Artinian algebras and Jordan type *

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Abstract

There has been much recent work in the commutative algebra community on strong and weak Lefschetz conditions for graded Artinian algebras A, especially those that are Artinian Gorenstein (AG). A more general invariant of an Artinian algebra A or A-module M that we consider here is the set of Jordan types of elements of the maximal ideal $\mathfrak{m}$ of A. Here, the Jordan type of $\ell \in \mathfrak{m}$ is the partition giving the Jordan blocks of the multiplication map $m_\ell : M \to M$. In particular, we consider the Jordan type of a generic linear element ℓ in A_1, or in the case of a local ring $\mathcal{A}$ of a generic element $\ell \in \mathfrak{m}_{\mathcal{A}}$, the maximum ideal.

The strong Lefschetz property of an element, as well as the weak Lefschetz property can be expressed simply in terms of its Jordan type. Despite substantial recent work on Lefschetz properties of Artinian algebras there has not been a systematic study of Jordan types for a given Artinian algebra A or A-module M, except, importantly, in modular invariant theory, and also in work involving commuting Jordan types.

We first show some basic properties of the Jordan type and their loci, for modules over graded or local Artinian algebras. We as well show their relation to Hilbert functions, and we survey related topics, such as the Jordan types for tensor products.

Our goal is, first, to expose some basic properties of Jordan type, second, to survey some different threads of research on Jordan type. We discuss background, and as well propose some open problems.

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1 Introduction.

Throughout the paper k will be an arbitrary field unless otherwise specified – except that we will assume k is infinite when we discuss "generic" Jordan type or parametrization. We denote by $R = \mathsf{k}[x_1, x_2, \ldots, x_r]$ the polynomial ring over k, and by $\mathcal{R} = \mathsf{k}\{x_1, x_2, \ldots, x_r\}$ the regular local ring over k. We denote by $\mathfrak{m}_R$ or $\mathfrak{m}_{\mathcal{R}}$, respectively, the maximal ideal $(x_1, \ldots, x_r)$ in R or $\mathcal{R}$, respectively. We denote by $\mathfrak{D} = \mathsf{k}_{DP}[X_1, \ldots, X_r]$ the divided power ring that is the Macaulay dual of R or $\mathcal{R}$. We will consider an Artinian algebra $A = R/I$ in the graded case or $\mathcal{A} = \mathcal{R}/I$ in the local case. We denote its maximum ideal by $\mathfrak{m}_A$, or $\mathfrak{m}_{\mathcal{A}}$, respectively, or just $\mathfrak{m}$ when the algebra is understood. We say A has *highest socle degree* j or *formal dimension* j (as in [MS]) when $A_j \neq 0$ but $A_i = 0$ for $i > j$.[1]

We fix an algebra A over k and a finite A-module M of dimension $\dim_{\mathsf{k}} M = n$. We consider multiplication maps m_ℓ on M by elements ℓ in $\mathfrak{m}_A$. The Jordan type $P_\ell = P_{\ell,M}$ (to make M explicit) is the partition of n specifying the lengths of blocks in the Jordan block matrix determined by m_ℓ in a suitable k-basis for M (Definition 2.1 ff.). When k is infinite, the *generic Jordan type* P_M is $P_M = P_{\ell,M}$ for a general enough or generic element ℓ in $\mathfrak{m}_A$. We make the analogous definitions for a module M over a local algebra $\mathcal{A}$.

The set of Jordan types of elements of A acting on M is a poset $\mathfrak{P}_M$ under the "dominance" partial order (Equation (2.5)); the poset $\mathfrak{P}_M$ is an invariant of the A-module M. We consider the projective space $\mathbb{P}(\mathfrak{m}_A)$. Given a partition P of $m = \dim_{\mathsf{k}}(M)$, we may consider the locus $\mathfrak{Z}_{P,M} \subset \mathbb{P}(\mathfrak{m}_A)$ parametrizing those elements $\ell \in \mathfrak{m}_A$ such that the action of m_ℓ on M has Jordan type $P_\ell = P$. The closures $\overline{\mathfrak{Z}_{P,M}}$ form a poset under inclusion that is isomorphic to $\mathfrak{P}_M$.

There have been studies in representation theory of modules having a constant Jordan type, and of the generic Jordan type of a module, and as well the connection of Jordan type loci to vector bundles [CFP, FPS, Be, Pev1, Pev2]. In addition there has been progress in the study of the modular case of tensor products of Jordan types, generalizing the Clebsch-Gordan formula in characteristic zero (Section 3.2, Remark 3.12). There are subtle conditions, investigated by P. Oblak, T. Košir and others, on which pairs of Jordan types $P_\ell, P_{\ell'}$, partitions of n, "commute" – can both occur for an Artinian graded algebra A or a local algebra $\mathcal{A}$ of length n [Kh1, Kh2, KOb, Ob1, Ob2, IKVZ, Pan] (Section 3.3).

We will recall the definitions of a strong Lefschetz/weak Lefschetz element of a graded Artinian algebra A in Section 2.4 (Definitions 2.29, 2.30) and will interpret them in terms of Jordan type. It is well known that if A is a standard graded Artinian algebra with Hilbert function $H = H(A)$ then $\ell \in A_1$ is a strong Lefschetz element for A if and only if $P_\ell = H(A)^\vee$, the conjugate partition (exchange rows and columns in the Ferrers diagram) of $H(A)$ regarded as a partition (Lemma 2.31, [H-W, Prop. 3.64]). The element ℓ is weak Lefschetz for A if the number of parts of P_ℓ is the Sperner number of A, the maximum value of the Hilbert function $H(A)$ (Lemma 2.33). The generic Jordan type P_A (sometimes written "generic linear Jordan type") is defined using a generic linear element ℓ; it thus tells whether "ℓ is strong Lefschetz" (SL), or "ℓ is weak Lefschetz" (WL). When A is not strong Lefschetz the generic Jordan type of A may convey rather more information than the absence of SL.

[1] We have followed [H-W, Remark 2.11] in this definition. When A is not standard graded, this might not agree with the largest j such that $(\mathfrak{m}_A)^j \neq 0$.

However, we will also consider Jordan type for more general elements $\ell \in \mathfrak{m}_{\mathcal{A}}$ or for non-homogeneous elements of $\mathfrak{m}_A$ when A is graded. We say ℓ has "strong Lefschetz Jordan type" (SLJT) if $P_\ell = H(A)^\vee$ (Definition 2.34).

Let A be a non-standard-graded Artinian algebra. We note that the Jordan type P_ℓ of a non-homogeneous element $\ell \in \mathfrak{m}_A$ may be the same as that expected for a strong Lefschetz element, even though A may have no linear strong Lefschetz elements? This we at first noticed on an example of relative covariants proposed by Chris McDaniel, and discussed in [IMM2, Example 1.2].

Definition 1.1. [H-W, Section 3.5] Given a nilpotent linear homomorphism m_ℓ (multiplication by $\ell \in A$ acting on a finite A-vector space V, there is a (non-unique) direct sum decomposition $V = \oplus V_i$ into cyclic m_ℓ invariant subspaces. We call bases of these cyclic subspaces "strings" of m_ℓ. The cardinalities of these strings for a single decomposition are the *Jordan type* partition $P_{\ell,V}$ of the integer $n = \dim_{\mathsf{k}} V$.

Example 1.2. We let $R = \mathsf{k}[y,z]$, with weights $\mathsf{w}(y,z) = (1,2)$, and let $A = R/I, I = (yz, y^7, z^3)$, of k-basis $A = \langle 1, y, y^2, z, y^3, y^4, z^2, y^5, y^6$ of Hilbert function $H(A) = (1,1,2,1,2,1,1)$. The only linear element of A, up to non-zero constant multiple is y and the partition given by the multiplication m_y is $P_y = (7,1,1)$, so A is not strong-Lefschetz. However the non-homogeneous element $\ell = y + z$, has strings $\{1, y+z, y^2+z^2, y^3, y^4, y^5, y^6\}$ and $\{z, z^2\}$ so $P_\ell = (7,2)$, which is the maximum possible given $H(A)$, so ℓ has strong Lefschetz Jordan type.

A related local algebra is $\mathcal{A} = \mathcal{R}/(yz, z^3 + y^6)$, where here we set $\mathsf{w}'(y,z) = (1,1)$, of Hilbert function $H(\mathcal{A}) = (1,2,2,1,1,1,1)$ where the element $\ell' = (y+z)$ has Jordan type $(7,2)$, so $\mathcal{A}$ is strong Lefschetz.

Overview

In Part 2 we state and prove the basic properties for Jordan types for elements of Artin algebras. In section§2.1 well known equivalent definitions of the Jordan type of the multiplication map m_ℓ for an element $\ell \in \mathfrak{m}$, the maximal ideal, and we present some properties of Artinian algebras that we will need. We show a semicontinuity result for Jordan type, and discuss generic Jordan type in §2.2. In §2.3 we fix the Hilbert function $H(A)$ and prove sharp inequalities $P_\ell \le \mathsf{P}(H)$ for $\ell \in A_1$, as well as the analogous result for a local algebra $\mathcal{A}$ and $\ell \in \mathfrak{m}_{\mathcal{A}}$ (Theorem 2.19). In §2.4 we connect Jordan type with Lefschetz properties.

In Part 3 we survey several aspects of Jordan types studied by others. §3.1 the generic Jordan types for algebras constructed by idealization or by partial idealization: these include some well known non-WL examples of H. Sekiguchi (H. Ikeda), idealization examples of J. Watanabe and R. Stanley, as well as partial idealization examples of several authors. These latter have Jordan types that are weak Lefschetz but not strong Lefschetz. In §3.2 we report briefly on the Jordan type of tensor products, in both non-modular (char k is zero or char $\mathsf{k} = p$ is large) and modular cases; and in §3.3 we discuss which Jordan types are compatible. In Section 3.4 we propose further problems and possible directions of study.

In sequel papers we study free extensions, applying results concerning Jordan type to rings of relative coinvariants [IMM2, McDCIM] (latter with S. Chen).

Guide. A reader may wish to check out the basic definitions in Sections 2.1-2.4, understanding that some, such as the poset $\mathfrak{P}_M$ are not further developed in this article, partly for lack of space. The sections 3.1-3.3 survey relevant literature.

This article is a development of Part 1 of [IMM1], and supercedes it.

Contents

2 Basic properties of Jordan type.

2.1 Jordan type of a multiplication map.

We consider a (possibly non-standard) graded Artinian algebra A over an arbitrary field k, or a local Artinian k algebra $\mathcal{A}$, and a finite A-module (or $\mathcal{A}$-module) M. Recall that we denote by $\mathfrak{m}_A = \sum_{i\geq 1} A_i$ when A is graded; and by $\mathfrak{m}_{\mathcal{A}}$ the unique maximal ideal of a local algebra $\mathcal{A}$. We will assume throughout the paper that $\mathcal{A}/\mathfrak{m}_{\mathcal{A}} = \mathsf{k}$ in the local case, and that $A_0 = \mathsf{k}$ for a graded algebra A.

Definition 2.1 (Jordan type)**.** For a nilpotent element $\ell \in A$ or $\mathcal{A}$ we denote by $m_{\ell,M}$ or $\times\ell$ the multiplication map of ℓ on M, and by $P_{\ell,M}$ the Jordan type of $m_{\ell,M}$: this is the partition of $\dim_{\mathsf{k}} M$ giving the sizes of the Jordan blocks of the multiplication map in a suitable basis of M.

For an increasing sequence $d_\ell = (d_0 \leq d_1 \leq \cdots \leq d_j)$ we let $\delta_{d_\ell,i} = d_i - d_{i-1}$ for $0 \leq i$ with $d_{-1} = 0$. For a decreasing sequence $r_\ell = (r_0 \geq r_1 \geq \cdots \geq r_j)$ we let $\delta_{r_\ell,i} = r_i - r_{i+1}$.

The following result is well-known [H-W, Lemma 3.60].

Lemma 2.2. *i. Let A be an Artin graded or local algebra with maximum ideal $\mathfrak{m}_A$ and highest socle degree j_A (so $A_j \neq 0$ but $A_i = 0$ for $i > j$); and assume $\ell \in \mathfrak{m}_A$. Let M be a finite length A-module. The increasing dimension sequence*

$$d_\ell\colon (0 = d_0, d_1, \ldots, d_j, d_{j+1}), \text{ where } d_i = \dim_{\mathsf{k}} M/\ell^i M, \tag{2.1}$$

has first difference $\Delta(d_\ell) = (\delta_{d_\ell,1}, \delta_{d_\ell,2}, \dots, \delta_{d_\ell,j+1})$, *which satisfies*

$$P_\ell = \Delta(d_\ell)^\vee, \tag{2.2}$$

the conjugate (exchange rows and columns in the Ferrers diagram) of $\Delta(d_\ell)$.

ii. The (decreasing) rank sequence

$$r_\ell \colon (r_0, r_1, \dots, r_j, 0), \text{ where } r_i = \dim_{\mathsf{k}}(\ell^i \cdot A) = \text{ rank } m_{\ell^i} \text{ on } A). \tag{2.3}$$

has first difference $\Delta(r_\ell) = (\delta_{r_\ell,1}, \delta_{r_\ell,2}, \dots, \delta_{r_\ell,j})$ *which satisfies*

$$(P_\ell)^\vee = \Delta(r_\ell) = \Delta(d_\ell). \tag{2.4}$$

Definition 2.3. We recall the *dominance partial order* on partitions of n. Let $P = (p_1, \dots, p_s)$ and $P' = (p'_1, \dots, p'_t)$ with $p_1 \geq \cdots \geq p_s$ and $p'_1 \geq \cdots p'_t$. Then

$$P \leq P' \text{ if for all } i \text{ we have} \sum_{k=1}^{i} p_i \leq \sum_{k=1}^{i} p'_i. \tag{2.5}$$

Thus, $(2,2,1,1) < (3,2,1)$ but $(3,3,3)$ and $(4,2,2,1)$ are incomparable.

Lemma 2.4. *Let V be a k-vector space of dimension n. Let T be a parameter variety (as $T = \mathbb{A}_1$). Let $\ell(t) \in \mathrm{Mat}_n(V), t \in T$ be a family of nilpotent linear maps, and let P be a partition of n. Then the condition on Jordan types, $P_{\ell(t)} \leq P$ is a closed condition on T.*

Proof. This is straightforward to show from Lemma 2.2 and the semicontinuity of the rank of $\ell(t)^i$ (see [CM, Thm. 6.25], or [IKh, Lemma 3.1]) □

Note. The Jordan type partition P_ℓ has sometimes been called the Segre characteristic of ℓ [Sh]. The Weyr canonical form of a multiplication map is a block decomposition "dual" to the Jordan canonical form [OCV, §2.4]; the Weyr characteristic is the partition giving the sizes of the blocks in the Weyr form, and is just the conjugate $P_\ell^\vee$ of P_ℓ. For further discussion of the Weyr form, which may have advantages for some problems, see [Sh, OCV, OW].

2.2 Generic Jordan type of a module and deformation.

We assume that k is an infinite field when discussing either generic Jordan type or any parametrization. We will assume that our ring is either Artinian local ($\mathcal{A}$) or a graded ring $A = \oplus_{i\geq 0} A_i$, where $A_0 = \mathsf{k}$. Recall that the elements of a finite Artinian algebra A are parametrized by the affine space $\mathbb{A}^n$, and with our assumption $\mathfrak{m}_A$ is parametrized by the affine space $\mathbb{A}^{n-1}$. So $\mathfrak{m}_A$ is an irreducible variety. Since the rank of each power of m_ℓ acting on M is semicontinuous, and since by Lemma 2.2 these ranks determine the Jordan type of m_ℓ we have

Lemma 2.5 (Generic Jordan type of M)**.** *Given an A or $\mathcal{A}$ module M, there is an open dense subset $U_M \subset \mathfrak{m} = \mathbb{A}^{n-1}$ for which $\ell \in U_M$ implies that the partition P_ℓ satisfies $P_\ell \geq P_{\ell'}$ for any other element $\ell' \in \mathfrak{m}$.*

Likewise, when A is graded (standard or non-standard), with elements of degree 1, there is an open dense set $U_{1,M}$ of elements $\ell \in A_1$ for which $\ell \in U_{1,M}$ implies that $P_\ell \geq P_{\ell'}$ for any other element $\ell' \in A_1$.

