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## States

$INV_t$ - amount of inventory held by the market maker

$IMB_t$ - order imbalance defined as the share difference buy and sell orders within a period of time

$QLT_t$ - market quality measure, which could be the size of the bid-ask spread or other metrics

## Actions

$\Delta Bid_t$ - change the bid price

$\Delta Ask_t$ - change the ask price

## Rewards

$\Delta PRO_t$ - change in profit

$\Delta INV_t$ - change in inventory level

$QLT_t$ - market quality measure

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Reward at each time step:

$$r_t = w_{pro}\,\Delta PRO_t + w_{inv}\,\Delta INV_t + w_{qlt}\,QLT_t$$

where $w_{pro}$, $w_{inv}$, and $w_{qlt}$ are weights that control the tradeoff between profit, inventory risk, and market quality.

+ The <u>basic model</u> simulates the behavior of traders given the quotes set by the market maker. The basic model includes a single price quote for both buy and sell orders; there is no spread in the basic model.

+ Two types of traders:

  - <u>Informed traders</u> - have superior information and only trade when they have an advantage. They buy when the market-maker's prices are too low and sell when the market-maker's prices are too high.

  - <u>Uninformed traders</u> - possess only public information. They trade for liquidity reasons regardless of the price set by the market-maker.

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- There are two prices at any given time period:

  - $p^m$ - market-maker price visible to all

  - $p^*$ - true price visible only to the informed traders.

At each time step, a trader can choose to buy one share, sell one share, or choose not to trade.

- At each time step, one of the following events occur:

  - The true price moves up one unit
  - The true price moves down one unit
  - An uninformed trader buys
  - An uninformed trader sells
  - An informed trader chooses to buy, sell, or take no action

$\lambda_p$ = probability that true price moves up / down
$\lambda_u$ = probability that uninformed trader buys / sells
$\lambda_i$ = probability that informed trader takes action

There is a guaranteed arrival of an event, so all probabilites add up to one:

$$2\lambda_p + 2\lambda_u + \lambda_i = 1$$

- Order imbalance is the excess demand since the last change of quote by the market-maker.

$IMB = x - y$, where $x$ are the buy orders and $y$ are the sell orders of one share

Strategy 1 - raise $p^m$ by one unit when $IMB = +1$ and lower $p^m$ when $IMB = -1$.

Strategy 2 - adjust $p^m$ when $|IMB| = 2$

Strategy 3 - adjust $p^m$ when $|IMB| = 3$

- $\Delta p = p^m - p^* \Rightarrow$ deviation of market price from the true price.

When $\Delta p = 0$, $p^*$ may jump to $p^* + 1$ or $p^* - 1$ with probability $\lambda_p$

When $\Delta p = 0$, $p^m$ may be adjusted to $p^m + 1$ or $p^m - 1$ with probability $\lambda_u$

When $p^m \neq p^*$ or $\Delta p \neq 0$, $p^m$ moves towards $p^*$ at a faster rate than $p^m$ moves away from $p^*$. Specifically:

$p^m$ moves towards $p^*$ at a rate of $\lambda_u + \lambda_i$
$p^m$ moves away $p^*$ at a rate of $\lambda_u$

$q_k$ is the probability that $\Delta p = k$

By symmetry $q_k = q_{-k}$ for $k = 1, 2, \ldots$

Consider transition from $\Delta p = k$ to $\Delta p = k+1$

$$q_{k+1} \cdot (\lambda_p + \lambda_u + \lambda_i) = q_k \cdot (\lambda_p + \lambda_u)$$

$$q_{k+1} = q_k \cdot \left( \frac{\lambda_p + \lambda_u}{\lambda_p + \lambda_u + \lambda_i} \right)$$

$$q_{k+1} = q_0 \cdot \left( \frac{\lambda_p + \lambda_u}{\lambda_p + \lambda_u + \lambda_i} \right)^{|k|} \quad \forall \ k \neq 0$$

- All probabilities $q_k$ sum up to 1 for all $k$

$$\sum_{k=-\infty}^{\infty} q_k = 1$$

$$q_0 + 2\sum_{k=1}^{\infty} q_k = 1$$

$$q_0 = \frac{\lambda_i}{2\lambda_p + 2\lambda_u + \lambda_i}$$

- Expected profit:

$$EP = \sum_{k=-\infty}^{\infty} -q_k \lambda_i |k|$$

$$= -2\sum_{k=1}^{\infty} q_k \lambda_i |k|$$

Note that $q_k \lambda_i |k| = 0$ when $k = 0$

$$= -2\sum_{k=1}^{\infty} q_k \lambda_i k$$

$$= -2\lambda_i \sum_{k=1}^{\infty} k q_k$$

$$= -2\lambda_i \sum_{k=1}^{\infty} k \cdot q_0 \left(\frac{\lambda_p + \lambda_u}{\lambda_p + \lambda_u + \lambda_i}\right)^k$$

(cont.)

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$$EP = -2 g_0 \lambda_i \sum_{k=1}^{\infty} k \left( \frac{\lambda p + \lambda u}{\lambda p + \lambda u + \lambda_i} \right)^k$$

Since $\left| \frac{\lambda p + \lambda u}{\lambda p + \lambda u + \lambda_i} \right| < 1$, the geometric series converges

Geometric power series:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

$$\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$$

Let $x = \frac{\lambda p + \lambda u}{\lambda p + \lambda u + \lambda_i}$

$$EP = -2 g_0 \lambda_i \sum_{k=1}^{\infty} k x^k$$

Let $k = n-1$

$\sum_{k=1}^{\infty} k x^k = \sum_{k=0}^{\infty} k x^k$ because $k x^k = 0$ when $k=0$

$$= \sum_{n-1=0}^{\infty} (n-1) x^{(n-1)} = \sum_{n=1}^{\infty} (n-1) x^{(n-1)}$$

(cont.)

