Review on Topology

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[4] 우리의 철학I. 같은 것은 같도다. (Isomorphism 의 철학)
우리의 철학II. 같은 것은 정말 똑같다. (Identification 의 철학)

Definition. [3, Section 3.1] 함수 $f : X \to Y$와 $x_0 \in X$가 주어져 있을 때, 임의의 $\epsilon > 0$ 에 대하여 다음의 성질

$$x \in X, \|x - x_0\| < \delta \Longrightarrow \|f(x) - y_0\| < \epsilon$$

이 성립하는 $\delta > 0$가 존재하면, 함수 f가 점 x_0에서 *연속*이라 한다.

만일 집합 $A \subset X$의 모든 점에서 f가 연속이면 A 위에서 연속이라 하고, 정의역 위에서 연속인 함수를 *연속함수*라고 한다.

Topological Spaces, Continuous Functions, and Homeomorphisms

Definition ([2, Section 12])**.** A topology on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\emptyset$ and X are in $\mathcal{T}$.
2. If $\{U_\alpha\}_{\alpha \in I} \subset \mathcal{T}$, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$.
3. If $U_1, \ldots, U_n \in \mathcal{T}$, then $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$.

Example. Let X be a three-element set, $X = \{a, b, c\}$. All the possible topologies are schematically represented in Figure 1. For example, the diagram in the upper right corner indicates the topology $\mathcal{T} = \{X, \emptyset, \{a, b\}, \{b\}, \{b, c\}\}$. All the topologies can be obtained by permuting a, b, c.

Definition ([2, Section 13])**.** If X is a set, a basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) such that $\emptyset$ and X are in $\mathcal{T}$.

1. For each $x \in X$, there is at least one $B \in \mathcal{B}$ containing x.
   (a) If $B_1, B_2 \in \mathcal{B}$ with $x \in B_1 \cap B_2$, then there is $B_3 \in \mathcal{B}$ containing x such that $B_3 \in B_1 \cap B_2$.

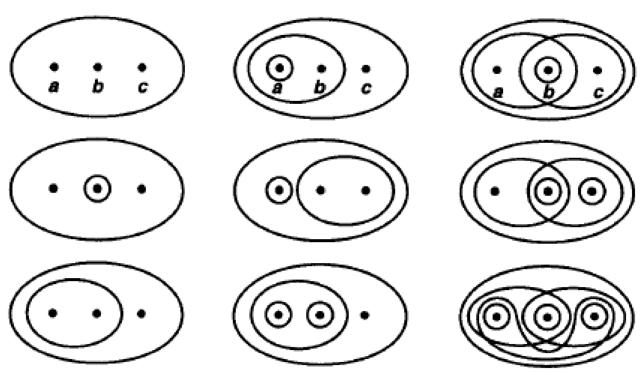

Figure 1: [2, Figure 12.1] Example of topologies of a three-element set.

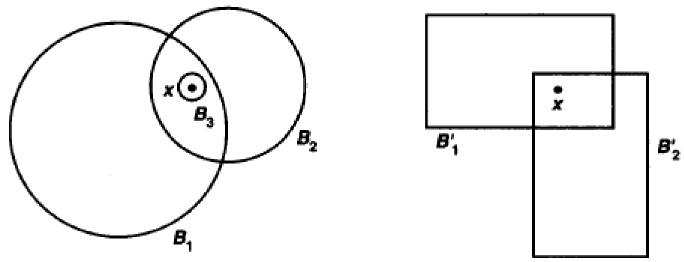

Figure 2: [2, Figure 13.1, 13.2] Example of bases of circular regions or rectangular regions.

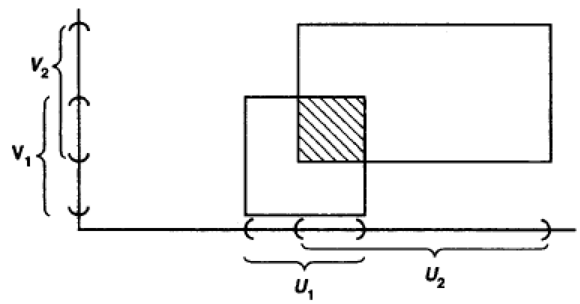

Figure 3: [2, Figure 15.1] Product topology.