Definition 2.6. For M a finite-length module (not necessarily graded) over an Artinian local ring $\mathcal{A}$ we define the *generic Jordan type* P_M by $P_M = P_\ell$ where ℓ is a generic element of the maximal ideal of A. For M a finite-length graded module over a graded Artin algebra $A = \mathsf{k}[x_1, \ldots, x_r]$ (with weights $\mathsf{w}(x_i) = \mathsf{w}_i > 0$ for $i \in [1, \ldots, r]$), then the *generic linear Jordan type* for M is $P_{lin,M} = P_{1,M} = P_\ell$ for ℓ a generic element of A_1 (it is not defined when $A_1 = 0$). More generally we denote by $P_{i,M} = P_\ell$ for ℓ a generic element of the vector space A_i, if $A_i \neq 0$.

Evidently $P_M \geq P_{lin,M}$ and for $i \in [1, \ldots, j]$ we have $P_M \geq P_{i,M}$ in the dominance partial order. As we have seen in Example 1.2, when A is a non-standard graded algebra the generic Jordan type P_A may not equal $P_{lin,A}$ even when $A_1 \neq 0$.

Question 2.7. Under what conditions on a graded module M over a graded Artin algebra A does its generic Jordan type satisfy $P_M = P_{lin,M}$, the generic linear Jordan type?

A *deformation* of a local Artinian algebra $\mathcal{A}$ will be a flat family $\mathcal{A}_t, t \in T$ of Artinian algebras with special fibre $\mathcal{A}_{t_0} = \mathcal{A}$; then $\mathcal{A}_{t_0}$ is a *specialization* of the family $\mathcal{A}_t$. We note that an algebraic family $\mathcal{A}_t, t \in T$ of Artinian algebras over a parameter space is flat if the fibres $\mathcal{A}_t, t \in T$ have constant length. If also, for $t \neq t_0$ the algebras $\mathcal{A}_t$ have constant isomorphism type, we will say $\mathcal{A}_t, t \neq t_0$ is a jump deformation of $\mathcal{A}_{t_0}$. We use the dominance partial order on partitions of n (Definition 2.3). The following result is well known in other contexts.

Lemma 2.8 (Semicontinuity of Jordan type)**.** *i. Let M_t for $t \in T$ be a family of constant length modules over a parameter space T. Then for a neighborhood $U_0 \subset T$ of $t = 0$, we have that the generic Jordan types satisfy $t \in U_0 \Rightarrow P_{M_t} \geq P_{M_0}$.*

ii. Let $\mathcal{A}_t, t \in T$ be a constant length family of local or graded Artinian algebras. Then for a neighborhood $U_0 \subset T$ of $t = t_0$, we have $t \in U_0 \Rightarrow P_{\mathcal{A}_{t_0}} \geq P_{\mathcal{A}_{t_0}}$.

iii. Let $\ell_t \in \mathcal{M}_n(\mathsf{k})$ for $t \in T$ be a family of $n \times n$ nilpotent matrices, and let P_t be their Jordan type. Then there is a neighborhood $U \subset T$ of t_0 such that $P_t \geq P_{t_0}$ for all $t \in U$.

Proof. Similarly to Lemma 2.4, the results are immediate from Lemma 2.2 and the semicontinuity of the dimension of $\ell^k M_t$ or of $\ell^k \mathcal{A}_t$ as t varies. $\square$

Applying this result to the deformation from the associated graded algebra $\mathcal{A}^*$ to the local Artinian algebra $\mathcal{A}$ we have

Corollary 2.9. *Suppose that $\mathcal{A}$ is a local Artinian algebra with maximum ideal $\mathfrak{m}$ and $\ell \in \mathfrak{m}$. Then $P_\ell(\mathcal{A}) \geq P_\ell(\mathcal{A}^*)$.*

Proof. In the natural deformation from $\mathcal{A}^*$ to $\mathcal{A}$, for $t \neq 0, \mathcal{A}_t$ has constant isomorphism type: this is a jump deformation, so the open neighborhood U of Lemma 2.8 includes elements where $P_{\ell,\mathcal{A}_t} = P_{\ell,\mathcal{A}}$. $\square$

Recall that a local Artin algebra $\mathcal{A}$ is *curvilinear* if $H(\mathcal{A}) = (1, 1, \ldots, 1_j, 0)$, in which case $\mathcal{A}$ is isomorphic to $\mathsf{k}\{x\}/(x^{j+1})$.

Example 2.10. Let $\mathcal{A}(t) = \mathsf{k}[x,y]/I_t$ where for $t \in \mathsf{k}, t \neq 0$ we let $I_t = (tx - y^2, y^3)$ and where $\mathcal{A}(0) = \lim_{t\to 0} \mathcal{A}(t) = \mathsf{k}[x,y]/(x^2, xy, y^2)$ (since for $t \neq 0$ the ideal $I_t \supset \{tx - y^2, xy, x^2, y^3\}$). For $t \neq 0$ the local algebra $\mathcal{A}(t)$ is a complete intersection with Hilbert function $(1,1,1)$. This family specializes to $\mathcal{A}(0)$ which is non-Gorenstein, with Hilbert function $(1,2)$. Here for $t \neq 0$, taking $\ell = x + y$, we have that the generic linear Jordan type $P_{\ell,\mathcal{A}(t)} = (3) > (2,1) = P_{\ell,\mathcal{A}(0)}$.

Example 2.11. Let $B_t = \mathsf{k}[x,y]/\operatorname{Ann} F_t$, where $F_t = tX^{[5]} + X^{[2]}Y$. Then for $t \in \mathsf{k}, t \neq 0$ the algebra B_t is a curvilinear complete intersection, as in the previous example, with Hilbert function $(1,1,1,1,1,1)$. The family specializes to $B_0 = \mathsf{k}[x,y]/(y^2, x^3)$, also a complete intersection (CI), with Hilbert function $(1,2,2,1)$. Here $\ell = x + y$ determines the generic (also generic linear) Jordan type $P_{\ell.B_t} = (6) > (4,2) = P_{\ell,B_0}$.

Example 2.12. Let $B_t = \mathsf{k}[x,y,z]/\operatorname{Ann} F_t$, where $F_t = t^2X^{[3]}Y^{[2]} + tX^{[2]}YZ + XZ^{[2]}$. Then for $t \neq 0$ the algebra $B_t = \mathsf{k}\{x,y,z\}/(tz - xy, y^3, x^4)$ is an Artinian complete intersection with Hilbert function $(1,2,3,3,2,1)$. The family specializes to the complete intersection $B_0 = \mathsf{k}[x,y,z]/(y^2, z^2, x^3)$, which has Hilbert function $(1,3,4,3,1)$. Here for $t \neq 0$ $P_{B_t} = (6,4,2) > (5,3,3,1) = P_{B_0}$.

Definition 2.13 (The poset $\mathfrak{P}_M$)**.** Let M be a finite-length $\mathcal{A}$ module for an Artinian algebra $\mathcal{A}$. We denote by $\mathfrak{P}_M$ the poset $\{P_\ell \mid \ell \in \mathfrak{m}_\mathcal{A}\}$, with the dominance partial order. We denote by $\mathfrak{P}_{i,M}$ the sub-poset $\mathfrak{P}_{i,M} = \{P_\ell \mid \ell \in (\mathfrak{m}_\mathcal{A})^i\}$

Example 2.14. For $\mathcal{A}_t = \mathsf{k}[x,y]/(tx - y^2, y^3), t \neq 0$ from Example 2.10, we have $\mathfrak{P}_{\mathcal{A}_t} = \{P_y = (3), P_x = (2,1), P_0 = (1,1,1)\}$. For $\mathcal{A}_0 = \mathsf{k}[x,y]/(x^2, xy, y^2)$, we have $\mathfrak{P}_{\mathcal{A}_0} = \{(2,1),(1,1,1)\}$.

2.3 Jordan types consistent with the Hilbert function.

For a positively graded module $M = \oplus_{i\geq 0} M_i$ over a graded ring A, the Hilbert function is the sequence $H(M) = (h_0, h_1, \ldots)$ where $h_i = \dim_\mathsf{k} M_i$. The *Hilbert function polynomial* is $\mathsf{p}(M,t) = \sum h_i t^i$. For a finite-length module M over a local ring $\mathcal{A}$ with maximal ideal $\mathfrak{m}_\mathcal{A}$ we first define the associated graded module

$$M^* = \operatorname{Gr}_{\mathfrak{m}_\mathcal{A}} M = \oplus_{i=0} M_i \text{ where } M_i = {\mathfrak{m}_\mathcal{A}}^i M/{\mathfrak{m}_\mathcal{A}}^{i+1} M. \tag{2.6}$$

Again, the Hilbert function $H(M) = (1, h_1, \ldots, h_j, 0, \ldots, 0)$ where $h_i = \dim_\mathsf{k} M_i, h_j \neq 0$ and $h_i = 0$ for $i > j = j_M$ we term j_M the *socle degree* of M. When $M = \mathcal{A}$, $H(A)$ is also the Hilbert function of the dualizing module $\mathcal{A}^\vee \subset \mathfrak{D}$ where $\mathfrak{D} \cong \operatorname{Hom}(\mathcal{R}, \mathsf{k})$, the dualizing module of $\mathcal{R}$.[2]

Let j be the socle degree of A (graded) or $\mathcal{A}$ (local). When $\mathcal{A}$ is local, we have a stratification

$$M \supset \mathfrak{m}_\mathcal{A} M \supset \cdots \supset (\mathfrak{m}_\mathcal{A})^i M \supset \cdots \supset (\mathfrak{m}_\mathcal{A})^{j+1} M = 0. \tag{2.7}$$

The *order* of an element x of an $\mathcal{A}$-module M is the smallest i such that $x \in (\mathfrak{m}_\mathcal{A})^i M$.

Question. Given the Hilbert function $H(A)$ of an Artinian graded algebra A or the Hilbert function $H(M)$ of a graded Artinian module M over A (or the Hilbert function of a module M over a local algebra $\mathcal{A}$) what restrictions are there for the Jordan type $P_{\ell,M}$?

[2]In finding $H(A^\vee)$ we use as initial form the top degree term f_j of $f = f_j + f_{j-1} + \cdots \in \mathcal{A}^\vee \subset \mathfrak{D}$, rather than the lowest degree term we take as initial form of an element $h \in R$. This is implicit in [I, Lemma 4.8ff].

Since the composite multiplication $(m_\ell)^k = m_{\ell^k}$, we have the equality $\operatorname{rk} m_\ell^k = \operatorname{rk} m_{\ell^k}$ on M. The following result is immediate.

Lemma 2.15 (Hilbert function $H(M)$ of a finite-length module over R and P_ℓ)**.** *Let $M = M_0 \oplus M_1 \oplus \cdots \oplus M_j$ be an Artinian graded module over the polynomial ring $R = k[x_1, \ldots, x_r]$, satisfying $H(M) = (h_0, \ldots, h_j)$, and let $\ell \in R_1$. Then*

$$\begin{aligned}
\operatorname{rk} m_\ell &\leq \sum_{i=0}^{j-1} \min\{h_i, h_{i+1}\} \\
\operatorname{rk} m_\ell^2 &\leq \sum_{i=0}^{j-2} \min\{h_i, h_{i+1}, h_{i+2}\} \\
&\cdots \\
\operatorname{rk} m_\ell^k &\leq \sum_{i=0}^{j-k} \min\{h_i, h_{i+1}, \ldots h_{i+k}\} \qquad (2.8)\\
&\cdots \\
\operatorname{rk} m_\ell^j &\leq \min\{h_0, h_1, \ldots h_j\}
\end{aligned}$$

Also (2.8) *holds for Artinian modules M over the local ring $\mathcal{R} = \mathsf{k}\{x_1, \ldots, x_r\}$.*

Definition 2.16. Recall that we denote by $P(H) = H^\vee$, the conjugate partition to the Hilbert function H (exchange rows and columns in the Ferrers diagram of the set $\{H\}$ of values of H). Assume that A is graded (possibly non-standard). Let M be a finite-length non-negatively graded A-module, and let $H = H(M)$ be its Hilbert function. We denote by $\mathsf{P}(H)$ the partition of n corresponding to equality in (2.8).

Recall that the *socle degree* j_M of an A-module M is the highest degree such that $M_j \neq 0$ but $M_u = 0$ for $u > j$; and A has maximal ideal $\mathfrak{m}_A = A_1 \oplus A_2 \oplus \cdots \oplus A_j$ where $j = j_A$ is the highest socle degree of A. We denote by $H_{\mathfrak{m}_A}(A)$ the Hilbert function of $\mathrm{Gr}_{\mathfrak{m}_A}(A)$ with respect to the grading by powers of $\mathfrak{m}_A$. For a local ring $\mathcal{A}$ with maximal ideal $\mathfrak{m}_{\mathcal{A}}$, the Hilbert function $H(\mathcal{A})$ is with respect to the $\mathfrak{m}_{\mathcal{A}}$-adic grading. (See Example 2.22)

A Hilbert function $H(A)$ of finite length is *unimodal* if there is an integer c such that $1 = H(A)_0 \leq H(A)_1 \leq \cdots \leq H(A)_c \geq H(A)_{c+1} \geq \cdots \geq H(A_j)$, where $j = j_A$, the highest socle degree of A.

Note. When A has the standard grading, the partition $\mathsf{P}(H)$ can be read off conveniently from the bar graph of H, as the lengths of the maximal contiguous row segments. When H is unimodal then $\mathsf{P}(H) = P(H)$ which is just $H^\vee$. When H is not unimodal then $\mathsf{P}(H)$ has parts that can be combined to make $P(H)$.

Example 2.17. For $H = (1, 3, 2, 3, 3, 1)$ we have $\mathsf{P}(H) = (6, 4, 2, 1)$ but $P(H) = (6, 4, 3)$. For $H' = (1, 3, 1, 3, 3, 2)$ we have $\mathsf{P}(H') = (6, 3, 2, 1, 1)$, while $P(H') = (6, 4, 3)$. See Figure 1.

In the dominance partial order of Definition 2.3 we have for two partitions P, P' of n (see [CM, Lemma 6.31])

$$P \leq P' \Leftrightarrow P^\vee \geq P'^\vee. \qquad (2.9)$$

We will need a definition:

$H = (1, 3, 2, 3, 3, 1)$ $\qquad\qquad$ $H' = (1, 3, 1, 3, 3, 2)$

Figure 1: $\mathsf{P}(H) = (6, 4, 2, 1)$; and $\mathsf{P}(H') = (6, 3, 2, 1, 1)$.

Definition 2.18 (Jordan strings of multiplication by ℓ)**.** When ℓ is an element of $\mathfrak{m}_A$ (so nilpotent) acting on an Artinian A-module M, we can regard M as a $\mathcal{T} = \mathsf{k}[t]$-module, where t acts by multiplication by ℓ. When the partition $P_\ell = P = (p_1, p_2, \ldots, p_r)$ with $p_1 \geq \cdots \geq p_r)$ we may write M as the direct sum

$$M = \oplus_1^r \langle S_i \rangle \text{ of } \mathcal{T} \text{ cyclic submodules } \langle S_1 \rangle, \ldots, \langle S_r \rangle \text{ of lengths } \dim_{\mathsf{k}} \langle S_i \rangle = p_i. \tag{2.10}$$

Here we are denoting by S_i a homogeneous choice of k basis for $\langle S_i \rangle$. We will term each S_i a *string* of the action of ℓ on M.

