Review on Geometry

Definition. Let M be a topological space. A chart (U, φ) on M consists of an open set $U \subset M$ and a homeomorphism φ from U to an open subset of $\mathbb{R}^n$.

Definition ([Munkres, 2000, Section 36])**.** A topological manifold of dimension n is a Hausdorff space M with a countable basis such that there is a collection of charts $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^n\}_{\alpha \in A}$ such that $\bigcup_{\alpha \in A} U_\alpha = M$.

Definition ([do Carmo, 1992, Ch.0, 2.1 Definition, modified])**.** A differentiable (resp, C^k, C^∞) manifold of dimension n is a topological manifold of dimension n such that the collection of charts $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^n\}_{\alpha \in A}$ satisfy that

1. $\bigcup_{\alpha \in A} U_\alpha = M$.
2. for any pair $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$, the mapping $\varphi_\beta \circ \varphi_\alpha^{-1}$ is differentiable (resp, C^k, C^∞).
3. The family $\{(U_\alpha, \varphi_\alpha)\}$ is maximal relative to the conditions (1) and (2).

Remark ([do Carmo, 1992, Ch.0, 2.3 Remark]). $A \subset M$ is open if and only if $\varphi_\alpha^{-1}(A \cap U_\alpha)$ is open in $\mathbb{R}^n$ for all $\alpha \in A$. Sometimes, a differentiable manifold is defined without a topological manifold (i.e., M is just a set), and then the topology is defined in this way.

Definition. A topological manifold with boundary of dimension n is a Hausdorff space M with a countable basis such that there is a collection of maps $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^n_+\}_{\alpha \in A}$ where φ_α is a homeomorphism onto its image such that $\bigcup_{\alpha \in A} U_\alpha = M$, where $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$ is a Euclidean half space.

Definition ([do Carmo, 1992, Ch.0, 2.5 Definition])**.** Let M and N be differentiable manifolds of dimensions m and n. A mapping $f : M \to N$ is differentiable at $p \in M$ if there exist local charts (U, φ) of $p \in M$ and (V, ϕ) of $f(p)$ respectively, such that the mapping $\phi \circ f \circ \varphi^{-1} : \varphi^{-1}(U) \subset \mathbb{R}^m \to \mathbb{R}^n$ is differentiable.

Definition ([do Carmo, 1992, Ch.0, 3.1 Definition, modified])**.** Let M and N be topological (resp, differentiable) manifolds. If $M \subset N$ and the inclusion $\imath : M \subset N$ is an embedding, i.e., if $\imath : M \to N$ yields a homeomorphism between M and $\imath(M) \subset N$, then we say M is a submanifold of N.

Remark. When a manifold M of dimension m is a submanifold of a manifold N of dimension n, then $m \leq n$.

Definition ([do Carmo, 1992, Ch.0, 2.6 Definition, modified])**.** Let M be a differentiable manifold of dimension n. A differentiable curve is a function $\alpha : (-\epsilon, \epsilon) \to M$. For $p \in M$, let

$$\mathrm{Curves}_p M \coloneqq \{\alpha : (-\epsilon, \epsilon) \to M : \alpha(0) = p\}$$

be the smooth curves of M centered at p. Pick a chart (U, φ) of $p \in M$, and then $\alpha, \beta : (-\epsilon, \epsilon) \to M$ are equivalent, written as $\alpha \sim \beta$, if

$$\frac{d}{dt}(\varphi \circ \alpha)(0) = \frac{d}{dt}(\varphi \circ \beta)(0).$$

