Homology Inference

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Homology is a classical concept in algebraic topology, providing a powerful tool to formalize and handle the notion of the topological features of a topological space or of a simplicial complex in an algebraic way. For any dimension k, the k-dimensional "holes" are represented by a vector space (or more generally R-module) H_k, whose dimension is intuitively the number of such independent features. For example, the zero-dimensional homology group H_0 represents the connected components of the complex, the one-dimensional homology group H_1 represents the one-dimensional loops, the two-dimensional homology group H_2 represents the two-dimensional cavities, and so on.

We first start with the definition of group, subgroup, and quotient group:

Definition. An abelian group $(G, +)$ is a set G and a binary operation $+ : G \times G \to G$ satisfying

(a) for all $a, b, c \in G$, $(a + b) + c = a + (b + c)$

(b) there exists $0 \in G$ such that $a + 0 = 0 + a = a$ for all $a \in G$

(c) for all $a \in G$, there exists $-a \in G$ such that $a + (-a) = -a + a = 0$.

(d) for all $a, b \in G$, $a + b = b + a$.

Definition. For an abelian group $(G, +)$ and $H \subset G$, H is a subgroup of G if $(H, +)$ is itself a group, and denote as $H \leq G$.

Definition. Let $(G, +)$ be an abelian group and $H \leq G$. For each $a \in G$, we define its coset as $a + H \coloneqq \{a + h : h \in H\} \subset G$. Then the quotient group is a set defined as

$$G/H \coloneqq \{a + H : a \in G\}.$$

Write $[a] = a + H$ for convenience. We define the group structure on G/H as $[a] + [b] \coloneqq [a + b]$.

Note that $[a] = [b] \in G/H$ if and only if $a - b \in H$. So G/H is defined as like "all the members in H are announced as zero".

Example. $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$ be a group of integer with binary operator $+$, and $2\mathbb{Z} = \{\ldots, -2, 0, 2, \ldots\}$ be a set of even integers. Then $2\mathbb{Z}$ is a subgroup of $\mathbb{Z}$.The quotient group $\mathbb{Z}_2 \coloneqq \mathbb{Z}/2\mathbb{Z}$ can be characterized as

$$\mathbb{Z}_2 = \{[0], [1]\},$$

where $[0] + [0] = [1] + [1] = [0]$ and $[0] + [1] = [1] + [0] = [1]$.

Then recall the simplicial complex:

Given a set V, an *(abstract) simplicial complex* is a set K of subsets of V such that $\alpha \in K$ implies $\text{card}\alpha < \infty$, and $\alpha \in K$ and $\beta \subset \alpha$ implies $\beta \in K$. Each set $\alpha \in K$ is called its *simplex*. The *dimension* of a simplex α is $\dim \alpha = \text{card}\alpha - 1$, and the dimension of the simplicial complex is the maximum dimension of any of its simplices. Note that a simplicial complex of dimension 1 is a graph. See Figure .

Definition. Let K be a simplicial complex, $k \geq 0$ be a nonnegative integer, and G be an abelian group. The space of k-chains on K, $C_k(K; G)$, is the set whose elements are a finite formal sum of k-simplices of K with coefficients from G, i.e.,

$$C_k(K; G) = \left\{ \sum_i n_i \sigma_i : n_i \in G, \sigma_i \in K_k \right\},$$

where $K_k \subset K$ is the set of k-simplices of K. We write $C_k(K)$ if the coefficient group G is understood from the context.

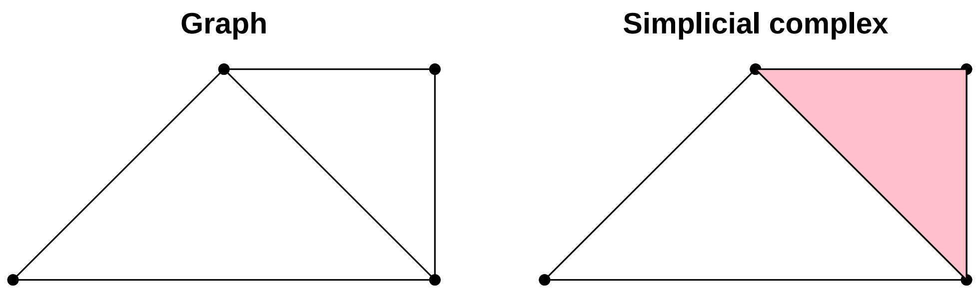


Figure 1: Graph (left) and simplicial complex (right).

For an integer $k \leq -1$, we define $C_k(K) = 0$ for convenience.

Remark. Typical examples of G are $G = \mathbb{Z}$ and $G = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. For $G = \mathbb{Z}_2$, $C_k(K; \mathbb{Z}_2)$ becomes a vector space.

Remark. $C_k(K; G)$ has an abelian group structure as for $\sum_i n_i\sigma_i, \sum_i n_i'\sigma_i \in C_k(K; G)$,

$$\left(\sum_i n_i\sigma_i\right) + \left(\sum_i n_i'\sigma_i\right) := \sum_i (n_i + n_i')\sigma_i.$$

When G is a field, $C_k(K; G)$ has a natural vector space structure as for $\sum_i n_i\sigma_i \in C_k(K; G)$ and $\lambda \in G$,

$$\lambda \cdot \left(\sum_i n_i\sigma_i\right) = \sum_i (\lambda \cdot n_i)\sigma_i.$$

To relate chain groups of different dimensions, we define the boundary map as sending each -simplex to the sum of its $(k-1)$-dimensional faces. We write $\sigma = [v_0, \ldots, v_k]$ for an ordered simplex, i.e., $[v_0, v_1] = -[v_1, v_0]$.