The following is a key result.

Theorem 2.19 (Jordan type and Hilbert function)**.** *i. Let A be a (possibly non-standard) graded Artinian algebra with $A_0 = \mathsf{k}$ and $A_1 \neq 0$. Fix a finite-length graded A-module M of Hilbert function $H = H(M)$. Then for $\ell \in A_1$ the Jordan type of m_ℓ satisfies*

$$P_{\ell,M} \leq \mathsf{P}(H) \leq P(H) \text{ in the dominance partial order.} \tag{2.11}$$

i'. Furthermore, for an arbitrary $\ell \in \mathfrak{m}_A = A_1 \oplus A_2 \oplus \cdots \oplus A_j$ we have

$$P_{\ell,A} \leq (H_{\mathfrak{m}_A}(A))^\vee. \tag{2.12}$$

ii. Let $\mathcal{A}$ be a local Artinian k-algebra, with $\mathsf{k} = \mathcal{A}/\mathfrak{m}_{\mathcal{A}}$. Consider the standard $\mathfrak{m}_{\mathcal{A}}$-adic grading for its associated graded algebra. Fix a finite-length $\mathcal{A}$ module M having Hilbert function $H = H(M)$ with respect to the $\mathfrak{m}_{\mathcal{A}}$ grading. Then for $\ell \in \mathfrak{m}_{\mathcal{A}}$

$$P_\ell \leq P(H) \text{ in the dominance partial order.} \tag{2.13}$$

Proof of (i). A proof of (2.12) will be given in the Appendix.
We show (2.11) by induction on the highest socle degree j_M of M. It is evident for socle degree one. We assume (2.11) is shown for all triples (A, M, ℓ) where $\ell \in A_1, M$ is a graded Artinian A-module and where M has socle degree less than j; and suppose that M is a graded A-module of socle degree j. We may assume that M is the direct sum of the spans of strings of the form $m_i, \ell m_i, \ell^2 m_i, \ldots$ in M where each m_i is homogeneous. Each string is comprised of elements having different degrees. We have for the A-module $M' = M/M_j$ that $P_{\ell,M'} \leq \mathsf{P}(H(M'))$. Let $H(M)_j = h_j$, and $H(M)_{j-1} = h_{j-1}$. By definition $\mathsf{P}(H(M))$ is obtained from $\mathsf{P}(H(M'))$ by adding 1 to each of the largest $\min\{h_{j-1}, h_j\}$ parts of $\mathsf{P}(H(M'))$ and then

adjoining $(\max\{h_{j-1}, h_j\} - \min\{h_{j-1}, h_j\})$ more parts equal to one. But an examination of strings shows that $P_{\ell,M}$ is obtained from $P_{\ell,M'}$ by adding one to some set of $k \leq \min\{h_{j-1}, h_j\}$ parts of $P_{\ell,h,M'}$, then by adjoining $(\max\{h_{j-1}, h_j\} - k)$ parts equal to 1. We thus have that $P_{\ell,M'} \leq \mathsf{P}(H(M')) \Rightarrow P_{\ell,M} \leq \mathsf{P}(H(M))$, completing the induction step.

Proof of (ii). Let ℓ be a generic element of $\mathfrak{m}_{\mathcal{A}}$ and let $\mathsf{k}[x]$ act on the Artinian $\mathcal{A}$ module M via $x = \times\ell$. We will show

Claim. For any $T = \mathsf{k}[x]$ submodule N of the $\mathcal{A}$-module M we have $P_\ell(N) \leq H(N)^\vee$.
Proof of Claim. We proceed by induction on the pairs (m, n), $m = \dim_{\mathsf{k}} M$ where M is an $\mathcal{A}$ module, and $n = \dim_{\mathsf{k}} N$ (so $n \leq m$) where N is a T-submodule of M. We let $(m, n) < (m', n')$ if $m < m'$ or if $m = m'$ and $n < n'$. The Claim is true for all pairs (m, n) with $m = 1$ or $n = 1$. Suppose the claim is true for all pairs $(m, n) < (m', n')$, let M be an $\mathcal{A}$-module of length m' and N a T submodule of length n. Let $S = (a, \ell a, \ell^2 a, \ldots) \subset N$ be a longest string (basis of a cyclic T-submodule) in N: then S has length $p_{1,\ell}$ no greater than $j(N) + 1$, the largest part of $P(N) = H(N)^\vee$, and S is a direct T-summand of N (as it has maximum length). Consider a complementary T submodule $N' \subset N$ with $N' \cong N/S$ and $N' \oplus_T S = N$ and choose N' of maximum possible order. Denote by $\{H(N)\}$ the set of integers $H(N)$. Then $\{H(N')\}$ is obtained from $\{H(N)\}$ by decreasing $p = p_{1,\ell}$ largest entries of $\{H(N)\}$ by one. No entry $H(N)_i$ is decreased by 2 in $H(N')_i$ as the orders of $a, \ell a, \ell^2 a, \ldots$ are strictly increasing. Evidently, $P_{\ell,N} = (p, P_{\ell,N'})$ - we simply adjoin a largest part p to the Jordan partition $P_{\ell,N'}$. Since $P_{\ell,N'} \leq H(N')^\vee$ by the induction assumption, we have by (2.9)

$$P^\vee_{\ell,N'} \geq (H(N')^\vee)^\vee. \tag{2.14}$$

The partition $P^\vee_{\ell,N}$ is obtained from $P^\vee_{\ell,N'}$ by increasing the first p entries by one. The Hilbert function $H(N)$ (arranged in non-increasing order) is obtained from $H(N')$ by increasing some subset of p entries by one. Thus we have[3]

$$P^\vee_{\ell,N'} \geq (H(N')^\vee)^\vee \Rightarrow P^\vee_{\ell,N} \geq (H(N)^\vee)^\vee \tag{2.15}$$

in the dominance partial order, since the sum of the first k entries for $P^\vee_{\ell,N}$ remains greater than the analogous sum of the first k entries of $H(N)$, for each $k = 1, \ldots, j(N) + 1$. We use here that the difference $H(N)_i - H(N')_i \leq 1$ for each i. By conjugating the partitions in equation (2.15) and applying Equation (2.9) we have shown $P_{\ell,N} \leq H(N)^\vee$. This completes the induction step. □

Example 2.20. Let $S = \mathsf{k}[x, y]$ with weights $\mathsf{w}(x, y) = (1, 2)$ and let $A = S/(xy - x^3, y^2)$, a non-standard graded Artinian CI algebra. Let $\mathcal{R} = \mathsf{k}\{x, y\}$ (with standard grading) and let $\mathcal{A} = \mathcal{R}/(xy - x^3, y^2)$. Then the Hilbert function $H(A) = (1, 1, 2, 1, 1)$ and $H(\mathcal{A}) = (1, 2, 1, 1, 1)$ with conjugate partitions $P(H(A)) = (H(A))^\vee = P(H(\mathcal{A})) = (H(\mathcal{A}))^\vee = (5, 1)$.

A "standard" basis B for the local ring $\mathcal{A}$ is one such that the elements of $\mathsf{B} \cap \mathfrak{m}^i_{\mathcal{A}}$ are a basis for $\mathfrak{m}^i_{\mathcal{A}}$. Such a basis for $\mathcal{A}$ is $\{1, x; y, x^2; x^3, x^4\}$. The multiplication $\times x$ in this basis has Jordan strings

$$1 \to x \to x^2 \to x^3 \to x^4 \to 0 \text{ and } (y - x^2) \to 0. \tag{2.16}$$

[3] Note that $(H(N)^\vee)^\vee$ is just the integers in the set $\{H(N)\}$ rearranged in non-increasing order to form a partition.

Thus, $P_{x,\mathcal{A}} = (5,1)$.

A basis for the graded ring A are the classes of $\{1, x, y, x^2, xy, x^2y\}$ and the only linear element is x: the strings of m_x on this basis for A are $(1 \to x \to x^2 \to xy = x^3 \to x^2y = x^4 \to 0)$ and $((y - x^2) \to 0)$ so here we have $P_{x,A} = P_{x,\mathcal{A}} = (5,1) = P(H(A)) = P(H(\mathcal{A}))$.

However, the Example 1.2 shows that a generic linear element ℓ of the graded algebra A, and a generic element $\ell' \in \mathfrak{m}_{\mathcal{A}}$ of the related local algebra $\mathcal{A}$, may satisfy $P_{\ell,A} = (7,1,1) < (7,2) = P_{\ell',\mathcal{A}}$.

Remark 2.21. Instead of using $(\dim_{\mathsf{k}} M, \dim_{\mathsf{k}} M/\ell, \dim_{\mathsf{k}} M/\ell^2, \ldots)$ as in (2.1) to define $P_{\ell,M}$ we may replace each ℓ^k by $(\ell_1 \cdot \ell_2 \cdots \ell_k)$, a product of possibly different linear forms, yielding a partition $Q(M)$. It can be shown similarly to the proof of equation (2.13) that

$$P_{\ell,M} \leq Q(M) \leq P(H(M)). \tag{2.17}$$

We can ask similar questions for $Q(M)$ to those we ask about $P_{\ell,M}$. When is $P_{\ell,M} = Q(M)$?

Question. The partition $Q(M)$ appears to be related to the concepts of "k-Lefschetz" [H-W, §6.1] and "mixed Lefschetz" [Cat]. What is the relation of these to Jordan type?

Example 2.22. We thank Lorenzo Robbiano for pointing out that we needed to make explicit our assumption that $A_0 = \mathsf{k}$ in Theorem 2.19. He provided the following example when $A_0 \neq \mathsf{k}$. Let $\mathsf{k} = \mathbb{Q}$ be the rationals, set $P = \mathbb{Q}[x]$, let M be the maximal ideal $M = (x^2+1) \subset P$ and denote by $K = P/M$ the quotient field. Consider the Artinian ring $A = P/M^2$ over $\mathbb{Q}$. Let $\mathfrak{m}_A$ be the maximal ideal of A, and consider the associated graded algebra $G = \mathrm{Gr}_{\mathfrak{m}_A}(A)$. It satisfies $\dim_K G = 2$, with Hilbert function $H_K(G) = (1,1)$; the Jordan type of the mutiplication m_x is $P_{x,G} = (2) = (H_K(G))^\vee$. However, over $\mathbb{Q}$ we have $\dim_{\mathbb{Q}}(A) = 4$, $\dim_{\mathbb{Q}} \mathfrak{m}(A) = 2$, so the Hilbert function $H_{\mathbb{Q}}(A) = (2,2)$. The multiplication m_x on A has the string $1 \to x \to x^2 \to x^3$, so the Jordan type $P_{x,A} = 4 > (H_{\mathbb{Q}}(A))^\vee = (2,2)^\vee = (2,2)$.

Jordan degree type and Hilbert function

We next introduce a finer invariant than Jordan type, state a basic result, and pose some questions, but we do not develop it further.

Definition 2.23 (Jordan degree type)**.** Let M be an Artinian A-module (or $\mathcal{A}$-module), let $\ell \in \mathfrak{m}$. Denote by $P_{\ell,i}$ the partition giving the lengths of those strings of m_ℓ acting on A that begin in degree i. The set $\{P_{\ell,i}, i = 0, 1, \ldots, j\}$ stratifies the partition P_ℓ by the initial degree i of the strings. We denote by $\mathcal{P} = \mathcal{P}_{\deg,\ell}$ or by $\mathcal{P}_{\ell,M}$ (to specify the module M) the sequence

$$\mathcal{P}_{\deg,\ell} = (P_{\ell,0}, \ldots, P_{\ell,j-1}). \tag{2.18}$$

Given a sequence $\mathcal{P}$ of partitions as in (2.18) we denote by $H(\mathcal{P})$ the sequence $H = (h_0, \ldots, h_j)$ where h_{k} counts the number of beads of strings of $\mathcal{P}$ in degree k: it is the minimal Hilbert function compatible with $\mathcal{P}$.

We will say for $\mathcal{P}, \mathcal{P}'$ with the same Hilbert function $H(\mathcal{P}) = H(\mathcal{P}')$ that $\mathcal{P} \leq \mathcal{P}'$ if the strings of $\mathcal{P}$ can be concatenated so as to form $\mathcal{P}'$.

Remark 2.24. For a module M having unimodal Hilbert function, the Jordan degree type can be related to the central simple modules (CSM) invariant of T. Harima and J. Watanabe: each CSM $V_{i,\ell}$ of m_ℓ is the vector space span of the initial elements of length-i strings of M under the multiplication map m_ℓ. See [H-W, §3.1], [HW2, §5.1]; the latter treats non-standard grading.

For an integer a we denote by $a^+ = \max\{a, 0\}$.

Definition 2.25 (Degree type associated to a Hilbert function)**.** Let $H = (h_0, h_1, \dots, h_j)$ be a sequence of non-negative integers.. We denote by $\mathsf{P}_{\deg,i}(H)$ the partition having $[h_i - h_{i-1}]^+$ parts, whose lengths are the lengths of those bars (strings) of the bar graph of H beginning in degree i. The *Jordan degree-type* $\mathsf{P}_{\deg}(H)$ is

$$\mathsf{P}_{\deg}(H) = (\mathsf{P}_{\deg,0}(H), \mathsf{P}_{\deg,1}(H) \dots, \mathsf{P}_{\deg,j}(H)). \tag{2.19}$$

Evidently, the degree-type of H determines H, so is equivalent to H – in contrast to $P(H)$ or even $\mathsf{P}(H)$ which, when H is non-unimodular, may not determine the Hilbert function H.

Lemma 2.26. *Let M be a finite-length graded A-module over the Artinian graded algebra A, or let M be an finite length module over the Artinian local ring $\mathcal{A}$. Let $\ell \in A_1$ in the first case and $\ell \in \mathfrak{m}_{\mathcal{A}}$ in the second. Let $\mathcal{P} = \mathcal{P}_{deg,\ell}$ be the sequence of equation* (2.18)*, Then*

$$\mathcal{P} \leq \mathsf{P}_{\deg}(H(\mathcal{P})). \tag{2.20}$$

Let M be a fixed finite-length A-module, then there is a generic linear Jordan degree type $\mathcal{P}_{deg}(M) = \mathcal{P}_{deg,\ell}(M)$ for $\ell \in U_1$ a dense open set of A_1.

Proof. The first statement is evident. For the second, begin with the generic Jordan type $P(M)$, consider the set of highest length parts of $P(M)$, and their initial degrees: that these initial degrees are minimal is an open condition on ℓ. Now fix this open set U_1 and go to the set of next highest-length parts for $\ell \in U_1$, forming an open U_2. In a finite number of steps one shows that $P_{deg}(M)$ is achieved for an open dense set of $\ell \in A_1$. □

Example 2.27. For $H = (1, 3, 2, 3, 3, 1)$ we have $\mathsf{P}_{\deg}(H) = ((6_0), (4_1, 1_1), (2_3))$, a decomposition of $\mathsf{P}(H) = (6, 4, 2, 1)$. For $H' = (1, 3, 1, 3, 3, 2)$ we have $\mathsf{P}_{\deg}(H) = ((6_0), (1_1, 1_1), (3_3, 2_3))$ and $\mathsf{P}(H') = (6, 3, 2, 1, 1)$. See Example 2.17 and Figure 1.

Question 2.28. How does the degree type $P_{deg,\ell}(M)$ behave under

i. deformation of $\ell \in A_1$?

ii. deformation of M within the family of A-modules of fixed Hilbert function H?

Also, does the inequality of equation (2.20) of Lemma 2.26 extend to finite length modules over a local Artinian $\mathcal{A}$, taking $\ell \in \mathfrak{m}_{\mathcal{A}}$?

