The ABC of KLM

optical quantum information processing

- qubit: polarization $|H\rangle, |V\rangle$; dual rail $|0,1\rangle, |1,0\rangle$ } equivalent

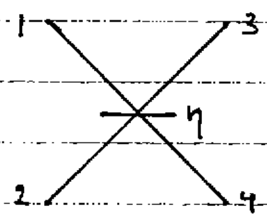
- single-qubit operations are easy:
  we can use phase shifters, polarization rotations and beam splitters.

optical modes are populated by photons that are created and annihilated by $\hat{a}^\dagger$ and $\hat{a}$:

$$\hat{a}_k |n\rangle_k = \sqrt{n}\, |n-1\rangle_k \quad ; \quad \hat{a}_k^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle_k$$

and $\hat{a}_k^\dagger \hat{a}_k |n\rangle_k \equiv \hat{n}_k |n\rangle_k = n |n\rangle_k$; $[\hat{a}_k, \hat{a}_j^\dagger] = \delta_{jk}$

Beam splitter:



$$\left.\begin{aligned} \hat{a}_1 &\to \sqrt{\eta}\, \hat{a}_3 + \sqrt{1-\eta}\, \hat{a}_4 \\ \hat{a}_2 &\to -\sqrt{1-\eta}\, \hat{a}_3 + \sqrt{\eta}\, \hat{a}_4 \end{aligned}\right\} \text{linear transformation}$$

This is what we mean by Linear Optics

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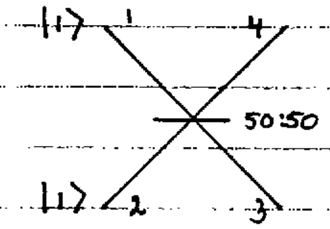
- two-qubit operations are hard because photons do not interact directly with each other

we can make use of bosonic symmetry relations:



$$\hat{a}_1 \to \frac{1}{\sqrt{2}}\left(\hat{a}_3 + \hat{a}_4\right)$$

$$\hat{a}_2 \to \frac{1}{\sqrt{2}}\left(-\hat{a}_3 + \hat{a}_4\right)$$

the output of the beam splitter is then:

$$\hat{a}_1^\dagger \hat{a}_2^\dagger |0\rangle_{12} \to \tfrac{1}{2}\left(\hat{a}_3^\dagger + \hat{a}_4^\dagger\right)\left(-\hat{a}_3^\dagger + \hat{a}_4^\dagger\right)|0\rangle_{34}$$

$$= \tfrac{1}{2}\left(-\hat{a}_3^{\dagger 2} + \hat{a}_3^\dagger \hat{a}_4^\dagger - \hat{a}_4^\dagger \hat{a}_3^\dagger + \hat{a}_4^{\dagger 2}\right)|0\rangle_{34}$$

$$= \tfrac{1}{2}\left(-\sqrt{2}\,|2,0\rangle + \sqrt{2}\,|0,2\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|2,0\rangle - |0,2\rangle\right) \text{ up to a global phase}$$

The absence of the $|1,1\rangle_{34}$ term is called the Hong-Ou-Mandel effect. The photons do seem to "know" that they come in a pair.

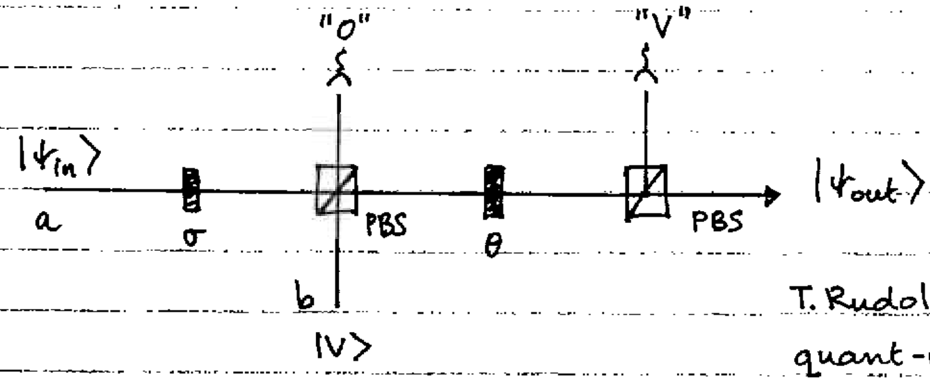
However, this is not enough to create two-photon gates universal for quantum computing.