Then $\sim$ is regardless of the choice of a chart, and gives an equivalence relation. The set of tangent vectors of M at p is defined by

$$T_p M := \mathrm{Curves}_p M / \sim .$$

To define a vector space structure on $T_p M$, again pick a chart (U, φ) of $p \in M$, and define a map $d\varphi_p : T_p M \to \mathbb{R}^n$ by

$$d\varphi_p([\alpha]) := \frac{d}{dt}(\varphi \circ \alpha)(0).$$

Then $d\varphi_p$ is a bijection, and we use this to transfer the vector-space operations on $\mathbb{R}^n$ over to $T_p M$, i.e., we set

$$\begin{aligned}
[\alpha] + [\beta] &:= d\varphi_p^{-1}(d\varphi_p([\alpha]) + d\varphi_p([\beta])), \\
\lambda[\alpha] &:= d\varphi_p^{-1}(\lambda d\varphi_p([\alpha])).
\end{aligned}$$

Remark. T_pM acts on any real valued function $f : M \to \mathbb{R}$ as follows:

$$[\alpha] \in T_pM : f \mapsto [\alpha]f := \left.\frac{d(f \circ \alpha)}{dt}\right|_{t=0}.$$

Remark. Consider the coordinate curve: when $\varphi(p) = 0$, let $\frac{\partial}{\partial x_i}$ be the eqiuivalent class of the following curve

$$x_i \mapsto \varphi^{-1}(0, \ldots, 0, x_i, 0, \ldots, 0).$$

Then $\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\}$ forms a basis in T_pM.

Definition ([do Carmo, 1992, Ch.0, 2.7 Proposition])**.** Let M and N be differentiable manifolds of dimensions m and n, and let $f : M \to N$ be a differentiable mapping. For every $p \in M$ and $v \in T_pM$, choose a differentiable curve $\alpha : (-\epsilon, \epsilon) \to M$ with $\alpha(0) = p$, $[\alpha] = v$. Take $\beta = f \circ \alpha$. The mapping $df_p : T_pM \to T_{f(p)}N$ given by $df_p(v) = [\beta]$ is a linear mapping that does not depend on the choice of α. The linear map df_p is called the differential of f at p.

Remark. When $M = \mathbb{R}^n$, for any $p \in M$ a chart can be always chosen as $(\mathbb{R}^n, \mathrm{id})$, and $\alpha \sim \beta$ if $\alpha'(0) = \beta'(0)$. Hence the tangent space T_pM is just the vector space of the velocities in the calculus, i.e.

$$T_pM = \{\alpha'(0) \mid \alpha : (-\epsilon, \epsilon) \to M,\ \alpha(0) = p\}.$$

Remark. When M is a differentiable submanifold of N, we have a natural characterization of the tangent space T_pM of M as a linear subspace of the tangentspace T_pN of N, since the inclusion $\imath : M \to N$ induces an injective linear map

$$d\imath_p : T_pM \to T_pN,$$

by

$$[\alpha] \in T_pM \to d\imath_p([\alpha]) = [\alpha] \in T_pN.$$

In particular, when $N = \mathbb{R}^n$, then $[\alpha]$ can be identified by $\alpha'(0) \in \mathbb{R}^n$, and hence T_pM is again just the vector space of the velocities in the calculus, i.e.,

$$T_pM = \{\alpha'(0) \mid \alpha : (-\epsilon, \epsilon) \to M,\ \alpha(0) = p\}.$$

Remark. In this sense, we also use $\alpha'(0)$ for $[\alpha] \in T_pM$ from now on.

Definition ([do Carmo, 1992, Ch.0, 3.1 Definition, modified])**.** Let M and N be differentiable manifolds. A differentiable mapping $f : M \to N$ is called an immersion if $df_p : T_pM \to T_{f(p)}N$ is injective for all $p \in M$. In addition if if $f : M \to N$ yields a homeomorphism between M and $f(M) \subset N$, then f is an embedding. This coincides with the previous definition of the embedding.

Remark. When there is an immersion $f : M \to N$ between a manifold M of dimension m and a submanifold of a manifold N of dimension n, then $m \le n$.