2.4 Lefschetz properties and Jordan type.

We recall first the traditional, "narrow sense" of strong Lefschetz for a graded algebra ([H-W, Definition 3.18] and [HW2, Definition 2.1].)

Definition 2.29 (Narrow SL)**.** Let A be a graded algebra of socle degree j (not necessarily standard-graded). A linear form $\ell \in A_1$ is (narrow) *strong Lefschetz* [nSL] on A if for each $i \in [0, \lfloor \frac{j}{2} \rfloor]$, the multiplication $\times \ell^{j-2i}$ is an isomorphism from A_i to A_{j-i}.

We say the graded algebra A (or the pair (A, ℓ)) is *strong Lefschetz* if there is a linear form $\ell \in A_1$ that is narrow strong Lefschetz on A.

This requires $H(A)$ to be unimodal and symmetric; that is, $\ell \in A_1$ is nSL on the graded algebra A implies

$$\text{for } 0 \leq i \leq j/2, \quad \dim_{\mathsf{k}} A_i \leq \dim_{\mathsf{k}} A_{i+1} \text{ and } \dim_{\mathsf{k}} A_{j-i} = \dim_{\mathsf{k}} A_i. \tag{2.21}$$

For a discussion of nSL in terms of T. Harima and J. Watanabe's central-simple modules see [HW3] and the summary in [H-W, §4.1].

A more general concept of strong Lefschetz applies to a (not-necessarily standard) graded algebra with non-symmetric Hilbert function [H-W, Definition 3.8ff], [MiNa, Definition 2.4]:

Definition 2.30 (General SL or WL)**.** Let A be a graded algebra of socle degree j. An element $\ell \in A_1$ (or the pair (A, ℓ) is (general) strong Lefschetz [gSL] if each induced homomorphism $\times \ell^d \colon A_i \to A_{d+i}$ has maximum rank, for every pair of integers (i, d) satisfying $0 \leq d \leq j - i$. For A standard graded we say the element $\ell \in A_1$ is *weak Lefschetz* if $m_\ell : A_i \to A_{i+1}$ is of maximum rank for each $i \in [0, j-1]$. We say that A is nSL or gSL if it has an nSL or gSL element $\ell \in A_1$.

It is easy to see that A has a gSL element implies that its Hilbert function is unimodal, i.e.

$$\exists k \text{ such that } i \leq k \Rightarrow h_i \leq h_{i+1} \text{ and } i > k \Rightarrow h_i \geq h_{i+1}. \tag{2.22}$$

When an element $\ell \in A_1$ is nSL or gSL, we term ℓ a "strong Lefschetz element" of A.
Recall that $H(A)^\vee$ is the conjugate partition of the sequence $H(A)$: switch rows and columns in the Ferrers graph of $H(A)$, regarded as a partition of $n = \dim_{\mathsf{k}}(A)$ (Definition 2.16). The following result is straightforward (see, for example, [H-W, Proposition 3.64]).

Lemma 2.31. *Assume that A is standard graded, and that $\ell \in A_1$ is strong Lefschetz in the narrow or general sense, Then the Jordan type P_ℓ of the multiplication map m_ℓ satisfies $P_\ell = H(A)^\vee$.*

We show the following, stronger result.

A special case of Theorem 2.19(i') shows that for a graded algebra A, the Jordan type of any homogeneous element of strictly positive degree is bounded above by the conjugate partition of the Hilbert function of A. The next result shows that the strong Lefschetz property for A is equivalent to the Jordan type of a linear form in A actually achieving this bound. The case that A is standard graded is shown in [H-W, Proposition 3.64].

Proposition 2.32. *Let A be a (possibly non-standard) graded Artin algebra and $\ell \in A_1$. Then the following statements are equivalent:*

i. For each integer b, the multiplication maps $\times\ell^b\colon A_i \to A_{i+b}$ have maximal rank in each degree i.

ii. The Jordan type of ℓ is equal to the conjugate partition of the Hilbert function, i.e.

$$P_\ell = P_H^\vee.$$

iii. For each degree j and each integer i we have the equivalence

$$\dim_{\mathsf{k}}(A_j) \geq i \quad \Leftrightarrow \quad A_j \cap S_a \neq \emptyset \;\forall\; a \leq i. \tag{2.23}$$

Proof. $(i) \Rightarrow (ii)$: Assume $\times\ell^b\colon A_i \to A_{i+b}$ has maximal rank for each i, b. For each integer $i \in [0, j_A]$ define the set of indices $T_i = \{u \,|\dim(A_u) \geq i\}$, and let $n_i = \#T_i$. Let $S_1, \ldots, S_r$ as in Equation (2.10) be strings for the action of ℓ on A, arranged so that their lengths $n_i = \#S_i$ are non-increasing. To these we associate their degree sequences $\deg(S_1), \ldots, \deg(S_r)$, where

$$\deg(S_i) = \left\{\deg(z_i), \ldots, \deg(\ell^{p_i-1}z_i)\right\}.$$

We want to show that $t = r$ and $p_i = n_i$ for $1 \leq i \leq r$.

Claim. There is an injective function $\sigma\colon \{1, \ldots, t\} \to \{1, \ldots, r\}$ such that for each index $i \in [1, t]$, we have

$$T_i \subseteq \deg(S_{\sigma(i)}).$$

Note that the claim implies that $n_i \leq p_{\sigma(i)}$ for $1 \leq i \leq t \leq r$. Then we have

$$\dim_{\mathsf{k}}(A) = \sum_{i=1}^{t} n_i \leq \sum_{i=1}^{t} p_{\sigma(i)} \leq \sum_{i=1}^{r} p_i = \dim_{\mathsf{k}}(A)$$

which implies that $t = r$ and $n_i = p_i$ for all $1 \leq i \leq t$, as desired.

Proof of Claim. We proceed by induction on i. We have $T_1 = \{j \,|\dim_{\mathsf{k}}(A_j) \geq 1\} = \{0, \ldots, d\}$. Since $\times\ell^d\colon A_0 \to A_d$ has full rank, we conclude that T_1 must belong to the degree sequence of some Jordan string, say $S_{\sigma(1)}$. Inductively, assume that we have defined an injective function $\sigma\colon \{1, \ldots, i-1\} \to \{1, \ldots, r\}$ for which

$$\begin{cases} T_1 \subseteq & \deg(S_{\sigma(1)}) \\ \vdots & \\ T_{i-1} \subseteq & \deg(S_{\sigma(i-1)}) \end{cases}. \tag{2.24}$$

Write $T_i = \{j \,|\dim_{\mathsf{k}}(A_j) \geq i\} = \{j_1 < \cdots < j_z\}$. By our assumption, the multiplication map

$$\times\ell^{j_z - j_1}\colon A_{j_1} \to A_{j_z}$$

has rank at least i, hence there are at least i distinct Jordan strings which meet both A_{j_1} and A_{j_z}. Since there are only $i-1$ strings appearing in Equation (2.24), there must be one not listed, call it $S_{\sigma(i)}$ for which $T_i \subseteq \deg(S_{\sigma(i)})$. This completes the induction step and proves the claim. $\square$

$(ii) \Rightarrow (iii)$: Assume that $P_\ell = P_H^\vee$. Then $r = t$ and $p_i = \#\{j \,|\dim_{\mathsf{k}}(A_j) \geq i\}$. Clearly, if $A_j \cap S_a \neq \emptyset\ \forall\ a \leq i$ then $\dim_{\mathsf{k}}(A_j) \geq i$. We prove the other implication by downward induction on the integers $i \leq r$. For the base case, if $\dim_{\mathsf{k}}(A_j) \geq r$, then A_j must contain exactly one element from each Jordan string, hence $A_j \cap S_a \neq \emptyset\ \forall\ a \leq r$. For the inductive step, assume the statement for indices greater than i, and suppose that $\dim_{\mathsf{k}}(A_j) \geq i$. If $\dim_{\mathsf{k}}(A_j) \geq i+1$ then by the induction hypothesis $A_j \cap S_a \neq \emptyset$ for all $a \leq i \leq i+1$. On the other hand if $\dim_{\mathsf{k}}(A_j) = i$, then $A_j \cap S_a = \emptyset$ for all $a \geq i+1$. Indeed, for each index $m \geq i+1$,

$$\dim_{\mathsf{k}}(A_j) \geq m \ \Rightarrow\ A_j \cap S_m \neq \emptyset.$$

By our assumption there are exactly p_m such indices j, hence if $\dim_{\mathsf{k}}(A_j) = i < m$ then $A_j \cap S_m = \emptyset$. So if $\dim_{\mathsf{k}}(A_j) = i$ we must have $A_j \cap S_a \neq \emptyset$ for all $a \leq i$. This completes the induction step.

$(iii) \Rightarrow (i)$: Assume that for each $1 \leq i \leq t$ we have the equivalence

$$\dim_{\mathsf{k}}(A_j) \geq i \ \Leftrightarrow\ A_j \cap S_a \neq \emptyset\ \forall\ a \leq i.$$

Fix an index $i \in [1,t]$ and an integer b, and consider the multiplication map $\times\ell^b \colon A_i \to A_{i+b}$. Let $m = \min\{\dim_{\mathsf{k}}(A_i), \dim_{\mathsf{k}}(A_{i+b})\}$. Then

$$\dim_{\mathsf{k}}(A_i), \dim_{\mathsf{k}}(A_{i+b}) \geq m$$

implies that the m Jordan strings $S_1, \ldots, S_m$ each intersect both A_i and A_{i+b}, which in turn implies that $\times\ell^b \colon A_i \to A_{i+b}$ has rank m. $\square$

Note also that if the Hilbert function $H(A)$ is symmetric, the condition that A is nSL is equivalent to A is gSL.

Recall that for A graded Artinian the Sperner number $\mathrm{Sperner}(A) = \max\{H(A)_i$ for $i \in [0,j]\}$ [H-W, §2.3.4]. For a local ring $\mathcal{A}$, the $\mathrm{Sperner}(\mathcal{A}) = \max\{\mu(\mathfrak{m}_{\mathcal{A}})^i\}$ for $i \in [0,j]\}$ where $\mu(I) = \#$ minimal generators of I.

Lemma 2.33. *[H-W, Proposition 3.5] When the Hilbert function $H(A)$ for a standard graded Artinian algebra A is unimodal and symmetric then $\ell \in A_1$ is weak Lefschetz for A if and only if $\dim_{\mathsf{k}} A/\ell A =$Sperner (A) or, equivalently, if P_ℓ has Sperner(A) parts.*

Strong Lefschetz Jordan type

We introduce here a main concept of the paper. For a graded algebra A we denote by $A^+ = \oplus_{i=1}^{j} A_i$, where j is the socle degree of A.

Definition 2.34 (SLJT)**.** i. Suppose that the graded Artinian algebra A has Hilbert function $H = H(A)$ with respect to the given (possibly non-standard) grading, and that $\ell \in A^+$ (possibly non-homogeneous). We say that ℓ has *strong Lefschetz Jordan type* (SLJT) for A if $P_\ell = H(A)^\vee$. If also $\ell \in A_1$ we say that ℓ has linear SLJT.

ii. Suppose that $\mathcal{A}$ is a local Artinian algebra with maximal ideal $\mathfrak{m}_{\mathcal{A}}$, of Hilbert function $H = H(\mathcal{A})$ with respect to the $\mathfrak{m}_{\mathcal{A}}$-adic grading. We say that $\ell \in \mathfrak{m}_{\mathcal{A}}$ has *strong Lefschetz Jordan type* [SJLT] if $P_\ell = H(\mathcal{A})^\vee$.

Recall from Definition 2.29 that we say that the graded Artinian algebra A is strong-Lefschetz if it has a linear element that is narrow strong Lefschetz. We say that the local algebra $\mathcal{A}$ (or the pair $(\mathcal{A}, \ell)$) is *strong-Lefschetz* if it has an element $\ell \in \mathfrak{m}_{\mathcal{A}}$ that has SLJT.

Remark. Here we have adopted the conventions of [HW2, H-W] concerning the definition of a non-standard graded A being strong Lefschetz – it must be with respect to a linear form $\ell \in A_1$; so the ring A of Example 1.2 is not strong Lefschetz, even though $y+z$ is an element in A having SLJT. But the corresponding local ring $\mathcal{A}$ is strong Lefschetz. Conceivably, with our definitions, we could have a local ring $\mathcal{A}$ defined by a homogeneous ideal, that is strong Lefschetz when regarded as a local ring, but not strong Lefschetz as a graded ring. We don't know of a specific example of this.

Example 2.35. For the algebra A of Example 1.2, of Hilbert function $H(A) = (1,1,2,1,2,1,1)$ we have that for a generic linear element $\ell \in A_1$ the Jordan type $P_\ell = \mathsf{P}(H) = (7,1,1)$, the maximum possible by Theorem 2.19, equation (2.11), while $H(A)^\vee = (7,2)$. Here $\mathsf{P}(H) = (7,1,1)$, as $\times \ell^2 : A_2 \to A_4$ can at most have rank 1, since $\times \ell : A_2 \to A_3$ has rank 1. Thus, (A,ℓ) is not strong-Lefschetz (nor weak Lefschetz). But, as we have seen the element $(z+xy) \in \mathfrak{m}$ has SLJT, corresponding to equality in Theorem 2.19, equation (2.12).

Question. Assume that A is standard graded, and has an element of strong Lefschetz Jordan type. Then must A be strong Lefschetz (have a linear element of SLJT)?

The assumption of standard graded is needed as one could otherwise take $A = \mathsf{k}[x]/(x^2)$ with $\mathsf{w}(x) = 2$, where x has SLJT, but there are no linear elements. We answer "Yes" under the additional assumption that $H(A)$ is unimodal.

Proposition 2.36. *Assume that A is a standard graded Artin algebra and $H(A)$ is unimodal. Then A has an element of strong Lefschetz Jordan type if and only if A is strong Lefschetz.*

Proof. Assume that A has an element ℓ (possibly non-homogeneous) of strong Lefschetz Jordan type, so $P_\ell = H(A)^\vee = (p_1, p_2, \ldots, p_r)$ with $p_1 \geq p_2 \geq \cdots \geq p_r$. Consider Jordan strings $S_1, \ldots, S_r$ for ℓ of Definition 2.18. The orders of elements in a single string are distinct. Let ℓ' be the initial form of ℓ, which, as we will see, must be linear. We will modify the strings, if needed, to a set of Jordan strings for ℓ whose initial forms $S'_1, \ldots, S'_r$ are Jordan strings for ℓ': this will show $P_{\ell'} = P_\ell = H(A)^\vee$, and prove that A is strong Lefschetz.

Given that $H(A)$ is unimodal, we claim that we may choose the strings so that

i. The first s strings together contain $\min\{H(A)_i, s\}$ elements of order i for each $i \in [0, j_A]$; and the initial forms of these elements are linearly independent;

ii. The order $\nu(\ell^i z_{k,1}) = \nu(z_{k,1}) + i$ for each pair (k,i) satisfying $0 \leq k \leq r$ and $0 \leq i \leq p_k - 1$.

We prove (i) and (ii) by induction on u and for all pairs (k,s) satisfying $1 \leq k, s \leq u$. For $u = 1$, the longest string $S_1 = (z_{1,1}, z_{1,2}, \ldots z_{1,p_1})$ where $z_{1,i} = \ell^{i-1} z_{1,1}$ satisfies $p_1 = j_A$, the socle degree of A; and we may choose, after scaling by a non-zero constant, $z_{1,1} = 1 + \alpha, \alpha \in \mathfrak{m}_A$. It follows (since A has standard grading) that the leading term ℓ' of ℓ is linear, and that the elements $S'_1 = (1, \ell', \ldots, (\ell')^{p_1 - 1})$ satisfy $S' = \pi(S_1)$, where π is the projection of the elements of S_1 onto their initial forms, and form a string of length p_1 for ℓ'.

