, first find the points where they intersect, then use the above formula as such

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} [f_{outer}(\theta)]^2 - [f_{inner}(\theta)]^2 d\theta.$$

Remember to draw the picture (and remember those graph shapes!)

Mass

- We want to find little slices of mass, i.e. $dM(x) = \delta(x)dV(x)$.
- Think very carefully about each strip/slice and then sum them up (integrate).
- Remember to always check units.
- Be careful of 1D vs 2D vs 3D.

Mass Example

- Consider the region bounded by $y = x$ and $y = x^2$.
- Write and evaluate an integral that shows the mass of the region given that its density is $\delta(y) = 4 + 5y \frac{kg}{m^2}$.
- Is our region 1D, 2D, or 3D?

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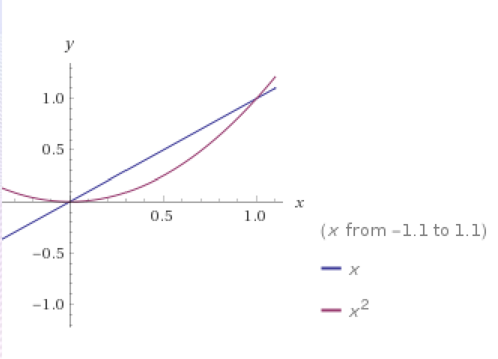
- Will we want vertical or horizontal slices? Horizontal, so we want dy
- Now we need to know what our strips look like.

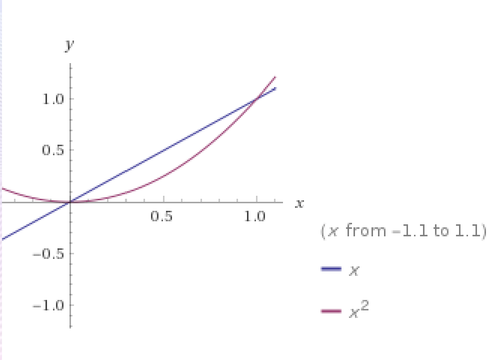
Mass Example

What is our right-most function?



Mass Example

What is our outermost function? $y = x^2$, or $y = \sqrt{x}$. This means that $y = x$ is our inner most function.



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- What are our bounds?

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- Will we want vertical or horizontal slices? Horizontal, so we want dy
- What is the length of one of our strips? $\sqrt{y} - y$
- What are our bounds? We can set our two equations equal to each other to yield the points $(0, 0)$ and $(1, 1)$. So we want to integrate from 0 to 1.
- What does our final integral look like?

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- Is our region 1D, 2D, or 3D? 2D
- Will we want vertical or horizontal slices? Horizontal, so we want dy
- What is the length of one of our strips? $\sqrt{y} - y$
- What are our bounds? 0 to 1.
- What does our final integral look like?

$$M = \int_0^1 \delta(y)(\sqrt{y} - y)dy$$

$$= \int_0^1 (4 + 5y)(\sqrt{y} - y)dy$$

Work

- Everything builds on itself! $W = F \cdot d = m \cdot a \cdot d = \delta \cdot V \cdot a \cdot d$.
- Think very carefully about each strip/slice and then sum them up (integrate).
- Your bounds and your distance function should make sense together.
- Remember to always check units. Acceleration due to gravity is incorporated into the density at times.

Work Example

- We are in Antopia, exploring Ant civilizations to learn more about their culture. For some reason, they build pyramids upside down and fill them with a strange toxic liquid. To explore the pyramid, we need to drain the liquid. Write an integral that determines the work necessary to remove all of the liquid from an upside down pyramid with base length 14 meters and height 29. The liquid has a density $\delta(h) = 17kg/m^3$ and fills the pyramid up to a height of 12 meters. Acceleration due to gravity is $9.8m/s^2$.
- What can we get out of this?

Work Example

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- What can easily we get out of this? $\delta(h) = 17$, $a = 3.7$
- What will our strips look like? What are their volumes?