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$$\sum_{n=1}^{\infty} (n-1)\, x^{(n-1)}$$

$$= \sum_{n=1}^{\infty} n x^{(n-1)} - x^{(n-1)}$$

$$= \sum_{n=1}^{\infty} n x^{(n-1)} - \sum_{n=1}^{\infty} x^{(n-1)}$$

$$= \sum_{n=1}^{\infty} n x^{(n-1)} - \sum_{n-1=0}^{\infty} x^{(n-1)}$$

$$= \sum_{n=1}^{\infty} n x^{(n-1)} - \sum_{k=0}^{\infty} x^{k}$$

$$= \frac{1}{(1-x)^2} - \frac{1}{1-x}$$

$$= \frac{1}{(1-x)^2} - \frac{(1-x)}{(1-x)^2}$$

$$= \frac{1-(1-x)}{(1-x)^2} = \frac{1-1+x}{(1-x)^2}$$

$$= \frac{x}{(1-x)^2} = x \cdot \frac{1}{(1-x)^2}$$

(cont.)

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Replace $x$ with $\dfrac{\lambda p + \lambda u}{\lambda p + \lambda u + \lambda i}$

$$= \frac{\lambda p + \lambda u}{\lambda p + \lambda u + \lambda i} \cdot \left[ \frac{1}{1 - \left( \frac{\lambda p + \lambda u}{\lambda p + \lambda u + \lambda i} \right)^2} \right]$$

$$= \frac{\lambda p + \lambda u}{\lambda p + \lambda u + \lambda i} \cdot \left[ \frac{1}{\left( \frac{\lambda p + \lambda u + \lambda i - \lambda p - \lambda u}{\lambda p + \lambda u + \lambda i} \right)^2} \right]$$

$$= \frac{\lambda p + \lambda u}{\lambda p + \lambda u + \lambda i} \cdot \frac{(\lambda p + \lambda u + \lambda i)^2}{\lambda i^2}$$

$$= \frac{(\lambda p + \lambda u)(\lambda p + \lambda u + \lambda i)}{\lambda i^2}$$

$$\therefore \sum_{k=0}^{\infty} k \left( \frac{\lambda p + \lambda u}{\lambda p + \lambda u + \lambda i} \right)^k$$

$$= \frac{1}{\lambda i^2} (\lambda p + \lambda u)(\lambda p + \lambda u + \lambda i)$$

$$EP = -2 g_0 \lambda i \cdot \frac{1}{\lambda i^2} (\lambda p + \lambda u)(\lambda p + \lambda u + \lambda i)$$

$$g_0 = \frac{\lambda i}{2 \lambda p + 2 \lambda u + \lambda i}$$

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$$EP = -2\left(\frac{\lambda_i}{2\lambda_p + 2\lambda_u + \lambda_i}\right)\lambda_i\left(\frac{1}{\lambda_i^2}\right)(\lambda_p + \lambda_u)(\lambda_p + \lambda_u + \lambda_i)$$

$$= \frac{-2(\lambda_p + \lambda_u)(\lambda_p + \lambda_u + \lambda_i)}{(2\lambda_p + 2\lambda_u + \lambda_i)}$$

Let $\lambda_u = \alpha_u \lambda_i$

$\lambda_p = \alpha_p \lambda_i$

$$2\lambda_p + 2\lambda_u + \lambda_i = 1$$

$$2\alpha_p \lambda_i + 2\alpha_u \lambda_i + \lambda_i = 1$$

$$\lambda_i(2\alpha_p + 2\alpha_u + 1) = 1$$

$$\lambda_i = \frac{1}{2\alpha_p + 2\alpha_u + 1}$$

$$EP = \frac{-2(\alpha_p\lambda_i + \alpha_u\lambda_i)(\alpha_p\lambda_i + \alpha_u\lambda_i + \lambda_i)}{2\alpha_p\lambda_i + 2\alpha_u\lambda_i + \lambda_i}$$

$$= \frac{-2\lambda_i^2(\alpha_p + \alpha_u)(\alpha_p + \alpha_u + 1)}{\lambda_i(2\alpha_p + 2\alpha_u + 1)}$$

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$$= \frac{-2\lambda_i(\alpha_p + \alpha_u)(\alpha_p + \alpha_u + 1)}{(2\alpha_p + 2\alpha_u + 1)}$$

Since $\lambda_i = \dfrac{1}{(2\alpha_p + 2\alpha_u + 1)}$

$$EP = \frac{-2(\alpha_p + \alpha_u)(\alpha_p + \alpha_u + 1)}{(2\alpha_p + 2\alpha_u + 1)^2}$$

The expected profit is negative because the informed traders take advantage of the market-maker because they have superior knowledge of the true price.

A higher frequency of price jumps or a higher frequency of uninformed traders increases the expected profit, although it still remains negative.

The market-maker can compensate for losses by charging a fee. The fee appears in the form of a bid-ask spread.

Computing the expected profit for alternate strategies can be mathematically difficult to solve. Use Monte Carlo simulations instead.