For the induction step we will need several facts. Denote by $m(s,H)$ the smallest integer i such that $H(A)_i \geq s$, and $n(s,H)$ the largest integer i such that $H(A)_i \geq s$. That $H(A)$ is unimodal is equivalent to the inequalities:

$$m(1,H) \leq m(2,H) \leq \cdots \leq m(r,H) \leq n(r,H) \leq n(r-1,H) \leq \cdots \leq n(1,H). \qquad (2.25)$$

We have $p_u = 1 + n(u,H) - m(u,H)$.

Given $s \in [1,r]$ the condition (i) above for all pairs (s,i) satisfying $i > n(s,H)$ implies

(iii) Let $i < n(s,H)$ then the initial forms of all elements of $S_1 \cup \cdots \cup S_s$ having order no greater than i are a basis for $A/\mathfrak{m}_A{}^{i+1}A \cong \oplus_{k=0}^{i} A_k$.

(iv) Let $i > n(s,H)$. The union $\bigcup_{k=1}^{s} (\mathfrak{m}_A)^i \cap S_k$ is a basis for $(\mathfrak{m}_A)^i A = \oplus_{k=i}^{j_A} A_k$.

Induction step: Fix $u \in [1, r-1]$ and assume that a set $S_1, \ldots, S_r$ of Jordan strings for m_ℓ has been chosen satisfying (i) and (ii) for all pairs (k,s) such that $1 \le k, s \le u$. We will keep the strings $S_1, S_2, \ldots, S_u$ fixed and will modify the chain S_{u+1} to obtain a set of r Jordan chains for ℓ so that the conditions (i),(ii) will be satisfied for all pairs k,s such that $1 \le k, s \le u+1$.

Consider the next string S_{u+1}, of length p_{u+1}; by assumption, its elements are linearly independent of the span of those from $S_1 \cup \cdots \cup S_u$, using that (i) and (ii), hence (iii),(iv) are satisfied for u, we may adjust the generator $z_{u+1,1}$ for the string S_{u+1} by linear combinations of elements from the previous strings to obtain a possibly new generator within the span of $S_1, \ldots, S_{u+1}$ having order $m(u+1,H)$, and whose initial form $z'_{u+1,1} = \pi(z_{u+1,1})$ is linearly independent of the degree $m(u+1,H)$ initial forms from elements of the strings $S_1, \ldots, S_u$. Using (iv), we may adjust the generator $z_{u+1,1}$ further by suitable elements of order at least $m(u+1,H)$ from the previous u strings so that $\ell^{p_{u+1}} \cdot z_{u+1,1} = 0$. It follows that $\ell'^{p_{u+1}} \cdot z'_{u+1,1} = 0$, and $z'_{u+1,1}$ is generator of an ℓ' string of length p_{u+1}, linearly independent from the ℓ' strings $S'_1, \ldots, S'_u$ determined by the initial elements from $S_1, \ldots, S_u$. It follows that (i) and (ii) are satisfied for $1 \le k, s \le u+1$. This completes the induction step.

We have shown (i) and (ii) for S_1 and the induction step. It follows that $P_{\ell'} = P_\ell = H(A)^\vee$, as claimed. ∎

Question Let A be a standard-graded Artinian algebra A with unimodal Hilbert function. Is the generic linear Jordan type of A always the same as the generic Jordan type of A.

The following result is well known (and has been reproved several times). Recall that for a graded algebra with $A_0 \cong \mathsf{k}$ we set $\mathfrak{m}_A = \oplus_{i \ge 1} A_i$.

Lemma 2.37 (Height two Artinian algebras are strong Lefschetz). *Let $A = \mathsf{k}[x,y]/I$ be Artinian standard graded of socle degree j, or $\mathcal{A} = \mathsf{k}\{x,y\}/I$ be local Artinian, and suppose* char $\mathsf{k} = 0$ *or* char $\mathsf{k} \ge j$. *Let ℓ be a general element of $\mathfrak{m}_A$ in the first case, or of $\mathfrak{m}_{\mathcal{A}}$ in the second. Then ℓ has strong Lefschetz Jordan type and A (or $\mathcal{A}$) is strong Lefschetz.*

Proof. These statements follow readily from a standard basis argument for ideals in $\mathbb{C}[x,y]$ [Bri], that extends to the case char $\mathsf{k} = p \ge j$ [BaI, Theorem 2.16].[4] ∎

Calculating Jordan type of (ℓ, A) from quotients $A, A/(I:\ell), A/(I:\ell^2), \ldots$.

When the graded A or local $\mathcal{A}$ is Artinian Gorenstein, with dual generator f (see Lemma 3.3) then $I : \ell^i = \mathrm{Ann}(\ell^i \circ \mathsf{f})$, so the Jordan type $P_{\ell,\mathcal{A}}$ can be calculated readily using the dimensions of the Artin algebras $\mathcal{A}(i) = \mathcal{R}/(I : \ell^i)$, and directly from appropriately differentiating f

[4]Proofs occur also in [HMNW, Proposition 4.4] and [Co2], see [MiNa, Theorem 2.27].

Lemma 2.38. *Let $\mathcal{A} = \mathcal{R}/I$ be local Artinian Gorenstein of socle degree j with Macaulay dual generator $F \in \mathfrak{D}$ and let $\ell \in \mathfrak{m}_{\mathcal{A}}$. The conjugate $(P_\ell)^\vee$ to the Jordan type P_ℓ satisfies*

$$(P_\ell)^\vee = \Delta\left(\dim_{\mathsf{k}} \mathcal{A}, \dim_{\mathsf{k}} \mathcal{A}(1), \ldots, \dim_{\mathsf{k}} \mathcal{A}(i), \ldots, \dim_{\mathsf{k}} \mathcal{A}(j)\right) \tag{2.26}$$

where $\mathcal{A}(i) = \mathcal{R}/(I : \ell^i) = \mathcal{A}/(0 : \ell^i) = \mathcal{R}/\operatorname{Ann}(\ell^i \circ F)$.

Proof. This result is standard and follows from Lemma 2.2. See also [H-W, Lemma 3.60]. □

3 Examples, tensor products, commuting Jordan types.

3.1 Idealization and Macaulay dual generator.

The principle of idealization, introduced by M. Nagata to study modules, has been used to "glue" an Artinian algebra to its dual, and so to construct Artinian Gorenstein algebras either having non-unimodal Hilbert functions; or with unimodal Hilbert functions but not having a strong or a weak Lefschetz property [St, BI, BoLa, MiZa]. H. Ikeda and J. Watanabe [Se, IkW] gave examples of Artinian Gorenstein algebra having unimodal Hilbert functions, but not satisfying even weak Lefschetz. Similar examples involving a partial idealization, also not strong Lefschetz or not weak Lefschetz have been constructed more recently by R. Gondim and G. Zappalá [Gon, GonZ].

An example where m_L has Jordan type strictly between weak and Strong Lefschetz is given in [H-W, Section 5.4], referencing [IkW].

We will give here several idealization or partial idealization examples where we calculate the generic Jordan type. First we review Macaulay duality, as idealizations arise from a particular structure for a homogeneous dual generator.

Dual generator of an Artinian Gorenstein algebra

Definition 3.1 (Macaulay dual generator)**.** An Artinian Gorenstein (AG) algebra quotient of $\mathcal{R} = \mathsf{k}\{x_1, \ldots, x_r\}$ satisfies $\mathcal{A} = \mathcal{R}/\operatorname{Ann}\mathsf{f}$ where $\mathsf{f} \in \mathfrak{D} = \mathsf{k}_{DP}[X_1, \ldots, X_r]$ is called the *dual generator* of $\mathcal{A}$. Here R acts on $\mathfrak{D}$ by contraction: $x_i^k \circ X_j^{[k']} = \delta_{i,j} X_j^{[k'-k]}$ for $k' \geq k$, extended multilinearly. The module $\hat{\mathcal{A}} = \mathcal{R} \circ \mathsf{f}$ is the Macaulay dual of $\mathcal{A}$, equivalent to the Macaulay *inverse system* of $\mathcal{A}$.[5] The *socle* of $\mathcal{A}$ is $\operatorname{Soc}(\mathcal{A}) = (0 : \mathfrak{m}_{\mathcal{A}}) \subset \mathcal{A}$, the unique minimal non-zero ideal of $\mathcal{A}$, and $\dim_{\mathsf{k}} \operatorname{Soc}(\mathcal{A}) = 1$.

For a more general Artinian algebra $A = R/I$ (graded) or $\mathcal{A} = \mathcal{R}/I$ (local), a set of *Macaulay dual generators* of A are a minimal set of A (or $\mathcal{A}$) module generators in $\mathfrak{D}$ of $I^\perp = \mathfrak{h} \in \mathfrak{D} \mid I \circ h = 0$.

Example 3.2. (i). Let $R = \mathsf{k}[x, y], A = R/I, I = \operatorname{Ann}\mathsf{f}$ with $\mathsf{f} = XY \in \mathfrak{D} = \mathsf{k}_{DP}[X, Y]$. Then $I = (x^2, y^2)$ and $A = R/(x^2, y^2)$ of Hilbert function $H(A) = (1, 2, 1)$. Here $x^2 \circ XY = 0$ is the contraction analogue of $\partial^2(XY)/(\partial X)^2 = 0$ and the dualizing module $A^\vee = R \circ \mathsf{f} = \langle 1, X, Y, XY \rangle$.

[5] F.H.S. Macaulay used the notation x_i^{-s} for the element $X_i^{[s]}$ in $\mathfrak{D}$.

(ii). Let $\mathcal{R} = \mathsf{k}\{x, y\}$, the regular local ring, and take $\mathsf{f} = X^{[4]} + X^{[2]}Y$. Then $\mathcal{A} = \mathcal{R}/I$, $I = \operatorname{Ann}\mathsf{f} = (xy - x^3, y^2)$, and the Hilbert function $H(\mathcal{A}) = (1, 2, 1, 1, 1)$. The dualizing module $A^\vee = \mathcal{R} \circ \mathsf{f} = \langle 1, X, Y, X^{[2]}, X^{[3]} + XY, \mathsf{f}\rangle$.

Letting $\mathfrak{m}_{\mathcal{A}}$ be the maximal ideal of $\mathcal{A}$, we have $\operatorname{Soc}(\mathcal{A}) = {\mathfrak{m}_{\mathcal{A}}}^j\mathcal{A} \neq 0$ and ${\mathfrak{m}_{\mathcal{A}}}^{j+1}\mathcal{A} = 0$, so $\operatorname{Soc}(\mathcal{A}) = \mathcal{A}_j$. Then we have ([Mac, §60-63],[I, Lemma 1.1], or, in the graded case, [MS, Lemma 1.1.1]),

Lemma 3.3. *i. Assume that $\mathcal{A} = \mathcal{R}/I$ is Artinian Gorenstein of socle degree j. Then there is a degree-j element $\mathsf{f} \in \mathfrak{D}$ such that $I = \operatorname{Ann}\mathsf{f}$. Furthermore f is uniquely determined up to action of a differential unit $u \in \mathcal{R}$: that is*

$$I_{\mathsf{f}} = I_{u\circ\mathsf{f}} \text{ and } I_{\mathsf{f}} = I_{\mathsf{g}} \Leftrightarrow \mathsf{g} = u \circ \mathsf{f} \text{ for some unit } u \in \mathcal{R}. \tag{3.1}$$

The $\mathcal{R}$-module $(I_{\mathsf{f}})^\perp = \{h \in \mathfrak{D}$ such that $I \circ h = 0\}$ satisfies $I^\perp = R\circ\mathsf{f}$. When f is homogeneous, it is uniquely determined by I_{f} up to nonzero constant multiple.

ii. Denote by $\phi : \operatorname{Soc}(\mathcal{A}) \to \mathsf{k}$ a fixed non-trivial isomorphism, and define the pairing $\langle\cdot,\cdot\rangle_\phi$ on $\mathcal{A} \times \mathcal{A}$ by $\langle(a, b)\rangle_\phi = \phi(ab)$. Then the pairing $\langle(\cdot,\cdot)\rangle_\phi$ is an exact pairing on $\mathcal{A}$, for which $(\mathfrak{m}^i)^\perp = (0 : {\mathfrak{m}_{\mathcal{A}}}^i)$. We have $0 : {\mathfrak{m}_{\mathcal{A}}}^i = \operatorname{Ann}({\mathfrak{m}_{\mathcal{A}}}^i \circ \mathsf{f})$. Also $\operatorname{Ann}(\ell^i \circ \mathsf{f}) = I : \ell^i$.

When $\mathcal{A} = \mathcal{R}/\mathcal{I}$ is a local ring then in general f is not homogeneous. When the AG algebra A is (perhaps non-standard) graded, then the dual generator $\mathsf{f} \in \mathfrak{D}$ may be taken homogeneous in a suitable grading of $\mathfrak{D}$.

Examples of Idealization and Jordan type

Our first is the example of R. Stanley in codimension 13. We will use the notation m^k to represent the partition $(m, \dots, m)$ with k parts.

Example 3.4. [St, Example 4.3] We let $R = \mathsf{k}[x, y, z]$ and $S = \mathsf{k}[x, y, z, u_1, \dots u_{10}]$, $\mathfrak{D} = \mathsf{k}_{DP}[X, Y, Z]$ and $\mathfrak{F} = \mathsf{k}_{DP}[X, Y, Z, U_1, \dots, U_{10}]$. Stanley's example results from idealization of $R/{\mathfrak{m}_R}^4$. We let $I \subset S$, $I = \operatorname{Ann} F$, $F = \sum U_i\Xi_i \in \mathfrak{F}_4$ where $\Xi_1, \dots \Xi_{10}$ is a basis for $\mathfrak{D}_3 \subset \mathfrak{D}$. Then $A = S/I$ has dual module $S \circ F$ satisfying

$$\begin{aligned} S_1 \circ F &= \langle R_1 \circ F, \Xi_1, \dots, \Xi_{10}\rangle \\ S_2 \circ F &= \langle \mathfrak{D}_2, R_2 \circ F\rangle \\ S_3 \circ F &= \langle X, Y, Z, U_1, \dots, U_{10}\rangle. \end{aligned} \tag{3.2}$$

Consequently, $H(A) = (1, 13, 12, 13, 1)$ of length 40. Taking a general element $\ell \in S_1$, - up to action of $(\mathsf{k}^*)^{\times 13}$, take $\ell = x + y + z + u_1 + \cdots + u_{10}$, it is straightforward to calculate

$$\begin{aligned} H(S/\operatorname{Ann}(\ell \circ F)) = (1, 9, 9, 1),\ H(S/\operatorname{Ann}(\ell^2 \circ F)) = (1, 6, 1), \\ H(S/\operatorname{Ann}(\ell^3 \circ F)) = (1, 1),\ H(S/\operatorname{Ann}(\ell^4 \circ F)) = (1). \end{aligned}$$

By Lemma 2.38 equation (2.26), the conjugate $(P_\ell)^\vee$ is the first differences Δ of the lengths of the modules $S/\operatorname{Ann}(\ell^k \circ F)$ for $0 \le k \le 4$, so here

$$(P_\ell)^\vee = \Delta(40, 20, 8, 2, 1) = (20, 12, 6, 1, 1). \tag{3.3}$$

Thus, $P_\ell = (5, 3^5, 2^6, 1^8)$ with 20 parts. This is related to the following:

$m_{\ell^2}\colon A_1 \to A_3$ has rank 6 (from the five parts 3, and one part 5), but
$m_\ell\colon A_1 \to A_2$ has rank 9, and kernel rank 4.