- projective measurements can make probabilistic two-qubit gates.

Let's look at a simple probabilistic gate to do the following transformation:

$$\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle \rightarrow \alpha|0\rangle + \beta|1\rangle - \gamma|2\rangle$$

This is a so-called NS-gate, and it plays an important role in LOQC. It cannot be made to work deterministically.



T. Rudolph & J.-W. Pan
quant-ph/0108056

The polarization rotation is given by $\hat{a}_H \rightarrow \cos\sigma\, \hat{a}_H + \sin\sigma\, \hat{a}_V$

$$\left(\alpha + \beta\, \hat{a}_H^\dagger + \frac{\gamma}{\sqrt{2}}\, \hat{a}_H^{\dagger 2}\right) \hat{b}_V^\dagger\, |0\rangle \rightarrow$$

$$\left(\alpha + \beta\cos\sigma\, \hat{a}_H^\dagger + \beta\sin\sigma\, \hat{a}_V^\dagger + \frac{\gamma}{\sqrt{2}}\left(\cos^2\sigma\, \hat{a}_H^{\dagger 2} + \sin 2\sigma\, \hat{a}_H^\dagger \hat{a}_V^\dagger + \sin^2\sigma\, \hat{a}_V^{\dagger 2}\right)\hat{b}_V^\dagger\right)|0\rangle$$

detecting "0" at the first output port:

$$\rightarrow \left(\alpha + \beta\cos\sigma\, \hat{a}_H^\dagger + \frac{\gamma}{\sqrt{2}}\cos^2\sigma\, \hat{a}_H^{\dagger 2}\right)\hat{a}_V^\dagger\, |0\rangle$$

$$\left.\begin{array}{l} \hat{a}_H \rightarrow \cos\theta\, \hat{a}_H + \sin\theta\, \hat{a}_V \\ \hat{a}_V \rightarrow -\sin\theta\, \hat{a}_H + \cos\theta\, \hat{a}_V \end{array}\right] \text{ second rotation}$$

$$\left(\alpha + \beta\cos\sigma\left(\cos\theta\, \hat{a}_H^\dagger + \sin\theta\, \hat{a}_V^\dagger\right) + \frac{\gamma}{\sqrt{2}}\cos^2\sigma\left(\cos\theta\, \hat{a}_H^\dagger + \sin\theta\, \hat{a}_V^\dagger\right)^2\right)\left(-\sin\theta\, \hat{a}_H^\dagger + \cos\theta\, \hat{a}_V^\dagger\right)|0\rangle$$

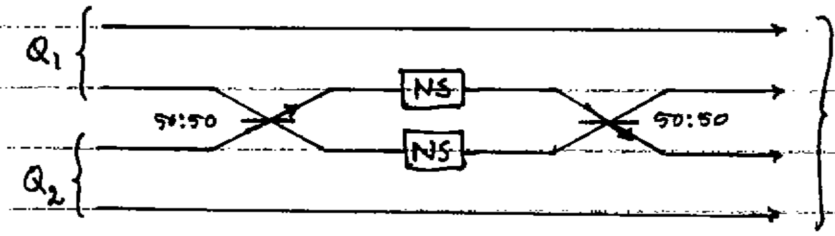
detecting "V" at the second output port:

$$\left[\alpha\cos\theta + \beta\cos\sigma\cos 2\theta\,\hat{a}_H^\dagger + \frac{\gamma}{\sqrt{2}}\cos^2\sigma\left(\cos^3\theta - 2\cos\theta\sin^2\theta\right)\hat{a}_H^{\dagger 2}\right]|0\rangle$$

$$= \alpha\cos\theta\,|0\rangle + \beta\cos\sigma\cos 2\theta\,|1\rangle + \gamma\cos^2\sigma\cos\theta\left(1-3\sin^2\theta\right)|2\rangle$$