Example ([do Carmo, 1992, Ch.0, 4.1 Example])**.** Let M be a differentiable manifold of dimension n. A tangent bundle of M is $TM = \{(p, v) : p \in M, v \in T_pM\}$ with a differentiable structure of dimension $2n$, described below:

Let $\{(U_\alpha, \varphi_\alpha)\}$ be the maximal dfferentiable structure on M. For each α, define $\phi_\alpha : TM \to \varphi_\alpha^{-1}(U_\alpha) \times \mathbb{R}^n$ as

$$\phi_\alpha(p, v) = (\varphi_\alpha(p), (d\varphi_\alpha)_p(v)).$$

Then $\left\{\left(\phi_\alpha^{-1}\left(\varphi_\alpha^{-1}(U_\alpha) \times \mathbb{R}^n\right), \phi_\alpha\right)\right\}$ becomes maximal charts for TM.

Definition ([do Carmo, 1992, Ch.0, 5.1 Definition])**.** A vector field X on a differentiable manifold M is a correspondence that associates to each point $p \in M$ a vector $X(p) \in T_pM$. X can be viewed as a mapping of M into the tangent bundle TM. The vector field is differentiable if the mapping $X : M \to TM$ is differentiable.

Definition ([do Carmo, 1992, Ch.1, 2.1 Definition])**.** A Riemannian metric on a differential manifold M assigns to each $p \in M$ an inner product (that is, a symmetric, bilinear, positive-definite) $\langle , \rangle_p : T_pM \times T_pM \to \mathbb{R}$, that varies differentiably in the following sense: If (U, φ) is a chart with $q \in U$ and $\frac{\partial}{\partial x_i}(q) = d\varphi_q^{-1}(0, \ldots, 1, \ldots, 0)$, then $\left\langle \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q) \right\rangle_q = g_{ij}(x_1, \ldots, x_n)$ is a differentiable function on $\varphi(U)$.

Example ([do Carmo, 1992, Ch.1, 2.5 Example]). Let $f : M \to N$ be an immersion. If N N has a Riemannian structure, f induces a Riemannian structure of M by defining $\langle u, v \rangle_p = \langle df_p(u), df_p(v) \rangle_{f(p)}$, $u, v \in T_pM$. Since df_p is injective, $\langle , \rangle_p$ is positive definite as well. This metric in M is called the metric induced by f, and f is an isometric immersion. In particular, when M is a submanifold of N, we assume that M also has the metric induced from N as well.

Proposition ([do Carmo, 1992, Ch.1, 2.10 Proposition]). *A differentiable manifold M has a Riemannian metric.*

Definition ([do Carmo, 1992, Ch.1, 2.8 Definition]). A differentiable mapping $c : I \to M$ of an open interval $I \subset \mathbb{R}$ into a differentiable manifold M is called a curve.

Definition ([do Carmo, 1992, Ch.1, 2.9 Definition]). When M is a differentiable manifold, a vector field V along a curve $c : I \to M$ is a differentiable mapping that associates to every $t \in I$ a tangent vector $V(t) \in T_{c(t)}M$. To say that V is differentiable means that for any differentiable function f on M, the function $t \to V(t)f$ is differentiable on I.

The vector field $dc(\frac{d}{dt})$, denoted by $\frac{dc}{dt}$, is called the velocity field of c, and written as c' as well.

The restriction of a curve c to a closed interval $[a, b] \subset I$ is called a segment. We define the length of a segment by

$$l_a^b(c) = \int_a^b \left\langle \frac{dc}{dt}, \frac{dc}{dt} \right\rangle_{c(t)}^{1/2} dt.$$

Definition. Let $R \subset M$ be a region (open connected subset), whose closure is compact. Suppose for convenience that R is contained in a coordinate neighborhood U for a chart (U, φ). We define the volume of R as the integral

$$\mathrm{vol}(R) = \int_{\varphi(R)} \sqrt{\det(g_{ij})} dx_1 \cdots dx_n.$$

Definition ([Bridson and Haefliger, 1999, 1.3 Definitions]). Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$,$c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$.

