Optional Sampling Theorems

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We first recall:

Theorem ([1, Theorem 4.2.9]). *If N is a stopping time (정지시간) and X_n is a supermartingale, then $X_{N\wedge n}$ is a supermartingale.*

In this lecture note, we will prove a number of results that allow us to conclude that if X_n is a submartingale and $M \leq N$ are stopping times, then $\mathbb{E}X_M \leq \mathbb{E}X_N$.

We first recall the related previous results:

[1, Example 4.2.13] shows that this is not always true.

Example ([1, Example 4.2.13]). Let $S_0 = 1$ and $\{S_n,\ n \geq 1\}$ be i.i.d. symmetric simple random walk. That is, $S_n = S_{n-1} + \xi_n$ where $\xi_1, \xi_2, \cdots$ are i.i.d. with $P(\xi_i = 1) = P(\xi_i = -1) = \frac{1}{2}$.

In fact, S_n does not converge and $\limsup S_n = \infty$ and $\liminf S_n = -\infty$, by [1, Exercise 5.4.1].

Let $N = \min\{n :\ S_n = 0\}$, then by above $N < \infty$ a.s., and $S_N = 0$ a.s..

Hence $\mathbb{E}S_0 = 1 > \mathbb{E}S_N = 0$.

But [1, Exercise 4.4.2] shows this is true if N is bounded:

Exercise ([1, Example 4.4.2]). (generalized version of [1, Theorem 4.4.1]) If X_n is a submartingale and $M \leq N$ are stopping times with $P(N \leq k) = 1$, then $\mathbb{E}X_M \leq \mathbb{E}X_N$.

So our attention will be focused on the case of unbounded N.

Theorem ([1, Theorem 4.8.1]). *If X_n is a uniformly integrable submartingale then for any stopping time N, $X_{N\wedge n}$ is uniformly integrable.*

As in [1, Theorem 4.2.5], the last result implies one for supermartingales with $\geq$ and one for martingales with $=$. This is true for the next two theorems as well.

Proof. X_n^+ is a submartingale, so [1, Theorem 4.4.1] implies $\mathbb{E}X_{N\wedge n}^+ \leq \mathbb{E}X_n^+$. Since $\{X_n^+\}$ is uniformly integrable, it follows from the remark after the definition that

$$\sup_n \mathbb{E}X_{N\wedge n}^+ \leq \sup_n \mathbb{E}X_n^+ < \infty.$$

From [1, Theorem 4.2.9] we have that $\{X_{N\wedge n}\}_{n\in\mathbb{N}\cup\{0\}}$ is a submartingale. Using the martingale convergence theorem (마팅게일 수렴정리) [1, Theorem 4.2.11] now gives $X_{N\wedge n} \to X_N$ a.s. (here $X_\infty = \lim_n X_n$) and $\mathbb{E}|X_N| < \infty$. With this established, the rest is easy. We write

$$\mathbb{E}\left[|X_{N\wedge n}|; |X_{N\wedge n}| > K\right] = \mathbb{E}\left[|X_N|; |X_N| > K, N \leq n\right] + \mathbb{E}\left[|X_n|; |X_n| > K, N > n\right].$$

Since $\mathbb{E}|X_N| < \infty$ and X_n is uniformly integrable, if we can choose large enough K so that each term is $< \varepsilon/2$. □

From the last computation in the proof of [1, Theorem 4.8.1], we get:

Theorem ([1, Theorem 4.8.2])**.** *If $\mathbb{E}|X_N| < \infty$ and $X_n 1_{\{N>n\}}$ is uniformly integrable, then $X_{N\wedge n}$ is uniformly integrable and hence $\mathbb{E}X_0 \le \mathbb{E}X_N$.*

Theorem ([1, Theorem 4.8.3])**.** *If X_n is a uniformly integrable submartingale then for any stopping time $N \le \infty$, we have*

$$\mathbb{E}X_0 \le \mathbb{E}X_N \le \mathbb{E}X_\infty,$$

where $X_\infty = \lim X_n$.

Proof. [1, Theorem 4.4.1] implies $\mathbb{E}X_0 \le \mathbb{E}X_{N\wedge n} \le \mathbb{E}X_n$. Observe that [1, Theorem 4.6.4] imply $X_n \to X_\infty$ in L^1, and [1, Theorem 4.8.1] and [1, Theorem 4.6.4] imply $X_{N\wedge n} \to X_N$ in L^1. Hence by letting $n \to \infty$ gives the desired result. □

This has a following useful corollary.

