

# Review on Topology

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[4] 우리의 철학I. 같은 것은 같도다. (Isomorphism 의 철학)

우리의 철학II. 같은 것은 정말 똑같다. (Identification 의 철학)

**Definition.** [3, Section 3.1] 함수  $f : X \rightarrow Y$ 와  $x_0 \in X$ 가 주어져 있을 때, 임의의  $\epsilon > 0$  에 대하여 다음의 성질

$$x \in X, \|x - x_0\| < \delta \implies \|f(x) - y_0\| < \epsilon$$

이 성립하는  $\delta > 0$ 가 존재하면, 함수  $f$ 가 점  $x_0$ 에서 연속이라 한다.

만일 집합  $A \subset X$ 의 모든 점에서  $f$ 가 연속이면  $A$  위에서 연속이라 하고, 정의역 위에서 연속인 함수를 연속함수라고 한다.

## Topological Spaces, Continuous Functions, and Homeomorphisms

**Definition** ([2, Section 12]). A topology on a set  $X$  is a collection  $\mathcal{T}$  of subsets of  $X$  having the following properties:

1.  $\emptyset$  and  $X$  are in  $\mathcal{T}$ .
2. If  $\{U_\alpha\}_{\alpha \in I} \subset \mathcal{T}$ , then  $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$ .
3. If  $U_1, \dots, U_n \in \mathcal{T}$ , then  $\bigcap_{i=1}^n U_i \in \mathcal{T}$ .

**Example.** Let  $X$  be a three-element set,  $X = \{a, b, c\}$ . All the possible topologies are schematically represented in Figure 1. For example, the diagram in the upper right corner indicates the topology  $\mathcal{T} = \{X, \emptyset, \{a, b\}, \{b\}, \{b, c\}\}$ . All the topologies can be obtained by permuting  $a, b, c$ .

**Definition** ([2, Section 13]). If  $X$  is a set, a basis for a topology on  $X$  is a collection  $\mathcal{B}$  of subsets of  $X$  (called basis elements) such that  $\emptyset$  and  $X$  are in  $\mathcal{T}$ .

1. For each  $x \in X$ , there is at least one  $B \in \mathcal{B}$  containing  $x$ .
  - (a) If  $B_1, B_2 \in \mathcal{B}$  with  $x \in B_1 \cap B_2$ , then there is  $B_3 \in \mathcal{B}$  containing  $x$  such that  $B_3 \subset B_1 \cap B_2$ .

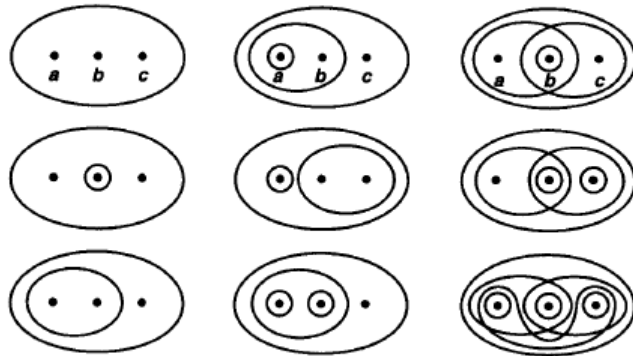


Figure 1: [2, Figure 12.1] Example of topologies of a three-element set.

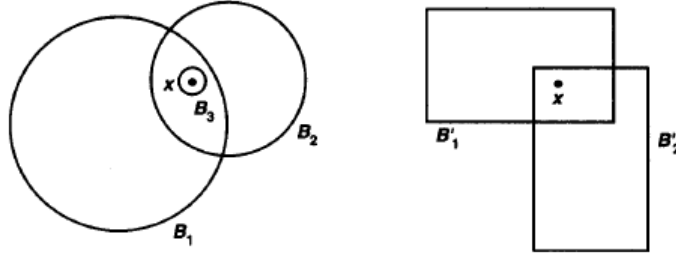


Figure 2: [2, Figure 13.1, 13.2] Example of bases of circular regions or rectangular regions.

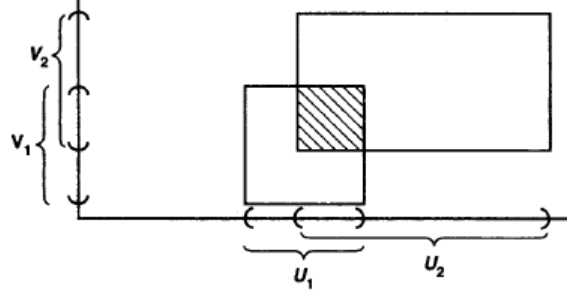


Figure 3: [2, Figure 15.1] Product topology.