Let $I \subset \mathbb{R}$ be an interval. A map $c : I \to X$ is a linearly reparameterized geodesic or a constant speed geodesic, if there exists a constant λ such that $d(c(t), c(t')) = \lambda |t - t'|$ for all $t, t' \in I$.

A local geodesic in X is a map c from an interval $I \subset \mathbb{R}$ to X with the property that for every $t \in I$ there exists $\epsilon > 0$ such that $d(c(t'), c(t'')) = |t' - t''|$ for all $t', t'' \in (t - \epsilon, t + \epsilon)$.

Definition ([Bridson and Haefliger, 1999, 1.3 Definitions]). Let (X, d) be a metric space. (X, d) is said to be a geodesic metric space (or, more briefly, a geodesic space) if every two points in X are joined by a geodesic. We say that (X, d) is uniquely geodesic if there is exactly one geodesic joining x to y, for all $x, y \in X$.

Definition. When M is a differentiable manifold, a vector field V along a geodesic $c : I \to M$ is called parallel if $\langle c', V \rangle =$ constant along c.

Definition. Let V be a vector field along a curve $c : I \to M$. The Levi-Civita connection $\bar{\nabla}_{c'} V$ of $\mathbb{R}^n$ along c is defined as

$$\left(\bar{\nabla}_{c'} V\right)(c(t)) \coloneqq \frac{d}{dt} V(c(t)) \in T_{c(t)}\mathbb{R}^n.$$

Definition ([do Carmo, 1992, Ch.2, Exercise 3]). Let M be a differentiable submanifold of $\mathbb{R}^n$, and let V be a vector field along a curve $c : I \to M$. The Levi-Civita connection $\nabla_{c'} V$ of M along c is defined as

$$\left(\nabla_{c'} V\right)(c(t)) \coloneqq \left(\left(\bar{\nabla}_{c'} V\right)(c(t))\right)^{\top} \in T_{c(t)}M,$$

where $\left(\left(\bar{\nabla}_{c'} V\right)(c(t))\right)^{\top}$ is the projection of $\left(\bar{\nabla}_{c'} V\right)(c(t)) \in T_{c(t)}\mathbb{R}^n$ to $T_{c(t)}M$.

Definition ([do Carmo, 1992, Ch.2, 2.5 Definition]). Let M be a differentiable submanifold of $\mathbb{R}^n$, and let V be a vector field along a curve $c : I \to M$. V is called parallel if $\nabla_{c'} V = 0$.

Proposition ([do Carmo, 1992, Ch.2, 2.6 Proposition]). *Let M be a differentiable manifold and $c : I \to M$ be a curve. Let$V_0 \in T_{c(t_0)}M$ for some $t_0 \in I$. Then there exists a unique parallel vector field V along c such that $V(t_0) = V_0$. $V(t)$ is called the parallel transport of $V(t_0)$ along c.*

Definition ([do Carmo, 1992, Ch.2, 2.5 Definition]). Let M be a differentiable submanifold of $\mathbb{R}^n$. A parametrized curve $c : I \to M$ is a *(local) geodesic at* $t_0 \in I$ if $\nabla_{c'} c' = 0$ at the point t_0; if γ is a geodesic at t for all $t \in I$, we say that γ is a *(local) geodesic.*

References

Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. ISBN 3-540-64324-9. doi: 10.1007/978-3-662-12494-9. URL https://doi.org/10.1007/978-3-662-12494-9.

Manfredo Perdigão do Carmo. *Riemannian geometry*. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1992. ISBN 0-8176-3490-8. doi: 10.1007/978-1-4757-2201-7. URL https://doi.org/10.1007/978-1-4757-2201-7. Translated from the second Portuguese edition by Francis Flaherty.

James R. Munkres. *Topology*. Prentice Hall, Inc., Upper Saddle River, NJ, 2000. ISBN 0-13-181629-2. Second edition of [MR0464128].