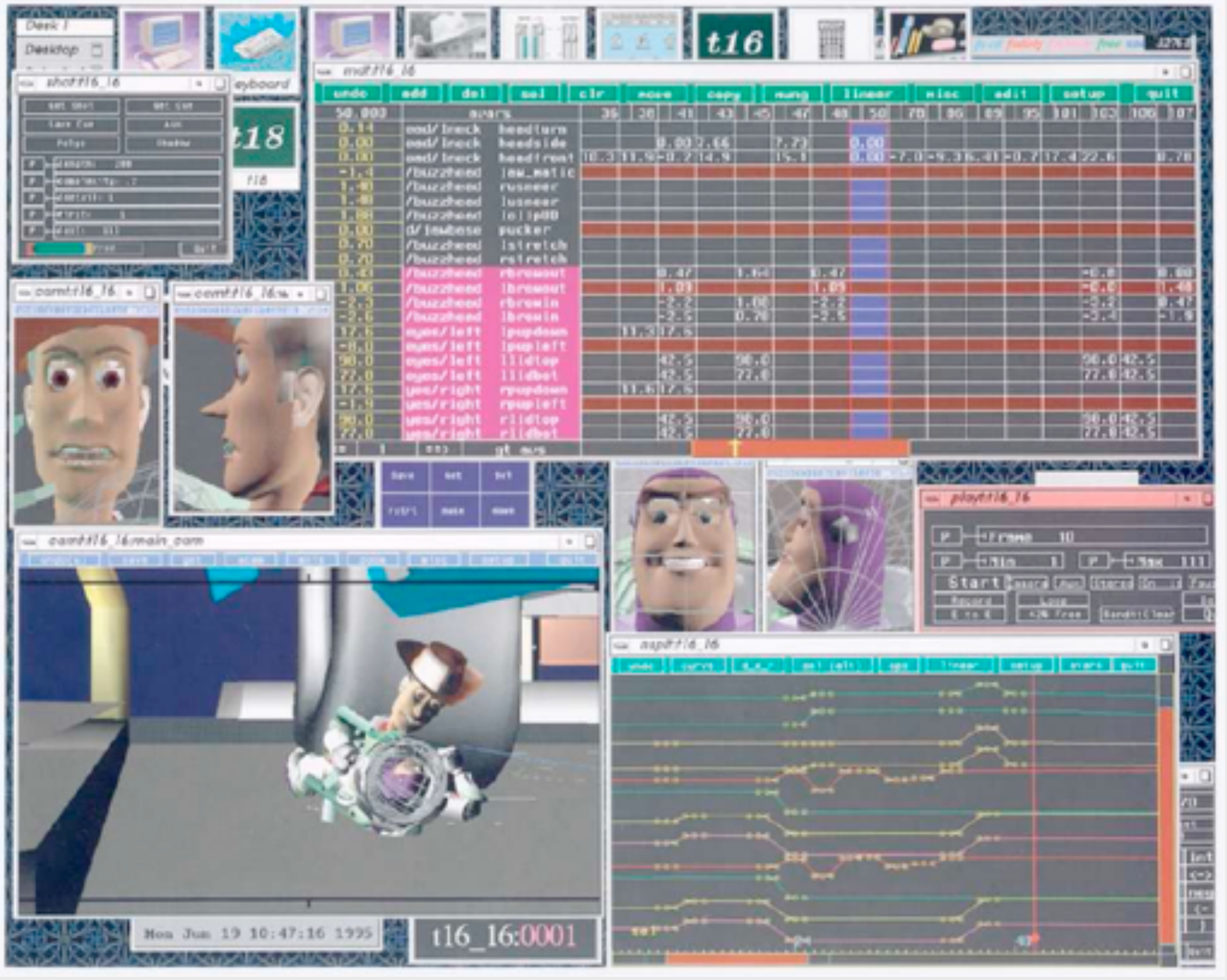
Keyframe animation



- Process of keyframing
- Keyframe interpolation
- Hermite and Bezier curves
- Splines
- Speed control

Oldest keyframe animation

- Two conditions to make moving images in 19th century
  - at least 10 frames per second
  - a period of blackness between images

2D animation

- Highly skilled animators draw the keyframes
- Less skilled (lower paid) animators draw the in-between frames
- Time consuming process
- Difficult to create physically realistic animation

3D animation

- Animators specify important keyframes in 3D
- Computers generates the in-between frames
- Some dynamic motion can be done by computers (hair, clothes, etc)
- Still time consuming; Pixar spent four years to produce Toy Story

General pipeline

- Story board
- Keyframes
- Inbetweens
- Painting

Storyboards

- The film in outline form
  - specify the key scenes
  - specify the camera moves and edits
  - specify character gross motion
- Typically paper and pencil sketches on individual sheets taped on a wall

"A bug's life"

http://www.pixar.com/featurefilms/abl/behind_pop4.html

The process of keyframing

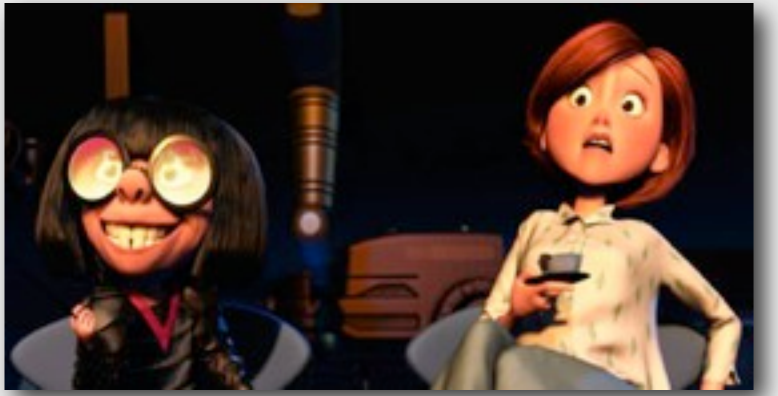
- Specify the keyframes
- Specify the type of interpolation
  - linear, cubic, parametric curves
- Specify the speed profile of the interpolation
  - constant velocity, ease-in-ease-out, etc
- Computer generates the in-between frames

A keyframe

- In 2D animation, a keyframe is usually a single image
- In 3D animation, each keyframe is defined by a set of parameters

Keyframe parameters

- What are the parameters?
  - position and orientation
  - body deformation
  - facial features
  - hair and clothing
  - lights and cameras