Choose $\sigma \cong 150.5^\circ$ and $\theta \cong 61.5^\circ$. This yields the NS gate with probability $\cos^2\theta \approx 0.227$: slightly less than the optimal $1/4$.

How can we use the NS gate to make a conditional sign flip: $CZ = \mathrm{diag}(1,1,1,-1)$? In dual-rail logic:



CZ on Q_1 and Q_2

I flipped the two spatial modes of the lower qubit

$Q_1 = \alpha|01\rangle + \beta|10\rangle$ and $Q_2 = \gamma|01\rangle + \delta|10\rangle$

$$(\alpha|01\rangle + \beta|10\rangle)(\gamma|01\rangle + \delta|10\rangle) =$$

$$\alpha\gamma|0101\rangle + \alpha\delta|0110\rangle + \beta\gamma|1001\rangle + \beta\delta|1010\rangle$$

Apply the beam splitter to the two indicated modes:

$$\alpha\gamma|0101\rangle + \frac{\alpha\delta}{\sqrt{2}}\left(|0110\rangle - |1100\rangle\right) + \frac{\beta\gamma}{\sqrt{2}}\left(|1001\rangle + |0011\rangle\right) + \frac{\beta\delta}{\sqrt{2}}\left(|2000\rangle - |0020\rangle\right)$$

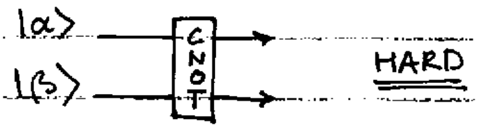
NS gates and second beam splitter:

$$\alpha\gamma|0101\rangle + \alpha\delta|0110\rangle + \beta\gamma|1001\rangle - \beta\delta|1010\rangle,$$

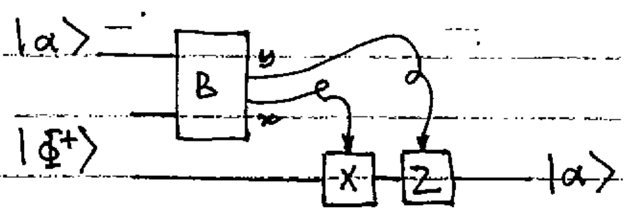
which is nolonger separable. However, this works at best with probability 1/16, and when we use this quantum circuit many times, the overall success rate decreases exponentially.

the teleportation trick by Gottesman & Chuang.

a)

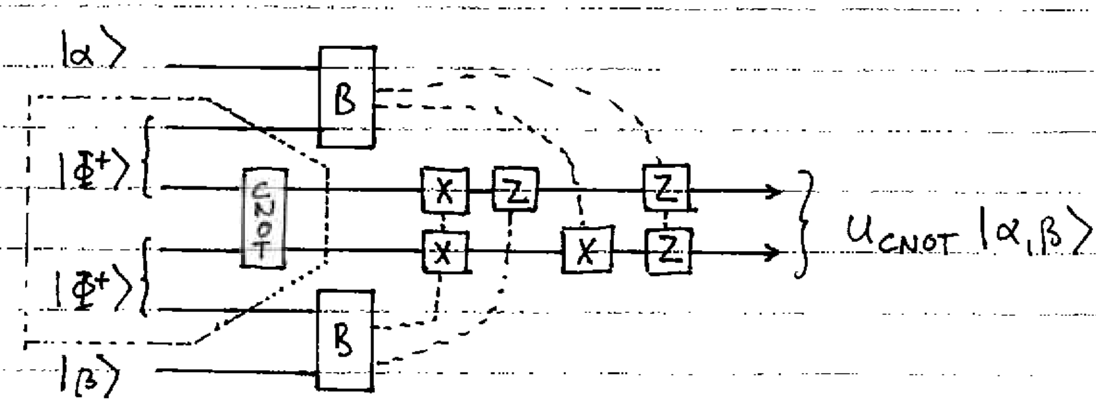


b)