Theorem. *The optional stopping theorem*

If $L \le M$ are stopping times and $\{Y_{M\wedge n}\}$ is uniformly integrable submartingale, then $\mathbb{E}Y_L \le \mathbb{E}Y_M$.

Proof. Use the inequality $\mathbb{E}X_N \le \mathbb{E}X_\infty$ in [1, Theorem 4.8.3] with $X_n = Y_{M\wedge n}$ and $N = L$. □

The next result does not require uniform integrability.

Theorem ([1, Theorem 4.8.4])**.** *If X_n is a nonnegative supermartingale and $N \le \infty$ is a stopping time, then $\mathbb{E}X_0 \ge \mathbb{E}X_N$ where $X_\infty = \lim X_n$, which exists by [1, Theorem 4.2.12].*

Proof. Using [1, Theorem 4.4.1] and Fatou's Lemma,

$$\mathbb{E}X_0 \ge \liminf_{n\to\infty} \mathbb{E}X_{N\wedge n} \ge \mathbb{E}X_N.$$

□

The next result is useful in some situations.

Theorem ([1, Theorem 4.8.5])**.** *Suppose X_n is a submartingale and $\mathbb{E}\left[|X_{n+1} - X_n| \mid \mathcal{F}_n\right] \le B$ a.s.. If N is a stopping time with $\mathbb{E}N < \infty$ then $X_{N\wedge n}$ is uniformly integrable and hence $\mathbb{E}X_N \ge \mathbb{E}X_0$.*

Proof. We begin by observing that

$$\begin{aligned} X_{N\wedge n} &= X_n 1_{\{N\ge n\}} + \sum_{m=0}^{n-1} X_m (1_{\{N\ge m\}} - 1_{\{N>m\}}) \\ &= X_0 + \sum_{m=0}^{n-1} (X_{m+1} - X_m) 1_{\{N>m\}}, \end{aligned}$$

and hence

$$|X_{N\wedge n}| \le |X_0| + \sum_{m=0}^{\infty} |X_{m+1} - X_m| 1_{\{N>m\}}.$$

To prove uniform integrability, it suffices to show that the right-hand side has finite expectation for then $|X_{N\wedge n}|$ is dominated by an integrable random variable. Now, $\{N > m\} \in \mathcal{F}_m$, so

$$\mathbb{E}\left[|X_{m+1} - X_m|; N > m\right] = \mathbb{E}\left[\mathbb{E}\left[|X_{m+1} - X_m| \mid \mathcal{F}_m\right]; N > m\right] \le BP(N > m),$$

and

$$\mathbb{E}\sum_{m=0}^{\infty}|X_{m+1}-X_m|1_{\{N>m\}} \le B\sum_{m=0}^{\infty}P(N>m) = B\mathbb{E}N < \infty.$$

□

Applications to Random Walks

Let $\xi_1, \xi_2, \dots$ be i.i.d., $S_n = S_0 + \xi_1 + \cdots + \xi_n$, where S_0 is a constant, and let $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$.

Example ([1, Example 4.2.1])**.** Linear martingale. If we let $\mu = \mathbb{E}\xi_i$ then $X_n = S_n - n\mu$ is a martingale.

Theorem ([1, Theorem 4.8.6])**.** *Wald's Equation. If $\xi_1, \xi_2, \dots$ are i.i.d. with $\mathbb{E}\xi_i = \mu$, $S_n = \xi_1 + \cdots + \xi_n$ and N is a stopping time with $\mathbb{E}N < \infty$ then $\mathbb{E}S_N = \mu\mathbb{E}N$.*

Proof. Let $X_n = S_n - n\mu$ and note that $\mathbb{E}\left[|X_{n+1} - X_n| \mid \mathcal{F}_n\right] = \mathbb{E}|\xi_i - \mu|$. □

Example ([1, Example 4.2.2])**.** Quadratic martingale. Suppose $\mathbb{E}\xi_i = 0$ and $\mathbb{E}\xi_i^2 = \sigma^2 \in (0,\infty)$. Then $X_n = S_n^2 - n\sigma^2$ is a martingale.

Example ([1, Example 4.2.3])**.** Exponential martingale. Suppose that $\varphi(\theta) = \mathbb{E}e^{\theta\xi_i} < \infty$. Then $X_n = e^{\theta S_n}/\varphi(\theta)^n$ is a martingale.