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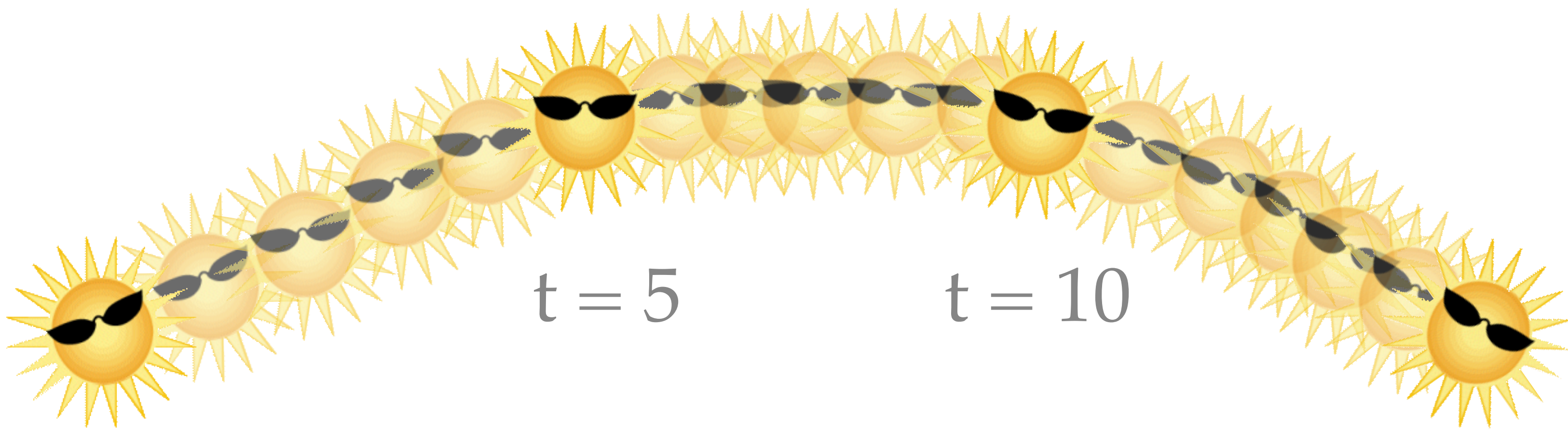
In-between frames

- Linear interpolation
- Cubic curve interpolation

Linear interpolation

Linearly interpolate the parameters between keyframes



end key start key

$$x = x_0 + \frac{t - t_0}{t_1 - t_0}(x_1 - x_0)$$

end time start time

Cubic curve interpolation

We can use three cubic functions to represent a 3D curve

Each function is defined with the range $0 \leq t \leq 1$

$$\mathbf{Q}(t) = [x(t) \;\; y(t) \;\; z(t)]$$

or

$$Q_x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$

$$Q_y(t) = a_y t^3 + b_y t^2 + c_y t + d_y$$

$$Q_z(t) = a_z t^3 + b_z t^2 + c_z t + d_z$$

bold: a vector or a matrix
italic: a scalar

vectors
- $\mathbf{a} \cdot \mathbf{b}$: inner product
- $\mathbf{a} \times \mathbf{b}$: cross product
- $\mathbf{ab}$: multiplication

matrices
- $\mathbf{A} \cdot \mathbf{B}$: multiplication
- $\mathbf{AB}$: multiplication

Compact representation

$$\mathbf{Q}(t) = [Q_x(t) \;\; Q_y(t) \;\; Q_z(t)]$$

$$Q_x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$

$$Q_y(t) = a_y t^3 + b_y t^2 + c_y t + d_y$$

$$Q_z(t) = a_z t^3 + b_z t^2 + c_z t + d_z$$

$$\mathbf{C} = \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix} \qquad \mathbf{T} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}$$

Compact representation

$$\mathbf{Q}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix} = \mathbf{TC}$$

$$\dot{\mathbf{Q}} = \frac{d}{dt}\mathbf{Q}(t) = \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix} \mathbf{C}$$

Constraints on the cubics

How many constraints do we need to determine a cubic curve? 4

Redefine **C** as a product of the basis matrix **M** and the geometry matrix **G**

$$\mathbf{C} = \mathbf{M} \cdot \mathbf{G}$$

$$\mathbf{Q}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} G_{1x} & G_{1y} & G_{1z} \\ G_{2x} & G_{2y} & G_{2z} \\ G_{3x} & G_{3y} & G_{3z} \\ G_{4x} & G_{4y} & G_{4z} \end{bmatrix}$$

$$= \mathbf{T} \cdot \mathbf{M} \cdot \mathbf{G}$$

- Process of keyframing
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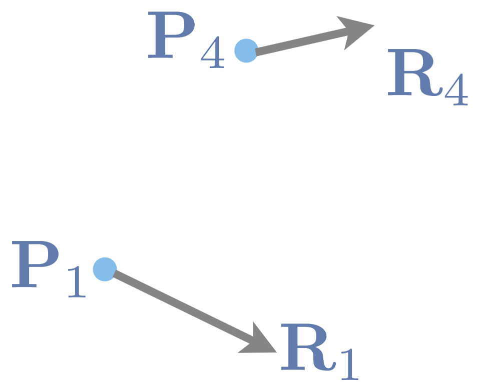
Hermite curves

- A Hermite curve is determined by
  - endpoints $\mathbf{P}_1$ and $\mathbf{P}_4$
  - tangent vectors $\mathbf{R}_1$ and $\mathbf{R}_4$ at the endpoints
- Use these elements to construct geometry matrix



$$\mathbf{G}_h = \begin{bmatrix} P_{1x} & P_{1y} & P_{1z} \\ P_{4x} & P_{4y} & P_{4z} \\ R_{1x} & R_{1y} & R_{1z} \\ R_{4x} & R_{4y} & R_{4z} \end{bmatrix}$$

Hermite basis matrix

Given desired constraints:

1. endpoints meet $\mathbf{P}_1$ and $\mathbf{P}_4$

$$\mathbf{Q}(0) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \cdot \mathbf{M}_h \cdot \mathbf{G}_h = \mathbf{P}_1$$

$$\mathbf{Q}(1) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \cdot \mathbf{M}_h \cdot \mathbf{G}_h = \mathbf{P}_4$$