Example. Let $\mathcal{B}$ be the collection of all circular regions (interiors of circles) in the plane $\mathbb{R}^2$, as in Figure 2 left, then $\mathcal{B}$ is a basis. Let $\mathcal{B}'$ be the collection of all rectangular regions (interiors of rectangles) in the plane $\mathbb{R}^2$, as in Figure 2 right, then $\mathcal{B}'$ is also a basis. And in fact, two bases $\mathcal{B}$ and $\mathcal{B}'$ generate the same topology for $\mathbb{R}^2$.

Definition ([2, Section 15])**.** Let X and Y be topological spaces. The product topology on $X \times Y$ is the topology having the basis as (see Figure 3):

$$\mathcal{B} = \{U \times V \subset X \times Y : U \text{ is open in } X,\ V \text{ is open in } Y\}.$$

Remark. This definition of the product topology can be naturally extended to a finite product space $X_1 \times \cdots \times X_n$.

Definition ([2, Section 16])**.** Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$$

is a topology on Y, called the subspace topology.

Definition ([2, Section 17])**.** A subset A of a topological space X is said to be closed if the set $X \backslash A$ is open.

Definition ([2, Section 17])**.** The closure of A, denoted by $\bar{A}$, is the intersection of all closed sets containing A.

Definition ([2, Section 17])**.** We say U is a neighborhood (neighbor) of x if U is an open set containing x.

Definition ([2, Section 17])**.** If A is a subset of a topological space X, We say x is a limit point of A if every neighborhood of x intersects A in some point other than x itself.

Theorem. *Let A be a subset of a topological space X, and A' be the set of all limit points of A, then*

$$\bar{A} = A \cup A'.$$

Definition ([2, Section 17])**.** A topological space X is called a Hausdorff space if for each pair $x_1 \neq x_2 \in X$, there exists neighborhoods U_1, U_2 of x_1, x_2, respectively, that $U_1 \cap U_2 = \emptyset$.

Definition ([2, Section 18])**.** A function $f : X \to Y$ is continuous if for each open set V of Y, $f^{-1}(V)$ is an open subset of X.

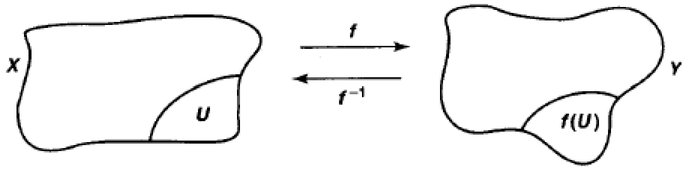

Figure 4: [2, Figure 18.1] Homeomorphism.

Remark. It suffices to show that the inverse image of every basis element is open.

Theorem ([2, Theorem 18.1])**.** *Let X, Y be topological spaces; let $f : X \to Y$. Then the followings are equivalent:*

1. *f is continuous.*
2. *For every closed set B of Y, $f^{-1}(B)$ is closed in X.*
3. *For each $x \in X$ and each neighborhood V of $f(x)$, there is an neighborhood U of x such that $f(U) \subset V$.*

Definition ([2, Section 18])**.** Let $f : X \to Y$ be a bijection. If both the function f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is called a homeomorphism.

X and Y are homeomorphic if such a homeomorphism f exists, and denoted as $X \cong Y$.

Remark. Another way to define a homeomorphism is to say that $f : X \to Y$ is a bijection such that $f(U)$ is open if and only if U is open (see Figure 4).

Remark. A homeomorphism gives us a bijective correspondence not only between X and Y but also between the collections of open sets of X and Y. As a result, any property of X that is entirely expreesed in terms of the topology of X yields, via f, the property of Y. Such a property of X is called a topological property of X.