Definition. A boundary map $\partial_k : C_k(K) \to C_{k-1}(K)$ is defined for each simplex as (see Figure)

$$\partial_k[v_0, \ldots, v_k] = \sum_{j=0}^{k} (-1)^j [v_0, \ldots, \hat{v}_j, \ldots, v_k],$$

where $[v_0, \ldots, \hat{v}_j, \ldots, v_k] = [v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_k] \in K_{k-1}$, i.e., $\hat{v}_j$ means that v_j is omitted. The definition is extended to entire k-chain as

$$\partial_k \left(\sum_i n_i\sigma_i\right) = \sum_i n_i \partial_k \sigma_i.$$

Remark. ∂_k satisfies that for $c, c' \in C_k(K)$, $\partial_k(c + c') = \partial_k c + \partial_k c'$, so $\partial_k : C_k(K) \to C_{k-1}(K)$ is a homomorphism.

Lemma ([2, Lemma 2.1])**.** $\partial_{k-1} \circ \partial_k = 0$.

Definition. Cycles and boundaries

(a) A k-cycle group $Z_k = Z_k(K)$ is the k-cycle whose boundary is 0, i.e.,

$$Z_k(K) = \ker \partial_k = \{c \in C_k(K) : \partial_k c = 0\}.$$

(b) A k-boundary group $B_k = B_k(K)$ is the k-cycle that is a boundary of $(k+1)$-chain,

$$B_k(K) = \mathrm{im}\partial_{k+1} = \{\partial_{k+1} d \in C_k(K) : d \in C_{k+1}(K)\}.$$

Then the above Lemma implies that $B_k(K)$, $Z_k(K)$, $C_k(K)$ are interleaved as subgroups (see Figure):

$$B_k(K) \subset Z_k(K) \subset C_k(K).$$

Definition. The k-th homology group is the k-th cycle group modulo the k-th boundary group,

$$H_k = H_k(K) := Z_k(K)/B_k(K).$$

The k-th Betti number is the rank of this group, $\beta_k = \mathrm{rank} H_k$.

Example. Suppose K is given as the right of Figure , and use $G = \mathbb{Z}$. Then for $k = 1$, its cycle group, boundary group, homology group, and betti number is computed as in Figure .

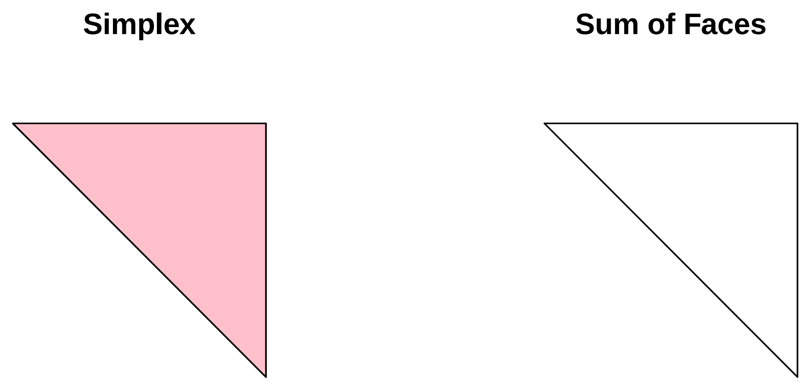


Figure 2: Boundary map.

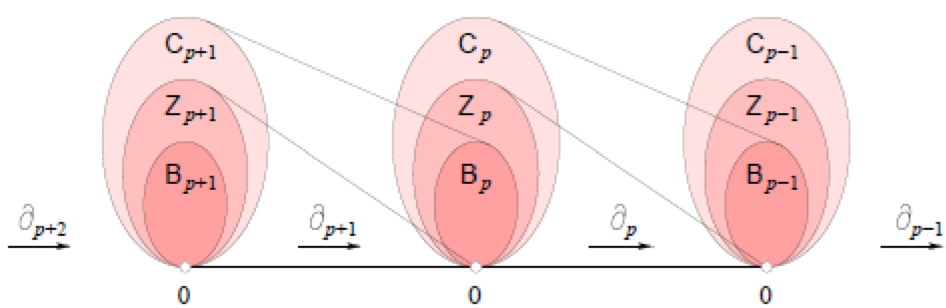
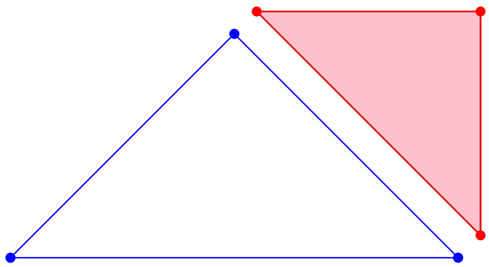


Figure 3: [1, Figure IV.1] Interleaving relations between cycle groups and boundary groups via boundary map.



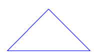
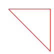
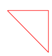
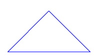
- $Z_1(K) = \ker \partial_1 = \mathbb{Z}^2 =<$  ,  $>$
- $B_1(K) = \mathrm{im}\partial_2 = \mathbb{Z} =<$  $>$
- $H_1(K) = Z_1(K)/B_1(K) = \mathbb{Z} =<$  $>$, $\beta_1(K) = 1$

Figure 4: Homology example for Figure .

References

[1] Herbert Edelsbrunner and John L. Harer. *Computational topology.* American Mathematical Society, Providence, RI, 2010. An introduction.

[2] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.