2. tangent vectors meet $\mathbf{R}_1$ and $\mathbf{R}_4$

$$\dot{\mathbf{Q}}(0) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \cdot \mathbf{M}_h \cdot \mathbf{G}_h = \mathbf{R}_1$$

$$\dot{\mathbf{Q}}(1) = \begin{bmatrix} 3 & 2 & 1 & 0 \end{bmatrix} \cdot \mathbf{M}_h \cdot \mathbf{G}_h = \mathbf{R}_4$$

Hermite basis matrix

We can solve for basis matrix $\mathbf{M}_h$

$$\begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_4 \\ \mathbf{R}_1 \\ \mathbf{R}_4 \end{bmatrix} = \mathbf{G}_h = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \cdot \mathbf{M}_h \cdot \mathbf{G}_h$$

$$\mathbf{M}_h = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

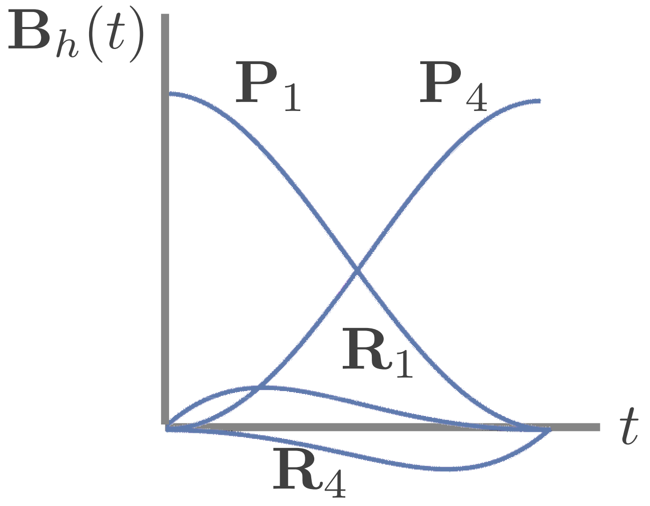
Hermite Blending functions

Let's define **B** as a product of **T** and **M**

$$\mathbf{B}_h(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$\mathbf{B}_h(t)$ indicates the weight of each element in $\mathbf{G}_h$

$$\mathbf{Q}(t) = \mathbf{B}_h(t) \cdot \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_4 \\ \mathbf{R}_1 \\ \mathbf{R}_4 \end{bmatrix}$$

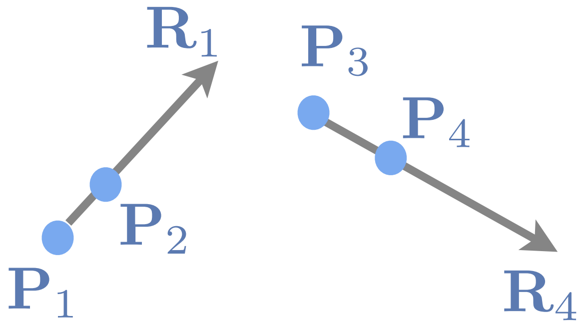


Bézier curves

Indirectly specify tangent vectors by specifying two intermediate points

$$\mathbf{R}_1 = 3(\mathbf{P}_2 - \mathbf{P}_1)$$

$$\mathbf{R}_4 = 3(\mathbf{P}_4 - \mathbf{P}_3)$$



$$\mathbf{G}_b = \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{P}_4 \end{bmatrix}$$

Bézier basis matrix

Establish the relation between Hermite and Bezier geometry vectors

$$\mathbf{G}_h = \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_4 \\ \mathbf{R}_1 \\ \mathbf{R}_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{P}_4 \end{bmatrix} = \mathbf{M}_{hb} \cdot \mathbf{G}_b$$

Bézier basis matrix

$$\mathbf{Q}(t) = \mathbf{T} \cdot \mathbf{M}_h \cdot \mathbf{G}_h = \mathbf{T} \cdot \mathbf{M}_h \cdot (\mathbf{M}_{hb} \cdot \mathbf{G}_b)$$

$$= \mathbf{T} \cdot (\mathbf{M}_h \cdot \mathbf{M}_{hb}) \cdot \mathbf{G}_b = \mathbf{T} \cdot \mathbf{M}_b \cdot \mathbf{G}_b$$

$$\mathbf{M}_b = \mathbf{M}_h \mathbf{M}_{hb} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

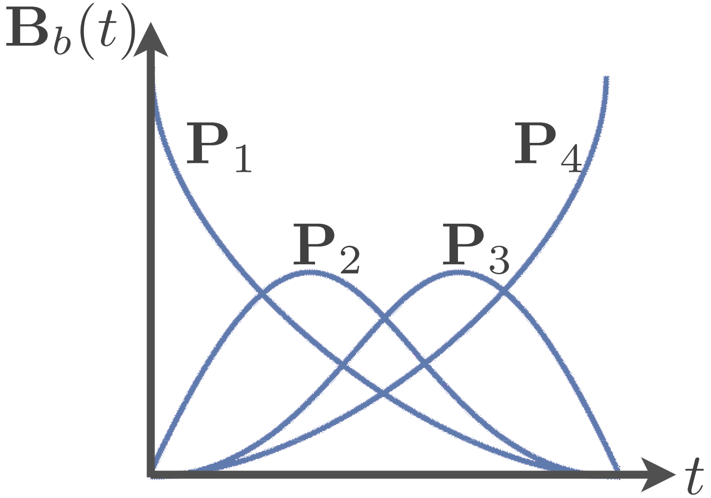
http://www.math.ucla.edu/~baker/java/hoefer/Bezier.htm

Bézier blending functions

Bezier blending functions are also called Bernstein polynomials

$$\mathbf{B}_b(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{Q}(t) = \mathbf{B}_b(t) \cdot \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{P}_4 \end{bmatrix}$$



Complex curves

What if we want to model a curve that passes through these points?