Definition ([2, Section 18])**.** Suppose $f : X \to Y$ is an injective continuous, and let $Z \coloneqq f(X) \subset Y$ be the image of f equipped with the subspace topology. If the function $f' : X \to Z$ obtained by restricting the range of f is a homeomorphism of X with Z, we say that $f : X \to Y$ is a topological embedding (imbedding) of X in Y.

Definition ([2, Section 20])**.** A metric on a set X is a function

$$d : X \times X \to \mathbb{R}$$

having the following properties:

1. $d(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. (Triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

Given a metric d on X, the number $d(x, y)$ is often called the distance between x and y. Given $\epsilon > 0$, consider the set

$$B_d(x, \epsilon) = \{y : d(x, y) < \epsilon\}$$

of all points y whose distance from x is less than ϵ. It is called the ϵ-ball centered at x. Sometimes we omit d and write $B(x, \epsilon)$.

Definition ([2, Section 20])**.** If d is a metric on the set X, then the collection of all ϵ-balls $B_d(x, \epsilon)$, for $x \in X$ and $\epsilon > 0$, is a basis for a topology on X, called the metric topology induced by d.

A metric space X is a topological space X together with a specific metric d that gives the topology of X.

Example. Given $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ and for $1 \leq p \leq \infty$, we define the p-norm of x by

$$\|x\|_p \coloneqq (x_1^p + \cdots + x_n^p)^{1/p}$$

for $p \in [1, \infty)$, and $\|x\|_\infty \coloneqq \max_{1 \leq i \leq n} |x_i|$. And then the induced distance d_p on $\mathbb{R}^n$ is defined as

$$d_p(x, y) = \|x - y\|_p .$$

All the metrics d_p induce the same topology on $\mathbb{R}^n$ for $1 \leq p \leq \infty$, and this is the usual topology on $\mathbb{R}^n$. This also coincides with the product topology on $\mathbb{R}^n$ as well.

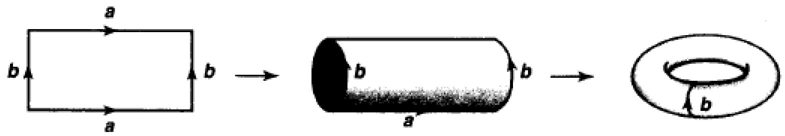

Figure 5: [2, Figure 22.1] Torus as a quotient space.

Theorem ([2, Theorem 21.1])**.** *Let (X, d_X), (Y, d_Y) be metric spaces and let $f : X \to Y$. Then continuity of f is equivalent to the requirement that given $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d_X(x, y) < \delta \Longrightarrow d_Y(f(x), f(y)) < \epsilon.$$

Definition ([2, Section 22])**.** Let X and Y be topological spaces; let $p : X \to Y$ be a surjective map. The map p is a quotient map if $U \subset Y$ is open in Y if and only if $p^{-1}(U)$ is open in X.

Definition ([2, Section 22])**.** If X is a topological space, A is a set, and $p : X \to A$ is a surjective map, then there exists exactly one topology $\mathcal{T}$ on A relative to which p is a quotient map; i.e., $U \subset A$ is open if and only if $p^{-1}(U) \subset X$ is open. $\mathcal{T}$is called the quotient topology induced by p.

Connectedness and Compactness

Definition ([2, Section 23])**.** Let X be a topological space. A separation of X is a pair U, V of disjoint nonempty subsets of X whose union is X. The space X is said to be connected if there does not exist a separation of X.

Theorem ([2, Theorem 23.4])**.** *Let A be a connected subspace of X. If $A \subset B \subset \bar{A}$, then B is also connected.*

Theorem ([2, Theorem 23.5])**.** *The image of a connected space under a continuous map is conencted.*

Corollary ([2, Corollary 24.2])**.** *The real line $\mathbb{R}$ is connected and so are intervals and rays in $\mathbb{R}$.*

Definition ([2, Section 24])**.** Given points x and y of the space X, a path in X from x to y is a continuous map $f : [a, b] \to X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$. A space X is said to be path connected if every pair of points of X can be joined by a path in X.