By symmetry $m_\ell\colon A_2 \to A_3$ also has rank 9 and cokernel rank 4.
Note that the contraction $R_1 \circ \mathfrak{D}_3 = \mathfrak{D}_2$ takes a 10-dimensional space to a 6 dimensional space: thus any multiplication map $m_\ell : \mathfrak{D}_3 \to \mathfrak{D}_2$ has kernel rank at least 4. The Jordan degree type of this ℓ is

$$\mathcal{P}_\ell = \big(P_{\ell,0} = 5, P_{\ell,1} = (3^5, 2^3, 1^4), P_{\ell,2} = (2^3), P_{\ell,3} = (1^4)\big). \tag{3.4}$$

According to Lemma 2.26 the maximum Jordan type of a multiplication map m_ℓ for $\ell \in \mathfrak{m}_A$ (non-homogeneous) consistent with the Hilbert function $H = H(A)$ would be $\mathsf{P}(H) = (5, 3^{11}, 1^2)$ with 14 parts for a linear form. For an element $\ell \in \mathfrak{m}$ the upper bound would be $P(H) = (5, 3^{11}, 2)$ with 13 parts expected – if a quadratic term in ℓ takes the kernel of the linear part m_{ℓ_1} on A_1 to an element of A_3 not in $\ell_1 \cdot A_2$. Here the generic Jordan type $P_\ell = (5, 3^5, 2^6, 1^8)$ with 20 parts for $\ell \in A_1$ (see above) or even for $\ell \in \mathfrak{m}_A$ (verified for several random $\ell \in \mathfrak{m}_A$, by calculation in Macaulay) is very far from these bounds.

R. Gondim applying work of T. Maeno and J. Watanabe [MW] relating higher Hessians and Lefschetz properties, exhibited Gorenstein algebras A with bihomogeneous dual generators of the form $F = \sum \mu_i \nu_i$, in $\mathfrak{F} = \mathsf{k}_{DP}[X, U]$, such that A does not satisfy weak Lefschetz, or, sometimes, has generic Jordan type strictly between WL and SL [Gon]. Here are two examples from R. Gondim.[6]

Example 3.5. (R. Gondim) Consider the cubic $f \in \mathfrak{F}$

$$f = X_1U_1U_2 + X_2U_2U_3 + X_3U_3U_4 + X_4U_4U_1. \tag{3.5}$$

The associated algebra $A = R/I$, of Hilbert function $H(A) = (1, 8, 8, 1)$ with $I = \mathrm{Ann}(f)$ does not have the WLP: the map $\ell : A_1 \to A_2$ is not injective for any $\ell \in A_1$. The algebra A is presented by 28 quadrics:

$$\begin{aligned} I =(&\mathfrak{m}_x^2, u_1^2, u_2^2, u_3^2, u_4^2, u_1u_3, u_2u_4, x_1u_3, x_1u_4, x_2u_4, x_2u_1, x_3u_1, x_3u_2, x_4u_2, x_4u_3 \\ &x_1u_1 - x_2u_3, x_2u_2 - x_3u_4, x_3u_3 - x_4u_1, x_4u_4 - x_1u_2). \end{aligned}$$

and has Jordan type $(4, 2, 2, 2, 2, 2, 2, 1, 1)$ so strictly less than $H(A)^\vee = (4, 2, 2, 2, 2, 2, 2, 2, 2)$. A random element (non-homogeneous) gives the same Jordan type (calculation in Macaulay 2).

Example 3.6. (R. Gondim) Let $F = XU^{[3]} + YUV^{[2]} + ZU^{[2]}V \in \mathfrak{F} = \mathsf{k}_{DP}[U, V, X, Y, Z]$. Consider $R = \mathsf{k}[u, v, x, y, z]$ and the algebra $A = R/I, I = \mathrm{Ann}\, F$, where

$$I = \left\langle x^2, y^2, z^2, u^4, v^3, xy, xz, yz, xv, zv^2, yu^2, u^2v^2, u^3v, xu - zv, zu - yv \right\rangle.$$

Since A is a bigraded idealization it is easy to see that $H(A) = (1, 5, 6, 5, 1)$. Since the partial derivatives $x \circ F = U^{[3]}, y \circ F = UV^{[2]}$ and $z \circ F = U^{[2]}V$ are algebraically dependent, by

[6] The first example is from R. Gondim's talk at the workshop "Lefschetz Properties and Artinian algebras" at BIRS on March 15 2016, at "https://www.birs.ca/workshops/2016/16w5114/files/Gondim.pdf". The second is a private communication from R. Gondim, following a discussion there with the first author.

the Gordan-Noether criterion ([GorN, Gon, MW]) the Hessian $\mathrm{Hess}_F = 0$. By the Maeno-Watanabe criterion ([MW], [H-W, Theorem 3.76]) this implies that A fails the strong Lefschetz Property. On the other hand it is easy to see that $u + v$ is a WL element for A.

Since A is not strong Lefschetz, the Jordan decomposition P_ℓ for $\ell = u + v + x + y + z$ (a generic-enough linear form) is by Theorem 2.19(i) less in the dominance order than the conjugate $H(A)^\vee = (5, 3, 3, 3, 3, 1)$; since A has WLP, P_ℓ has the same number of parts as $H(A)^\vee$, namely the Sperner number $H(A)_{\max} = 6$. That $\ell^4 \neq 0$ in A the string $S_1 = (1, \ell, \ell^2, \ell^3, \ell^4)$ so P_ℓ has a part 5; since $P_\ell < (5, 3^4, 1)$ and has 6 parts the only possibility is $P_\ell = (5, 3, 3, 3, 2, 2)$.

By Proposition 2.36 since A is standard graded, has unimodal Hilbert function, and is not strong Lefschetz, A cannot have an element – even non-homogeneous – that has strong Lefschetz Jordan type.

R. Gondim gives many further examples, using special bihomogeneous forms. R. Gondim and G. Zappalà have determined further graded Gorenstein algebras that are non-unimodal, sometimes completely non-unimodal (with decreasing Hilbert function from h_1 to $h_{j/2}$, then increasing to degree $j - 1$): they accomplish this by using properties of complexes to choose a suitable bihomogeneous dual generator $f \in \mathfrak{F}$ [GonZ].

3.2 Tensor products, monomial ideals, and Jordan type.

The goal of this section is to show the connection of several separate threads in the literature related to Clebsch-Gordan formulas and Jordan type, in arbitrary characteristic.

Define the two-variable graded complete intersection $R(m, n) = \mathsf{k}[x, y]/(x^n, y^m)$ of socle degree $j = m + n - 2$ and Hilbert function (we assume $m \leq n$ and indicate degree with a subscript)

$$H(R(m, n)) = (1, 2, 3, \ldots, m_{m-1}, m_m, \ldots, m_{n-1}, m - 1, m - 2, \ldots, 1_j). \tag{3.6}$$

The Clebsch-Gordan formula for the Jordan decomposition of the tensor product $[m] \otimes [n]$ of two Jordan blocks is well known in characteristic zero or char $\mathsf{k} = p$ large. It is equivalent to knowing the Jordan type P_ℓ of a generic linear form $\ell = x + y$ (after scaling) in $R(m, n)$.

The notation $[m]$ below means an $m \times m$ nilpotent Jordan block: it is the generic Jordan type of $\mathsf{k}[x]/(x^m)$.

Lemma 3.7 (Clebsch-Gordan)**.** *Consider the CI algebra $R(m, n)$. Assume that* char $\mathsf{k} = 0$ *or* char $k \geq j = m+n-2$, *and let $\ell = ax+by$ with $a, b \neq 0$. Then the Jordan type $[m] \otimes [n] = P_\ell$ satisfies $P_\ell = (H(R(m, n)))^\vee$, so*

$$P_\ell = (n + m - 1, n + m - 3, \ldots, n - m + 1) = \oplus_{k=1}^{m}(n + m + 1 - 2k), \tag{3.7}$$

and ℓ is strong Lefschetz (SL)s. Also, $\dim_{\mathsf{k}} \mathrm{Ker}(\times\ell) = m$. *The Jordan degree type $\mathcal{P}_\ell$ satisfies $P_{\ell,k} = m + n - 1 - 2k$ for $0 \leq k \leq m - 1$.*

Proof. This follows from Lemma 2.37 that implies $P_A = H(A)^\vee$ under the hypotheses. For a different proof of the Clebsch-Gordan portion see [Ait, Sr] or [H-W, Theorem 3.29]. □

This Clebsch-Gordan formula (3.7) extends simply by summing over Jordan blocks.

Proposition 3.8. *[H-W, Prop. 3.66].*[7] *Let $z \in A, w \in B$ be two non-unit elements of Artin algebras A, B. Set $P_z = (d_1, d_2, \dots, d_t)$ and $P_w = (f_1, f_2, \dots f_s)$. Denote by $\ell = z \otimes 1 + 1 \otimes w \in A \otimes B$. Assume* char $\mathsf{k} = 0$, *or* char $\mathsf{k} > \max\{d_i + f_j\}$. *Then*

$$P_\ell = \oplus_{i,j} \oplus_{k=1}^{\min\{d_i, f_j\}} (d_i + f_j + 1 - 2k). \tag{3.8}$$

Also, $dim_{\mathsf{k}} Ker(\times \ell) = \sum_{i,j} \min\{d_i, f_j\}$.

Recall that we denote by j_A the socle degree of A. The following corollary of Proposition 3.8 is not hard to show, and we leave the proof to the reader.

Corollary 3.9. *Assume that A, B are Artin algebras with symmetric unimodal Hilbert functions and that* char $\mathsf{k} = 0$ *or* char $\mathsf{k} > j_A + j_B$. *Then the element $\ell = z \otimes 1 + 1 \otimes w \in A \otimes B$ is SL if and only if z and w are both SL, respectively, in A and in B.*

For a different proof of Corollary 3.9, resting on the connection between the strong Lefschetz property of C and the weak Lefschetz properties of $C \otimes \mathsf{k}[t]/(t^i)$ see T. Harima and J. Watanabe's [HW2, Theorem 3.10]. Recently, J. Watanabe has used the Clebsch-Gordon formula to show that $A \otimes B$ is SL implies both A and B are SL, without prior assumption on the Hilbert functions of A, B.[8] We may use Proposition 3.8 to determine the Jordan types of other, special elements $\ell \in \mathfrak{m}_A$ for certain Artinian algebras A.

Example 3.10. [ADIKSS, Cor. 0.4] Consider $\ell = x^2 + y^2 \in A = \mathsf{k}[x, y]/(x^3, y^3)$ and suppose char $\mathsf{k} \neq 2, 3$. Then $P_\ell = (3, 2, 2, 1, 1)$. Here $P_x = (3)$ on $\mathsf{k}[x]/(x^3)$, so $P_{x^2} = [3]^2 = (2, 1)$; likewise P_{y^2} on $\mathsf{k}[y]/(y^3) = (2, 1)$. We have by Proposition 3.8

$$P_\ell = ([2] \otimes [2], [2] \otimes [1], [1] \otimes [2], [1] \otimes [1]) = ((3, 1), (2, 1), (2, 1), 1) = (3, 2, 2, 1, 1).$$

We found that for char $\mathsf{k} \neq 2, 3$ we could achieve all Jordan types of $\ell \in \mathfrak{m}_A$ from elements of the form $\ell = x^a + y^b$ or $\ell = x^a$ using Proposition 3.8, except for $P_{x^2y} = (2, 2, 1^5)$ ([ADIKSS, Cor. 0.4]).

Corollary 3.11. *Under the same assumptions on characteristic as Lemma 3.7, we have for Jordan degree type,*

$$\begin{aligned} m_s \otimes n_t &= ((n + m - 1)_{s+t}, (n + m - 3)s + t + 1, \dots, (n - m + 1)_{s+t+m-1}) \\ &= \oplus_{k=1}^m (n + m + 1 - 2k)_{s+t+k-1} \end{aligned} \tag{3.9}$$

Remark 3.12. There is substantial work determining Clebsch Gordan formulas in the modular case char $p \leq j$. S.P. Glasby, C.E. Praeger, and B. Xia in [GPX1] summarize previous algorithmic results of Kei-ichiro Iima and Ryo Iwamatsu [IiIw] using Schur functions, and of J.-C. Renaud [Re]; they obtain formulas that in principle allow one to compute the generic Jordan type of $R(m, n)$ in arbitrary characteristic p – they term this the Jordan type $\lambda(m, n, p)$ of $R(m, n, p)$, which always has m parts – so $R(m, n, p)$ is always weak Lefschetz (a result they ascribe to T. Ralley [Ra]). In [GPX2, Theorem 2] they show that $R(m, n, p)$ has SLP (in their language, "is standard") if $n \not\equiv \pm 1, \pm 2, \cdots \pm m \mod p$. They define a deviation vector $\epsilon(m, n, p) = \lambda(m, n, p) - (n, n, \dots, n)$, and show

[7] Although [H-W] restricts to char $\mathsf{k} = 0$, there is no change in showing it for char $\mathsf{k} = p$ large enough.
[8] Private communication, May 18, 2018.

Lemma 3.13. *[GPX2, Theorems 4,6,7] Let* $m \leq \min\{p^k, n, n'\}$.

1. *(periodicity) If* $n \equiv n' \mod p^k$ *then* $\epsilon(r, n, p) = \epsilon(r, n', p)$.
2. *(duality) If* $n' = -n \mod p^k$ *then* $\epsilon(m, n', p) = (-\epsilon_r, \ldots, -\epsilon_1)$*, the "negative reverse" of* $\epsilon(m, n, p)$.
3. *(bound) There are at most* 2^{m-1} *different deviation vectors* $\epsilon(m, n, p)$ *for all* $n \geq m$ *and characteristics* p.
4. *(computation) For fixed* m*, a finite computation suffices to compute the values of* $\epsilon(m, n, p)$ *for all* n *with* $n \geq m$*, and all primes* p.

The authors warn that, in contrast, determining $\lambda(m, n, p)$ is not a finite computation as it involves considering $n \mod p$ for infinitely many n.

Such tables of $\lambda(m, n, p)$ for $m = 3$ and $m = 4$ have been calculated by Jung-Pil Park and Yong-Su Shin in [ParSh].

We can similarly define for sequences $M = (m_1, \ldots, m_r), m_1 \leq m_2 \leq \cdots \leq m_r$ deviation vectors $\epsilon(M, p) = \epsilon(m_1, \ldots, m_r; p)$ for the Jordan types of CI algebras $R(M) = \mathsf{k}[x_1, \ldots, x_r]/(x_1^{m_1}, \ldots, x_r^{m_r})$ when char $\mathsf{k} = p$.

Question. What does the Lemma 3.13 tell us about determining modular Jordan types $\lambda(m_1, m_2, m_3, p)$ (three variables) or in more variables? This is a problem that has been studied and appears quite complex. For example in three variables work on it has involved tilings by lozenges [CoNa1, CoNa2], see also [BrK, Co1, Co2, KuVr]. Further studies of the weak Lefschetz properties of monomial ideals and the relation to algebraic-geometric Togliatti systems have been made by E. Mezzetti, R, G. Ottaviano, R. M. Miró-Roig, and others [MezMO, MezM, MicM]

G. Benkart and J. Osborn studied representations of Lie algebras in characteristic p [BO]. Subsequently, A. Premet [Pr, Lemma 3.4,3.5], and J. Carlson, E. Friedlander and J. Pevtsova in [CFP, §10] Appendix "Decomposition of tensor products of $k[t]/t^p$ modules" gave formulas for such products that apply to $R(m, n, p)$ when $m, n \leq p$. J. Carlson et al regard $T = \mathsf{k}[t]/t^p$ as a self-dual Hopf algebra, with coproduct $t \to 1 \otimes t + t \otimes 1$ and determine the tensor product of irreducible modules over T.

More general than the tensor product $A \otimes B$ are the free extensions C of A with fibre B, introduced by T. Harima and J. Watanabe in [HW1, HW1', HW2], and discussed in [H-W, §4.2-4.4]; we will study these in a sequel [IMM2].