Problem with higher order polynomials:

Wiggly curves

No local control

- Process of keyframing
- Keyframe interpolation
- Hermite and Bezier curves
- Splines
- Speed control

Splines

- A piecewise polynomial that has locally very simple form, yet be globally flexible and smooth
- There are three nice properties of splines we'd like to have
  - Continuity
  - Local control
  - Interpolation

Continuity

- Cubic curves are continuous and differentiable
- We only need to worry about the derivatives at the endpoints when two curves meet

Continuity

C^0: points coincide, velocities don't

C^1: points and velocities coincide

What's C^2?

points, velocities and accelerations coincide

Local control

- We'd like to have each control point on the spline only affect some well-defined neighborhood around that point
- Bezier and Hermite curves don't have local control; moving a single control point affects the whole curve

Interpolation

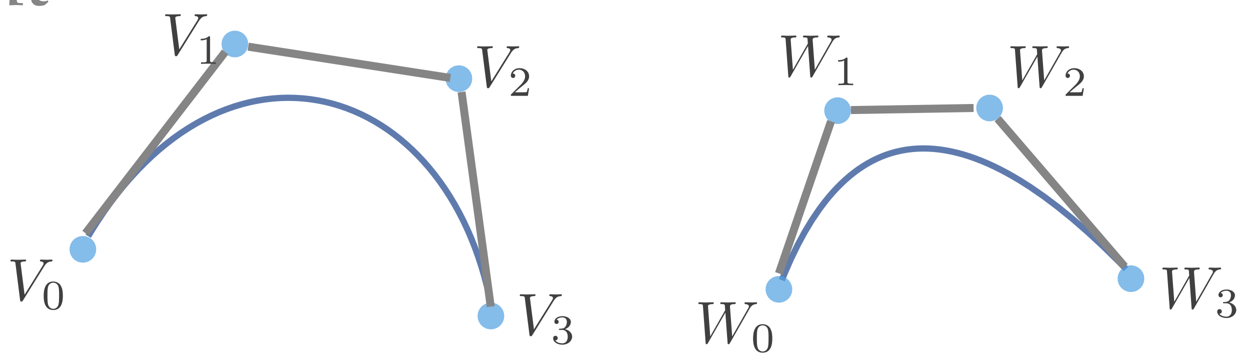
- We’d like to have a spline interpolating the control points so that the spline always passes through every control points
- Bezier curves do not necessarily pass through all the control points

B-splines

- We can join multiple Bezier curves to create B-splines
- Ensure C^2 continuity when two curves meet

Continuity in B-splines

Suppose we want to join two Bezier curves (V_0, V_1, V_2, V_3) and (W_0, W_1, W_2, W_3) so that C^2 continuity is met at the joint



$$Q_v(1) = Q_w(0)$$

$$\dot{Q}_v(1) = \dot{Q}_w(0)$$

$$\ddot{Q}_v(1) = \ddot{Q}_w(0)$$

$$V_3 = W_0$$

$$V_3 - V_2 = W_1 - W_0$$

$$V_1 - 2V_2 + V_3 = W_0 - 2W_1 + W_2$$

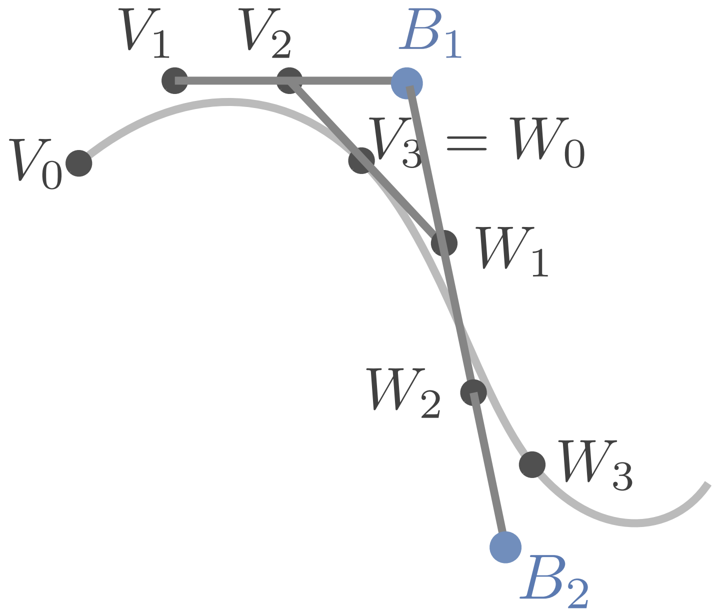
$$\boxed{W_2 = V_1 + 4V_3 - 4V_2}$$

Continuity in B-splines

What does this derived equation mean geometrically?

$$W_2 = V_1 + 4V_3 - 4V_2$$

What is the relationship between a, b and c, if a = 2b - c?

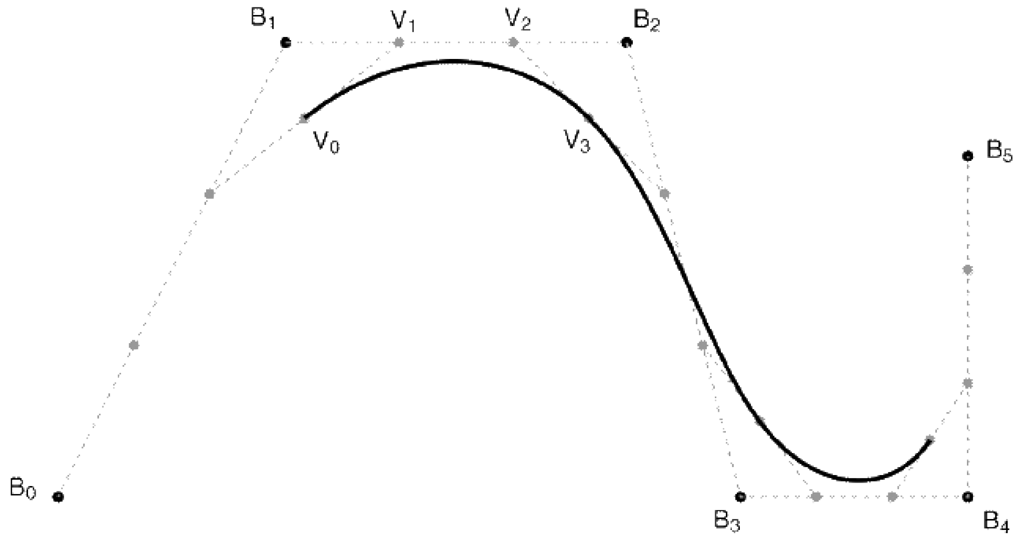


$$\begin{aligned} W_0 &= V_3 \\ W_1 &= 2V_3 - V_2 \\ W_2 &= V_1 + 4V_3 - 4V_2 \\ &= 2(2V_3 - V_2) - (2V_2 - V_1) \\ &= 2W_1 - B_1 \end{aligned}$$

What is B_2?

de Boor points

Instead of specifying the Bezier control points, let's specify the corners of the frames that form a B-spline



These points are called de Boor points and the frames are called A-frames

de Boor points

What is the relationship between Bezier control points and de Boor points?

$$V_0 = \frac{1}{2}\left(B_0 + \frac{2}{3}(B_1 - B_0) + B_1 + \frac{1}{3}(B_2 - B_1)\right)$$

$$V_1 = B_1 + \frac{1}{3}(B_2 - B_1)$$

$$V_2 = B_1 + \frac{2}{3}(B_2 - B_1)$$

$$V_3 = \frac{1}{2}\left(B_1 + \frac{2}{3}(B_2 - B_1) + B_2 + \frac{1}{3}(B_3 - B_2)\right)$$

Verify this by yourself

de Boor points

What about the next set of Bezier control points, W_0, W_1, W_2, and W_3? What de Boor points do they depend on?