$$\text{if} \begin{cases} x = 1 \rightarrow \text{apply } \sigma_x \\ y = 1 \rightarrow \quad " \quad \sigma_z \end{cases}$$

c)



This trick works only with operators that are part of the Clifford group and transform Pauli gates to Pauli gates.

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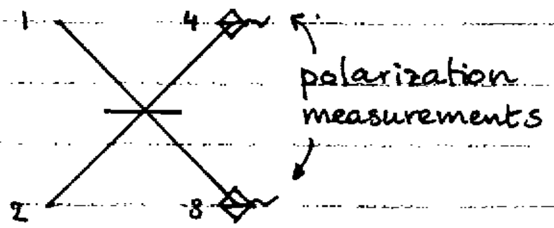
Unfortunately, there is a problem when we want to implement this with linear optics: the Bell-state measurement is not complete!

probabilistic Bell measurement



$$\hat{a}_{1k} \to \frac{1}{\sqrt{2}}\left(\hat{a}_{3k} + \hat{a}_{4k}\right)$$

$$a_{2k} \to \frac{1}{\sqrt{2}}\left(-\hat{a}_{3k} + \hat{a}_{4k}\right)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|H,V\rangle_{12} - |V,H\rangle_{12}\right) \to \frac{1}{\sqrt{2}}\left(|H,V\rangle_{34} - |V,H\rangle_{34}\right)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|H,V\rangle_{12} + |V,H\rangle_{12}\right) \to \frac{1}{\sqrt{2}}\left(|0,HV\rangle_{34} - |HV,0\rangle_{34}\right)$$

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}\left(|H,H\rangle_{12} \pm |V,V\rangle_{12}\right)$$

$$\to \frac{1}{2}\left(-|H^2,0\rangle \mp |V^2,0\rangle + |0,H^2\rangle \pm |0,V^2\rangle\right)$$

This means that we cannot distinguish $|\Phi^+\rangle$ and $|\Phi^-\rangle$. In general, you can only two out of four Bell states.

But if the teleportation procedure is itself probabilistic, does the trick still work?

Bell measurements of qubits (with optics) fail 50% of the time, but when we enlarge the Hilbert space, we can bring this number down.

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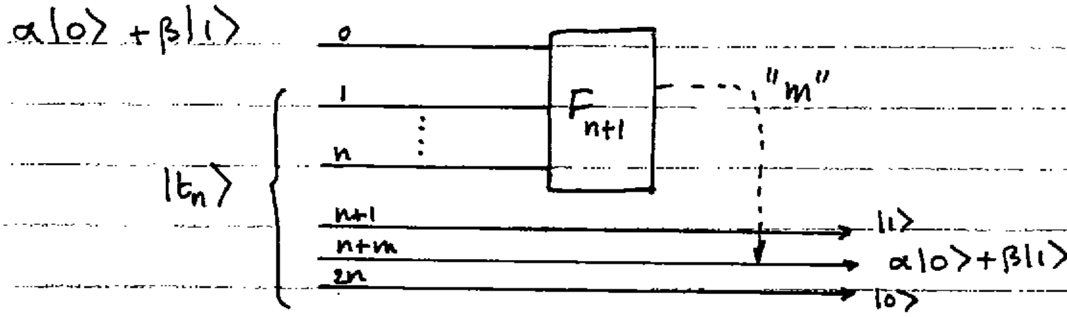
Let's construct the auxiliary state:

$$|t_n\rangle = \frac{1}{\sqrt{n+1}} \sum_{j=0}^{n} |1\rangle^{j} |0\rangle^{n-j} \; |0\rangle^{j} |1\rangle^{n-j}$$