Example ([2, Section 24] Topologist's sine curve)**.** Let S denote the following subset of the plane

$$S = \{(x, \sin(1/x)) : 0 < x \leq 1\}.$$

The set $\bar{S} = S \cup \{0\} \times [-1, 1]$ is a classical example in the topology called the topologist's sine curve (see Figure 6). The set $\bar{S}$ is connected but not path connected.

Since S is a continuous image of $(0, 1]$, S is connected, and then $\bar{S}$ is connected as well. Now we show $\bar{S}$ is not path connected. Suppose there is a path $\gamma : [a, c] \to \bar{S}$ with $\gamma(a) = (0, 0)$ and $\gamma(c) = (1, \sin 1)$. Since $\gamma^{-1}(\{0\} \times [-1, 1])$ is closed in $[a, c]$, it has the largest element b. Then $\gamma : [b, c]$ is a path that maps b into the vertical interval $\{0\} \times [-1, 1]$ and maps $(b, c]$ into S. Since γ is continuous, there exists $\delta > 0$ such that

$$\gamma[b, b + \delta] \subset B_{d_2}(\gamma(b), 0.5).$$

However, if we write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, then $\gamma_1[b, b + \delta]$ is a connected subset of $[0, 1]$ containing $\gamma_1(b) = 0$ and $\gamma_1(b + \delta) > 0$, so

$$[0, \gamma_1(b + \delta)] \subset \gamma_1[b, b + \delta].$$

But since $\gamma_2(t) = \sin(1/\gamma_1(t))$ if $\gamma_1(t) > 0$, so

$$\gamma_2[b, b + \delta] \supset \sin(1/(0, \gamma_1(b + \delta)]) = [-1, 1].$$

This contradicts with $\gamma[b, b + \delta] \subset B_{d_2}(\gamma(b), 0.5)$, so such path γ cannot exist and $\bar{S}$ is not path connected.

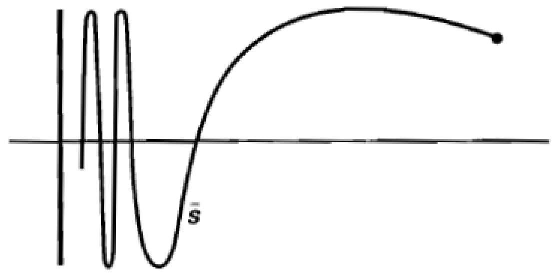

Figure 6: [2, Figure 24.5] Homeomorphism.

Definition ([2, Section 25]). Given X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y. The equivalence classes are called the components (or the "connected components") of X.

Theorem ([2, Theorem 25.1]). *The components of X are connected disjoint subspaces of X whose union is X, such that each nonempty conencted subspaces of X intersects only one of them.*

Definition ([2, Section 25]). Given X, define an equivalence relation on X by setting $x \sim y$ if there is a path in X from x to y. The equivalence classes are called the path components of X.

Theorem ([2, Theorem 25.2]). *The path components of X are path connected disjoint subspaces of X whose union is X, such that each nonempty path conencted subspaces of X intersects only one of them.*

Definition ([2, Section 25]). A space X is said to be locally connected at x if for every neighborhood U of x, there is a connected neighborhood V of x contained in U. X is locally connected if it is locally connected at each of its points. Similarly, a space X is said to be locally path connected at x if for every neighborhood U of x, there is a path connected neighborhood V of x contained in U. X is locally path connected if it is locally path connected at each of its points.

Theorem ([2, Theorem 25.5]). *If X is a topological space, then each path component of X lies in a component of X. If X is locally path connected, then the components and the path components of X are the same.*

Definition ([2, Section 26]). A collection $\mathcal{A}$ of subsets of a space X is said to cover X, or to be a covering of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called an open covering of X if its elements are open subsets of X.