B1, B2, B3 and B4.

Verify it by yourself

B-spline basis matrix

$$\mathbf{Q}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \mathbf{M}_{bs} \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \\ \mathbf{B}_4 \end{bmatrix}$$

$$\mathbf{M}_{bs} = \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

http://www.siggraph.org/education/materials/HyperGraph/modeling/splines/demoprog/curve.html

B-Spline properties

- C^2 continuity?
- Local control?
- Interpolation?

B-Spline properties

- ☑ C^2 continuity?
- ☑ Local control?
- ☐ Interpolation?

Catmull-Rom splines

- If we are willing to sacrifice C^2 continuity, we can get interpolation and local control
- If we set each derivative to be a constant multiple of the vector between the previous and the next controls, we get a Catmull-Rom spline

Catmull-Rom splines

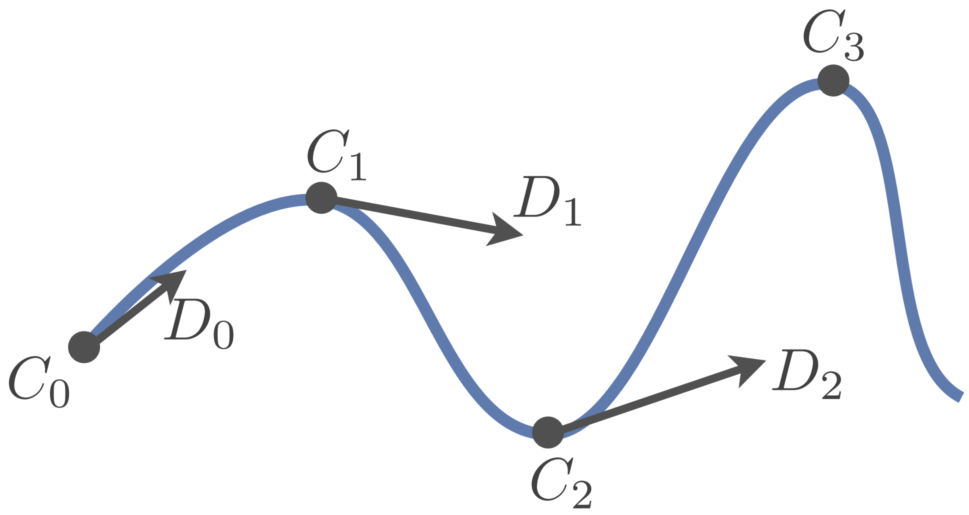
$$D_0 = C_1 - C_0$$

$$D_1 = \frac{1}{2}(C_2 - C_0)$$

$$D_2 = \frac{1}{2}(C_3 - C_1)$$

$$\vdots$$

$$D_n = C_n - C_{n-1}$$



Catmull-Rom Basis matrix

$$\mathbf{Q}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{P}_4 \end{bmatrix}$$

Derive Catmull-Rom basis matrix by yourself

Catmull-Rom properties

- C^2 continuity?
- Local control?
- Interpolation?

Catmull-Rom properties

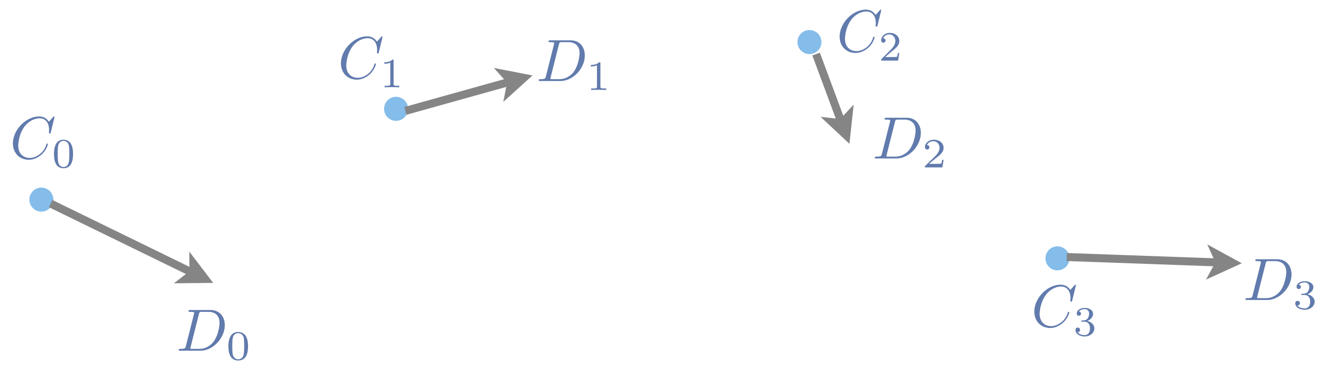
- ☐ C^2 continuity?
- ☑ Local control?
- ☑ Interpolation?

C^2 interpolating splines

- How can we keep the C^2 continuity of B-splines but get interpolation property as well?
- Suppose we have a set of Bezier control points, our goal is to find a C^2 spline that passes through all the points

C^2 interpolating splines

We know the control points C's, but we don't know the tangents D's



If we want to create a Bezier curve between each pair of these points, what are the V's and W's control points in terms of C's and D's?