3.3 Commuting Jordan types.

Work of the last ten years has shown that there are strong restrictions on the pairs $P_{\ell,A}, P_{\ell',A}$ that can coexist for an Artinian algebra A [Ob1, Ob2, KOb, IKh, Kh1, Kh2, Pan]. We state several such results. For a more complete discussion, including open questions, see [IKVZ, Ob2].

We say that a partition $P = (p_1, p_2, \ldots, p_s)$ where $p_1 \geq p_2 \geq \cdots p_s$ of n is *stable* if its parts differ pairwise by at least two: if $p_i - p_{i+1} \geq 2$ for $1 \leq i \leq s - 1$. Let B be a nilpotent $n \times n$

matrix over an infinite field k having Jordan type $P = P_B$. Denote by $\mathcal{C}_B$ the commutator of B in $\mathrm{Mat}_n(\mathsf{k})$, and by $\mathcal{N}_B$ the nilpotent elements of C_B. It is well known that $\mathcal{N}_B$ is an irreducible variety, hence there is a generic Jordan type $Q(B)$ of matrices in $\mathcal{N}_B$.

Theorem 3.14 (P. Oblak and T.Košir [KOb])**.** [9] *Assume* char $k = 0$ *or* char $k = p > n$*, let B be an $n \times n$ nilpotent matrix of Jordan type P. Then $Q(B)$ is stable and depends only on the Jordan type P.*

Their proof relied on showing that for a general enough matrix $A \in \mathcal{N}_B$, the local ring $\mathcal{A} = \mathsf{k}\{A, B\}$ is Gorenstein (note that, in general, it is non-homogeneous). A result of F.H.S. Macaulay shows that a Gorenstein (so complete intersection) quotient of $\mathsf{k}\{x, y\}$ has a Hilbert function $H(\mathcal{A})$ whose conjugate $H(\mathcal{A})^\vee$ is stable; and by Lemma 2.37 the generic Jordan type $P_{\mathcal{A}} = H(\mathcal{A})^\vee$. See also [BaI, Theorem 2.27] for a discussion of these steps and [BIK, Section 2.4] for a discussion of the P. Oblak-T. Košir result that $\mathsf{k}[A, B]$ is Gorenstein for A general enough in $\mathcal{N}_B$.

Corollary 3.15. *There can be at most one stable partition among the partitions $P_\ell, \ell \in \mathfrak{m}_{\mathcal{A}}$ for a local Artinian algebra $\mathcal{A}$.*

For example no two of $\{8, (7, 1), (6, 2), (5, 3)\}$ can occur for partitions P_ℓ for the same (commutative) local algebra $\mathcal{A}$.

P. Oblak has made a conjecture giving a recursive way to determine $Q(P)$ from P: it is still open in general. The largest and smallest part of $Q(P)$ is known [Ob1, Kh2], and "half" of the conjecture is shown in [IKh]. Showing the other half is equivalent to proving a combinatorial result about a certain poset associated to $\mathcal{N}_B$ [Kh2, IKh], that would be independent of characteristic.

Given an Artin graded algebra A of local algebra $\mathcal{A}$ there is a generic Jordan type P_A or $P_{\mathcal{A}}$ (Definition 2.6) by Lemma 2.5, simply because $\mathfrak{m}_A$ is an affine space. However, P_A or $P_{\mathcal{A}}$ need not be stable.

Example 3.16. For $A = \mathsf{k}[x, y]/(x^2, xy, y^2)$, or, more generally, for $A_{r,k} = \mathsf{k}[x_1, \ldots, x_r]/\mathfrak{m}^k$ for $r, k \geq 2$ the algebra A has generic Jordan type $P = H(A)^\vee$, which is non-stable. For example $H(A_{2,k})$ for $k > 1$ satisfies $H = (1, 2 \ldots, k)$, whose conjugate $H^\vee$ is $(1, 2, \ldots, k)$.

These are examples of algebras having *constant Jordan type* (CJT): the Jordan type $P_{\ell,A}$ is the same for each linear element $\ell \in A$. Modules of CJT have been extensively studied, and connected to vector bundles over projective space [CFP].

When the partition P_ℓ occurs for a pair (ℓ, M) where M is a finite A-module and A is Artinian, and $\ell \in A$ is nilpotent, then P_{ℓ^k} for a power ℓ^k can be simply described in terms of P_ℓ, and of course must also occur for M or for A (and likewise for $\mathcal{A}$ local and $\ell \in \mathfrak{m}_{\mathcal{A}}$). We briefly describe this.

Definition 3.17 (Almost rectangular partition)**.** [KOb] A partition P of n is *almost rectangular* if its parts differ pairwise by at most 1. We denote by $[n]^k$ the unique almost rectangular partition of n having k parts. If $n = qk + r, 0 \leq r < k$ then

$$[n]^k = \left((q + 1)^r, q^{k-r}\right). \tag{3.10}$$

[9] This is shown in [KOb] over an algebraically closed field of char $k = 0$, but their proof carries through for any infinite field of char $k = 0$ or char $k = p > n$. See [IKVZ, Remark 2.7].

Given a partiton $P = (p_1, p_2, \ldots, p_s), p_1 \geq p_2 \geq \cdots \geq p_s)$ we denote by $[P]^k$ the partition $\big([p_1]^k, [p_2]^k, \ldots, [p_s]^k\big)$ having ks parts.

For example $[7]^2 = (4,3), [7]^3 = (3,2,2), [7]^4 = (2,2,2,1), [7]^5 = (2,2,1,1,1)$, and if $P = (7,5)$ then $[P]^2 = (4,3,3,2)$.

It is easy to see,

Lemma 3.18. *Suppose a nilpotent $n \times n$ matrix M is* regular*: has Jordan type $[n]$. Then M^k has Jordan type $[n]^k$. Suppose that the Jordan type of M is P_M, then $P_{M^k} = [P_M]^k$.*

The term "almost rectangular partition" was introduced by T. Košir and P. Oblak; the notion is key in studying $Q(P)$, and occurs earlier in R. Basili's [Ba], who showed that the number of parts of $Q(P)$ is r_P, the smallest number of almost-rectangular subpartitions needed to cover P.

The more general question of determining which Jordan types may commute (may coexist in the same Artin local algebra A) is quite open in general. See J. Britnell and M. Wildon's [BrWi, §4] and P. Oblak's [Ob2] for some discussion. The former shows that the problem of when two nilpotent matrices commute is in general characteristic dependent: when k is infinite the orbits (d,d) and $(d+1, d-1)$ are commuting, but when k is a finite field, whether those orbits commute depend on the residue classes of d mod powers of p ([BrWi, Prop. 4.12, Rem. 4.15]). A result of G. McNinch [McN, Lemma 22] shows that the Jordan type of a generic element in the pencil of matrices $M + tN$ can depend on the characteristic. See also [IKh, Remark 3.16].

There appears to be substantial structure to the set of partitions P having $Q(P) = Q$ where Q is a given stable partition – see [IKVZ] which shows this structure for stable partitions Q with two parts, and poses a "box conjecture" for general stable Q.

3.4 Problems.

We end with some further problems concerning Jordan type.

Additivity of Jordan type.

We introduce the notation $\mathsf{a} = \sum_{i=1}^n a_i[i]$ for the isomorphism type of a $\mathsf{k}[t]/t^n$ module M having a_i Jordan blocks of size $i, 1 \leq i \leq n$, that is, for the Jordan type $P_{\ell,M} = \mathsf{a}$ where t acts by m_ℓ. The length $|M| = \sum ia_i$. For a and $\mathsf{b} = \sum_{i=1}^n b_i[i]$ we define $(\mathsf{a}+\mathsf{b}) = \sum_{i=1}^n (a_i + b_i)[i]$. We order those types of the same length by $\mathsf{a} \leq \mathsf{b}$ if for all $s, \sum_{i=1}^s ia_i \leq \sum_{i=1}^s ib_i$: this is the standard dominance partial order on partitions. Evidently, if the exact sequence of $\mathsf{k}[t]//(t^n)$ modules $0 \to L \to M \to N \to 0$ is split we have $[M] = [L] + [N]$ (for this notation and context see [CFP]).

For partitions $P = (p_1, \ldots, p_s), Q = (q_1, \ldots, q_s)$ with $p_1 \geq \cdots \geq p_s$ and $q_1 \geq \cdots \geq q_t$ we define the dominance sum $P +_b Q = (p_1 + q_1, p_2 + q_2, \ldots)$.

We consider a finer additive category of $\mathsf{k}[t]/(t^n)$ modules-with-grading, and their related Jordan degree type.

Definition 3.19. An *elementary* class $[i_{\mathsf{k}}]$ denotes a length-i module beginning in degree k. We may assign a Jordan degree type $[M^\circ] = \mathsf{a}^\circ = \sum_{(i,k)} a_{i,k}[i_{\mathsf{k}}]$ to a module M as in Definition 2.23. We define a collapsing operation on two elementary classes by

$$i_{\mathsf{k}} +_c j_{k'} = \begin{cases} (i+j)_{\mathsf{k}} \text{ if } k' = i+k \\ i_{\mathsf{k}} + j_{k'} \text{ otherwise.} \end{cases} \tag{3.11}$$

We denote by κ the functor $\kappa : \mathsf{a}^\circ \to \mathsf{a}$ that forgets the degree information. We define $\mathsf{a}^\circ \geq \mathsf{b}^\circ$ if $\kappa(\mathsf{a}^\circ) \geq \kappa(\mathsf{b}^\circ)$.
Given a Jordan degree class a° we define its *closure*
$\overline{\mathsf{a}^\circ} =$ the maximum Jordan degree type obtained from a° by the operations of (3.11).

Question 3.20. Let $0 \to L \to M \to N \to 0$ be an exact sequence of finite A-modules (or $\mathcal{A}$-modules). How can we compare the generic Jordan types P_L, P_M, and P_N? Under what conditions could we have additivity $P_M = P_L + P_N$, in a suitable sense for the Jordan types, or have $\mathcal{P}_M = \mathcal{P}_L + \mathcal{P}_N$ for the Jordan degree types (Definition 2.23)?

Example 3.21 (Additivity)**.** Let $R = \mathsf{k}[x,y]$, and consider the ideals $I = (x^6, xy, y^6)$ and $J = (x^4, xy, y^3)$. Let $M = R/I$, $N = R/J$ and $L = J/I$. Then $\ell = x+y$ computes the generic Jordan type in each of the modules L, M, and N. We have for the Jordan degree types,

$$\begin{aligned} L = J/I &= \langle y^3, y^4, x^4, y^5, x^5\rangle, \text{ and } \mathcal{P}_{\ell,L} = (3_3, 2_4)\,; \\ N = R/J &= \langle 1, x, y, x^2, y^2, x^3\rangle, \text{ and } \mathcal{P}_{\ell,N} = (4_0, 2_1) \\ M = R/I &= \langle 1, x, y, x^2, y^2, x^3, y^3, x^4, y^4, x^5, y^5\rangle, \text{ and } \mathcal{P}_{\ell,M} = (6_0, 5_1)\,. \end{aligned}$$

Thus, $\mathcal{P}_M = (6_0 = 4_0 +_c 2_4, 5_1 = 2_1 +_c 3_3)$ showing that $\mathcal{P}_{\ell,M} = \overline{\mathcal{P}_{\ell,N} + P_{\ell,L}}$ in the sense of Definition 3.19. On the other hand we have $P_{\ell,M} = (6,5) \neq P_{\ell,L} + P_{\ell,N} = (4,3,2,2)$ nor is $P_{\ell,M}$ the dominance sum $P_{\ell,L} +_b P_{\ell,N} = (4+3, 2+2) = (7,4)$.

Question. What partitions $P_{\ell,M}$ and degree-partitions $\mathcal{P}_{\ell,M}$ can we obtain for M, fixing those invariants for L and N?

Loci in $\mathbb{P}(\mathfrak{m}_{\mathcal{A}})$ defined by Jordan type.

Recall that the set of Jordan types of elements of A acting on M is a poset $\mathfrak{P}_M$ under the "dominance" partial order; this poset $\mathfrak{P}_M$ is an invariant of the module M. Given a partition P of $m = \dim_{\mathsf{k}} M$, the locus $\mathfrak{Z}_{P,M} \subset \mathbb{P}(\mathfrak{m}_A)$, the projective space of the maximal ideal, parametrizes those elements $\ell \in \mathfrak{m}_A$ such that the action of m_ℓ on M has Jordan type $P_\ell = P$. The closures $\overline{\mathfrak{Z}_{P,M}}$ form a poset under inclusion that is isomorphic to $\mathfrak{P}_M$. Of course, the actual loci $\mathfrak{Z}_{P,M}$ in either the A graded or $\mathcal{A}$ local case gives more information than the poset. There have been some study of these in the general commutative algebra community, for example the preprint by M. Boij, J. Migliore, R. Miró-Roig, and U. Nagel, on the non-weak Lefschetz locus [BMMN], and the notes [ADIKSS]. On the other hand, there has been recent study of the Jordan type loci $\mathfrak{Z}_P$ in the nilpotent commutator $\mathcal{N}_B$ of an $n \times n$ matrix B [Ob1, Ob2]; when B is a Jordan matrix of stable Jordan type Q, then it is conjectured that the set $\mathcal{B}(Q)$ of loci in the nilpotent commutator $\mathcal{N}_B$ can be arranged in a rectangular r-box, whose dimensions are determined by the r parts of Q (see [IKVZ, Conjecture 4.11]), and that the equations for these loci are complete intersections [IKVZ, Remark 4.13]. The first conjecture is shown for stable Q having $r = 2$ parts [IKVZ, Theorem 1.1].

Question. Let $x \in \mathfrak{m}_{\mathcal{A}}$ have Jordan type Q. Denote the matrix of m_x by B and recall that $\mathcal{N}_B$ is the nilpotent commutator of B (Section 3.3). Is there a morphism $\tau_{x,\mathcal{A}}\colon \mathfrak{m}_{\mathcal{A}} \to \mathcal{N}_B$, such that the Jordan type of $y \in \mathfrak{m}_{\mathcal{A}}$ satisfies $P_y = P_{\tau_{x,\mathcal{A}}(y)}$?

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A Proof of Theorem 2.19.

We give a proof of Equation (2.12) in the case $M = A$ and an alternative proof of Equation(2.13) in Theorem 2.19i'; the proof also outlines an inductive procedure for computing a Jordan basis with respect to $\ell \in A$ for a graded Artinian algebra A. The proof of Equation (2.12) for an Artinian A-module M is essentially the same as that for A itself. Note that Equation (A.1) is for an arbitrary element $\ell \in \mathfrak{m}_A$ (possibly non-homogeneous) so when $H(\mathcal{A})^\vee \leq H(A)^\vee$ it is stronger than $P_\ell \leq P(H(A)) = H(A)^\vee$ in Equation (2.11), which is just for a linear element $\ell \in A_1$ (See[McDCIM, Remark 2.8] or [IMM1, Remark 3.10] where it is shown that there is no simple relation between the Hilbert functions $H(A)$ and $H(\mathcal{A})$ for the local algebra $\mathcal{A}$ related to the non-standard graded Artinian algebra A.)

Theorem 2.19i'.[Case $M = A$] Let A be a graded Artinian algebra (possibly non-standard) with $A_0 = \mathsf{k}$ and suppose A has socle degree j. Let $\mathfrak{m} = \mathfrak{m}_A = \oplus_{i=1}^d A_i$. Let $\mathrm{Gr}_{\mathfrak{m}}(A) = \oplus_{i\geq 0}\mathfrak{m}^i/\mathfrak{m}^{i+1}$ be the associated graded algebra for A with respect to $\mathfrak{m}$. Then for any element $\ell \in \mathfrak{m}$ we have

$$P_{\ell,A} \leq H(\mathrm{Gr}_{\mathfrak{m}}(A))^\vee \text{ in the dominance partial order.} \tag{A.1}$$

Proof. We use induction on the socle degree $j_A = j$ of A.