**Example.** Let  $\mathcal{B}$  be the collection of all circular regions (interiors of circles) in the plane  $\mathbb{R}^2$ , as in Figure 2 left, then  $\mathcal{B}$  is a basis. Let  $\mathcal{B}'$  be the collection of all rectangular regions (interiors of rectangles) in the plane  $\mathbb{R}^2$ , as in Figure 2 right, then  $\mathcal{B}'$  is also a basis. And in fact, two bases  $\mathcal{B}$  and  $\mathcal{B}'$  generate the same topology for  $\mathbb{R}^2$ .

**Definition** ([2, Section 15]). Let  $X$  and  $Y$  be topological spaces. The product topology on  $X \times Y$  is the topology having the basis as (see Figure 3):

$$\mathcal{B} = \{U \times V \subset X \times Y : U \text{ is open in } X, V \text{ is open in } Y\}.$$

*Remark.* This definition of the product topology can be naturally extended to a finite product space  $X_1 \times \cdots \times X_n$ .

**Definition** ([2, Section 16]). Let  $X$  be a topological space with topology  $\mathcal{T}$ . If  $Y$  is a subset of  $X$ , the collection

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$$

is a topology on  $Y$ , called the subspace topology.

**Definition** ([2, Section 17]). A subset  $A$  of a topological space  $X$  is said to be closed if the set  $X \setminus A$  is open.

**Definition** ([2, Section 17]). The closure of  $A$ , denoted by  $\bar{A}$ , is the intersection of all closed sets containing  $A$ .

**Definition** ([2, Section 17]). We say  $U$  is a neighborhood (neighbor) of  $x$  if  $U$  is an open set containing  $x$ .

**Definition** ([2, Section 17]). If  $A$  is a subset of a topological space  $X$ , We say  $x$  is a limit point of  $A$  if every neighborhood of  $x$  intersects  $A$  in some point other than  $x$  itself.

**Theorem.** Let  $A$  be a subset of a topological space  $X$ , and  $A'$  be the set of all limit points of  $A$ , then

$$\bar{A} = A \cup A'.$$

**Definition** ([2, Section 17]). A topological space  $X$  is called a Hausdorff space if for each pair  $x_1 \neq x_2 \in X$ , there exists neighborhoods  $U_1, U_2$  of  $x_1, x_2$ , respectively, that  $U_1 \cap U_2 = \emptyset$ .

**Definition** ([2, Section 18]). A function  $f : X \rightarrow Y$  is continuous if for each open set  $V$  of  $Y$ ,  $f^{-1}(V)$  is an open subset of  $X$ .

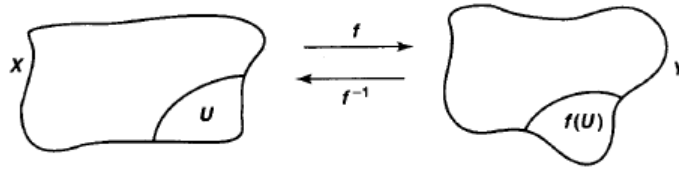


Figure 4: [2, Figure 18.1] Homeomorphism.

*Remark.* It suffices to show that the inverse image of every basis element is open.

**Theorem** ([2, Theorem 18.1]). *Let  $X, Y$  be topological spaces; let  $f : X \rightarrow Y$ . Then the followings are equivalent:*

1.  $f$  is continuous.
2. For every closed set  $B$  of  $Y$ ,  $f^{-1}(B)$  is closed in  $X$ .
3. For each  $x \in X$  and each neighborhood  $V$  of  $f(x)$ , there is an neighborhood  $U$  of  $x$  such that  $f(U) \subset V$ .

**Definition** ([2, Section 18]). Let  $f : X \rightarrow Y$  be a bijection. If both the function  $f$  and the inverse function  $f^{-1} : Y \rightarrow X$  are continuous, then  $f$  is called a homeomorphism.

$X$  and  $Y$  are homeomorphic if such a homeomorphism  $f$  exists, and denoted as  $X \cong Y$ .