For a superposition of the vacuum and a single photon we then have the following teleportation procedure:



We then mix up all the modes, i.e., erase all the which-path information and count the number of photons in the first $n+1$ modes. Because $|t_n\rangle$ is ordered, when we find "m" photons, the m^{th} mode of the remaining modes is the teleported state.

F_{n+1} is the discrete Fourier transform, or the $n+1$-mode generalization of the 50:50 beam splitter:

$$(F_n)_{jk} = \frac{1}{\sqrt{n}} \exp\left[2\pi i\,(j-1)(k-1)/n\right]$$

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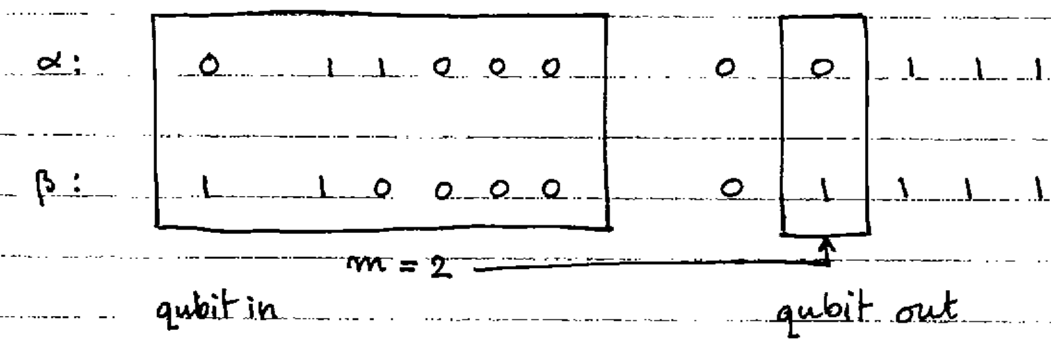
example: let's use $n=5$ and $m=2$.



when the photon number is $m=0$ or $m=n+1$, we collapse the input state, and the tellportation fails. The success rate is $n/(n+1)$.

Now we have to apply the two-qubit gate like CNOT or CZ to the state $|t_n\rangle$. For the CZ gate we need to make the state

$$|cs_n\rangle = \frac{1}{n+1} \sum_{i,j=0}^{n} (-1)^{(n-i)(n-j)} |1\rangle^i |0\rangle^{n-i} |0\rangle^i |1\rangle^{n-i} |1\rangle^j |0\rangle^{n-j} |0\rangle^j |1\rangle^{n-j}$$

You probably don't want to use n CZ gates with total success probability 16^{-n}, because that will be very costly. Nevertheless, the overall scaling is polynomial.

Preparing $|t_n\rangle$ and $|cs_n\rangle$ efficiently is still an open problem

Franson suppressed the average error rate

According to KLM: $|t_n\rangle \propto \sum_j |1\rangle^j |0\rangle^{n-j} |0\rangle^j |1\rangle^{n-j}$, so it is equally likely to find $|0..0\rangle|0..0\rangle$ as it is to find $|1..10..0\rangle|0..01..1\rangle$.

If the teleportation fails, we contribute to the error rate. However, if $m \neq 0, n+1$ then KLM is potentially error free.

Alternatively, we can add a small error to the teleported qubit such that the average error is lower than the KLM case.

Use $|t_n'\rangle = \sum_{j=0}^{n} f(j)\, |1\rangle^j |0\rangle^{n-j} |0\rangle^j |1\rangle^{n-j}$

$$\begin{array}{l|cccccc|ccccc}
\alpha \cdot f(2) & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \boxed{0} & 1 & 1 & 1 \\
\beta \cdot f(1) & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 1 & 1 & 1
\end{array}$$

$m = 2$

When $f(2) \neq f(1)$, the teleported qubit has an error.

