Review on Geometry

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Definition ([4, Section 9.1])**.** (매개화된) *곡선*이란 공간의 점이 시각에 따라 변하는 것을 뜻한다. 다시 말하면, 실수의 한 구간 I에서 정의된 연속함수

$$X : I \to \mathbb{R}^n$$

을 뜻한다. 이것을 좌표를 써서 표시하면

$$X(t) = (x_1(t), x_2(t), \ldots, x_n(t)), \qquad t \in I$$

와 같이 나타낼 수 있다. 이때 t를 *매개변수*(parameter)라고 부른다.

Definition ([4, Section 16.1.1])**.** (삼차원) 좌표공간 $\mathbb{R}^3$에서 (매개화된) *곡면*이란 좌표평면 $\mathbb{R}^2$의 한 영역 D에서 정의된 연속사상

$$X : D \to \mathbb{R}^3, \qquad (u, v) \mapsto X(u, v)$$

을 뜻한다.

Differentiable Manifolds

Definition. Let M be a topological space. A chart (U, φ) on M consists of an open set $U \subset M$ and a homeomorphism φ from U to an open subset of $\mathbb{R}^n$.

Definition ([3, Section 36])**.** A topological manifold of dimension n is a Hausdorff space M with a countable basis such that there is a collection of charts $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^n\}_{\alpha \in A}$ such that $\bigcup_{\alpha \in A} U_\alpha = M$.

Definition ([2, Ch.0, 2.1 Definition, modified])**.** A differentiable (resp, C^k, C^∞) manifold of dimension n is a topological manifold of dimension n such that the collection of charts $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^n\}_{\alpha \in A}$ satisfy that

1. $\bigcup_{\alpha \in A} U_\alpha = M$.
2. for any pair $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$, the mapping $\varphi_\beta \circ \varphi_\alpha^{-1}$ is differentiable (resp, C^k, C^∞) (see Figure 1).
3. The family $\{(U_\alpha, \varphi_\alpha)\}$ is maximal relative to the conditions (1) and (2).

Remark ([2, Ch.0, 2.3 Remark])**.** $A \subset M$ is open if and only if $\varphi_\alpha^{-1}(A \cap U_\alpha)$ is open in $\mathbb{R}^n$ for all $\alpha \in A$. Sometimes, a differentiable manifold is defined without a topological manifold (i.e., M is just a set), and then the topology is defined in this way.

Definition. A topological manifold with boundary of dimension n is a Hausdorff space M with a countable basis such that there is a collection of maps $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^n_+\}_{\alpha \in A}$ where φ_α is a homeomorphism onto its image such that $\bigcup_{\alpha \in A} U_\alpha = M$, where $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$ is a Euclidean half space.

Definition ([2, Ch.0, 2.5 Definition])**.** Let M and N be differentiable manifolds of dimensions m and n. A mapping $f : M \to N$ is differentiable at $p \in M$ if there exist local charts (U, φ) of $p \in M$ and (V, ϕ) of $f(p)$ respectively, such that the mapping $\phi \circ f \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^m \to \mathbb{R}^n$ is differentiable (see Figure 2).

Definition ([2, Ch.0, 3.1 Definition, modified])**.** Let M and N be topological (resp, differentiable) manifolds. If $M \subset N$ and the inclusion $\imath : M \subset N$ is an embedding (imbedding), i.e., if $\imath : M \to N$ yields a homeomorphism between M and $\imath(M) \subset N$, then we say M is a submanifold of N.

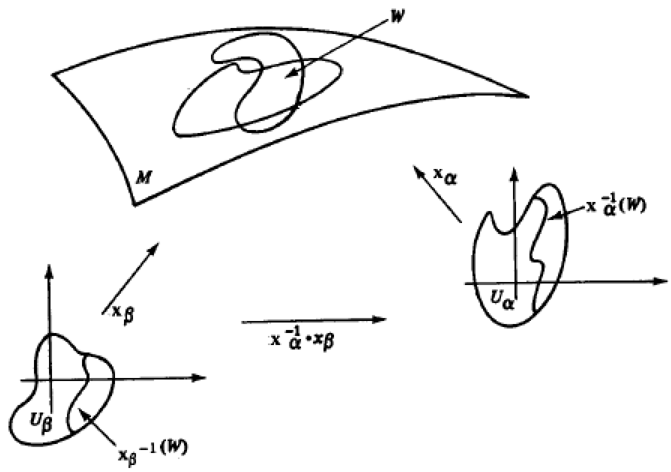

Figure 1: [2, Figure 1] Definition of a differentiable manifold.