Derivatives of splines

$$V_0 = C_0 \qquad W_0 = C_1$$

$$V_1 = C_0 + \frac{1}{3} D_0 \qquad W_1 = C_1 + \frac{1}{3} D_1$$

$$V_2 = C_1 - \frac{1}{3} D_1 \qquad W_2 = C_2 - \frac{1}{3} D_2$$

$$V_3 = C_1 \qquad W_3 = C_2$$

To solve for D's we apply C^2 continuity requirement

$$6(V_1 - 2V_2 + V_3) = 6(W_0 - 2W_1 + W_2)$$

$$\Downarrow$$

$$D_0 + 4D_1 + D_2 = 3(C_2 - C_0)$$

Derivatives of splines

$$D_0 + 4D_1 + D_2 = 3(C_2 - C_0)$$

$$D_1 + 4D_2 + D_3 = 3(C_3 - C_1)$$

$$\vdots$$

$$D_{m-2} + 4D_{m-1} + D_m = 3(C_m - C_{m-2})$$

How many equations do we have?

How many variables are we trying to solve?

Boundary conditions

We can impose more conditions on the spline to solve the two extra degrees of freedom

Natural C^2 interpolating splines require second derivative to be zero at the endpoints

$$6(V_0 - 2V_1 + V_2) = 0$$

Linear system

Collect m+1 equations into a linear system

$$\begin{bmatrix} 2 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & & \ddots & & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 2 \end{bmatrix} \begin{bmatrix} D_0 \\ D_1 \\ D_2 \\ \vdots \\ D_{m-1} \\ D_m \end{bmatrix} = \begin{bmatrix} 3(C_1 - C_0) \\ 3(C_2 - C_0) \\ 3(C_3 - C_1) \\ \vdots \\ 3(C_m - C_{m-2}) \\ 3(C_m - C_{m-1}) \end{bmatrix}$$

Use forward elimination to zero out every thing below the diagonal, then back substitute to compute D's

Choice of splines

Spline types	Continuity	Interpolation	Local control
B-Splines	C^2	No	Yes
Catmull-Rom Splines	C^1	Yes	Yes
C^2 interpolating splines	C^2	Yes	No

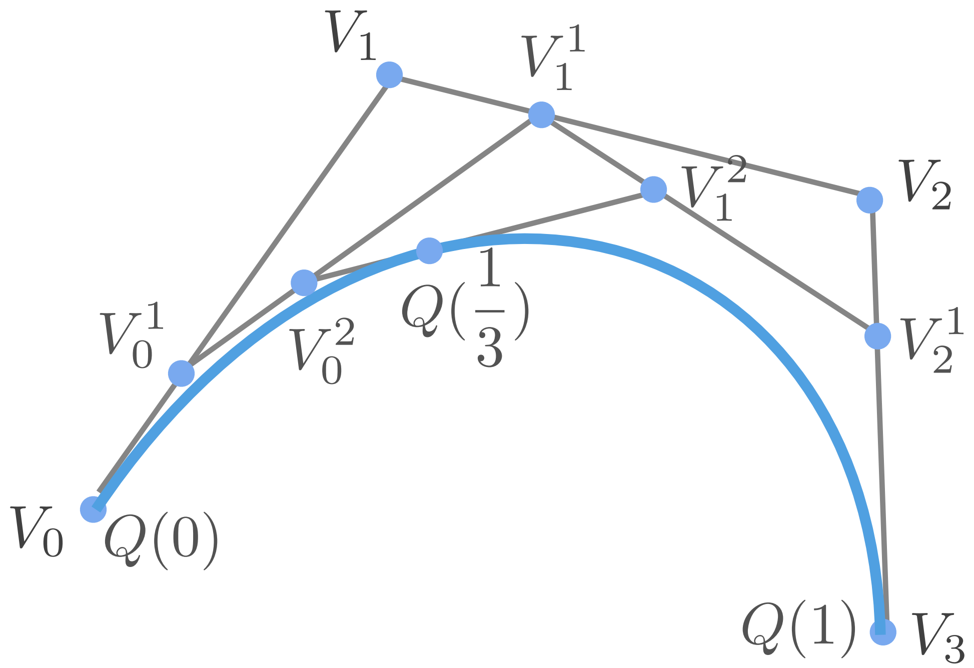
de Casteljau's algorithm

For each sample of t from 0 to 1, use de Casteljau's algorithm to compute $Q(t)$

Where is $Q(0)$?

Where is $Q(1)$?

Where is $Q(\frac{1}{3})$?



What is the equation for V_0^1 ?

de Casteljau's algorithm

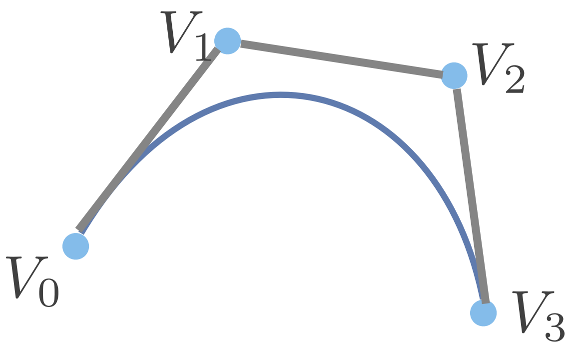
$$V_0^1 = (1-t)V_0 + tV_1$$
$$V_1^1 = (1-t)V_1 + tV_2$$
$$V_2^1 = (1-t)V_2 + tV_3$$

$$V_0^2 = (1-t)V_0^1 + tV_1^1$$
$$V_1^2 = (1-t)V_1^1 + tV_2^1$$

$$\begin{aligned} Q(t) &= (1-t)V_0^2 + tV_1^2 \\ &= (1-t)[(1-t)V_0^1 + tV_1^1] + t[(1-t)V_1^1 + tV_2^1] \\ &= (1-t)^3 V_0 + 3t(1-t)^2 V_1 + 3t^2(1-t)V_2 + t^3 V_3 \\ &= \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} V_i \end{aligned}$$

Displaying Bezier curves

```
DisplayBezier(V0, V1, V2, V3)
   begin
      if (FlatEnough(V0, V1, V2, V3))
            Line(V0, V3)
      else
            do something
```