(a). Suppose first that the socle degree of A is one. Then $\mathfrak{m}^2 = 0$ and $\mathrm{Gr}_{\mathfrak{m}}(A) = A/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2$. Note that in general we have

$$H(\mathrm{Gr}_{\mathfrak{m}}(A))^\vee = (m_1 \geq \cdots \geq m_s), \text{ where } m_i = \#\left\{u \,\middle|\, \dim(\mathfrak{m}^u/\mathfrak{m}^{u+1}) \geq i\right\}.$$

Hence when $\mathfrak{m}^2 = 0$ we have $H(\mathrm{Gr}_{\mathfrak{m}}(A))^\vee = (2, 1, \ldots, 1)$. We claim that the multiplication map $\times\ell\colon A \to A$ can have at most one Jordan string of length 2 or more. Indeed $\{1 \to \ell\}$ is a length-2 string, and suppose that $\{z \to \ell \cdot z\}$ is another one. If $z \in \mathfrak{m}$ then $\ell \cdot z = 0$ since $\mathfrak{m}^2 = 0$. Otherwise we can write $z = c + m$ where $m \in \mathfrak{m}$ and $c \in \mathsf{k}$. But then $\ell \cdot z = c \cdot \ell$, which would give a non-trivial dependence relation between elements of different strings, which cannot occur. Therefore when $\mathfrak{m}^2 = 0$ we must have $P_\ell \leq H(\mathrm{Gr}_{\mathfrak{m}}(A))^\vee$.

(b). Now assume that the result holds for all graded Artinian algebras B having socle degrees less than j. Let A be a graded Artinian algebra with socle degree j, and set $B = A/\mathfrak{m}^j$. Note that the socle degree of B is $j-1$ (since $\mathfrak{m}_B = \mathfrak{m}_A/\mathfrak{m}_A^j$ and therefore $\mathfrak{m}_B^{j-1} \neq 0$.) By a standard property of associated graded algebras (e.g. Exercise 5.3 in [Ei]) we see that

$$\mathrm{Gr}_{\mathfrak{m}_B} = \mathrm{Gr}_{\mathfrak{m}_A}/\mathfrak{m}^j.$$

Hence as graded vector spaces we have $\mathrm{Gr}_{\mathfrak{m}_A} = \mathrm{Gr}_{\mathfrak{m}_B} \oplus \mathfrak{m}_A^j$. Thus $H(\mathrm{Gr}_{\mathfrak{m}_A})^\vee$ is obtained by tacking on one to the *first* (largest) $H(A)_j = \dim_{\mathsf{k}}(\mathfrak{m}_A^j)$ parts of $H(\mathrm{Gr}_{\mathfrak{m}_B})^\vee$.

Fix $\ell \in \mathfrak{m} = \mathfrak{m}_A$, and let $\bar{\ell} \in \mathfrak{m}_B$ be its image under the natural projection $\pi\colon A \to B$. Fix a Jordan basis of strings (Definition 2.18) $\bar{S}_1, \ldots, \bar{S}_r$ for the action of $\bar{\ell}$ on A:

$$\bar{S}_i = \left\{\bar{z}_i, \ldots, \bar{\ell}^{n_i} \cdot \bar{z}_i\right\}.$$

Assume the strings are labeled so that their lengths satisfy $n_1 \geq \cdots \geq n_r$.

Let $z_i \in A$ be any π-lift of $\bar{z}_i \in B$ for $1 \leq i \leq r$. Then we have strings in A, $\langle S_1\rangle, \ldots, \langle S_r\rangle$ where $S_i = \{z_i, \ldots, \ell^{n_i} \cdot z_i\}$. Note that for each $1 \leq i \leq r$ we have $\ell^{n_i+1} z_i \in \mathfrak{m}^j$, hence in particular, $\ell^{n_i+2} z_i = 0$ since $\mathfrak{m}^{j+1} = 0$. We will transform these partial Jordan strings $S_1, \ldots, S_r$ to actual Jordan strings, $\hat{S}_1, \ldots, \hat{S}_s$ $(s \geq r)$, which form a full Jordan basis for A via the following claim. By $sp\{S\}$ we mean the k span $\langle\{S\}\rangle$.

Claim: Given the partial Jordan strings $S_1, \ldots, S_r$ with $S_i = \{z_i, \ldots, \ell^{n_i} z_i\}$, there exist strings $\hat{S}_1, \ldots, \hat{S}_r$ with $\hat{S}_i = \{\hat{z}_i, \ldots, \ell^{\epsilon_i} \hat{z}_i\}$ satisfying the following conditions for each $1 \leq i \leq r$:

(A$_i$). $\epsilon_i = n_i$ or $\epsilon_i = n_i + 1$ and $\ell^{\epsilon_i+1} \hat{z}_i = 0$

(B$_i$). $\mathrm{sp}(\hat{S}_1, \ldots, \hat{S}_i) = \mathrm{sp}\,(S_1, \ldots, S_i) \oplus \mathrm{sp}\,(\ell^{n_1+1} z_1, \ldots, \ell^{n_i+1} z_i)$

Proof of Claim. We establish the claim by induction on u for $1 \leq u \leq r$. For $u = 1$, set $\hat{z}_1 = z_1$ and define

$$\hat{S}_1 = \begin{cases} S_1 & \text{if } \ell^{n_1+1} z_1 = 0 \\ S_1 \cup \{\ell^{n_1+1} z_1\} & \text{if } \ell^{n_1+1} z_1 \neq 0 \end{cases} \tag{A.2}$$

Clearly $\hat{S}_1$ satisfies both conditions (A$_1$) and (B$_1$) given in the claim.

Now assume that the we have constructed strings $\hat{S}_1, \ldots, \hat{S}_{u-1}$ satisfying conditions (A$_i$) and (B$_i$) above for $1 \leq i \leq u-1$. Consider the partial string $S_u = \{z_u, \ldots, \ell^{n_u} z_u\}$. There are three separate cases to consider: First, if $\ell^{n_u+1} z_u = 0$, then S_u is already a Jordan string and we should set $\hat{S}_u = S_u$. Second, if $\ell^{n_u+1} z_u \neq 0$ and $\ell^{n_u+1} z_u \notin \mathrm{sp}(\hat{S}_1, \ldots, \hat{S}_{u-1})$, then we may simply tack on this element to form a Jordan string, i.e. set $\hat{S}_u = S_u \cup \{\ell^{n_u+1} z_u\}$. Third, it may happen that $\ell^{n_u+1} z_u \neq 0$ but $\ell^{n_u+1} z_u \in \mathrm{sp}(\hat{S}_1, \ldots, \hat{S}_{u-1})$. Note that in this case, we cannot simply tack this element onto S_u because the resulting set of strings $\hat{S}_1 \cup$

$\cdots \cup \hat{S}_{u-1} \cup (S_u \cup \{\ell^{n_u+1} z_u\})$ will not be linearly independent. On the other hand note that $\ell^{n_u+1} z_u \in \mathfrak{m}^j \cap \mathrm{sp}(\hat{S}_1, \dots, \hat{S}_{u-1}) = \langle \ell^{n_1+1} z_1, \dots, \ell^{n_{u-1}+1} z_{u-1} \rangle$: this follows from condition B_{u-1}. Thus, there exist scalars $c_1, \dots, c_{u-1} \in \mathsf{k} = A/\mathfrak{m}$ such that $\ell^{n_u+1} z_u = \sum_{k=1}^{u-1} c_{\mathsf{k}} \ell^{n_{\mathsf{k}}+1} z_{\mathsf{k}}$. Then we define

$$\hat{z}_u = z_u - \sum_{k=1}^{u-1} c_{\mathsf{k}} \ell^{n_{\mathsf{k}} - n_u} z_{\mathsf{k}}.$$

Note that the terms in this sum are well defined because we are assuming that $n_{\mathsf{k}} \geq n_u$ for all $k \leq u$. Note that $\ell^{n_u} \hat{z}_u \neq 0$ since $\ell^{n_u} z_u \notin \mathrm{sp}\,(S_1, \dots, S_{u-1})$, but that $\ell^{n_u+1} \hat{z}_u = 0$. Finally in this case, we get our desired Jordan string by setting $\hat{S}_u = \{\hat{z}_u, \dots, \ell^{n_u} \hat{z}_u\}$. To recapitulate, our u^{th} Jordan string is defined by the rule

$$\hat{S}_u = \begin{cases} S_u & \text{if } \ell^{n_u+1} z_u = 0 \\ S_u \cup \{\ell^{n_u+1} z_u\} & \text{if } \ell^{n_u+1} z_u \neq 0 \text{ and } \ell^{n_u+1} z_u \notin \mathrm{sp}\left(\hat{S}_1, \dots, \hat{S}_{u-1}\right) \\ \{\hat{z}_u, \dots, \ell^{n_u+1} \hat{z}_u\} & \text{if } \ell^{n_u+1} z_u \neq 0 \text{ and } \ell^{n_u+1} z_u \in \mathrm{sp}\left(\hat{S}_1, \dots, \hat{S}_{u-1}\right) \end{cases} \tag{A.3}$$

where $\hat{z}_u$ is defined as above. Certainly $\hat{S}_u$ satisfies condition (A_u). It remains only to see that condition (B_u) is satisfied. To see this note that

$$\begin{aligned} \mathrm{sp}(\hat{S}_1, \dots, \hat{S}_u) &= \mathrm{sp}(\hat{S}_1, \dots, \hat{S}_{u-1}) + \mathrm{sp}(\hat{S}_u) \\ &\subseteq \left[\mathrm{sp}(S_1, \dots, S_{u-1}) \oplus \langle \ell^{n_1+1} z_1, \dots, \ell^{n_{u-1}+1} z_{u-1} \rangle\right] \oplus \left[\mathrm{sp}\,(S_u) \oplus \mathrm{sp}(\ell^{n_u+1} z_u)\right] \\ &= \mathrm{sp}(S_1, \dots, S_u) \oplus \langle \ell^{n_1+1} z_1, \dots, \ell^{n_u+1} z_u \rangle. \end{aligned}$$

Since condition (B_{u-1}) holds, and since the subspaces $\mathrm{sp}(\hat{S}_u)$ and $\mathrm{sp}\,(S_u) \oplus \{\ell^{n_u+1} z_u\}$ evidently have the same dimension, it suffices to see that the sum on the first line is direct. This amounts to showing the equality

$$\begin{aligned} &\mathrm{sp}(\hat{S}_u) \cap\ \mathrm{sp}(\hat{S}_1, \dots, \hat{S}_{u-1}) = 0 \text{ where} \\ &\mathrm{sp}(\hat{S}_1, \dots, \hat{S}) = \mathrm{sp}(S_1, \dots, S_{u-1}) \oplus \langle \ell^{n_1+1} z_1, \dots, \ell^{n_{u-1}+1} z_{u-1} \rangle. \end{aligned} \tag{A.4}$$

But note that by Equation (A.3) we have that $\hat{z}_u - z_u \in \mathrm{sp}(\hat{S}_1, \dots, \hat{S}_{u-1})$. Suppose that $\alpha = \sum_{t=0}^{n_u+1} a_t \ell^t \hat{z}_u \in \mathrm{sp}(\hat{S}_1, \dots, \hat{S}_{u-1})$ for some $a_t \in \mathsf{k}$. Then $\sum_{t=0}^{n_u+1} a_t \ell^t z_u \in \mathrm{sp}(\hat{S}_1, \dots, \hat{S}_{u-1})$ too, and hence we have

$$\pi\left(\sum_{t=0}^{n_u+1} a_t \ell^t z_u\right) = \sum_{t=0}^{n_u} a_t \bar{\ell}^t \bar{z}_u \in \mathrm{sp}(\bar{S}_1, \dots, \bar{S}_{u-1}).$$

On the other hand since $\bar{S}_1 \cup \cdots \cup \bar{S}_u$ is a linearly independent set in B, we conclude that $a_t = 0$ for $0 \leq t \leq n_u$. Thus $\alpha = a_{n_u+1} \ell^{n_u+1} \hat{z}_u$. But this must be zero as well, because if not, Equation (A.3) tells us $\alpha \notin \mathrm{sp}(\hat{S}_1, \dots, \hat{S}_{u-1})$, contrary to our choice for α. Hence Equation (A.4) is satisfied, hence so is condition (B_u). This completes the proof of the claim, and the induction step of (b). □

Finally we have constructed Jordan strings for the multiplication map $\times\ell \colon A \to A$, $\hat{S}_1, \dots, \hat{S}_r$ satisfying conditions (A_i) and (B_i) for $1 \leq i \leq r$. We can complete this to a

Jordan basis for A by simply choosing a basis for a complementary subspace to the subspace $\mathrm{sp}(\hat{S}_1,\dots,\hat{S}_r)\cap\mathfrak{m}^j=\langle \ell^{n_1+1}z_1,\dots,\ell^{n_r+1}z_r\rangle\subset\mathfrak{m}^j$, say $\{\omega_{r+1},\dots,\omega_s\}\subset\mathfrak{m}^j$, and form the one element strings $\hat{S}_{r+1}=\{\omega_{r+1}\},\dots,\hat{S}_s=\{\omega_s\}$.

From our construction we finally see that $P_{\ell,A}$ is obtained from $P_{\bar{\ell},B}$ by adding one to *some* $\dim(\mathfrak{m}^d)$ strings. Since $H(\mathrm{Gr}_{\mathfrak{m}_A}(A))^\vee$ is obtained from $H(\mathrm{Gr}_{\mathfrak{m}_B}(B))^\vee$ by adding ones to the *first* (largest) $\dim(\mathfrak{m}^d)$ parts, we see that

$$P_{\bar{\ell},B}\le H(\mathrm{Gr}_{\mathfrak{m}_B})^\vee \;\Rightarrow\; P_{\ell,A}\le H(\mathrm{Gr}_{\mathfrak{m}_A}(A))^\vee.$$

□

Remark. i. The same proof works verbatim for the local case as well.

ii. A similar induction using this construction can also be used to show that if $\ell\in\mathfrak{m}$ is homogeneous, then there exists a homogeneous Jordan basis for A with respect to $\times\ell\colon A\to A$.

We give the following non-inductive proof of the second inequality in Equation (2.11) for an Artinian algebra.

Theorem 2.19$''$. Let A be a graded Artinian algebra with $A_0=\mathsf{k}$ and Hilbert function $H=H(A)$ and let $\ell\in\mathfrak{m}_A$ be homogeneous. Then

$$P_{\ell,A}\le P(H)=H(A)^\vee \text{ in the dominance partial order.}$$

Proof. For each $1\le i\le t$ and each integer u, define the new integer $m(i,u)$ to be the number elements of degree u in the disjoint union of strings $S_1\cup\cdots\cup S_i$. Then we certainly have the inequality

$$\dim_{\mathsf{k}}(A_u)\ge m(i,u). \tag{A.5}$$

On the other hand, for each $1\le i\le t$ define the set of indices $T_i=\{u\,|\dim_{\mathsf{k}}(A_u)\ge i\}$. Note that $n_i=\#T_i$ and $T_t\subseteq T_{t-1}\subseteq\cdots\subseteq T_1$. For each $1\le i\le t$ and each integer u, define the new integer $n(i,u)$ to be the number of times the index u appears in the multi-set $T_1\cup\cdots\cup T_i$. Since no two elements of the same string have the same degree, we see that $0\le m(i,u)\le i$. Since $\dim_{\mathsf{k}}(A_u)\ge m(i,u)$, the index u must appear in $T_{m(i,u)}$, as well as in $T_{m(i,u)-1},\dots,T_1$. Thus we see that

$$m(i,u)\le n(i,u). \tag{A.6}$$

Summing over all u, we get the desired result,

$$p_1+\cdots+p_i=\sum_u m(i,u)\le\sum_u n(i,u)=n_1+\cdots n_i$$

□

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