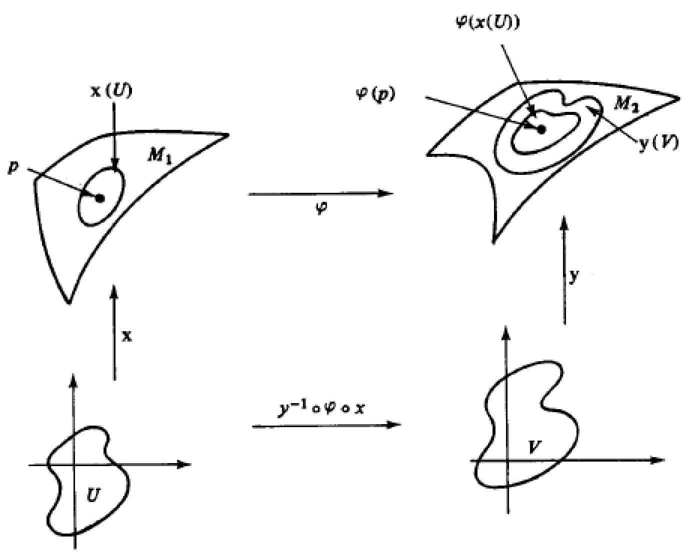

Figure 2: [2, Figure 2] Definition of a differentiable mapping.

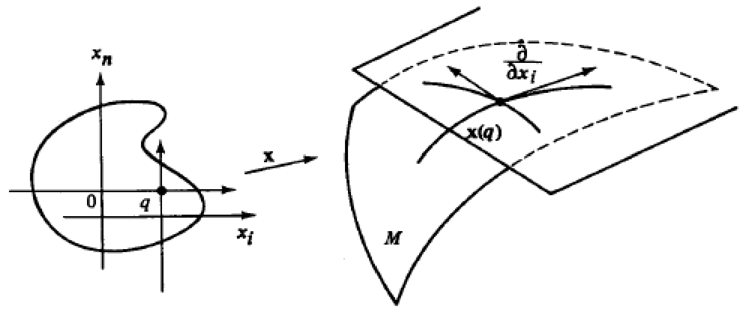

Figure 3: [2, Figure 3] Basis of a tangent space.

Remark. When a manifold M of dimension m is a submanifold of a manifold N of dimension n, then $m \leq n$.

Definition ([2, Ch.0, 2.6 Definition, modified])**.** Let M be a differentiable manifold of dimension n. A differentiable curve is a function $\alpha : (-\epsilon, \epsilon) \to M$. For $p \in M$, let

$$\mathrm{Curves}_p M := \{\alpha : (-\epsilon, \epsilon) \to M : \alpha(0) = p\}$$

be the smooth curves of M centered at p. Pick a chart (U, φ) of $p \in M$, and then $\alpha, \beta : (-\epsilon, \epsilon) \to M$ are equivalent, written as $\alpha \sim \beta$, if

$$\frac{d}{dt}(\varphi \circ \alpha)(0) = \frac{d}{dt}(\varphi \circ \beta)(0).$$

Then $\sim$ is regardless of the choice of a chart, and gives an equivalence relation. The set of tangent vectors of M at p is defined by

$$T_p M := \mathrm{Curves}_p M / \sim .$$

To define a vector space structure on $T_p M$, again pick a chart (U, φ) of $p \in M$, and define a map $d\varphi_p : T_p M \to \mathbb{R}^n$ by

$$d\varphi_p([\alpha]) := \frac{d}{dt}(\varphi \circ \alpha)(0).$$

Then $d\varphi_p$ is a bijection, and we use this to transfer the vector-space operations on $\mathbb{R}^n$ over to $T_p M$, i.e., we set

$$\begin{aligned} [\alpha] + [\beta] &:= d\varphi_p^{-1}(d\varphi_p([\alpha]) + d\varphi_p([\beta])), \\ \lambda[\alpha] &:= d\varphi_p^{-1}(\lambda d\varphi_p([\alpha])). \end{aligned}$$

Remark. $T_p M$ acts on any real valued function $f : M \to \mathbb{R}$ as follows:

$$[\alpha] \in T_p M : f \mapsto [\alpha] f := \left. \frac{d(f \circ \alpha)}{dt} \right|_{t=0} .$$

Remark. Consider the coordinate curve: when $\varphi(p) = 0$, let $\frac{\partial}{\partial x_i}$ be the eqiuivalent class of the following curve

$$x_i \mapsto \varphi^{-1}(0, \ldots, 0, x_i, 0, \ldots, 0).$$

Then $\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\}$ forms a basis in $T_p M$ (see Figure 3).