*Remark.* Another way to define a homeomorphism is to say that  $f : X \rightarrow Y$  is a bijection such that  $f(U)$  is open if and only if  $U$  is open (see Figure 4).

*Remark.* A homeomorphism gives us a bijective correspondence not only between  $X$  and  $Y$  but also between the collections of open sets of  $X$  and  $Y$ . As a result, any property of  $X$  that is entirely expressed in terms of the topology of  $X$  yields, via  $f$ , the property of  $Y$ . Such a property of  $X$  is called a topological property of  $X$ .

**Definition** ([2, Section 18]). Suppose  $f : X \rightarrow Y$  is an injective continuous, and let  $Z := f(X) \subset Y$  be the image of  $f$  equipped with the subspace topology. If the function  $f' : X \rightarrow Z$  obtained by restricting the range of  $f$  is a homeomorphism of  $X$  with  $Z$ , we say that  $f : X \rightarrow Y$  is a topological embedding (imbedding) of  $X$  in  $Y$ .

**Definition** ([2, Section 20]). A metric on a set  $X$  is a function

$$d : X \times X \rightarrow \mathbb{R}$$

having the following properties:

1.  $d(x, y) \geq 0$  for all  $x, y \in X$ ; equality holds if and only if  $x = y$ .
2.  $d(x, y) = d(y, x)$  for all  $x, y \in X$ .
3. (Triangle inequality)  $d(x, y) + d(y, z) \geq d(x, z)$  for all  $x, y, z \in X$ .

Given a metric  $d$  on  $X$ , the number  $d(x, y)$  is often called the distance between  $x$  and  $y$ . Given  $\epsilon > 0$ , consider the set

$$B_d(x, \epsilon) = \{y : d(x, y) < \epsilon\}$$

of all points  $y$  whose distance from  $x$  is less than  $\epsilon$ . It is called the  $\epsilon$ -ball centered at  $x$ . Sometimes we omit  $d$  and write  $B(x, \epsilon)$ .

**Definition** ([2, Section 20]). If  $d$  is a metric on the set  $X$ , then the collection of all  $\epsilon$ -balls  $B_d(x, \epsilon)$ , for  $x \in X$  and  $\epsilon > 0$ , is a basis for a topology on  $X$ , called the metric topology induced by  $d$ .

A metric space  $X$  is a topological space  $X$  together with a specific metric  $d$  that gives the topology of  $X$ .

**Example.** Given  $x = (x_1, \dots, x_n)$  in  $\mathbb{R}^n$  and for  $1 \leq p \leq \infty$ , we define the  $p$ -norm of  $x$  by

$$\|x\|_p := (x_1^p + \dots + x_n^p)^{1/p}$$

for  $p \in [1, \infty)$ , and  $\|x\|_\infty := \max_{1 \leq i \leq n} |x_i|$ . And then the induced distance  $d_p$  on  $\mathbb{R}^n$  is defined as

$$d_p(x, y) = \|x - y\|_p.$$

All the metrics  $d_p$  induce the same topology on  $\mathbb{R}^n$  for  $1 \leq p \leq \infty$ , and this is the usual topology on  $\mathbb{R}^n$ . This also coincides with the product topology on  $\mathbb{R}^n$  as well.

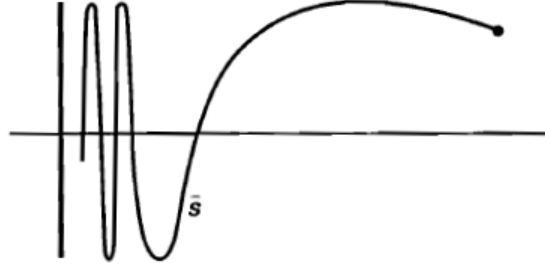


Figure 5: [2, Figure 24.5] Homeomorphism.

**Theorem** ([2, Theorem 21.1]). *Let  $(X, d_X)$ ,  $(Y, d_Y)$  be metric spaces and let  $f : X \rightarrow Y$ . Then continuity of  $f$  is equivalent to the requirement that given  $x \in X$  and  $\epsilon > 0$ , there exists  $\delta > 0$  such that*

$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon.$$

## Connectedness and Compactness

**Definition** ([2, Section 23]). Let  $X$  be a topological space. A separation of  $X$  is a pair  $U, V$  of disjoint nonempty subsets of  $X$  whose union is  $X$ . The space  $X$  is said to be connected if there does not exist a separation of  $X$ .