Theorem ([1, Theorem 4.8.7])**.** *Symmetric Simple Random Walk. Let $P(\xi_i = 1) = P(\xi_i = -1) = 1/2$. Suppose $S_0 = x$ and let $N = \min\{n : S_n \notin (a,b)\}$. Writing a subscript x to remind us of the starting point:*

(a) $P_x(S_N = a) = \dfrac{b-x}{b-a}, \quad P_x(S_N = b) = \dfrac{x-a}{b-a}.$

(b) $\mathbb{E}_0 N = -ab$ *and hence* $\mathbb{E}_x N = (b-x)(x-a)$.

Let $T_x = \min\{n : S_n = x\}$. Taking $a = 0$, $x = 1$, and $b = M$ we have

$$P_1(T_M < T_0) = \frac{1}{M}, \qquad P_1(T_0 < T_M) = \frac{M-1}{M}.$$

The first result proves the probability bound in [1, Example 4.4.5]:

$$P_1(\max S_{T_0 \wedge m} \ge M) = \frac{1}{M}$$

Letting $M \to \infty$ in the second we have $P_1(T_0 < \infty) = 0$.

Proof. (a)

To see that $P(N < \infty) = 1$ note that if we have $(b-a)$ consecutive steps of size $+1$ we will exit the interval. From this it follows that

$$P(N > m(b-a)) \le (1 - 2^{-(b-a)})^m,$$

so $\mathbb{E}N = \sum_{t=1}^{\infty} P(N \ge t) < \infty$.

Clearly $\mathbb{E}|S_N| < \infty$ and $S_n 1_{\{N>n\}}$ are uniformly integrable, so using [1, Theorem 4.8.2] we have

$$x = \mathbb{E}S_N = aP_x(S_N = a) + b[1 - P_x(S_N = a)].$$

Rearranging we have $P_x(S_N = a) = (b-x)/(b-a)$, and subtracting this from 1 gives $P_x(S_N = b) = (x-a)/(b-a)$.

(b)

Using the stopping theorem for the bounded stopping time $N \wedge n$ we have

$$0 = \mathbb{E}_0 S_{N\wedge n}^2 - \mathbb{E}_0(N \wedge n).$$

The monotone convergence theorem implies that $\mathbb{E}_0(N \wedge n) \uparrow \mathbb{E}_0 N$. Using the bounded convergence theorem and the result of (a) with $x = 0$ implies

$$\mathbb{E}_0 S_{N\wedge n}^2 \to a^2 \frac{b}{b-a} + b^2 \frac{-a}{b-a} = -ab,$$

which completes the proof. $\square$

Theorem ([1, Theorem 4.8.8]). *Let S_n be symmetric random walk with $S_0 = 0$ and let $T_1 = \min\{n : S_n = 1\}$.*

$$\mathbb{E}_s T_1 = \frac{1 - \sqrt{1-s^2}}{s}.$$

Inverting the generating function we find

$$P(T_1 = 2n-1) = \frac{1}{2n-1} \cdot \frac{(2n)!}{n!n!} 2^{-2n}.$$

Theorem ([1, Theorem 4.8.8]). *Asymmetric Simple Random Walk. Suppose $P(\xi_i = 1) = p$ and $P(\xi_i = -1) = q = 1-p$ with $p \neq q$.*

(a) If $\varphi(y) = \left(\frac{1-p}{p}\right)^y$ then $\varphi(S_n)$ is a martingale.

(b) If we let $T_z = \inf\{n : S_n = z\}$ then for $a < x < b$

$$P_x(T_a < T_b) = \frac{\varphi(b) - \varphi(x)}{\varphi(b) - \varphi(a)}, \quad P_x(T_b < T_a) = \frac{\varphi(x) - \varphi(a)}{\varphi(b) - \varphi(a)}.$$

(c) For $1/2 < p < 1$ and $a < 0$,

$$P\left(\min_n S_n \leq a\right) = P(T_a < \infty) = \left(\frac{1-p}{p}\right)^{-a}.$$

(d) If $b > 0$ then $P(T_b < \infty) = 1$ and $\mathbb{E}T_b = b/(2p-1)$.

References

[1] Rick Durrett. *Probability—theory and examples*, volume 49 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, 2019. Fifth edition.