Definition ([2, Ch.0, 2.7 Proposition])**.** Let M and N be differentiable manifolds of dimensions m and n, and let $f : M \to N$ be a differentiable mapping. For every $p \in M$ and $v \in T_p M$, choose a differentiable curve $\alpha : (-\epsilon, \epsilon) \to M$ with $\alpha(0) = p$, $[\alpha] = v$. Take $\beta = f \circ \alpha$. The mapping $df_p : T_p M \to T_{f(p)} N$ given by $df_p(v) = [\beta]$ is a linear mapping that does not depend on the choice of α. The linear map df_p is called the differential of f at p.

Remark. When $M = \mathbb{R}^n$, for any $p \in M$ a chart can be always chosen as $(\mathbb{R}^n, \mathrm{id})$, and $\alpha \sim \beta$ if $\alpha'(0) = \beta'(0)$. Hence the tangent space $T_p M$ is just the vector space of the velocities in the calculus, i.e.

$$T_p M = \{\alpha'(0) \mid \alpha : (-\epsilon, \epsilon) \to M, \ \alpha(0) = p\}.$$

Remark. When M is a differentiable submanifold of N, we have a natural characterization of the tangent space T_pM of M as a linear subspace of the tangentspace T_pN of N, since the inclusion $\imath : M \to N$ induces an injective linear map

$$d\imath_p : T_pM \to T_pN,$$

by

$$[\alpha] \in T_pM \to d\imath_p([\alpha]) = [\alpha] \in T_pN.$$

In particular, when $N = \mathbb{R}^n$, then $[\alpha]$ can be identified by $\alpha'(0) \in \mathbb{R}^n$, and hence T_pM is again just the vector space of the velocities in the calculus, i.e.,

$$T_pM = \{\alpha'(0) \mid \alpha : (-\epsilon, \epsilon) \to M,\ \alpha(0) = p\}.$$

Remark. In this sense, we also use $\alpha'(0)$ for $[\alpha] \in T_pM$ from now on.

Definition ([2, Ch.0, 3.1 Definition, modified])**.** Let M and N be differentiable manifolds. A differentiable mapping $f : M \to N$ is called an immersion if $df_p : T_pM \to T_{f(p)}N$ is injective for all $p \in M$. In addition if if $f : M \to N$ yields a homeomorphism between M and $f(M) \subset N$, then f is an embedding. This coincides with the previous definition of the embedding.

Remark. When there is an immersion $f : M \to N$ between a manifold M of dimension m and a submanifold of a manifold N of dimension n, then $m \leq n$.

Example ([2, Ch.0, 4.1 Example] Tangent bundle)**.** Let M be a differentiable manifold of dimension n. A tangent bundle of M is $TM = \{(p, v) : p \in M, v \in T_pM\}$ with a differentiable structure of dimension $2n$, described below:

Let $\{(U_\alpha, \varphi_\alpha)\}$ be the maximal dfferentiable structure on M. For each α, define $\phi_\alpha : TM \to \varphi_\alpha^{-1}(U_\alpha) \times \mathbb{R}^n$ as

$$\phi_\alpha(p, v) = (\varphi_\alpha(p), (d\varphi_\alpha)_p(v)).$$

Then $\left\{\left(\phi_\alpha^{-1}\left(\varphi_\alpha^{-1}(U_\alpha) \times \mathbb{R}^n\right), \phi_\alpha\right)\right\}$ becomes maximal charts for TM.

Example ([2, Ch.0, 3.1 Definition, modified] Regular surfaces in $\mathbb{R}^n$)**.** A subset $M \subset \mathbb{R}^n$ is a regular surface of dimension k if for every $p \in M$ there exists a neighborhood U of p and a mapping $\varphi : U \to \varphi(U) \subset \mathbb{R}^k$ such that

1. φ is a differentiable homeomorphism onto its image $\varphi(U)$
2. $(d\varphi^{-1})_q : \mathbb{R}^k \to \mathbb{R}^n$ is injective for all $q \in U$.

Example ([2, Ch.0, 3.1 Definition, modified] Inverse image of a regular value)**.** Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable mapping of an open set U of $\mathbb{R}^n$. A point $p \in U$ is defined to be a critical point of F if the differential $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ is not surjective. The image $F(p)$ of a critical point is called a critical value of F, and a point $a \in \mathbb{R}^m$ that is not a critical point is called a regular value of F.

For a regular value $a \in F(U)$ of F, the inverse image $F^{-1}(a) \subset \mathbb{R}^n$ is a regular surface of dimension $n - m$.