**Theorem** ([2, Theorem 23.4]). *Let  $A$  be a connected subspace of  $X$ . If  $A \subset B \subset \bar{A}$ , then  $B$  is also connected.*

**Theorem** ([2, Theorem 23.5]). *The image of a connected space under a continuous map is connected.*

**Corollary** ([2, Corollary 24.2]). *The real line  $\mathbb{R}$  is connected and so are intervals and rays in  $\mathbb{R}$ .*

**Definition** ([2, Section 24]). Given points  $x$  and  $y$  of the space  $X$ , a path in  $X$  from  $x$  to  $y$  is a continuous map  $f : [a, b] \rightarrow X$  of some closed interval in the real line into  $X$ , such that  $f(a) = x$  and  $f(b) = y$ . A space  $X$  is said to be path connected if every pair of points of  $X$  can be joined by a path in  $X$ .

**Example** ([2, Section 24] Topologist's sine curve). Let  $S$  denote the following subset of the plane

$$S = \{(x, \sin(1/x)) : 0 < x \leq 1\}.$$

The set  $\bar{S} = S \cup \{0\} \times [-1, 1]$  is a classical example in the topology called the topologist's sine curve (see Figure 5). The set  $\bar{S}$  is connected but not path connected.

Since  $S$  is a continuous image of  $(0, 1]$ ,  $S$  is connected, and then  $\bar{S}$  is connected as well. Now we show  $\bar{S}$  is not path connected. Suppose there is a path  $\gamma : [a, c] \rightarrow \bar{S}$  with  $\gamma(a) = (0, 0)$  and  $\gamma(c) = (1, \sin 1)$ . Since  $\gamma^{-1}(\{0\} \times [-1, 1])$  is closed in  $[a, c]$ , it has the largest element  $b$ . Then  $\gamma : [b, c]$  is a path that maps  $b$  into the vertical interval  $\{0\} \times [-1, 1]$  and maps  $(b, c]$  into  $S$ . Since  $\gamma$  is continuous, there exists  $\delta > 0$  such that

$$\gamma[b, b + \delta] \subset B_{d_2}(\gamma(b), 0.5).$$

However, if we write  $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ , then  $\gamma_1[b, b + \delta]$  is a connected subset of  $[0, 1]$  containing  $\gamma_1(b) = 0$  and  $\gamma_1(b + \delta) > 0$ , so

$$[0, \gamma_1(b + \delta)] \subset \gamma_1[b, b + \delta].$$

But since  $\gamma_2(t) = \sin(1/\gamma_1(t))$  if  $\gamma_1(t) > 0$ , so

$$\gamma_2[b, b + \delta] \supset \sin(1/(0, \gamma_1(b + \delta))) = [-1, 1].$$

This contradicts with  $\gamma[b, b + \delta] \subset B_{d_2}(\gamma(b), 0.5)$ , so such path  $\gamma$  cannot exist and  $\bar{S}$  is not path connected.

**Definition** ([2, Section 25]). Given  $X$ , define an equivalence relation on  $X$  by setting  $x \sim y$  if there is a connected subspace of  $X$  containing both  $x$  and  $y$ . The equivalence classes are called the components (or the “connected components”) of  $X$ .

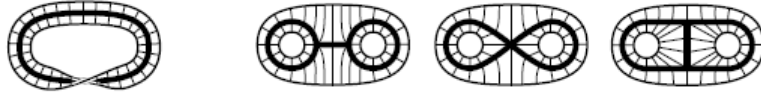


Figure 6: [1, Section 0] Example of a deformation retract.

**Theorem** ([2, Theorem 25.1]). *The components of  $X$  are connected disjoint subspaces of  $X$  whose union is  $X$ , such that each nonempty connected subspaces of  $X$  intersects only one of them.*

**Definition** ([2, Section 25]). Given  $X$ , define an equivalence relation on  $X$  by setting  $x \sim y$  if there is a path in  $X$  from  $x$  to  $y$ . The equivalence classes are called the path components of  $X$ .

**Theorem** ([2, Theorem 25.2]). *The path components of  $X$  are path connected disjoint subspaces of  $X$  whose union is  $X$ , such that each nonempty path connected subspaces of  $X$  intersects only one of them.*

**Definition** ([2, Section 26]). A collection  $\mathcal{A}$  of subsets of a space  $X$  is said to cover  $X$ , or to be a covering of  $X$ , if the union of the elements of  $\mathcal{A}$  is equal to  $X$ . It is called an open covering of  $X$  if its elements are open subsets of  $X$ .