For the optimal choice we can boost the average error rate from

$$\frac{1}{n+1} \longrightarrow \frac{1}{(n+1)^2}$$

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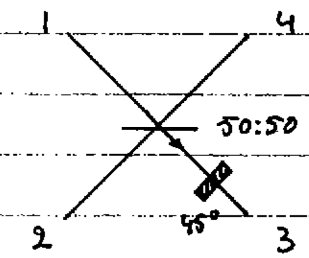
There is another way of doing teleportation with linear optics:

$$|H,1\rangle \to \frac{1}{\sqrt{2}}\left(|V,3\rangle + |H,4\rangle\right)$$

$$|V,1\rangle \to \frac{1}{\sqrt{2}}\left(|V,4\rangle - |H,3\rangle\right)$$

$$|H,2\rangle \to \frac{1}{\sqrt{2}}\left(|H,4\rangle - |V,3\rangle\right)$$

$$|V,2\rangle \to \frac{1}{\sqrt{2}}\left(|V,4\rangle + |H,3\rangle\right)$$



Formally, this is a Bell-state transformation, and it is <u>deterministic</u> (Popescu, quant-ph/9501020).

However, there is <u>only one photon</u> present!

<u>Yoran and Reznik</u> used this to improve the KLM scheme. Consider the so-called "chain state":

$$\left(\alpha|H\rangle_{p_1} + \beta|V\rangle_{p_1}\right)\left(|1\rangle_{p_1}|H\rangle_{p_2} + |2\rangle_{p_1}|V\rangle_{p_2}\right)\left(|3\rangle_{p_2}|H\rangle_{p_3} + |4\rangle_{p_2}|V\rangle_{p_3}\right)$$

$$\times\ldots\times\left(|2n-1\rangle_{p_n}|H\rangle_{p_{n+1}} + |2n\rangle_{p_n}|V\rangle_{p_{n+1}}\right)|2n+1\rangle_{p_{n+1}}$$

This is a highly entangled state.

If we now do a measurement on p_1, the qubit state $|\chi\rangle = \alpha|H\rangle + \beta|V\rangle$ is transferred to p_2.

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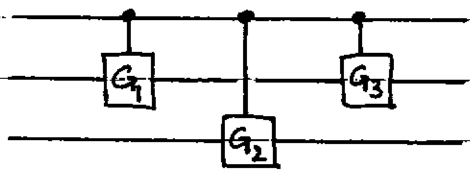
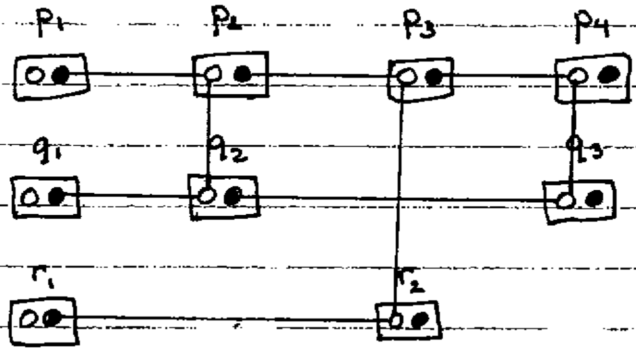
For quantum computation we can turn the following example circuit into an optical circuit:



In terms of chain states, this becomes:



The open circles represent polarization, and the dot represents the path. Applying gate G_1 then means pre-applying G_1 to photons p_2 and q_2 on the polarization degree of freedom:

$$|\chi\rangle_{p_1}\left(|1\rangle_{p_1}\boxed{|H\rangle_{p_2}} + |2\rangle_{p_1}\boxed{|V\rangle_{p_2}}\right)\ldots$$

$$|\phi\rangle_{q_1}\left(|1'\rangle_{q_1}\boxed{|H\rangle_{q_2}} + |2'\rangle_{q_1}\boxed{|V\rangle_{q_2}}\right)\ldots$$

After theleportation of the initial qubits $|\chi\rangle$ and $|\phi\rangle$, the polarization of p_2 and q_2 is in $G_1|\chi,\phi\rangle$.