Example ([2, Ch.0, 3.1 Definition, modified] Sphere)**.** The sphere $S^n := \{x \in \mathbb{R}^{n+1} : \|x\|_2 = 1\}$ is an inverse image of a regular value 1 of a function $\|\cdot\|_2 : \mathbb{R}^{n+1} \to \mathbb{R}$, so it is a manifold of dimension n.

Definition ([2, Ch.0, 5.1 Definition])**.** A vector field X on a differentiable manifold M is a correspondence that associates to each point $p \in M$ a vector $X(p) \in T_pM$. X can be viewed as a mapping of M into the tangent bundle TM. The vector field is differentiable if the mapping $X : M \to TM$ is differentiable.

Definition ([3, Section 41])**.** Let $\{U_\alpha\}$ be an indexed open covering of X. An indexed family of continuous functions

$$\rho_\alpha : X \to [0, 1]$$

is said to be a partition of unity on X, dominated by (or subordinate to) $\{U_\alpha\}$, if:

1. (support ρ_α) $\subset U_\alpha$ for each α, i.e., $\overline{\{x : \rho_\alpha(x) \neq 0\}} \subset U_\alpha$.
2. The indexed family {support ρ_α} is locally finite, that is, $\forall x \in X$, there is only finite ρ_α's such that $\rho_\alpha(x) > 0$.
3. $\sum \rho_\alpha(x) = 1$ for each $x \in X$.

Theorem ([2, Ch.0.5])**.** *A differentiable manifold M of dimension n (with Hausdorff and countable basis condition) can be immersed in $\mathbb{R}^{2n}$ and embedded in $\mathbb{R}^{2n+1}$.*

Theorem ([2, Ch.0, 5.6 Theorem])**.** *A differentiable manifold M (possibly without Hausdorff and countable basis condition) has a differentiable partition of unity if and only if every connected component of M is Hausdorff and has a countable basis.*

Riemannian Metrics

Definition ([2, Ch.1, 2.1 Definition])**.** A Riemannian metric on a differential manifold M assigns to each $p \in M$ an inner product (that is, a symmetric, bilinear, positive-definite) $\langle , \rangle_p : T_pM \times T_pM \to \mathbb{R}$, that varies differentiably in the following sense: If (U, φ) is a chart with $\varphi^{-1}(x_1, \dots, x_n) = q \in U$ and $\frac{\partial}{\partial x_i}(q) = d\varphi_q^{-1}(0, \dots, 1, \dots, 0)$, then $\left\langle \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q) \right\rangle_q = g_{ij}(x_1, \dots, x_n)$ is a differentiable function on $\varphi(U)$.

Example ([2, Ch.1, 2.4 Example])**.** $M = \mathbb{R}^n$ with $\frac{\partial}{\partial x_i}$ identified with $e_i = (0, \dots, 1, \dots, 0)$. The metric is given by $\langle e_i, e_j \rangle = \delta_{ij}$. $\mathbb{R}^n$ with this metric coincides with the usual Euclidean space of dimension n, and the Riemannian geometry is the usual metric Euclidean geometry.

Example ([2, Ch.1, 2.5 Example])**.** Let $f : M \to N$ be an immersion. If N has a Riemannian structure, f induces a Riemannian structure of M by defining $\langle u, v \rangle_p = \langle df_p(u), df_p(v) \rangle_{f(p)}$, $u, v \in T_pM$. Since df_p is injective, $\langle , \rangle_p$ is positive definite as well. This metric in M is called the metric induced by f, and f is an isometric immersion. In particular, when M is a submanifold of N, we assume that M also has the metric induced from N as well.

Proposition ([2, Ch.1, 2.10 Proposition])**.** *A differentiable manifold M has a Riemannian metric.*

Definition ([2, Ch.1, 2.8 Definition])**.** A differentiable mapping $c : I \to M$ of an open interval $I \subset \mathbb{R}$ into a differentiable manifold M is called a curve.

Definition ([2, Ch.1, 2.9 Definition])**.** When M is a differentiable manifold, a vector field V along a curve $c : I \to M$ is a differentiable mapping that associates to every $t \in I$ a tangent vector $V(t) \in T_{c(t)}M$. To say that V is differentiable means that for any differentiable function f on M, the function $t \to V(t)f$ is differentiable on I.

The vector field $dc(\frac{d}{dt})$, denoted by $\frac{dc}{dt}$, is called the velocity field of c, and written as c' as well.