Definition ([2, Section 26]). A space X is said to be compact if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Theorem ([2, Theorem 26.2]). *Every closed subspace of a compact space is compact.*

Theorem ([2, Theorem 26.3]). *Every compact subspace of a Hausdorff space is closed.*

Theorem ([2, Theorem 26.5]). *The image of a compact space under a continuous map is compact.*

Theorem ([2, Theorem 27]). *A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and is bounded in the metric induced by p-norm $\|\cdot\|_p$.*

Theorem ([2, Theorem 27] Extreme value theorem). *Let $f : X \to \mathbb{R}$ be continuous. If X is compact, then there exists points c and d in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.*

Definition ([2, Section 41]). Let $\{U_\alpha\}$ be an indexed open covering of X. An indexed family of continuous functions

$$\rho_\alpha : X \to [0, 1]$$

is said to be a partition of unity on X, dominated by (or subordinate to) $\{U_\alpha\}$, if:

1. (support ρ_α) $\subset U_\alpha$ for each α, i.e., $\overline{\{x : \rho_\alpha(x) \neq 0\}} \subset U_\alpha$.
2. The indexed family {support ρ_α} is locally finite, that is, $\forall x \in X$, there is only finite ρ_α's such that $\rho_\alpha(x) > 0$.
3. $\sum \rho_\alpha(x) = 1$ for each $x \in X$.

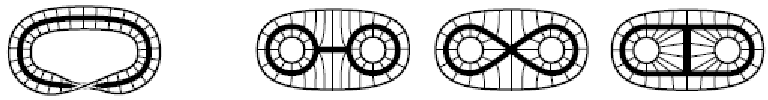

Figure 7: [1, Section 0] Example of a deformation retract.

Homotopy

Definition ([1, Chapter 0])**.** Let $f_0, f_1 : X \to Y$. A *homotopy* between f_0 and f_1 is a continuous function $F : X \times [0,1] \to Y$ such that for all $x \in X$, $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. Two functions f_0, f_1 are *homotopic* if such F exists, and we write $f_0 \simeq f_1$.

Definition ([1, Chapter 0])**.** A map $f : X \to Y$ is called a homotopy equivalence if there is a map $g : Y \to X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. The space X and Y are said to be homotopy equivalent or to have the same homotopy type, and write $X \simeq Y$, if such homotopy equivalence $f : X \to Y$ exists.

Definition ([1, Chapter 0])**.** Let $A \subset X$. Then A is a *deformation retract* of X if there exists a continuous function $F : X \times [0,1] \to Y$ such that for all $x \in X$ and $a \in A$, $F(x,0) = x$, $F(x,1) \in A$, and $F(a,1) = a$. In other words, A is a deformation retract of X if there exists $r : X \to A$ with $r|_A = id_A$ and r and $id_X : X \to X$ are homotopic. We additionally say A is a *strong deformation retract* of X if F also satisfies $F(a,t) = a$ for all $a \in A$ and $t \in [0,1]$.[1]

Example. See Figure 7. The left figure shows a (strong) deformation retract of a Möbius band onto its core circle. The three figures on the right show deformations in which a disk with two smaller open subdisks removed shrinks to three different subspaces. Also note that these three different subspaces are homotopy equivalent to each other but not homeomorphic.

Remark. The relationship between different equivalences of topology is as follows:

$$\begin{tikzcd}
X \text{ and } Y \text{ are homeomorphic } (X \cong Y) \arrow[d, Rightarrow] & Y \text{ is a deformation retract of } X \arrow[dl, Rightarrow] \\
X \text{ and } Y \text{ are homotopic } (X \simeq Y) \arrow[d, Rightarrow] & \\
X \text{ and } Y \text{ have the same homology } (H_*(X) = H_*(Y)) &
\end{tikzcd}$$

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[1] In [1], strong deformation retract is taken as the definition of deformation retract.