**Definition** ([2, Section 26]). A space  $X$  is said to be compact if every open covering  $\mathcal{A}$  of  $X$  contains a finite subcollection that also covers  $X$ .

**Theorem** ([2, Theorem 26.2]). *Every closed subspace of a compact space is compact.*

**Theorem** ([2, Theorem 26.3]). *Every compact subspace of a Hausdorff space is closed.*

**Theorem** ([2, Theorem 26.5]). *The image of a compact space under a continuous map is compact.*

**Theorem** ([2, Theorem 27]). *A subspace  $A$  of  $\mathbb{R}^n$  is compact if and only if it is closed and is bounded in the metric induced by  $p$ -norm  $\|\cdot\|_p$ .*

**Theorem** ([2, Theorem 27] Extreme value theorem). *Let  $f : X \rightarrow \mathbb{R}$  be continuous. If  $X$  is compact, then there exists points  $c$  and  $d$  in  $X$  such that  $f(c) \leq f(x) \leq f(d)$  for every  $x \in X$ .*

## Homotopy

**Definition** ([1, Chapter 0]). Let  $f_0, f_1 : X \rightarrow Y$ . A *homotopy* between  $f_0$  and  $f_1$  is a continuous function  $F : X \times [0, 1] \rightarrow Y$  such that for all  $x \in X$ ,  $F(x, 0) = f_0(x)$  and  $F(x, 1) = f_1(x)$ . Two functions  $f_0, f_1$  are *homotopic* if such  $F$  exists, and we write  $f_0 \simeq f_1$ .

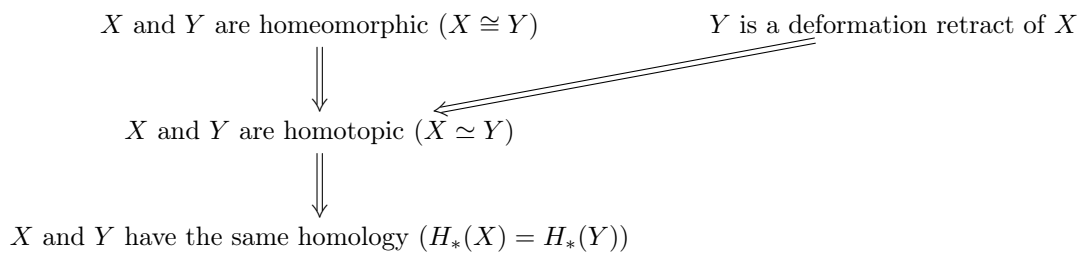
**Definition** ([1, Chapter 0]). A map  $f : X \rightarrow Y$  is called a homotopy equivalence if there is a map  $g : Y \rightarrow X$  such that  $f \circ g \simeq id_Y$  and  $g \circ f \simeq id_X$ . The space  $X$  and  $Y$  are said to be homotopy equivalent or to have the same homotopy type, and write  $X \simeq Y$ , if such homotopy equivalence  $f : X \rightarrow Y$  exists.

**Definition** ([1, Chapter 0]). Let  $A \subset X$ . Then  $A$  is a *deformation retract* of  $X$  if there exists a continuous function  $F : X \times [0, 1] \rightarrow Y$  such that for all  $x \in X$  and  $a \in A$ ,  $F(x, 0) = x$ ,  $F(x, 1) \in A$ , and  $F(a, 1) = a$ . In other words,  $A$  is a deformation retract of  $X$  if there exists  $r : X \rightarrow A$  with  $r|_A = id_A$  and  $r$  and  $id_X : X \rightarrow X$  are homotopic. We additionally say  $A$  is a *strong deformation retract* of  $X$  if  $F$  also satisfies  $F(a, t) = a$  for all  $a \in A$  and  $t \in [0, 1]$ .<sup>1</sup>

**Example.** See Figure 6. The left figure shows a (strong) deformation retract of a Möbius band onto its core circle. The three figures on the right show deformations in which a disk with two smaller open subdisks removed shrinks to three different subspaces. Also note that these three different subspaces are homotopy equivalent to each other but not homeomorphic.

*Remark.* The relationship between different equivalences of topology is as follows:

<sup>1</sup>In [1], strong deformation retract is taken as the definition of deformation retract.



## References

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