The restriction of a curve c to a closed interval $[a, b] \subset I$ is called a segment. We define the length of a segment by

$$l_a^b(c) = \int_a^b \left\langle \frac{dc}{dt}, \frac{dc}{dt} \right\rangle_{c(t)}^{1/2} dt.$$

Definition. Let $R \subset M$ be a region (open connected subset), whose closure is compact. For R being contained in a coordinate neighborhood U for a chart (U, φ). We define the volume of R as the integral

$$\mathrm{vol}(R) = \int_{\varphi(R)} \sqrt{\det(g_{ij})} dx_1 \cdots dx_n,$$

and for general R, choose a partition of unity $\{\rho_\alpha\}$ subject to charts $\{U_\alpha\}$ and define as

$$\mathrm{vol}(R) = \sum_\alpha \int_{\varphi_\alpha(R \cap U_\alpha)} (\rho_\alpha \circ \varphi_\alpha^{-1}) \sqrt{\det(g_{ij})} dx_1 \cdots dx_n.$$

Example. The integral of a function f on a manifold M with respect to the volume measure can be computed as, by choose a partition of unity $\{\rho_\alpha\}$ subject to charts $\{U_\alpha\}$,

$$\int_M f d\mathrm{vol} = \sum_\alpha \int_{\varphi_\alpha(U_\alpha)} (\rho_\alpha \circ \varphi_\alpha^{-1})(f \circ \varphi_\alpha^{-1}) \sqrt{\det(g_{ij})} dx_1 \cdots dx_n.$$

Geodesics

Definition ([1, 1.3 Definitions])**.** Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$,$c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$.

Let $I \subset \mathbb{R}$ be an interval. A map $c : I \to X$ is a linearly reparameterized geodesic or a constant speed geodesic, if there exists a constant λ such that $d(c(t), c(t')) = \lambda|t - t'|$ for all $t, t' \in I$.

A local geodesic in X is a map c from an interval $I \subset \mathbb{R}$ to X with the property that for every $t \in I$ there exists $\epsilon > 0$ such that $d(c(t'), c(t'')) = |t' - t''|$ for all $t', t'' \in (t - \epsilon, t + \epsilon)$.

Definition ([1, 1.3 Definitions])**.** Let (X, d) be a metric space. (X, d) is said to be a geodesic metric space (or, more briefly, a geodesic space) if every two points in X are joined by a geodesic. We say that (X, d) is uniquely geodesic if there is exactly one geodesic joining x to y, for all $x, y \in X$.

Definition. When M is a differentiable manifold, a vector field V along a geodesic $c : I \to M$ is called parallel if $\langle c', V \rangle =$ constant along c.

Definition. Let V be a vector field along a curve $c : I \to M$. The Levi-Civita connection $\bar{\nabla}_{c'} V$ of $\mathbb{R}^n$ along c is defined as

$$\left(\bar{\nabla}_{c'} V\right)(c(t)) \coloneqq \frac{d}{dt} V(c(t)) \in T_{c(t)} \mathbb{R}^n.$$

Definition ([2, Ch.2, Exercise 3])**.** Let M be a differentiable submanifold of $\mathbb{R}^n$, and let V be a vector field along a curve $c : I \to M$. The Levi-Civita connection $\nabla_{c'} V$ of M along c is defined as

$$\left(\nabla_{c'} V\right)(c(t)) \coloneqq \left(\left(\bar{\nabla}_{c'} V\right)(c(t))\right)^{\top} \in T_{c(t)} M,$$

where $\left(\left(\bar{\nabla}_{c'} V\right)(c(t))\right)^{\top}$ is the projection of $\left(\bar{\nabla}_{c'} V\right)(c(t)) \in T_{c(t)} \mathbb{R}^n$ to $T_{c(t)} M$.

Definition ([2, Ch.2, 2.5 Definition])**.** Let M be a differentiable submanifold of $\mathbb{R}^n$, and let V be a vector field along a curve $c : I \to M$. V is called parallel if $\nabla_{c'} V = 0$.

Proposition ([2, Ch.2, 2.6 Proposition])**.** *Let M be a differentiable manifold and $c : I \to M$ be a curve. Let$V_0 \in T_{c(t_0)} M$ for some $t_0 \in I$. Then there exists a unique parallel vector field V along c such that $V(t_0) = V_0$. $V(t)$ is called the parallel transport of $V(t_0)$ along c.*

Definition ([2, Ch.2, 2.5 Definition])**.** Let M be a differentiable submanifold of $\mathbb{R}^n$. A parametrized curve $c : I \to M$ is a *(local) geodesic at* $t_0 \in I$ if $\nabla_{c'} c' = 0$ at the point t_0; if γ is a geodesic at t for all $t \in I$, we say that γ is a *(local) geodesic.*

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