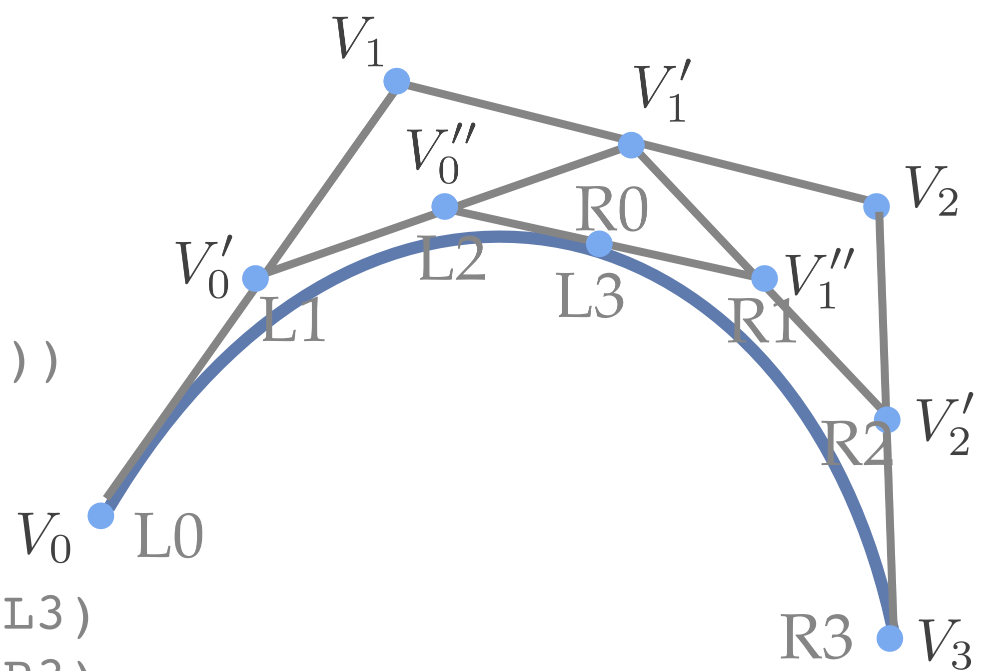


It would be nice if we had an adaptive algorithm that would take into account flatness

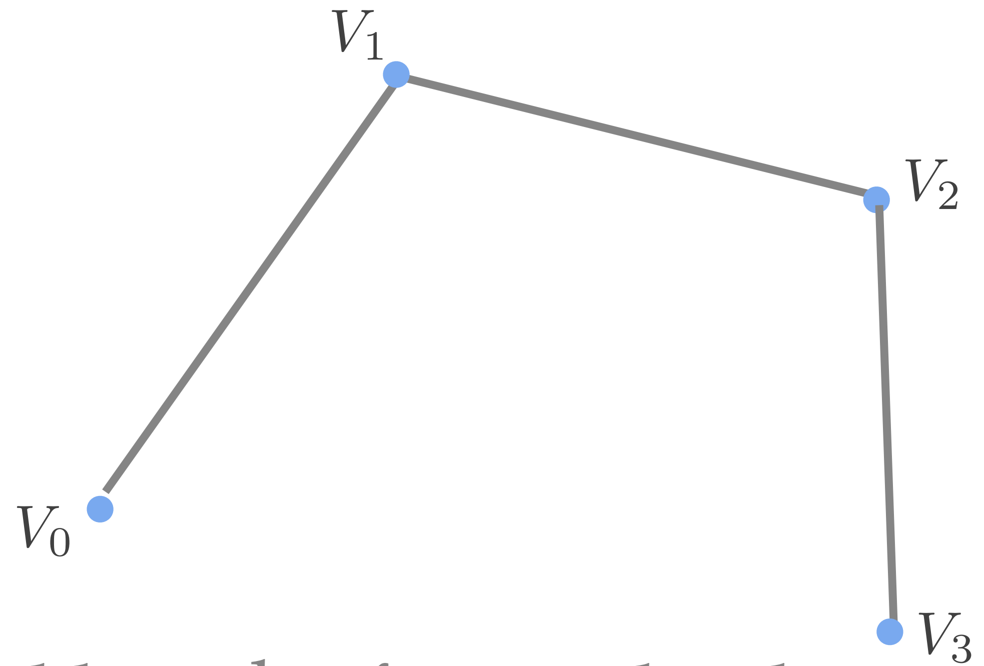
Subdivide and Conquer

```
DisplayBezier(V0, V1, V2, V3)
begin

   if (FlatEnough(V0, V1, V2, V3))
      Line(V0, V3)
   else
      Subdivide(V) -> L, R
      DisplayBezier(L0, L1, L2, L3)
      DisplayBezier(R0, R1, R2, R3)
```



Flatness Test



Compare total length of control polygon to length of line connecting endpoints:

$$\frac{|V_0 - V_1| + |V_1 - V_2| + |V_2 - V_3|}{|V_0 - V_3|} < 1 + \epsilon$$

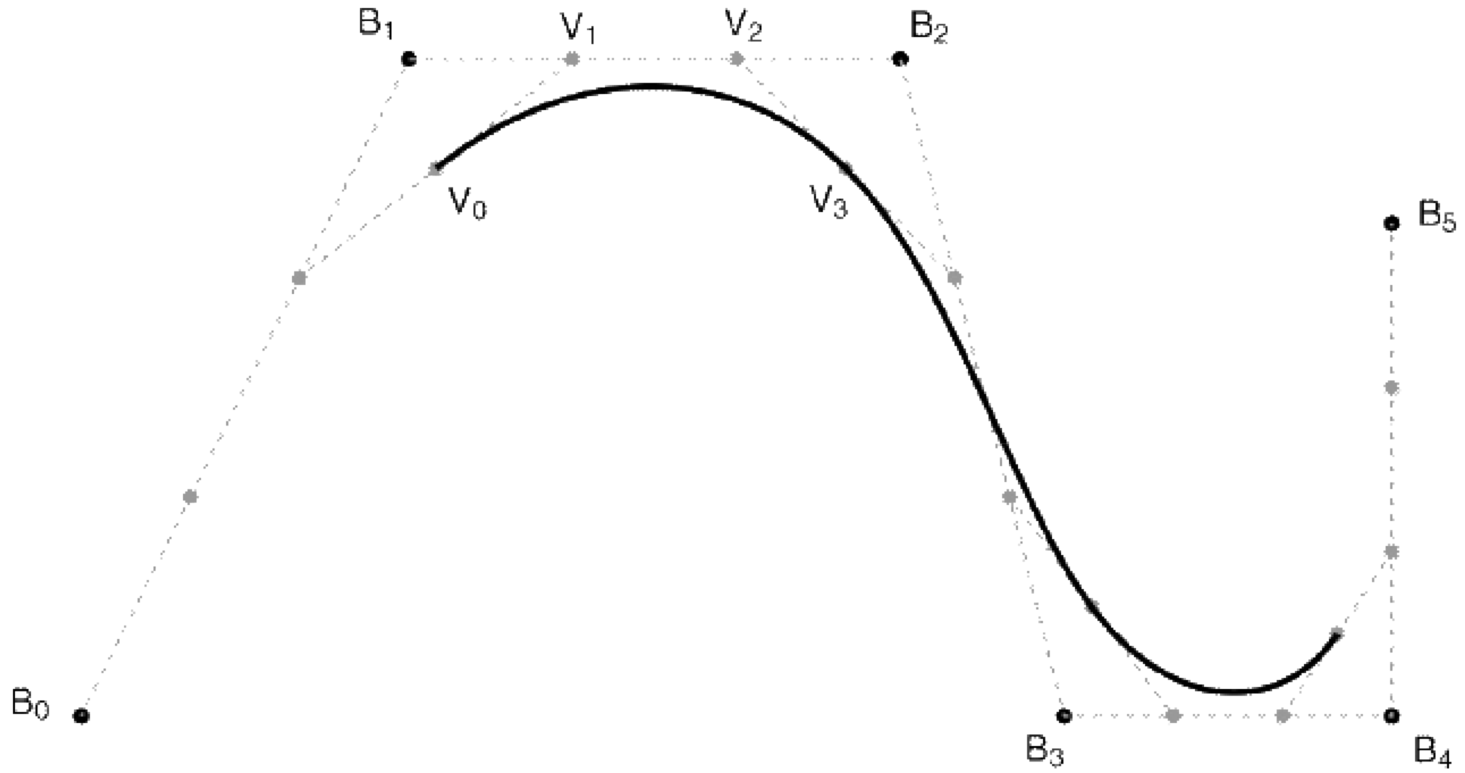
Endpoints of B-splines

- We can see that B-splines don't interpolate the de Boor points
- It would be nice if we could at least control the endpoints of the splines explicitly
- There's a trick to make the spline begin and end at the de Boor points by repeating them

Endpoints of B-splines

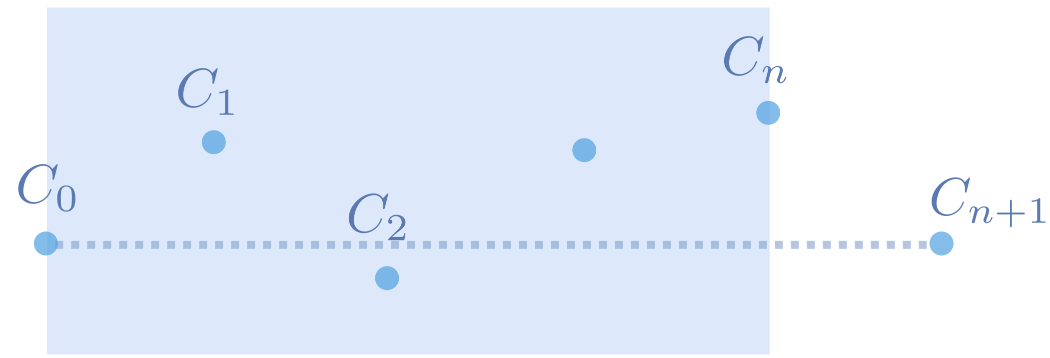
How many $\mathbf{B}_0$'s need to be repeated?

3 times. See slide 39.



Wrapping the curves

- Wrapping is an important feature that makes the animation restart smoothly when looping back to the beginning
- Create "phantom" control points before and after the first and the last control points



- Process of keyframing
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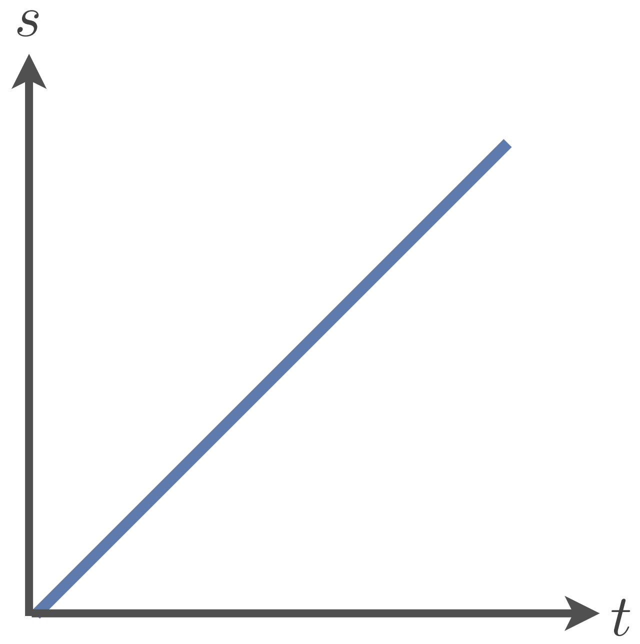
- Hermite and Bezier curves
- Splines
- Speed control

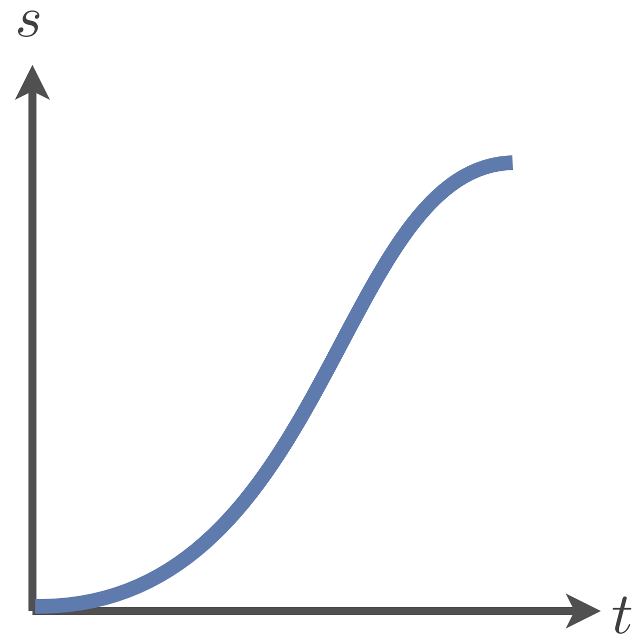
Speed control

- Simplest form is to have constant velocity along the path



Ease-in Ease-out curve

- Assume that the motion slows down at the beginning and end of the motion curve



Issues

- What kind of bad things can occur from interpolation? How do we prevent them?
  - Invalid configurations (pass through walls)
  - Unnatural motions (painful twists)

What’s next?

- What about rotation?
- Can we interpolate rotations using the these same techniques?