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- How do we find s?

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- How do we find s? similar triangles: $s = \frac{14}{29}h$, so $dV = (\frac{14}{29}h)^2\Delta h$
- What about distance and our bounds?

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- How do we find s? similar triangles: $s = \frac{14}{29}h$, so $dV = (\frac{14}{29}h)^2 \Delta h$
- What about distance and our bounds? Liquid at the bottom of the pyramid has to go a total distance of 29 meters and at the top of the liquid has to go $29 - 12 = 17$ meters. So if our bounds are from 0 to 12, our distance function is $29 - h$.
- What is our final integral?

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- How do we find s? similar triangles: $s = \frac{14}{29}h$, so $dV = (\frac{14}{29}h)^2\Delta h$
- What about distance and our bounds? If our bounds are from 0 to 12, our distance function is $29 - h$.
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$$W = \int_0^{12} (17)(\frac{14}{29}h)^2(9.8)(29 - h)dh$$

Sequences

- A **sequence** is an ordered set of values.
  - closed form: $a_n = n^2 - 1$
  - recursive form: $a_n = ns_{n-1}^2 + 3s_{n-2} - 1$
- We say a sequence converges to a number L if a_n is as close to L as we please whenever n is sufficiently large. We write $\lim_{n\to\infty} a_n = L$
- Bounded and monotone sequences are convergent.
  - If a sequence is monotone decreasing and bounded below, it is convergent.
  - If a sequence is monotone increasing and bounded above, it is convergent.

Geometric Series

- A **finite geometric series** has the form

$$S_n = a + ax + ax^2 + ... + ax^{n-1}$$

  and an **infinte geometric series** has the form

$$a + ax + ax^2 + ax^3 + \dots$$

  where a is the leading term and x is the common ratio.
- the **sum of a finite geometric series** has closed form

$$S_n = a\left(\frac{1-x^n}{1-x}\right).$$

- If $|x| < 1$, then the **sum of an infinte geometric series** exists and is

$$\lim_{n\to\infty} S_n = \frac{a}{1-x}.$$

nth Term Test and Partial Sums

Partial Sums

The n-th partial sum of a series $\sum_{n=1}^{\infty} a_n$ is the first n terms added together: $\sum_{k=1}^{n} a_k$ A series is convergent if the sequence of its partial sums is convergent.

n-th Term Test

If $\sum_{n=1}^{\infty} a_n$ is a series, then:

- if $\lim_{n\to\infty} \neq 0$, the series must be divergent
- if $\lim_{n\to\infty} = 0$, **we can't tell if the series converges or diverges** (try another test!)

Integral Test and P-Series

Integral Test

If we have a series $\sum_{n=1}^{\infty} a_n$ and notice that $a_n = f(n)$, we can use the integral test if $f(x)$ is decreasing and positive.
When these conditions are met, $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$ either both converge or both diverge (they do the same thing).
Always be sure to check the conditions!

p-series

Sometimes we will come across a series that looks like $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$. By applying the integral test, we can tell that this series converges when $p > 1$ and diverges when $p \leq 1$.

Limit Comparison Test

The limit comparison test comes in handy when a series looks like (but isn't exactly the same as) some other series that we already know either converges or diverges.

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two series with $a_n > 0$ and $b_n > 0$, then we can take the limit of the ratio between their terms:

$$\lim_{n\to\infty} \frac{a_n}{b_n}.$$

If this limit is a number that is greater than 0, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both diverge or both converge.

Direct Comparison Test

The direct comparison test is similar to the comparison test for improper integrals.
Say we're given a series $\sum_{n=1}^{\infty} a_n$. Then we can try to come up with a series $\sum_{n=1}^{\infty} b_n$ so that we can tell if $\sum_{n=1}^{\infty} a_n$ converges with the following rules:

- if $a_n \leq b_n$, and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- if $a_n \geq b_n$, and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.