This scheme is inspired by Rausschendorf and Briegel.

Q: How do we make these chain states?

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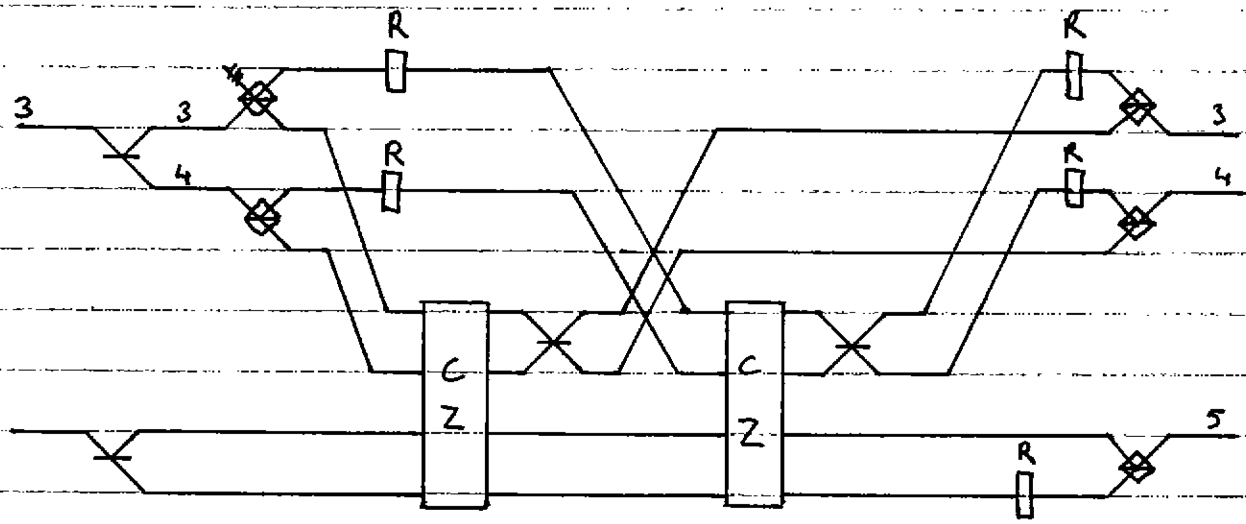
Suppose that b is the last photon in an existing chain:

$$(|1\rangle_a |H\rangle_b + |2\rangle_a |V\rangle_b) |3\rangle_b$$

The photon that is to be added is in the state:

$$(|5\rangle_c + |6\rangle_c) |V\rangle_c$$

The circuit for adding photon c to the chain is:



The state is then

$$(|1\rangle_a |H\rangle_b + |2\rangle_a |V\rangle_b) (|3\rangle_b |H\rangle_c + |4\rangle_b |V\rangle_c) |5\rangle_c$$

So we need two CZ operations. These are probabilistic. Also, we can enumerate the photons because they are never in the same mode.

For greater efficiency, we create the gates together with the chains.

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If a CZ fails, it might destroy a previous link. We therefore need:

- CZ-gates with success probability $> \frac{1}{2}$
- buffer chain links between gates

We can use the KLM scheme to make high probability CZ-gates. However they don't have to be near-deterministic anymore.

It turns out that every gate in the computation needs three CZ gates with success probability $n^2/(n+1)^2$.

$n=3$ is the smallest value for which forward movement in the chain is faster than backward movement.

summary

KLM: 200 successful KLM gates per logic gate with 5% error
YR : 5 " " " " perfect logic gate.

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